Nov 2003

Special Theory of Relativity

For an excellent reference text book, see `Introduction to special relativity, by Robert Resnick, John Wiley & sons`

Failure of Galilean Transformation

Classical (Newtonian) view:

- Space and time are absolute and not interfering each other
- Definition: Inertial frames are reference

frames moving at constant velocity with respect to some rest frame of reference, $\Rightarrow$ All inertial frames are equivalent

- According to the principle of Newtonian relativity (described by Galilean transformation), the laws of mechanics are the same in all inertial frames of reference

http://es.rice.edu/ES/humsoc/Galileo/index.html

- An event occurring in an inertial system, O, is specified by a set of coordinate $(x, y, z; t)$.

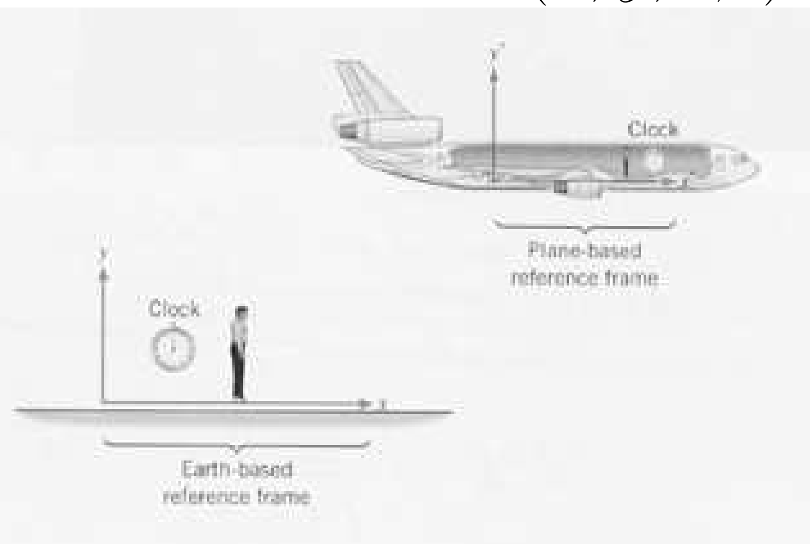
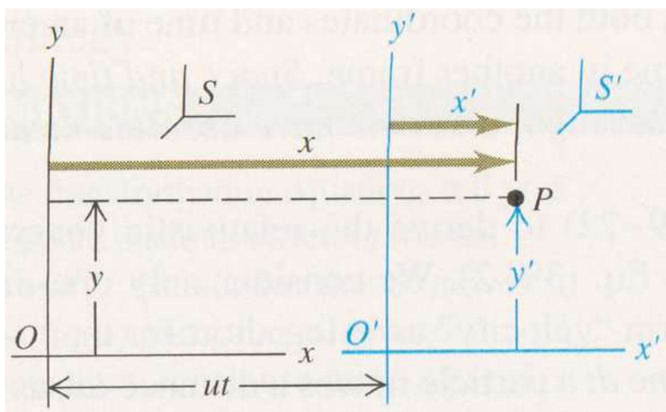


Figure 28.1 Using an earth-based reference frame, an observer standing on the earth records the location and time of an event (the space shuttle lift-off). Likewise, an observer in the airplane uses a plane-based reference frame to describe the event.



- Galilean transformation relates the coordinates from one initial frame, O, to another, O' which is moving with a relative velocity u:

$$x' = x - ut, \;\; y' = y, z' = z, t' = t \qquad (1)$$

- Galilean addition law for velocity (in 1-D):

$$v'_x = v_x - u \tag{2}$$

where O' moves with a relative constant speed of u wrp to O

- Differentiating Eq.(2),
  $F = m\frac{dv}{dt} = ma$ (in frame O) appears as $F' = m\frac{dv'}{dt} = ma'$ (in frame O'),
  $\Rightarrow$ Newton's law is the same in all inertial frames
- Note:
  - inertial frame means $\frac{du}{dt} = 0$
  - Newtonian view $\Rightarrow t' = t, m$ is the same in both frames
  - consistent with the absolute notion on space and time, the (temporal) interval between two events seen in two different inertial frames, O and O', is the same ($\Delta t = \Delta t'$)
- Nevertheless, Galilean transformation is only

valid if the scale of velocities (v) involved is much smaller than the speed of light ($c = 30 \times 10^6$m/s). It shall breakdown when the speed scale v approach c, and shall lead to logical contradiction.

- When Galilean transformation is applied to the propagation of light on free space, as in accordance to the Newtonian view of absolute space and time, some inconsistency arises (see the following section)

Revision: What are the Newton's law?

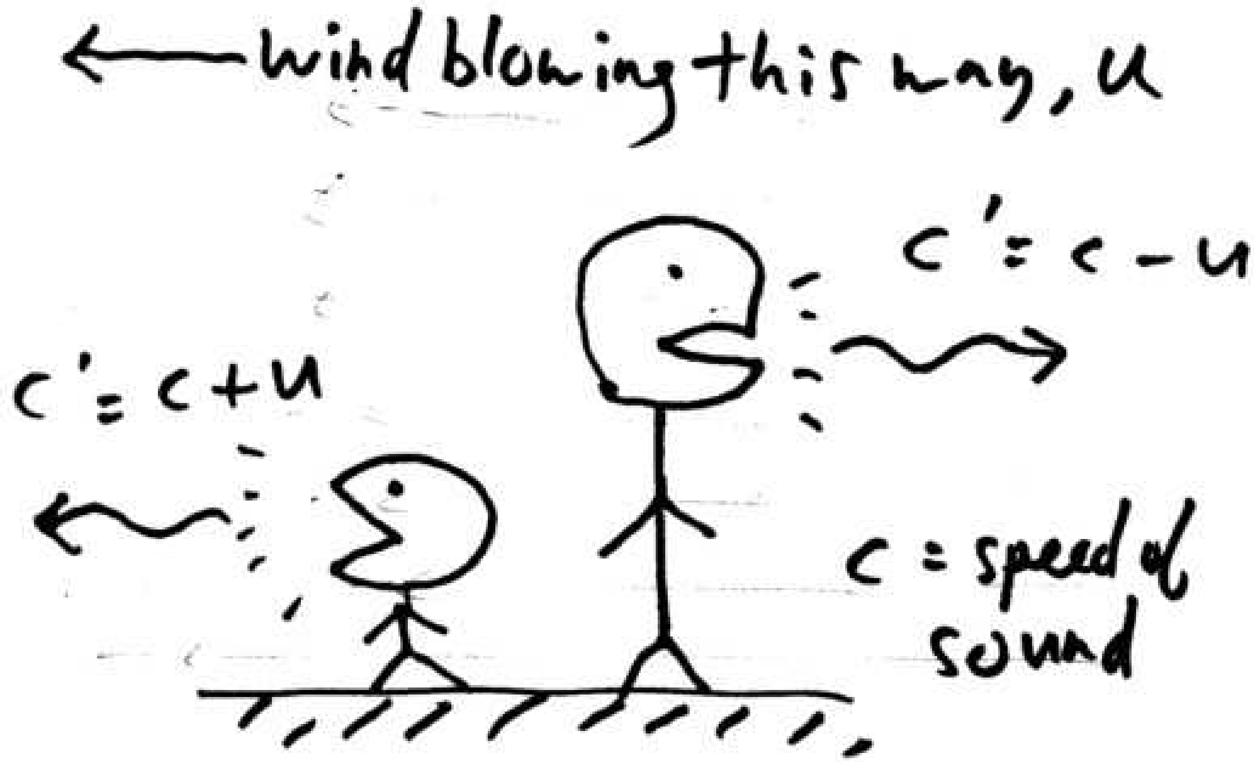
- An object at rest will always be in the wrong place
- An object in motion will always be headed in the wrong direction
- For every action, there is an equal and opposite criticism

Question: Can you discriminate a physical joke from a physical statement?

Ether and Michelson-Morley experiment

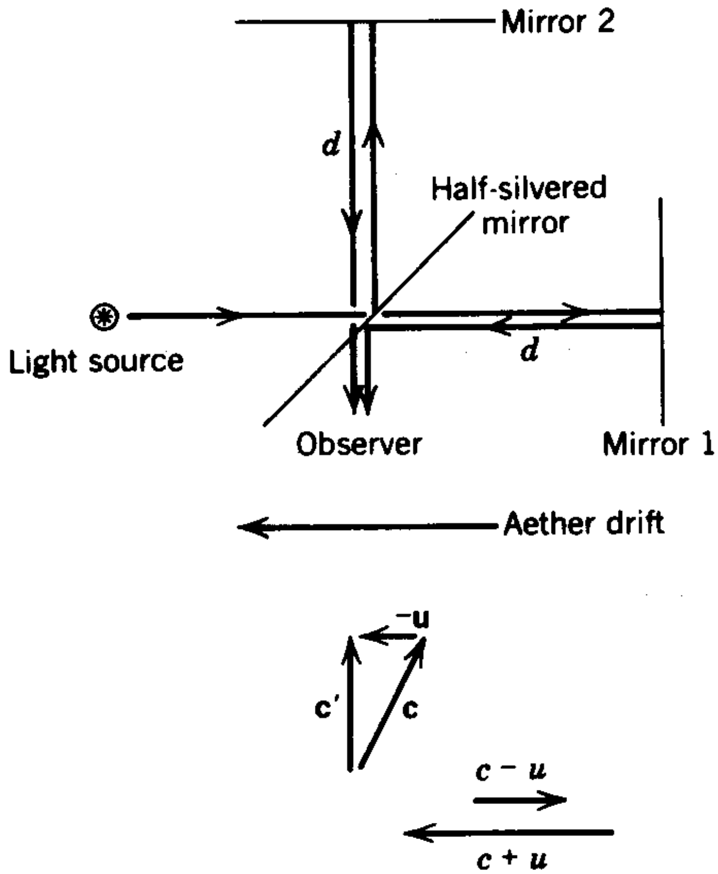
- In early 19th century, electromagnetic waves (light) as described by the Maxwell, is thought to propagate at a speed $c = 10^{-8}$ m/s with respect to a special medium that permeates the whole Universe called **Ether**
- Direct analogy to the propagation of mechanical wave (e.g. sound) through an elastic medium (air) at speed of $\tilde{3}30$ m/s
- The Ether defines a privileged inertial frames in which all motion in the Universe (including light and the motion of Earth in space) refers to it as an absolute frame of reference
- The Ether frame goes in accordance with Newton's notion of the absolute space which is independent of the motions and events that happen in it. The Ether frame is also though to exist independently from the flow of time

- In this classical view, Earth is thought to drift through the Ether background at some constant relative speed u
  Analogous to talking in the wind. Ether : medium air;
  light : sound propagating in the air;
  $u =$ speed of the wind wrp to you
- The speed u can be measured by observing the difference in the speed of light when Earth is moving in different direction in the Ether frame - Ether wind (in analogy, speed of **sound** measured by "talking" in different directions in the wind can reveal the speed of the **wind**)



- The difference in the speed of light as seen on Earth can be revealed in a Michelson-Morley interferometer experiment which can detect any change of interference pattern

Michelson interferometer with arms of equal length. For the Michelson–Morley experiment, the aether is presumed to be moving through the interferometer with speed u. The aether hypothesis implies that the light speeds differ over the two paths, as shown.



- A monochromatic beam of light is split in two; the two beams are made to travel different paths and are then recombined. Due to the differing path lengths an interference pattern will be produced

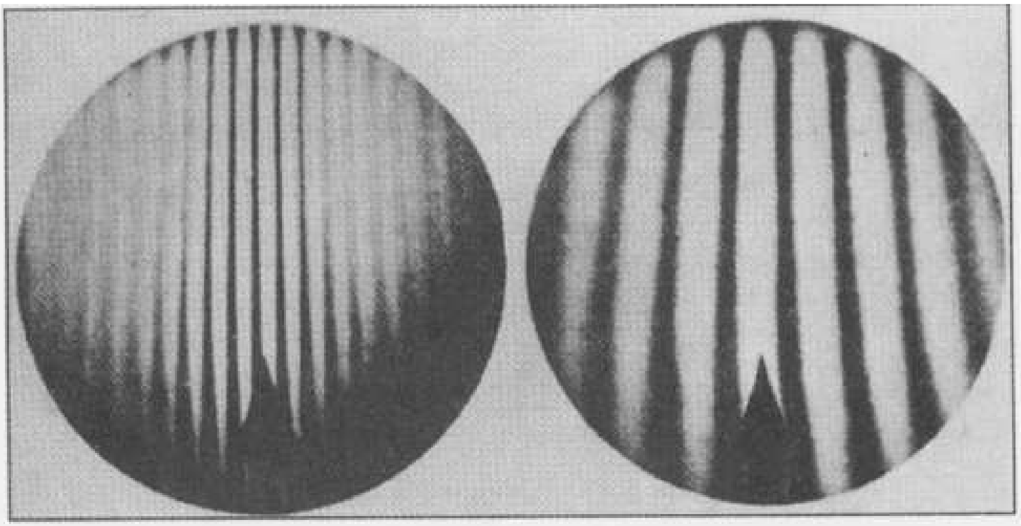


FIGURE 2.3 Interference fringes as they would appear in the eyepiece of the Michelson-Morley experiment. *From L. S. Swenson, Jr., Invention and Discovery, 43 (Fall, 1987).*

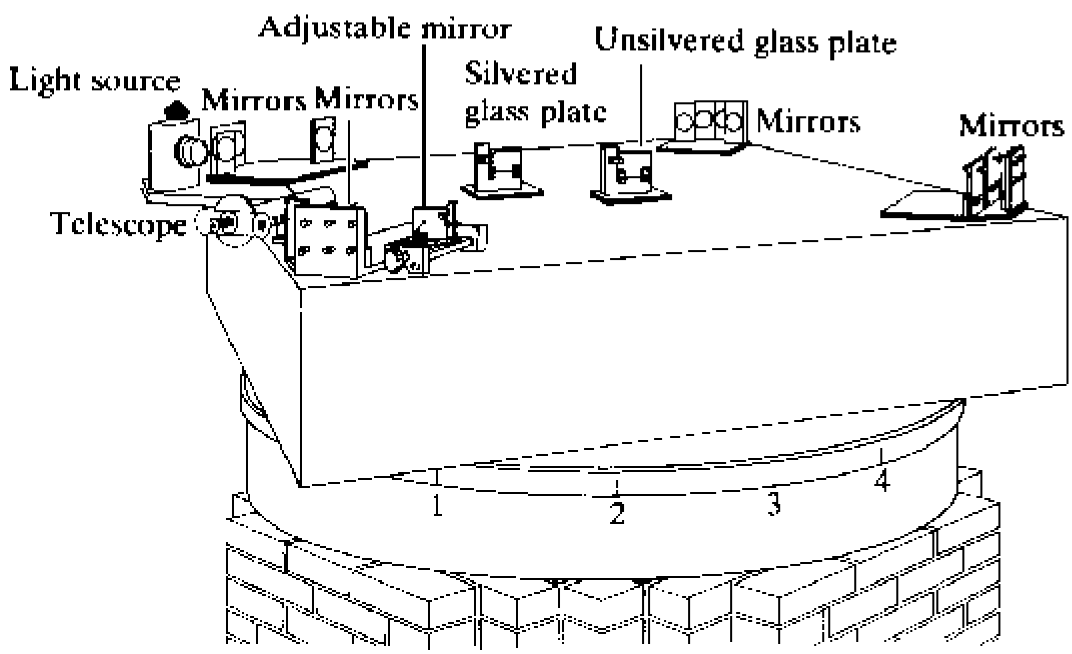


Figure 9.67 The Michelson–Morley experiment.

- In accordance with Galilean transformation, the velocity of light as seen from the Earth (or interferometer) is $c' = c - u$.

- Time taken for light to complete route 1 (parallel to ether drift) is

$$t_1 = \frac{d}{c-u} + \frac{d}{c+u} = \frac{2dc}{c^2-u^2}$$

- For light (perpendicular to ether drift), the speed of beam relative to Earth is $\sqrt{c^2 - u^2}$, then the time taken to complete route 2 is

$$t_2 = 2d/\sqrt{c^2 - u^2}$$

- The difference in the transit time $\Delta t = |t_1 - t_2| \approx \frac{du^2}{c^3}$ causes the beams to arrive at the observer out of phase, thus causing interference pattern.

- A difference in the pattern should be detected by rotating the apparatus through 90° as the two beams exchange role

- When rotating through 90°, the net time difference between the two beams is $2\Delta t$, corresponding to a optical path difference of $\delta p = c(2\Delta t)$.

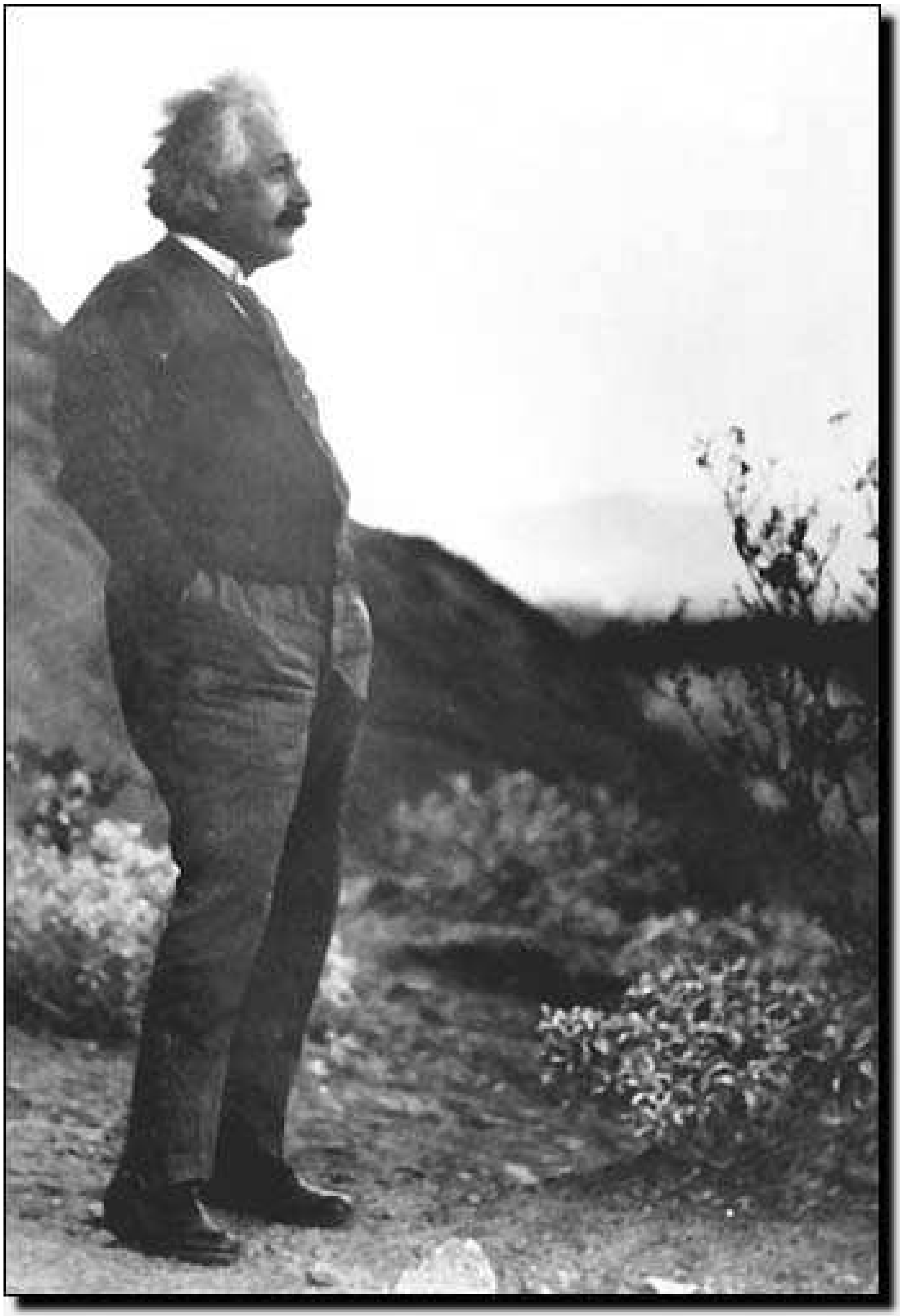
- the number of wavelengths contained in δp is $\delta n = \frac{\delta p}{\lambda} = 2\frac{d}{\lambda}(\frac{u}{c})^2$ = number of fringe shift of the interference pattern upon the 90° rotation
- $\frac{u}{c} \approx 10^{-4}$ assuming that u is of the order of speed of Earth moving around the Sun
- the experiment expects to detect a change in δn as small as 0.4. But Michelson and Morley observed only a **Null result**:
  $\Rightarrow$ the speed of light is the same for the two perpendicular paths

$\Rightarrow$ concept of the absolute frame of reference is inconsistent with the MM experiment

- From a theoretical viewpoint considered quite independently from the MM experiments, in the inertial reference frame c (which is considered as no different from any other inertial frame), the oscillation of the EM wave fronts would appear 'frozen' and the wave motion not detectable, hence Maxwell equations would be no more valid in this inertial frame

$\Rightarrow$ Newtonian view (that all inertial frame of references are equivalent and the law of mechanics are the same in all inertial frame) contradicts with the EM theory described by Maxwell's equations.

Einstein in 1905 proposed a consistent picture of mechanics and electromagetism – the special theory of relativity. In this theory, the Newtonian view of absolute space and time would have to be modified, and the Galilean transformation be revised



http://www.aip.org/history/einstein/

Principle of special relativity

1. The laws of EM are valid in all frame of reference in which the laws of mechanics hold (or alternatively, all laws of physics are the in all reference frames - this is a generalisation of Newtonian relativity)

2. The speed of light in vacuum is the same for all observers independent of their motion of the source

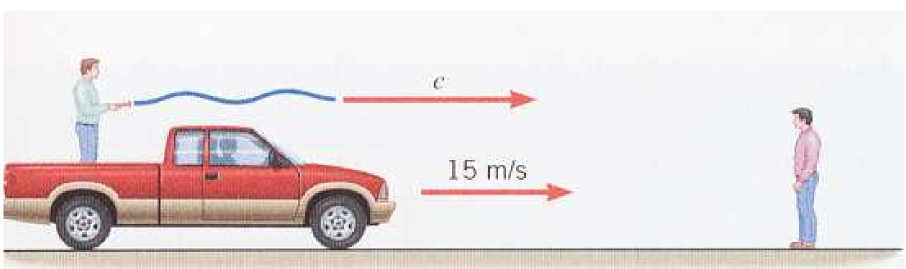


Figure 28.2 Both the person on the truck and the observer on the earth measure the speed of the light to be c, regardless of the speed of the truck.

Note:

- The premise of MM experiment is wrong according to Postulate (2), the null result of MM experiment is naturally expected in SR

- Postulate (1) infers that there are *no* experiment by which we can measure velocity with respect to absolute space – all we can measure is the *relative* speed of two inertial system. The notion of absolute reference frame is discarded (because we can in no way measure it's presence).

- In addition, in SR, simultaneity is a relative concept (not an absolute concept anymore), i.e two events that simultaneous in one reference frame are in general no simultaneous in a second frame moving with respect to the first.

Consequences of Einstein's postulate

- **time dilation**
  Consider the Gadenken (thought) experiment: O' is in the train (moving frame), O is on the ground (a stationary frame).

- A light pulse is sent from the source and received at the same point as the source after reflected by the mirror placed at a distance d away

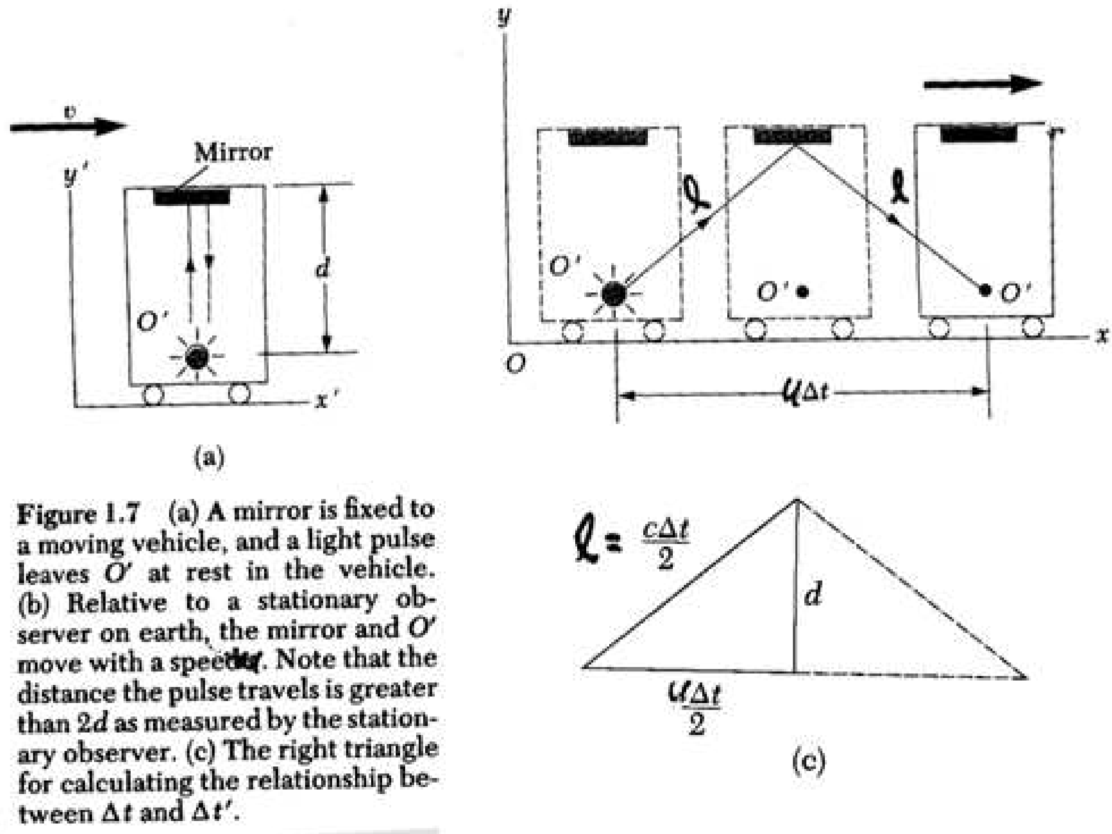


Figure 1.7 (a) A mirror is fixed to a moving vehicle, and a light pulse leaves O' at rest in the vehicle. (b) Relative to a stationary observer on earth, the mirror and O' move with a speed u. Note that the distance the pulse travels is greater than $2d$ as measured by the stationary observer. (c) The right triangle for calculating the relationship between Δt and $\Delta t'$.

- for observer in O', the time interval between the emission and detection of light pulse is

$$\Delta t' = 2d/c \tag{3}$$

- for observer in O, the locus of the light pulse appears not vertically, but slightly slanted due to the motion of the source travelling at a relative speed u away from O. The time interval between the emission and detection of light pulse is

$$\Delta t = 2\ell/c \tag{4}$$

where

$$\ell^2 = d^2 + (\frac{u\Delta t}{2})^2 \tag{5}$$

Eliminating ℓ and d from Eqs.(3,4,5), we find the relation between the time intervals as measured by observer in O and O':

$$\Delta t = \Delta t'\gamma; \;\; \gamma = \frac{1}{\sqrt{1 - u^2/c^2}} \tag{6}$$

$\Rightarrow$ **TIME DILATION**

γ (always ≥ 1)is called the Lorentz factor

- $\Delta t'$ is called **proper time**. It measures the time interval of the two events at the same point in space

- The elapsed time Δt between the same events in any other frame is *dilated* by a factor of γ compared to the proper time interval $\Delta t'$

- Or in other words, according to a stationary observer, a moving clock runs slower than an identical stationary clock

Example:
A spacecraft is moving past the Earth at a constant speed $0.92c$. The astronaut measures the time interval between succsessive "ticks" of the spacecraft clock to be $\Delta t_0 = 0.1$ s. What is the time interval Δt that an Earth observer measures between "ticks" of the astronaut's clock?

ANS:
Since the clock on the spacecraft is moving relative to the Earth observer, the Earth observer measures a greater time interval Δt "ticks" than does the astronaut, who is at rest relative to the clock. The dilated time interval Δt can be determined from the time dilation equation as per

$$\Delta t = \gamma \Delta t_0 = \frac{1.0 \text{ s}}{\sqrt{1 - (\frac{0.92c}{c})^2}} = 2.6 \text{ s}$$

From the point of view of the Earth-based observer, the astronaut is using a clock that is running slowly, because the Earth-based observer measures a time between "ticks" that is longer (2.6 s) that what the astronaut measures (1.0 s)

- **Example of muon decay lifetime**: *a muon is an unstable elementary particle which has a 2 μs average lifetime (as measured in the muon rest frame) and decays into lighter particles. Fast muons are created in the interactions of very high-energy particles as they enter the Earth's upper atmosphere. Observation has verified the relativistic effect.*

  A muon travelling at 99.99 % the speed of light has a Lorentz factor $\gamma = 22$. Hence, to an observer in the rest frame (e.g Earth) the lifetime of the muon is no more 2μs but $\gamma \times 2\mu$s = 44 μs. Thus the muon would appear to travel for 44μs before it decays. The distance it traversed as seen from Earth is $D' = (0.9999c) \times 44\mu\text{s} = 13$ km. If relativistic effect of time dilation not taken into account, it would appear to travel only for a distance $0.9999c \times 2\mu\text{s} = 0.6$ km

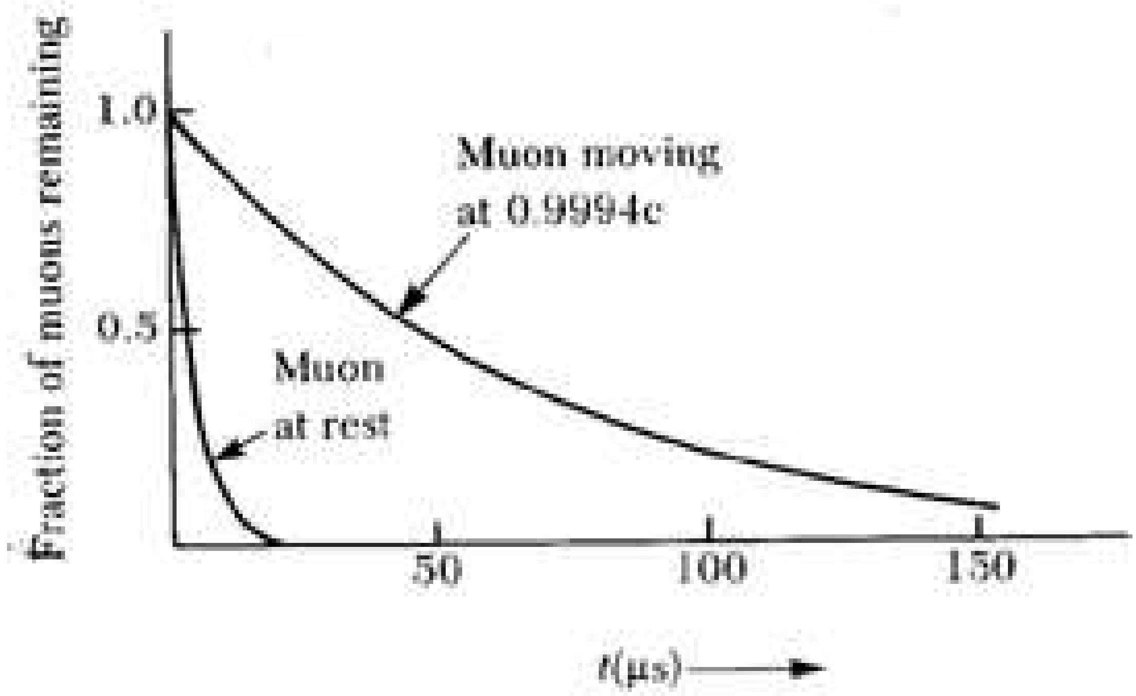


Figure 1.9 Decay curves for muons traveling at a speed of $0.9994c$ and for muons at rest.

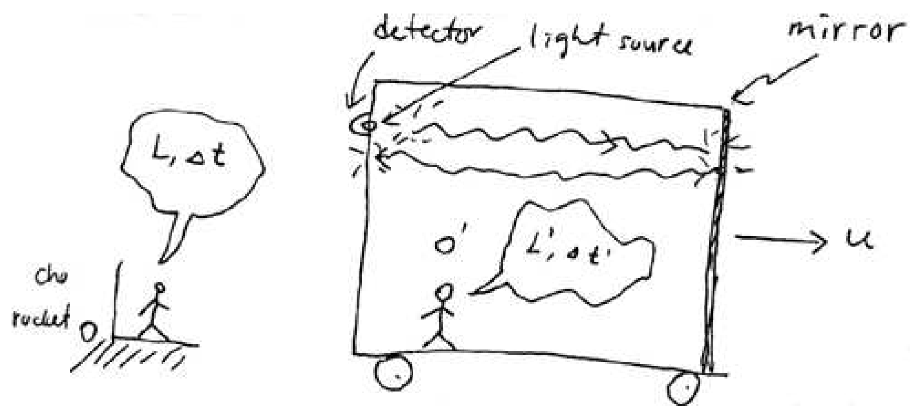
Twin Paradox

Two identical twin, E and S, undergo a controlled experiment. S sets out on a journey towards a star many light-years away from Earth in a spaceship that can be accelerated to near the speed of light. After reaching the star, S immediately returns to Earth at the same high speed. Upon arrival on Earth, would S sees his twin brother E

(a) becomes much older
(b) becomes much younger
(c) is still the same age as himself?

Length contraction

- Consider a box of is moving (with a speed u wrp to a rest frame O) in the direction along a light source (contained within the box) that shines a light pulse from one wall to the other. The light pulse is then mirrored back to the source.



Relativity-of-length box at rest in O' and is moving wrp to O with speed u.

- observer inside the frame, O', (that move together with the box) sees the length of the box as L', and the interval for the light pulse to complete the emission-reflection -receipt process (length $2L'$) is $\Delta t'$ ($\Delta t'$ is the proper time as the events occur at the same point in space)). L' is called the proper length because the length of the box is at rest when seen in this frame (i.e. O').

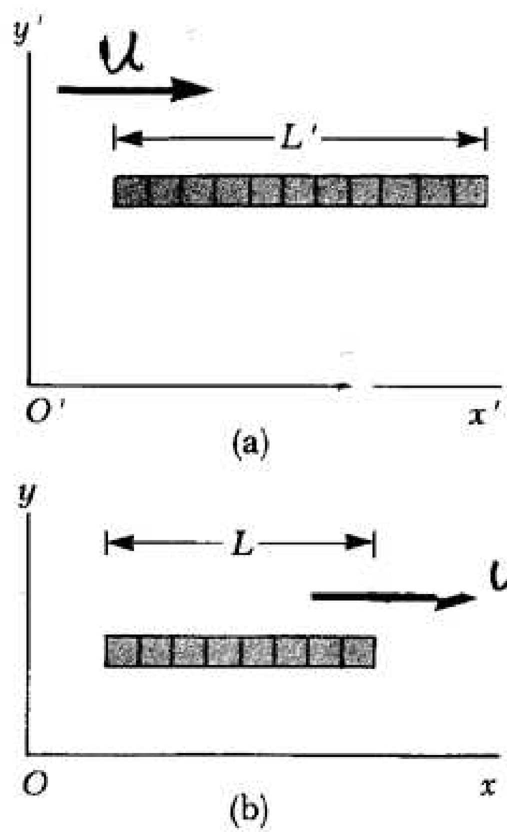
- observer in the stationary frame O (on the ground) sees the length of the moving box as L, and the interval for the light pulse to complete the emission-reflection -receipt process (lenght $2L$) is Δt.
- now apply constancy of light speed in both frames:

$$\begin{aligned} \text{in O: } , 2L &= c\Delta t; \\ \text{in O': } 2L' &= c\Delta t'; \end{aligned} \tag{7}$$

- Due to time dilation effect, $\Delta t = \gamma \Delta t'$
- Combining Eq.(7) and the time dilation effect, we arrive at

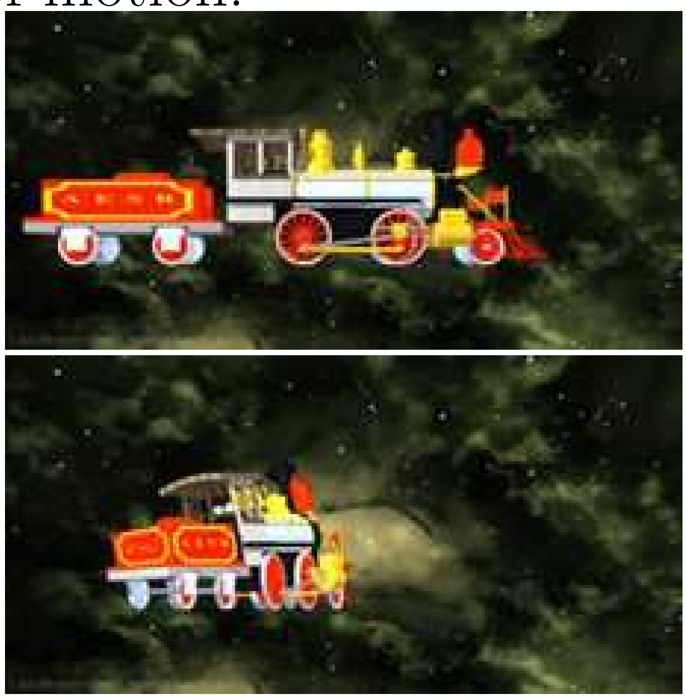
$$L = L'/\gamma \tag{8}$$

- **If an observer at rest wrp to an object measures its length to be L', an observer moving with a relative speed u wrp to the object will find it to be shorter than its rest length by the fact**



In (a), the stick is viewed by a frame travels along with the stick, (b) The stick is seen from a frame at rest relative to the stick. The length measured in the rest frame (i.e. frame (b)) is *shorter* than the proper length, L' by a factor $1/\gamma$

- length contraction takes place only along the direction of motion.



The first image in each pair is the standard non-relativistic photograph. The second image is the photograph taken from a camera moving at 95% the speed of light relative to the object.
http://www.maths.monash.edu.au/ leo/relativity/sr
photography/sr-photo.shtml

Example:

An observer on Earth sees a spaceship at an altitude of 435 m moving downward toward the Earth with a speed of $0.97c$. What is the altitude of the spaceship as measured by an observer in the spaceship?

ANS:

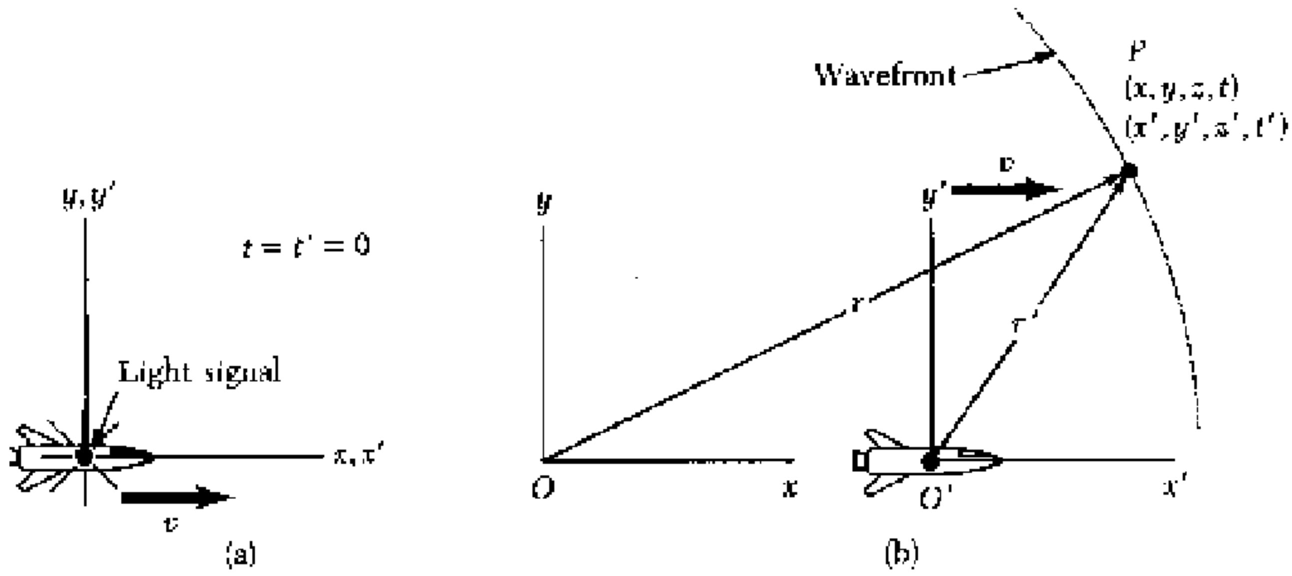
One can consider the altitude see by the stationary (Earth) observer as the proper length (L'). The observer in the spaceship should sees a contracted length, L, as compared to the proper length. Hence the moving observer in the ship finds the altitude to be

$$L = L'/\gamma = 435\text{m} \times \sqrt{1 - \frac{(0.97c)^2}{c^2}} = 106\text{m}$$

Lorentz Transformation

- Galilean transformation is not valid when $u \to c$. The relativistic version of the transformation law of the kinematic quantities is given by Lorentz transformation.

- Consider a rocket moving with a speed u (O' frame) along the xx' direction wrp to the stationary O frame. A light pulse is emitted at the instant $t' = t = 0$ when the two origins of the two reference frames coincide.
  O' frame uses (x', y', z', t') to denote the coordinate system, whereas O frame uses (x, y, z, t). How to related (x', y', z', t') with (x, y, z, t)? Lorentz transformation will do that.



- the light signal travels as a spherical wave at a constant speed c in both frames.

- After some times t, the origin of the wave, centered at O has a radius $r = ct$, where $r^2 = x^2 + y^2 + z^2$

- likewise, from the view point of O', After some times t', the origin of the wave, centered at O' has a radius $r' = ct'$, where $r'^2 = (x')^2 + (y')^2 + (z')^2$

- $y' = y, z' = z$ because the motion of O' is along the xx' axis. Taking the difference $(r' = ct')^2$ from $(r = ct)^2$ and solving for $\{x', t'\}$ in terms of the unprimed quantities i.e. $\{x, t\}$,

$$x' = \frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma(x - ut)$$

$$t' = \frac{t - (u/c^2)x}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma(t - (u/c^2)x) \qquad (9)$$

Note that, now, the space and time are mixed

- Lorentz transformation reduces to Galilean transformation when $u \ll c$.

Lorentz velocity transformation

How to relate the velocity in the O' frame $u'_x = \frac{du'_x}{dt'}$ with that of the O frame $u_x = \frac{du_x}{dt}$?

- From Eq.(9)

$$dx' = \gamma(dx - udt), \;\; dt' = \gamma(dt - \frac{u}{c^2}dx)$$

Combining, $u'_x = \frac{du'_x}{dt'} = \frac{u_x - u}{1 - \frac{u_x u}{c^2}}$,

where we have made used of the relation $u_x = \frac{du_x}{dt}$.

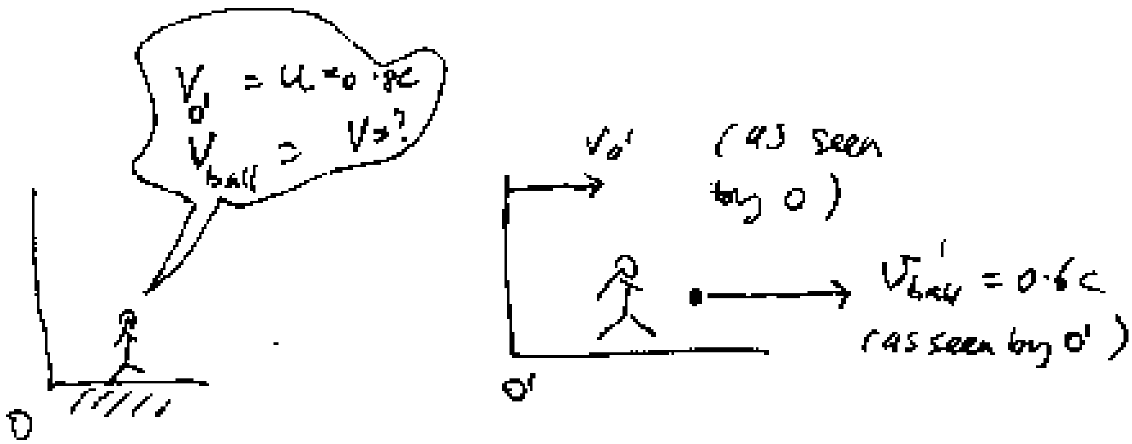
- To obtain u_x in terms of u'_x, we replace u by $-u$ to obtain $u_x = \frac{u'_x + u}{1 + \frac{u'_x u}{c^2}}$

- Note two limits:
  - as $u \ll c$, Galilean transformation $u'_x = u_x - u$ is recovered.

– when $u_x = c$, $u'_x = c \Rightarrow$ an object moving with a speed c relative to an observer in O also has a speed c relative to an observer in O' – *independent* of the relative motion of O and O', consistent with Einstein constancy of speed of light postulate

Exercise: recover time dilation and length contraction from Lorentz transformation

- **Example:**
  Imagine a driver in a car is moving with a speed of $0.8c$ past an stationary observer. If the driver throws an apple in the forward direction with a speed of $0.6c$ wrp to himself, what is the speed of the ball as seen by the stationary observer?

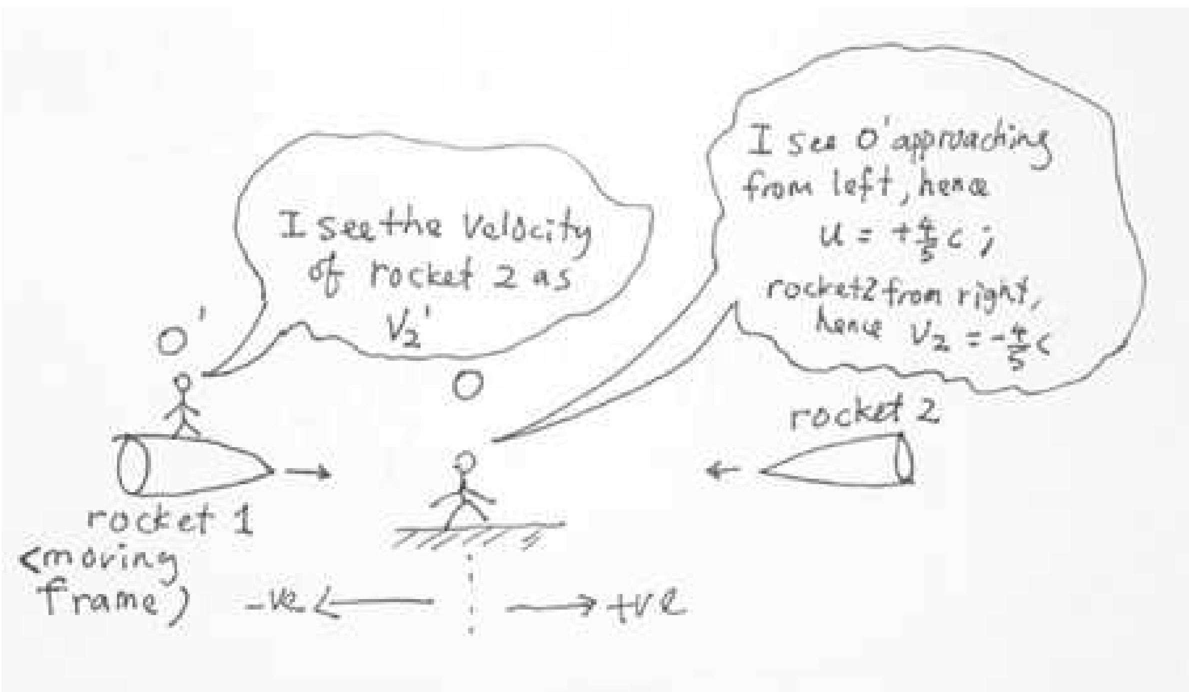


ANS:

Referring to $v' = v - u/(1 - uv/c^2)$, v' $(= 0.6c)$ is the velocity of the ball as seen by the moving frame (driver), u $(= 0.8c)$ is the relative speed of the moving frame as seen by the stationary observer. Plug u, v' into the Lorentz transformation for the velocities, we get $v = 0.946c$ as the speed of the ball as seen from the stationary observer point of view.

Example:

Rocket 1 is approaching rocket 2 on a head-on collision course. Each is moving at velocity $4c/5$ relative to an independent observer midway between the two. With what velocity does rocket 2 approaches rocket 1? Let the rockets have proper length of 20 m. What is the length of the one of the rocket as seen from the other?



ANS

Take rocket 1 as the moving frame O'. The velocity of rocket 2 is the quantity we are interested in. Frame O' is moving in the +ve direction as seen from O, so $u = +4c/5$. The velocity of rocket 2 as seen from O is in the

-ve direction, so $v_2 = -4c/5$. Now, what is the velocity of rocket 2 as seen from frame O', $v_2' =$?.
use

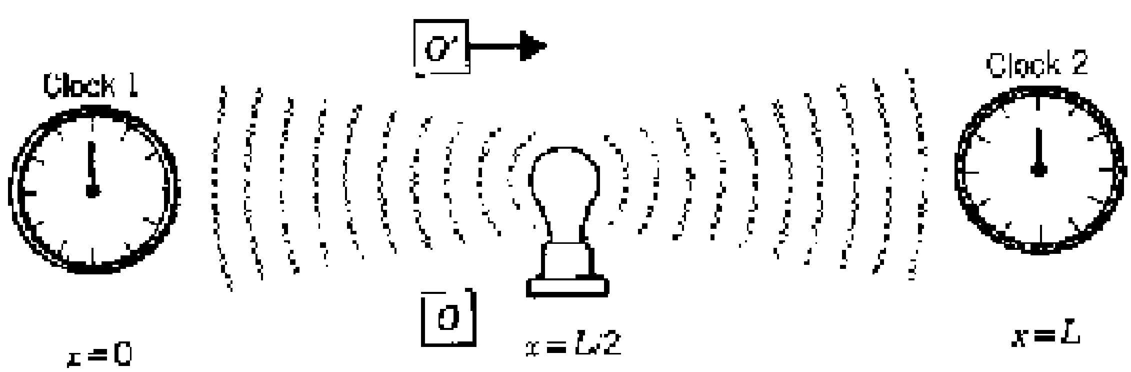
$$v_2' = \frac{v_2 - u}{1 - \frac{v_2 u}{c^2}} = \frac{(\frac{-4}{5}) - (\frac{4}{5})}{1 - (\frac{-4}{5})(\frac{4}{5})} c = -40c/41$$

From frame O' (i.e. rocket 1), the relative speed of rocket 2 is v_2', hence the corresponding Lorentz factor is $\gamma = 1/\sqrt{1 - {v_2'}^2/c^2} = 41/9$. Hence the contracted length, i.e the length of rocket 2 as seen from rocket 1, is $L'/\gamma = 20\text{m} \times \frac{9}{41} = \frac{180}{41}$ m.

Simultaneity in SR

As a consequence of SR, two events that are seen to be simultaneous in one reference frame (e.g. two clocks which are synchronised) are observed to be otherwise in other frame of reference – simultaneity is a relative concept and it depends on which frame of reference the events are observed

- Consider, in a rest frame O, two clocks which are located at $x = 0$ and $x = L$. A light source is located at the middle ($x = L/2$) btw the clocks.



- The two clocks can be synchronised (i.e to make the two clocks start at the same time and tick at the same rate as seen in frame

0) by sending a flash of light from the source to the two clocks which set them running "simultaneously" when light hits them after $t = L/2c$

- However, from an observer O' which is moving at velocity u wrp to O, the times when the flash of light hits clock 1 (located at $x_1 = 0$) and clock 2 (located at $x_2 = L/2$) are different, according to Lorentz transformation:

$$\begin{aligned} t'_1 &= \gamma\left[t_1 - (u/c)^2 x_1\right] = \gamma(L/2c) \\ t'_2 &= \gamma\left[t_2 - (u/c)^2 x_2\right] \\ &= \gamma\left[L/2c - (u/c)^2 L\right] \end{aligned}$$

so that the two clocks are out of synchronization by an amount

$$\Delta t' = t'_1 - t'_2 = \gamma(\frac{uL}{c^2}) \tag{10}$$

- Note that in Lorentz transformation of the time component,

$$t' = \gamma \left[t - (u/c)^2 x \right] \tag{11}$$

the first term is the time dilation effect. As far as O' is concerned, both clocks run slower at the same factor γ due to time dilation effect.

- The out-of-synchronisation effect as seen here is the result from the second term in Eq. (11)

Relativistic dynamics

- The Lorentz transformation of velocity would upset conservation of linear momentum if the Newtonian definition of momentum,

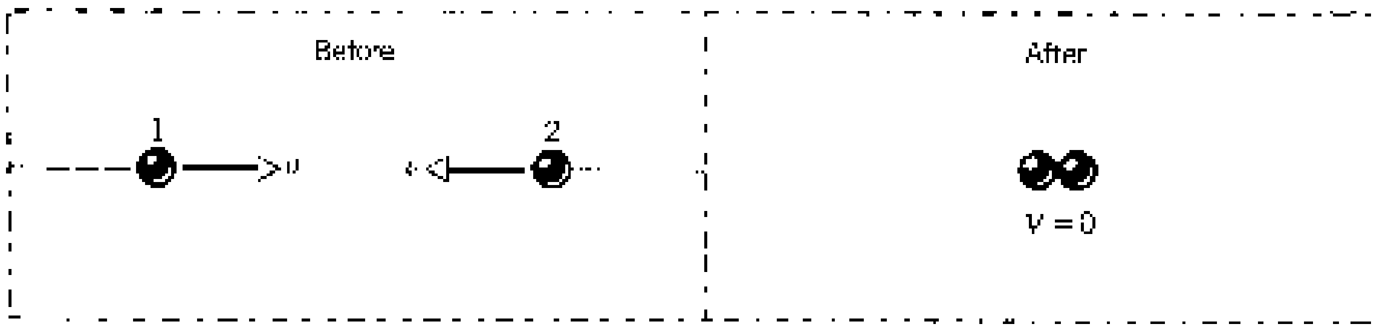
$$\text{momentum} = \text{mass} \times \text{velocity} \tag{12}$$

is not modified correspondingly in the light of the modification to the transformation law for the velocity (i.e. from Galilean to Lorentz). In order to preserve consistency of the definition of momentum as per equation Eq.(12) in relativistic regime, the mass term, m has to be modified and made dependent on the frame of reference.

- We will first see how inconsistency in conservation of momentum arises by analysing a collision problem in two different frames of reference, while retaining the mass as frame-independent.

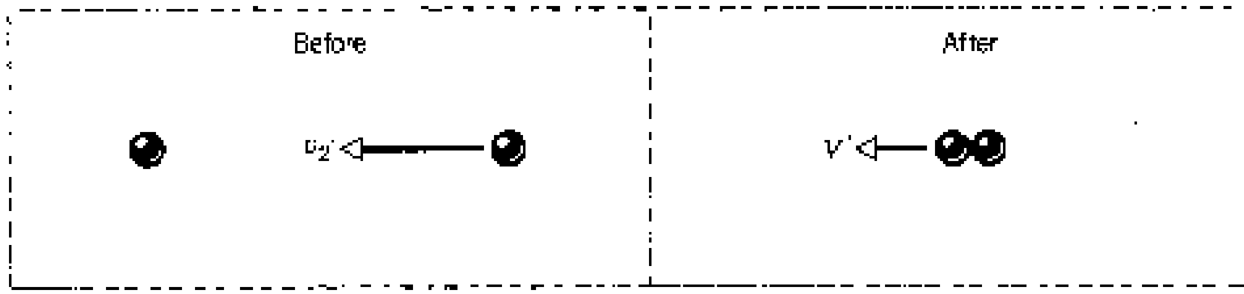
- Consider the collision of two identical particles approaching each other at the same speed v wrp to a rest frame.

  – In the lab (rest) frame:



  the final velocity of the combined mass, $2m$, is $V = 0$, and it is trivial to note that $p_i = p_f = 0$.

  – Now, lets analyse the same collision in a moving frame in which the particle 1 is at rest. In this frame, particle 2 is seen to approach the particle 1 (at rest initially) from the right (hence v_2 and v_2' are both negative).



– Before collision, according to the relativistic law of velocity addition, initial velocity of particle of 2 as seen by particle 1 is

$$\begin{aligned} v_2' &= v_2 - u/(1 - v_2 u/c^2) \\ &= (-v) - v/(1 - (-v)v/c^2) \\ &= -2v/(1 + v^2/c^2). \end{aligned} \quad (13)$$

so that $p_i' = 0 + mv_2' = -2mv/(1+v^2/c^2)$

– Note that $u(= +v)$ is the velocity of the moving frame (particle 1) as seen in the lab (rest) frame. It is $+v$ because the frame is moving to the right (we choose right as the positive direction, -ve for the direction pointing to the left).

– after collision, the final velocity V' (velocity as seen in the particle 1 moving frame) of the combined mass $M = 2m$ can be related to V (velocity as seen in the lab frame) as per

$$\begin{aligned} V' &= V - u/(1 - Vu/c^2) \\ &= (0) - v/(1 - (0)v/c^2) \\ &= -v \end{aligned}$$

The final momentum is is then $p'_f = (2m)V' = -2mv$

- Hence, by Lorentz transforming the velocity from the rest frame into a moving frame, the momentum before collision is not equal to the momentum after the collision in the moving frame
$\Rightarrow$ inconsistency in conservation of linear momentum with Lorentz transformation of velocity

- To preserve the conservation of momentum in relativistic speed, we must modify the classical definition of momentum to the one consistent with special relativity. The redefinition to relativistic must reduce back to the classical definition, $p = mu$ when $u \ll c$:

- To do so, we define the relativistic mass and relativistic momentum of a particle moving with constant speed u wrp to a given frame as

$$m = \gamma m_0, \;\; p = mu = \gamma m_0 u$$

where $\gamma = 1 - u^2/c^2{}^{-1/2}$, m_0 is called the *rest mass* = the mass measured in a frame where the object is at rest.

- the mass of a moving object at relativistic speed will increase compared to when it is at rest by a factor γ (analogous to relativistic modification of length and time)

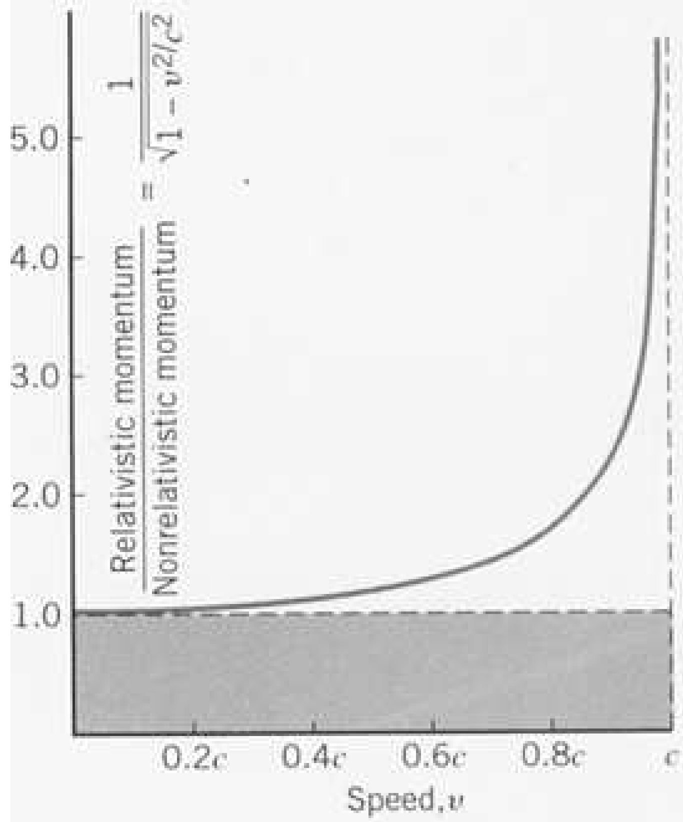


Figure 28.7 This graph shows how the ratio of the magnitude of the relativistic momentum to the magnitude of the nonrelativistic momentum increases as the speed of an object approaches the speed of light.

- upon re-definition of the classical momentum to the relativistic one, momentum conservation is "restored" in all frames of reference and at all u

- Now, lets revisit the same collision problem by taking into account of the redefinition of "Newtonian mass" $\longrightarrow$ relativistic mass. We shall restore conservation of momentum:

- In the moving frame,

$$
\begin{aligned}
p_i' &= m_2 \times v_2' \\
&= m_0\gamma'(u = v_2') \times (\,-\,2v/(1 + v^2/c^2)) \\
&= \frac{-2m_0 v}{1 - v^2/c^2}
\end{aligned}
$$

where $\gamma'(u = v_2') = (1 - v_2'/c^2)^{-1/2}$, $v_2' = -2v/(1 + v^2/c^2)$.

$$
\begin{aligned}
p_f' &= M \times V' \\
&= 2m_0\gamma(u = V') \times (-v) \\
&= \frac{-2m_0 v}{1 - v^2/c^2}
\end{aligned}
$$

where $\gamma'(u = V') = (1 - (-v'/c)^2)^{-1/2}$, $V' = -v \Rightarrow p_i' = p_f'$ in the moving frame.

- By definition, momentum = mass $\times$ velocity. If we only make the relativistic correction to the velocity but not to the mass, $p'_i = p'_f$ would not hold

Example:
The rest mass of an electron is $m_0 = 9.11 \times 10^{-31}$kg. If it moves with $u = 0.75c$, then $\gamma(u = 0.75c) = 1.511$ the relativistic momentum is

$$p = \gamma(u = 0.75c)m_0 \times u = 3.1 \times 10^{-22} \text{ N.}$$

In comparison, the incorrect classical expression would give $p(classical) = m_0 v = m_0 \times 0.75c = 2.05 \times 10^{-22}$ N, which is about 34% lesser than the correct relativistic result.

- *Note that the above example implicity assumes that the momentum is as seen from a rest frame wrp to the moving electron*

Relativistic energy

- relativistic correction to the definition of momentum must be followed by modification of the relation btw work and energy in order to preserve conservation of energy

- by definition, force = rate change in momentum, $F = \frac{dp}{dt}$

- by the law of conservation of mechanical energy, work done by external force on an object (W) = change in the kinetic energy of the object (ΔK)

- by definition, work done by external force from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F dx = \int_{x_1}^{x_2} \frac{dp}{dt} dx$$

$$= \int_{u_1}^{u} \frac{dp}{du} u du \qquad (14)$$

where

$$\frac{dp}{dt} dx = (\frac{dp}{dx}\frac{dx}{dt}) dx$$
$$= (\frac{dp}{du}\frac{du}{dx}) u dx$$
$$= \frac{dp}{du} u du,$$
$$\text{and } u = \frac{dx}{dt}$$

Explicitly

$$\frac{dp}{du} = m_0 (1 - \frac{u^2}{c^2})^{-3/2}$$

therefore

$$W = \int_{0}^{u} m_0 u (1 - \frac{u^2}{c^2})^{3/2} du$$
$$\Rightarrow K = m_0 \gamma c^2 - m_0 c^2$$
$$= mc^2 - m_0 c^2 \qquad (15)$$

where we assumed that the object is accelerated from rest, $K(x_1 = 0) = u_1 = 0$

- Eq.(15) is the relativistic kinetic energy K of an object of rest mass m_0 travelling at speed u

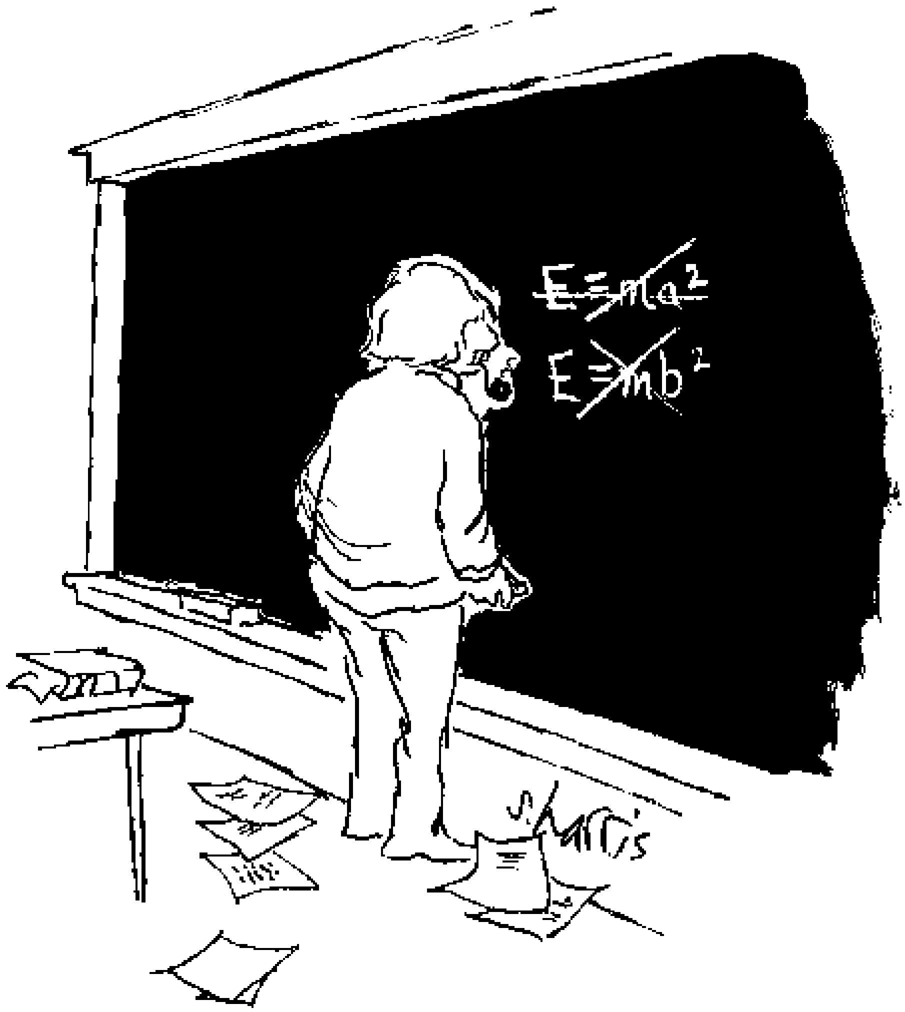
- $E_0 = m_0c^2$ is called the *rest energy* – a constant

- The total energy of a moving particle is
$$E = E_0 + K = m_0c^2 + K = mc^2$$
This is the famous Einstein equation for the equivalence of mass and energy

- $E = mc^2$ relates the mass of an object to the total energy released when the object is converted into pure energy

- using $E = mc^2$ we can express the mass of an object in unit of energy

Mass equivalent to Energy: $E = mc^2$



How did the formula $E = mc^2$ comes about ...

Example:
electron has rest mass
$m_0 = 9.1 \times 10^{-31}$ kg
$= 9.1 \times 10^{-31}$ kg $\times (3 \times 10^8)^2 (\mathrm{m/s})^2$
$= 8.19 \times 10^{-14}$ Joule
$= 8.19 \times 10^{-14} \times 6.25 \times 10^{18}$ eV
$= 0.51 \times 10^6$ eV $= 0.51$ MeV

- recall that the unit for energy, Joule, can be converted into the unit eV via 1 Joule $= (1/e)$ eV $= 6.25 \times 10^{18}$ eV

- In terms of momentum,

$$E^2 = (pc)^2 + (m_0 c^2)^2$$

- E, p are frame dependent (i.e. both quantities take different value at different frame), but $E^2 - (pc)^2$, which equals $(m_0 c^2)^2$ (=constant) is *invariant* in all frame
- we say that the term $E^2 - (pc)^2$ is Lorentz invariant

- The relativistic kinetic energy reduces to Newtonian form when $u \ll c$
- Expand the binomial of γ in $(u/c)^2$ as

$$\begin{aligned}\gamma &= (1 - \frac{u^2}{c^2})^{-1/2} \\ &= 1 + \frac{1}{2}(\frac{u}{c})^2 \\ &\qquad + \text{terms of order}(\frac{u}{c})^4 \text{ and higher}\end{aligned}$$

we arrive at the approximate result

$$\begin{aligned}K &= mc^2 - m_0c^2 \\ &= m_0c^2(\gamma - 1) \\ &= m_0c^2[1 + \frac{1}{2}(\frac{u}{c})^2 + \ldots - 1] \\ &\approx m_0u^2/2 \quad \text{(Newtonian limit)}\end{aligned}$$

- to summerise, we can preserve the (i) conservation of energy, (ii) conservation of momentum (iii) Newton's second law, $F = dp/dt$ if we introduce the following new relativistic concepts:

1. $p = m_0 u / \sqrt{1 - (\frac{u}{c})^2}$

2. $m = m_0 / \sqrt{1 - (\frac{u}{c})^2}$

3. $E = mc^2 = m_0 c^2 + K$

let's see some example

Example:
An electron moves with speed $u = 0.85c$. Find its total energy and kinetic energy in eV

Arial view of CERN in Geneva. The circular accelerator accelerates electron almost the speed of light.

ANS:

First calculate the Lorentz factor, $\gamma = [1 - (u/c)^2]^{-1/2} = 1.89$

The rest mass of electron is $m_0c^2 = 0.5$ MeV. Hence, the total energy is

$$E = m_0c^2\gamma = 0.5 \text{ MeV} \times 1.89 = 0.97 \text{ MeV}$$

Kinetic energy is the difference between the total energy and the rest energy,

$K = E - m_0c^2 = (0.97 - 0.51)$ MeV $= 0.46$ MeV

Binding energy of the Deuterium

- The atom of heavy hydrogen, the deuterium, is composed of one proton and one neutron
- However the mass of the deuterium is not equal to the sum of the two nucleons but slightly less
- The difference in the mass between $(m_n + m_p)$ and m_d, Δm, is actually converted, via

$E = \Delta mc^2$, into the binding energy (within the nucleus) that binds the proton to the neutron together

- We can calculate the energy that will be released when a proton and a neutron is *fused* in a nuclear reaction to form a deuterium (the same amount of energy is required if we want to separate the proton from the neutron in a deuterium nucleus)
- $m_n = 1.008665u$, $m_p = 1.007276u$, $m_d = 2.013553u$, where u is the standard atomic unit = 1/12 the mass of a carbon ^{12}C nucleus = 1.66×10^{-27} kg = 1.494 $\times 10^{-10}$ Joule = 933.75 MeV
- $\Delta m = (m_n + m_p) - m_d = 0.002388$ u = 0.002388×933.75 MeV = 2.23 MeV

Please attempts the problems assigned

Is time travel possible according to Einstein theory of relativity?

I happen to know something that is faster than light : Darkness.
Try this: lock yourself inside a darkened room. Then, slowly, open the door. You can surely see the light coming in, but you cant see the darkness going out. Thats how fast it is!!!

This is an amazing result, since darkness is known to be heavier than light: just ask any diver – they will tell you that the deeper you go, the darker it gets. Darkness sinks, while light floats.