UNIVERSITI SAINS MALAYSIA

Final Exam
Academic Session 2004/2005

November 2005

ZCA 110/4 – Kalkulus dan Aljabar Linear

Duration: 3 hours
[Masa: 3 jam]

Please check that the examination paper consists of 11 pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi 11 muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instruction:
Answer ALL questions in Section A. Answer **any 5** out of 6 questions in Section B. Answer **the only** question in Section C.

Please answer the objective questions from Section A in the objective answer sheet provided. Please submit the objective answer sheet and the answers to the structured questions separately.

Students are allowed to answer all questions in Bahasa Malaysia or in English.

*[**Arahan**: Jawab **SEMUA** soalan dalam Bahagian A. Jawab **mana-mana 5** daripada 6 soalan dalam Bahagian B. Jawab **soalan tunggal** dalam Bahagian C.*

Sila jawab soalan-soalan objektif daripada bahagian A dalam kertas jawapan objektif yang dibekalkan. Sila serahkan kertas jawapan objektif dan jawapan kepada soalan-soalan struktur berasingan.

Pelajar dibenarkan untuk menjawab samada dalam bahasa Malaysia atau bahasa Inggeris.]

Section A: Objectives. [20 marks]
[Bahagian A: Soalan-soalan objektif]

Instruction: Answer all 20 objective questions in this Section.
[Arahan: Jawab kesemua 40 soalan objektif dalam Bahagian ini.]

1. Which of the following limits are true?
[Limit manakah daripada yang berikut adalah benar?]

I. $\lim_{x\to\infty} \frac{1}{x} = \infty$

II. $\lim_{x\to 0} \frac{1}{x} = \infty$

III. $\lim_{x\to 0} \frac{1}{x} = 0$

IV. $\lim_{x\to\infty} \frac{1}{x} = 0$

A. I, III **B. II, IV** **C. I, IV**
D. Non of A, B, C (Jawapan tiada dalam A, B, C)

Ans: B (II, IV)

2. Which of the following statements is true?
[Kenyataan manakah daripada yang berikut adalah benar?]

A. If a function is discontinuous at a point, the limit of the function at that point exists.
[Jika suatu fungsi adalah tak selanjar pada suatu titik, limit fungsi tersebut pada titik tersebut wujud]

B. If a function is continuous at a point, the limit of the function at that point exists.
[Jika suatu fungsi adalah selanjar pada suatu titik, limit fungsi tersebut pada titik tersebut wujud]

C. If a function is continuous at a point, the limit of the function at that point must not exists.
[Jika suatu fungsi adalah selanjar pada suatu titik, limit fungsi tersebut pada titik tersebut pasti tidak wujud]

D. Non of A, B, C *(Jawapan tiada dalam A, B, C)*

Ans: B

3. Which of the following statements is true?
[Kenyataan manakah daripada yang berikut adalah benar?]

A. Every polynomial function is everywhere discontinuous except at $x = 0$.
[Semua fungsi polinomial adalah tak selanjar di mana-mana sahaja kecuali pada $x = 0$.]

B. Every polynomial function is everywhere continuous except at $x = 0$.
[Semua fungsi polinomial adalah selanjar di mana-mana sahaja kecuali pada $x = 0$.]

C. Every polynomial function is everywhere continuous, including $x = 0$.
[Semua fungsi polinomial adalah selanjar di mana-mana sahaja termasuk $x = 0$.]

D. Non of A, B, C (Jawapan tiada dalam A, B, C)

Ans: C

4. Which of the following statements is true?
[Kenyataan manakah daripada yang berikut adalah benar?]

A. If f is continuous on $[a, b]$, then f takes on a least value m and a greatest value M on the interval.
[Jika f adalah selanjar dalam [a,b], maka f mengambil satu nilai terkecil m and satu nilai terbesar M dalam selang tersebut.]

B. If f is discontinuous on $[a, b]$, then f takes on a least value m and a greatest value M on the interval.
[Jika f adalah tak selanjar dalam [a,b], maka f mengambil satu nilai terkecil m and satu nilai terbesar M dalam selang tersebut.]

C. If f is continuous on $[a, b]$, then f takes on a least value m but not a greatest value on the interval.
[Jika f adalah selanjar dalam [a,b], maka f mengambil satu nilai terkecil m tapi tidak mengambil apa-apa nilai terbesar dalam selang tersebut.]

D. Non of A, B, C **(***Jawapan tiada dalam A, B, C***)**

Ans: A

5. Which of the following statements is true?
[Kenyataan manakah daripada yang berikut adalah benar?]

A. If f is continuous on $[a, b]$ and $f(a) = f(b)$, then there exists a unique number x_o in the open interval (a, b) for which $f(x_o) = f(a) = f(b)$.
[Jika f selanjar dalam [a,b] dan $f(a) = f(b)$, maka, wujud satu nombor unik x_o dalam selang terbuka (a,b) supaya $f(x_o) = f(a) = f(b)$.]

B. If f is NOT continuous on $[a, b]$ and $f(a) \neq f(b)$, then, for any number c between $f(a)$ and $f(b)$, there must exist at least one number x_o in the open interval (a, b) for which $f(x_o) = c$.
[Jika f BUKAN selanjar dalam [a,b] dan $f(a) \neq f(b)$, maka, untuk sebarang nombor c di antara f(a) dan f(b), mesti wujud sekurang-kurangnya satu nombor x_o dalam selang terbuka (a,b) supaya $f(x_o)=c$.]

C. If f is continuous on $[a, b]$ and $f(a) \neq f(b)$, then, for any number c between $f(a)$ and $f(b)$, there must exist at least one number x_o in the open interval (a, b) for which $f(x_o) = c$.
[Jika f selanjar dalam [a,b] dan f(a) ≠ f(b), maka, untuk sebarang nombor c di antara f(a) dan f(b), mesti wujud sekurang-kurangnya satu nombor x_o *dalam selang terbuka (a,b) supaya* $f(x_o)=c$.*]*
D. Non of A, B, C (Jawapan tiada dalam A, B, C)

Ans: C

6. Given $f(x)=5$ when $0<x\leq 1$ $\quad f(x)=10$ when $1<x\leq 2$
$f(x)=15$ when $2<x\leq 3$ $\quad f(x)=20$ when $3<x\leq 4$ etc.,
which of the following statements is true regarding the function $f(x)$?
[Kenyataan manakah daripada yang berikut adalah benar?]

A. $\lim_{x\to 0} f(x)$ exists.
B. $\lim_{x\to 0} f(x)$ does not exist.
C. The function is continuous for all $x > 0$.
[Fungsi tersebut adalah selanjar untuk semua x>0.]

D. Non of A, B, C (Jawapan tiada dalam A, B, C)

Ans: A

7. Consider the function $f(x)=x^n$ where n is a non-negative integer. Which of the following statements is true regarding the function $f(x)$?
[Pertimbangkan fungsi $f(x)=x^n$*, dengan n integer bukan negatif. Kenyataan manakah daripada yang berikut adalah benar?].*

A. $f(x)$ is not one-to-one when n is even.
[f(x) adalah bukan satu-satu jika n ialah genap.]

B. $f(x)$ is not one-to-one when n is odd.
[f(x) adalah bukan satu-satu jika n ialah ganjil.]

C. $f(x)$ is one-to-one when n is even.
[f(x) adalah satu-satu jika n ialah genap.]
D. Non of A, B, C (Jawapan tiada dalam A, B, C)

ANS: A

8. Which of the following statements is true?
[Kenyataan manakah daripada yang berikut adalah benar?]

A. The derivative of $f(x)=|x|$ does not exist at all.
[Terbitan bagi $f(x)=|x|$ *tidak wujud sama sekali.]*

B. The derivative of $f(x) = |x|$ does not exist at $x = 0$. *[Terbitan bagi $f(x) = |x|$ tidak wujud pada x = 0.]*

C. The derivative of $f(x) = |x|$ exists for all x .
[Terbitan bagi $f(x) = |x|$ wujud untuk semua nilai x.]

D. Non of A, B, C (Jawapan tiada dalam A, B, C)

ANS: B

9. Which of the following statements is true?
[Kenyataan manakah daripada yang berikut adalah benar?]

A. A function that is differentiable in an open interval is not necessarily continuous in that interval.
[Suatu fungsi yang terbezakan dalam suatu selang terbuka tidak semestinya selanjar dalam selang tersebut.]

B. A function that is differentiable in an open interval is necessarily continuous in that interval.
[Suatu fungsi yang terbezakan dalam suatu selang terbuka pasti selanjar dalam selang tersebut.]

C. A function that is continuous in an open interval is necessarily differentiable in that interval.
[Suatu fungsi yang selanjar dalam suatu selang terbuka pasti terbezakan dalam selang tersebut.]

D. Non of A, B, C (Jawapan tiada dalam A, B, C)

ANS: B

10. Consider a function $f(x)$ whose second order derivative, $\frac{d^2 f(x)}{dx^2}$, vanishes at a point $x=x_0$.
Which of the following statements is true?
[Pertimbangkan suatu fingsi f(x) yang terbitan tertib keduanya, $\frac{d^2 f(x)}{dx^2}$, lenyap pada titik $x=x_0$. Kenyataan manakah daripada yang berikut adalah benar?]

A. It is possible that $(x_0, f(x_0))$ is a local maximum. *[Ada kemungkinan bahawa $(x_0, f(x_0))$ maksimum setempat.]*

B. It is not possible that $(x_0, f(x_0))$ is a local maximum. *[$(x_0, f(x_0))$ tidak mungkin menjadi maksimum setempat.]*

C. It is not possible that $(x_0, f(x_0))$ is a local mimimum. *[$(x_0, f(x_0))$ tidak mungkin menjadi minimum setempat.]*

D. Non of A, B, C (Jawapan tiada dalam A, B, C)

ANS: A

11. Which of the following statements is true regarding the function $f(x) = 4 + x^{2/3}$?
[Kenyataan manakah daripada yang berikut adalah benar mengenai fungsi $f(x) = 4 + x^{2/3}$?]

A. f is increasing for all x.
[f adalah menokok untuk semua x.]

B. f is decreasing for all x.
[f adalah menyusut untuk semua x.]

C. f is decreasing for $x < 0$ but increasing for $x > 0$.
[f adalah menyusut untuk $x < 0$ tapi menokok untuk $x > 0$.]

D. Non of A, B, C (Jawapan tiada dalam A, B, C)

ANS: C

12. Which of the following statements is true regarding the function $y = \dfrac{x^2}{\sqrt{x^2 - 4}}$?

[Kenyataan manakah daripada yang berikut adalah benar mengenai fungsi $y = \dfrac{x^2}{\sqrt{x^2 - 4}}$ *?]*

A. $y = \dfrac{x^2}{\sqrt{x^2 - 4}}$ is symmetric with respect to both of the x- and y-axis.

[$y = \dfrac{x^2}{\sqrt{x^2 - 4}}$ *adalah bersimetri terhadap kedua-dua paksi-x dan -y.]*

B. $y = \dfrac{x^2}{\sqrt{x^2 - 4}}$ is symmetric with respect to the y-axis.

[$y = \dfrac{x^2}{\sqrt{x^2 - 4}}$ *adalah bersimetri terhadap paksi-y.]*

C. $y = \dfrac{x^2}{\sqrt{x^2 - 4}}$ is symmetric with respect to the x-axis.

[$y = \dfrac{x^2}{\sqrt{x^2 - 4}}$ *adalah bersimetri terhadap paksi-x.]*

D. Non of A, B, C (Jawapan tiada dalam A, B, C)

ANS: B

13. Which of the following limits are true?
[Limit manakah daripada yang berikut adalah benar?]

I. $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 0$

II. $\lim_{\theta \to 0} \dfrac{\theta}{\sin\theta} = \infty$

III. $\lim_{\theta \to 0} \dfrac{1 - \cos\theta}{\theta} = 0$

A. I, II **B. I, II, III** **C. III**

D. Non of A, B, C (Jawapan tiada dalam A, B, C)

ANS: C

14. Let $f(x)$ be a continuous function for all x in [a, b], and let $F(x)=\int f(x)dx$. Which of the following statements is true?

[Biar f(x) suatu fungsi selanjar untuk semua x dalam [a,b], dan biar $F(x)=\int f(x)dx$*.*

Kenyataan manakah daripada yang berikut adalah benar?]

I. F is an antiderivative of f.
[F ialah suatu anti-terbitan bagi f(x)]

II. $F(x)$ and $\int_a^x f(t)dt$ have the same derivative.

[F(x) dan $\int_a^x f(t)dt$ *memiliki terbitan yang sama.]*

III. $F(b)-F(a) = \int_a^b f(t)dt$

A. I, II, III **B. I, III** **C. III**
D. Non of A, B, C (Jawapan tiada dalam A, B, C)

ANS: A

15. The general exponential function a^x is define as
[Fungsi ekponen am adalah ditakrifkan sebagai?]

A. $a^x = e^{x\ln a}$
B. $a^x = e^{a\ln x}$
C. $a^x = a^{x\ln e}$
D. Non of A, B, C (Jawapan tiada dalam A, B, C)

ANS: A (pg. 237)

16. Assume $f(x)$ and $g(x)$ are differentiable and $g'(x)\neq 0$ in some open interval (a,b), and $\lim_{x\to x_0} f(x)=0=\lim_{x\to x_0} g(x)$ for $a<x_0<b$. Then, if $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$ exists,
[Katakan $f(x)$ *dan* $g(x)$ *adalah terbezakan, dan* $g'(x)\neq 0$ *dalam sesuatu selang terbuka* (a,b)*, dan* $\lim_{x\to x_0} f(x)=0=\lim_{x\to x_0} g(x)$ *untuk* $a<x_0<b$*. Maka, jika* $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$ *wujud]*

I. $\frac{f(x)-f(x_0)}{g(x)-g(x_0)}=\frac{f'(x_0^*)}{g'(x_0^*)}$, where x_0^* lies between x and x_0

[$\frac{f(x)-f(x_0)}{g(x)-g(x_0)}=\frac{f'(x_0^*)}{g'(x_0^*)}$*, dengan* x_0^* *terletak di antara x dan* x_0*]*

II. $\dfrac{f(x)-f(x_0^*)}{g(x)-g(x_0^*)}=\dfrac{f'(x_0)}{g'(x_0)}$, where x_0^* lies between x and x_0

[$\dfrac{f(x)-f(x_0^*)}{g(x)-g(x_0^*)}=\dfrac{f'(x_0)}{g'(x_0)}$, *dengan* x_0^* *terletak di antara x dan* x_0 *]*

III. $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to x_0}\dfrac{f'(x)}{g'(x)}$

A. I, II, III **B. I, III** **C. III**

D. Non of A, B, C (Jawapan tiada dalam A, B, C)

ANS: B(I, III) (pg. 245, SP1)

17. Let *f* be a continuous function such that *f*(*x*) ≥0 for $a \le x \le b$. The volume of the solid of revolution obtained by revolving *f* about the *x*-axis in that interval is given by
[Biar f suatu fungsi selanjar sedemikian rupa sehingga f(x) ≥0 untuk $a \le x \le b$*. Isipadu jasad yang diperolehi dengan memutarkan f sekitar paksi-x dalam selang tersebut diberikan oleh]*

A. $V=\pi\int_a^b [f(x)]^2 dx$

B. $V=\pi\int_a^b [f(x)]^2 dy$

C. $V=2\pi\int_a^b xf(x)dy$

D. Non of A, B, C (Jawapan tiada dalam A, B, C)

ANS: A

18. Which of the following statements are true?
[Kenyataan manakah daripada yang berikut adalah benar?]

A. Any polynomial *D*(*x*) with leading coefficient 1 can be expressed as a product of linear factors of the form (*x*-*a*).
[Mana-mana polonomial D(x) dengan pekali pelopor1 boleh dinyatakan sebagai hasildarab faktor-faktor linear berbentuk (x-a)]

B. Any polynomial *D*(*x*) with leading coefficient 1 can be expressed as a product of linear factors of the form (*x*-*a*) and of irreducible quadratic factors of the form x^2+bx+c.
[Mana-mana polonomial D(x) dengan pekali pelopor1 boleh dinyatakan sebagai hasildarab faktor-faktor linear berbentuk (x-a) dan hasildarab faktor kuadratik tak terturunkan berbentuk x^2+bx+c *.]*

C. Any polynomial *D*(*x*) with leading coefficient 1 can be expressed as a product of irreducible quadratic factors of the form x^2+bx+c.
[Mana-mana polonomial D(x) dengan pekali pelopor1 boleh dinyatakan sebagai hasildarab faktor kuadratik tak terturunkan berbentuk x^2+bx+c *.]*

D. Non of A, B, C (*Jawapan tiada dalam A, B, C)*

ANS: B (pg. 305, theorem 1)

19. Which of the following statements are true?
[Kenyataan manakah daripada yang berikut adalah benar?]

A. Every increasing sequence is non decreasing.
[Setiap jujukan menokok adalah tak menyusut.]

B. Every nondereasing sequence is increasing.
[Setiap jujukan tak menyusut adalah menokok.]

C. Every noninreasing sequence is decreasing.
[Setiap jujukan tak menokok adalah menyusut.]

D. Non of A, B, C (Jawapan tiada dalam A, B, C)

ANS: A, page 387 (on monotonic sequence)

20. Which of the following statements are true?
[Kenyataan manakah daripada yang berikut adalah benar?]

A. Given a geometric series $\sum ar^{n-1}$, the series converges if $|r| > 1$.
[Diberi siri geometri $\sum ar^{n-1}$ *,siri tersebut menumpu jika* $|r| > 1$*.]*

B. If $\sum s_n$ converges, then $\lim_{n\to+\infty} s_n = 0$.
[Jika $\sum s_n$ *menumpu, maka* $\lim_{n\to+\infty} s_n = 0$.

C. Assume that two series $\sum s_n$ and $\sum t_n$ both converge. Their sum $\sum s_n + t_n$ does not necessarily converge.
[Katakan kedua-dua siri $\sum s_n$ *dan* $\sum t_n$ *menumpu. Hasiltambah mereka* $\sum s_n + t_n$ *tidak semestinya menumpu.]*

D. Non of A, B, C (Jawapan tiada dalam A, B, C)

ANS: B (for A, refer theorem 43.1 ; for B, refer theorem 43.5; for C, refer theorem 43.3)

ANSWERS:

1. **Ans: B (II, IV)**
2. **Ans: B,** pg. 71
3. *Ans: C*, Theorem 8.3, pg. 73
4. *Ans: A*, Theorem 8.7, pg. 75
5. **Ans: C (**Theorem 8.5, pg. 74).
6. **Ans: A,** SP 8, pg. 66
7. **ANS: A, (**My own question. Note: 0 is even by definition.**)**
8. **ANS: B,** SP 11, pg. 83. $f'(x)$ exists everywhere except at x=0
9. **ANS: B, (**a function that is differentiable is necessarily continuous.**)**
10. **ANS: A,** see example 2, pg. 116.
11. **ANS: C,** SP 5, pg. 120,
12. **ANS: B,** SP 10, pg. 137. The function is symmetric wrp to the y axis, not the x axis.
13. ANS: C, SP 1, pg. 158, Eq. 17.2, pg. 153.
14. ANS: A, Eqs. (24.2), (24.3), pg. 217.
15. ANS: A (Definition, pg. 237)
16. **ANS: B(I, III)** (pg. 245, SP1)
17. **ANS: A**
18. **ANS: B** (pg. 305, theorem 1)
19. **ANS: A**, page 387 (on monotonic sequence)
20. **ANS: B** (for A, refer theorem 43.1, pg. 395; for B, refer theorem 43.5, pg. 396; for C, refer theorem 43.3, pg. 395)

SECTION B

Instruction: Answer any FIVE (5) questions in this Section. Each question carries 11 marks.
[Arahan: Jawab mana-mana LIMA (5) soalan dalam Bahagian ini. Setiap soalan membawa 11 markah.

1(a) $\int x^{1/3} dx$

Solution 1(a) [3 marks]

(a) $\frac{3}{4}\left(\sqrt[3]{x}\right)^4 + c \quad (=\frac{3}{4}x^{4/3} + c)$

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1 (b) $\int x^2 \cos x^3 dx$

Solution 1(b) [4 marks]

Use $\int u'(x) f[u(x)]dx = \int f(u)du$: Let $u(x) = x^3, f(u) = \sin u, u'(x) = 2x^2$

$$\int x^2 \cos x^3 dx = \frac{1}{3}\int u'(x) f(u)dx = \frac{1}{3}\int f(u)du$$

$$= \frac{1}{3}\int \cos(u)du = \frac{1}{3}\sin u + c = \frac{1}{3}\sin x^3 + c$$

1(c) $\int \frac{x^2 dx}{\sqrt{1-(x-1)^2}}$ [*Hint: use* x - 1=sinθ]

Solution 1(c): SP 28, pg. 300. [4 marks]

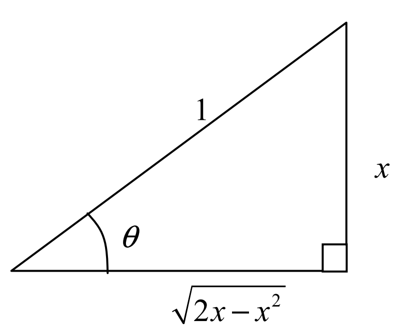
$x - 1 = \sin\theta, dx = \cos\theta d\theta$;

$$x^2 = (\sin\theta + 1)^2 = \sin^2\theta + 2\sin\theta + 1 = \frac{1}{2}(1-\cos 2\theta) + 2\sin\theta + 1 = \frac{3}{2} - \frac{1}{2}\cos 2\theta + 2\sin\theta$$

$$\int \frac{x^2 dx}{\sqrt{1-(x-1)^2}} = \int \frac{\left(\frac{3}{2} - \frac{1}{2}\cos 2\theta + 2\sin\theta\right)\cos\theta d\theta}{\sqrt{1-\sin^2\theta}} = \int \frac{3}{2} - \frac{1}{2}\cos 2\theta + 2\sin\theta d\theta$$

$$= \frac{3}{2}\theta - \frac{1}{4}\sin 2\theta - 2\cos\theta + c = \frac{3}{2}\theta - \frac{1}{2}\sin\theta\cos\theta - 2\cos\theta + c = \frac{3}{2}\theta - \frac{1}{2}\cos\theta(\sin\theta + 4) + c$$

$$= \frac{3}{2}\sin^{-1}(x-1) - \frac{1}{2}\sqrt{2x - x^2}(x+3) + c$$

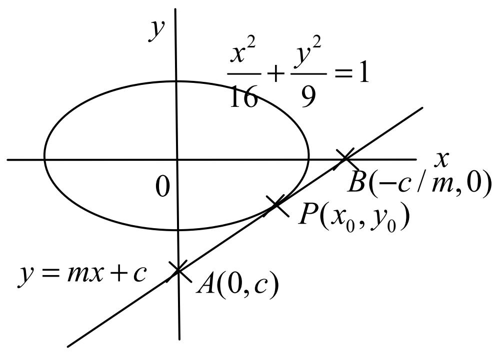


2. A tangent line is drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at some point $P(x_0, y_0)$. Refer to figure Q2 below. It cuts through the points *A* and *B* on the *y*- and the *x*-axis respectively.

[Suatu garis tangen dilukiskan pada elips $\frac{x^2}{16}+\frac{y^2}{9}=1$ *pada titik* $P(x_0, y_0)$*. Sila rujuk gambarajah Q2. Ia merentasi titik-titik A dan B pada paksi-y dan paksi-x masing-masing. Apakah nisbah* $\frac{y_0}{x_0}$ *yang meminimumkan panjang AB?]*

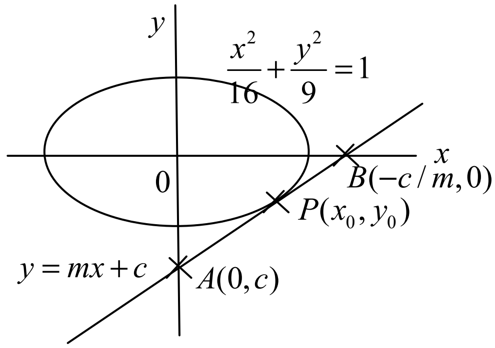
2(a) Show that the gradient, *m*, of the tangent line at $P(x_0, y_0)$ is given by $m=-\frac{9x_0}{16y_0}$.

Solution 2(a) (pg. 128, SupP 39): [3 marks]



At $P(x_0, y_0)$, $\frac{x^2}{16}+\frac{y^2}{9}=1$ take $\frac{d}{dx}\left(\frac{x^2}{16}+\frac{y^2}{9}=1\right)=\frac{2x}{16}+\frac{2y}{9}\frac{dy}{dx}=0$

$$\left.\frac{dy}{dx}\right|_{(x_0, y_0)}=-\frac{9x_0}{16y_0}\equiv m$$



2(b) Show that the coordinates of *A* and *B* are $A\left(0,\frac{9}{y_0}\right), B\left(\frac{16}{x_0},0\right)$ respectively.

Solution 2(b) [4 marks]

The tangent line's equation is $y=mx+c$, going through $P(x_0,y_0)$.
The tangent line cuts the *y*-axis at *A* when *x*=0. Hence the coordinate of point *A* is (0,*c*).
The tangent line cuts the *x*-axis at *B* when *y*=0. Hence the coordinate of point *B* is (-*c*/*m*,0).
Since the tangent line goes through $P(x_0,y_0)$, we slot $y=y_0, x=x_0$ into the tangent line equation $y=mx+c$:

$$y_0=-\frac{9x_0}{16y_0}x_0+c \Rightarrow c=\frac{16{y_0}^2+9{x_0}^2}{16y_0} \Rightarrow -\frac{c}{m}=\frac{16{y_0}^2+9{x_0}^2}{9x_0}.$$

y_0, x_0 is related via

$\frac{{x_0}^2}{16}+\frac{{y_0}^2}{9}=1 \Rightarrow 9{x_0}^2+16{y_0}^2=144 \Rightarrow -\frac{c}{m}=\frac{144}{9x_0}=\frac{16}{x_0}, c=\frac{144}{16y_0}=\frac{9}{y_0}$ Hence the coordinate of point *A* is $(0,c)=\left(0,\frac{9}{y_0}\right)$. The coordinate of point *B* is $\left(-\frac{c}{m},0\right)=\left(\frac{16}{x_0},0\right)$.

Alternatively:
Since $P(x_0,y_0)$ is located on the ellipse, x_0, y_0 is related via

$$\frac{x_0^2}{16}+\frac{y_0^2}{9}=1 \Rightarrow 9x_0^2+16y_0^2=144 \qquad \text{(Eq. 1)}$$

The equation of the tangent line at $P(x_0,y_0)$ with gradient $m=-\frac{9x_0}{16y_0}$ is

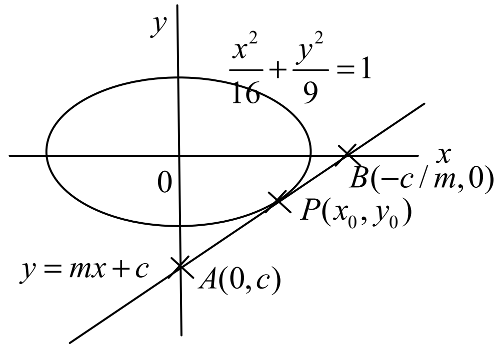
$$y-y_0=m(x-x_0) \Rightarrow y-y_0=-\frac{9x_0}{16y_0}(x-x_0). \qquad \text{(Eq.2)}$$

At *B*, $y=0$ and (Eq. 2) reads: $0-y_0=-\frac{9x_0}{16y_0}(x-x_0) \Rightarrow x=\frac{(16y_0^2+9x_0^2)}{9x_0}=\frac{144}{9x_0}=\frac{16}{x_0}$

(the last two steps followed from (Eq. 1). Thus, $B(16/x_0,0)$.

At *A*, $x=0$ and (Eq. 2) reads: $y-y_0=-\frac{9x_0}{16y_0}(0-x_0) \Rightarrow y=\frac{(16y_0^2+9x_0^2)}{16y_0}=\frac{144}{16y_0}=\frac{9}{y_0}$.

Hence, $A(0,9/y_0)$



2(c) What is the ratio that minimizes the length of *AB*?

Solution 2(c) [4 marks]

Let

$$l^2 = AB^2 = c^2 + (-c/m)^2 = \left(\frac{9}{y_0}\right)^2 + \left(\frac{16}{x_0}\right)^2$$

Set $\frac{d}{dx_0}\left(l^2\right) = 0$:

$$\frac{d}{dx_0}\left(l^2\right) = \frac{d}{dx_0}\left(\frac{16}{x_0}\right)^2 + \frac{d}{dx_0}\left(\frac{9}{y_0}\right)^2 = -2(16)^2\frac{1}{x_0{}^3} - 2(9)^2\frac{1}{y_0{}^3}\frac{dy_0}{dx_0}$$

$$= -2(16)^2\frac{1}{x_0{}^3} - 2(9)^2\frac{1}{y_0{}^3}\left(-\frac{9x_0}{16y_0}\right) = 0$$

$$\Rightarrow \left(\frac{16}{9}\right)^3 = \left(\frac{x_0{}^4}{y_0{}^4}\right)$$

$$\Rightarrow \frac{y_0}{x_0} = \pm\left(\frac{9}{16}\right)^{3/4}$$

3. (a) Given $y=\sin^{-1}(3x)$, find y'.

Solution 3(a): (pg 173, Supp 17) [3 marks]:

$$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}$$

$\Rightarrow$ Let $u=3x$

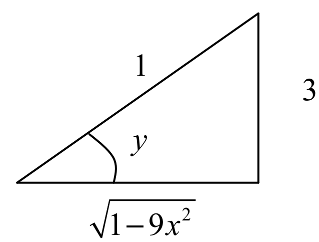
$$\frac{d}{dx}\left(\sin^{-1}3x\right)=\frac{d\left(\sin^{-1}u\right)}{dx}=\frac{d}{du}\left(\sin^{-1}u\right)\cdot\frac{du}{dx}=\frac{3}{\sqrt{1-u^2}}=\frac{3}{\sqrt{1-9x^2}}$$

Alternatively:

$y=\sin^{-1}(3x)\Rightarrow \sin y=3x$

Differentiate wrp to x: $\cos y\frac{dy}{dx}=3\Rightarrow\frac{dy}{dx}=\frac{1}{\cos y}$

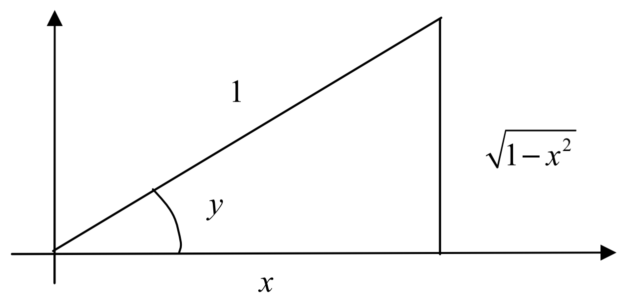
Hence $y'=\frac{dy}{dx}=\frac{3}{\sqrt{1-9x^2}}$



3 (b) Prove $\frac{d}{dx}\left(\cos^{-1}x\right)=-\frac{1}{\sqrt{1-x^2}}$

Solution 3(b): (pg 169,Eq. 18.6) [4 marks]

$$y=\cos^{-1}x$$
$$\Rightarrow \cos y=x$$
$$\Rightarrow -\sin y\frac{dy}{dx}=1$$
$$\frac{dy}{dx}=-\frac{1}{\sin y}$$
$$=-\frac{1}{\sqrt{1-x^2}}$$



3 (c) Sketch $y=e^x$ and its inverse on the same graph.

Solution 3(c): (pg 235, figure 26-1) [4 marks]

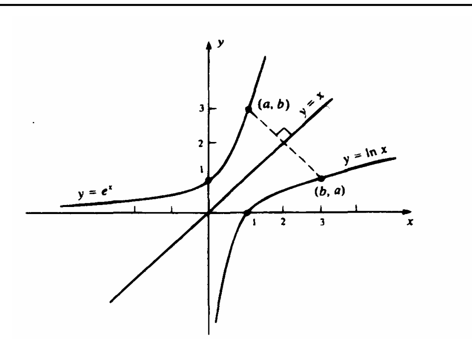


Fig. 26-1

4. (a) Find the derivative of $y=\sqrt{2x+1}$ using DEFINITION.
[Dapatkan terbitan bagi $y=\sqrt{2x+1}$ mengikut takrifan.]

Solution 4(a): SP7, pg 82. [5 marks]

Given $y(x)=\sqrt{2x+1}$,

$\Delta y = y(x+\Delta x)-y(x)$

$$=\sqrt{2(x+\Delta x)+1}-\sqrt{2x+1}=\sqrt{2(x+\Delta x)+1}-\sqrt{2x+1}\cdot\frac{\left(\sqrt{2(x+\Delta x)+1}+\sqrt{2x+1}\right)}{\left(\sqrt{2(x+\Delta x)+1}+\sqrt{2x+1}\right)}$$

$$=\frac{\left[2(x+\Delta x)+1\right]-\left[2x+1\right]}{\sqrt{2(x+\Delta x)+1}+\sqrt{2x+1}}=\frac{2\Delta x}{\sqrt{2(x+\Delta x)+1}+\sqrt{2x+1}}$$

$$\frac{\Delta y}{\Delta x}=\frac{2}{\sqrt{2(x+\Delta x)+1}+\sqrt{2x+1}}$$

$$\Rightarrow \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{2}{\sqrt{2x+1}+\sqrt{2x+1}}=\frac{1}{\sqrt{2x+1}}$$

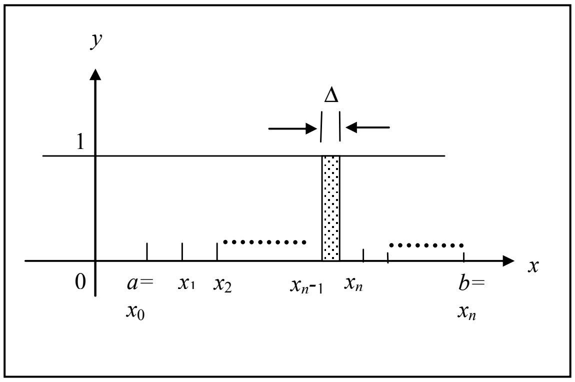
[4 marks for correct working + 1 mark for correct final answer = 5 marks]
[**1 marks only if not employ the definition.]

4(b) Evaluate $\int_a^b 1\,dx$ using RIEMANN INTEGRAL.

[Hitungkan $\int_a^b 1\,dx$ secara KAMIRAN RIEMANN.]

Solution 4(b): Example 2, page 209 [6 marks]

Subdivide close interval [a,b] into n tiny uniform subdivisions.



Let $a = x_0 < x_1 < x_2 < \ldots x_{n-1} < x_n = b$.

The width of the n-th interval is $\Delta=(b-a)/n$

The area of the strip in the n-th interval is $dA_n = \Delta \cdot f(x_n) = \Delta \quad \left(\because f(x_n) = 1\ \forall x_n\right)$

The sum of the area of individual strips is $\sum_{n=1}^{n} dA_n = \Delta \sum_{n=1}^{n} f\left(x_n\right) = \Delta \sum_{n=1}^{n} 1 = \Delta \cdot n = \frac{b-a}{n} \cdot n = b-a$

The Riemanian integral is defined as

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{n=1}^{n} dA_n$$
$$= \lim_{n\to\infty} \left(b-a\right) = b-a$$

[5 marks for correct working + 1 mark for correct final answer = 6 marks]
[**1 marks only if not employ the definition.]

5. **(a)** Find the volume of the solid generated by revolving about the y axis the region in the first quadrant inside the circle $x^2 + y^2 = r^2$, and between $y = a$ and $y = r$ (where $0 < a < r$). See figure Q5.
[*Hitungkan isipadu jasad terjanakan dengan memutarkan rantau di antara sukuan pertama dalam bulatan* $x^2 + y^2 = r^2$, *$y = a$ dan $y = r$ (dengan $0 < a < r$). Rujuk gambarajah Q5]*

Solution 5(a): SP 5, pg. 272. [4 marks]
(Accept any method of volume generation)

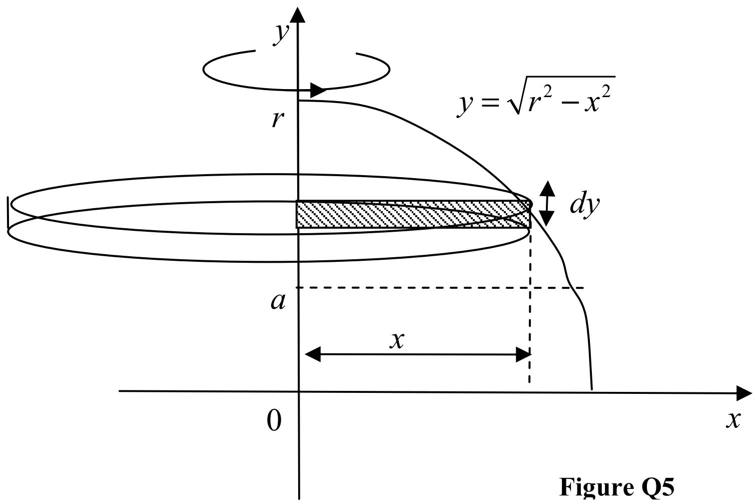


Figure Q5

The differential volume of the disk generated when revolved about the y axis is

$$dV = \pi x^2 dy = \pi\left(r^2 - y^2\right) dy$$

$$V = \int dV = \int_a^r \pi\left(r^2 - y^2\right) dy = \pi\left[r^2 y - \frac{y^3}{3}\right]_a^r = \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(ar^2 - \frac{a^3}{3}\right)\right]$$

$$= \frac{\pi}{3}\left[2r^3 - 3r^2 a + a^3\right]$$

5(b) (i) Evaluate $\lim_{x \to 0^+} x \ln x$

[Hitungkan $\lim_{x \to 0^+} x \ln x$ *]*

Solution 5(b) (i) Solution: Example 5, pg. 244: [3 marks]

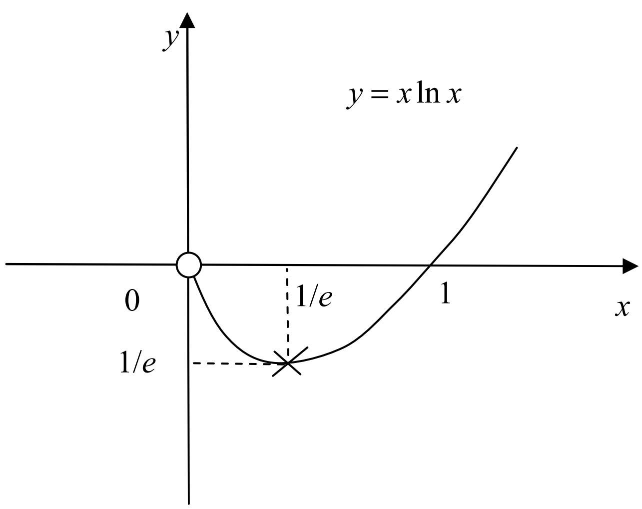
By L'Hopital's rule:

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = -\lim_{x \to 0^+} x = 0$$

5(b) (ii) Sketch the graph of $y = x \ln x$

[Lakarkan graf $y = x \ln x$*]*

Solution: 5(b) (ii) Solution: SupP 6, pg. 248 [4 marks]



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6. **(a)** Show that the following series diverges: $1+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}+\cdots$

[Tunjukkan bahawa siri yang berikut mencapah: $1+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}+\cdots$*]*

Solution 6(a): SP 2, pg 403 [5 marks]

Write the series as $\sum s_n = 1+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}+\cdots=\sum\frac{1}{\sqrt{2n-1}}$, where n = 1,2,3…

Let $f(x)=\frac{1}{\sqrt{2x-1}}; x\geq 1$ so that $f(n) = s_n$

$$\int_1^\infty f(x)\,dx=\int_1^\infty \frac{1}{\sqrt{2x-1}}dx=\left[(2x-1)^{1/2}\right]_1^\infty=\lim_{c\to\infty}\left[(2c-1)^{3/2}\right]-1=\infty$$

Hence, the series S_n diverges by integral test.

6(b)

Consider the harmonic series $\sum\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$.

[Pertimbangkan siri harmonik $\sum\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ *]*

(i) What is $\lim_{n\to\infty} s_n$? (s_n is the term in the series)

(ii) Show that the harmonic series diverges.
[Tunjukkan bahawa siri harmonik ini mencapah.]

Solution 6(b) (i) $\lim_{n\to\infty} s_n=\lim_{n\to\infty} 1/n=0$ **[2 marks]**

Solution 6b(ii) Example 4, pg 396. **[4 marks]**

Harmonic series $\sum 1/n=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ The trend of the partial sum:

$$S_2=1+\frac{1}{2}$$

$$S_4=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}>1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1+\frac{1}{2}+\frac{1}{2}=1+\frac{2}{2}$$

$$S_8=S_4+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>S_4+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=S_4+\frac{4}{8}=S_4+\frac{1}{2}>1+\frac{3}{2}$$

$$S_{16}=S_8+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}>S_8+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}$$

$$=S_8+\frac{1}{2}>1+\frac{4}{2}$$

So, in

$$S_{32}>1+\frac{5}{2}, S_{64}>1+\frac{6}{2}\cdots$$

general, the partial sum of goes like S_{2^k} > 1 + k/2 when k>1/

This implies that $\lim_{n\to\infty} S_n=+\infty$, and therefore diverges.