7.8 hyperbolic functions, pg 535, Thomas. 1/7

Every f^{n} can be decomposed into an even part and odd part uniquely:

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd part}}.$$

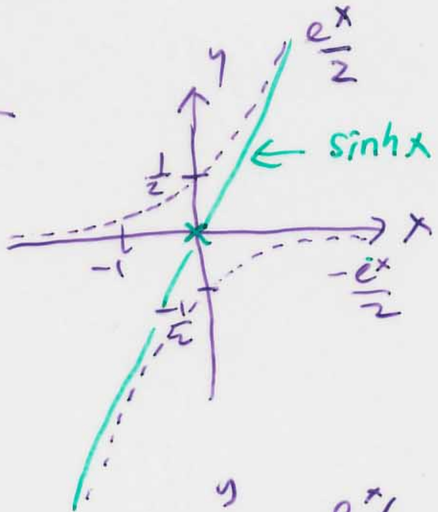
e.g. $f(x) = e^x = \underbrace{\frac{e^x+e^{-x}}{2}}_{\text{even}} + \underbrace{\frac{e^x-e^{-x}}{2}}_{\text{odd.}}$

$\frac{e^x+e^{-x}}{2}$ = hyperbolic cosine of x

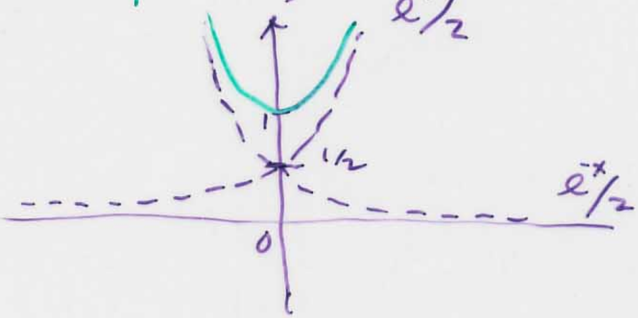
hyperbolic sine of x

six basic hyperbolic functions

$\sinh x = \frac{1}{2}(e^x - e^{-x})$



$\cosh x = \frac{1}{2}(e^x + e^{-x})$



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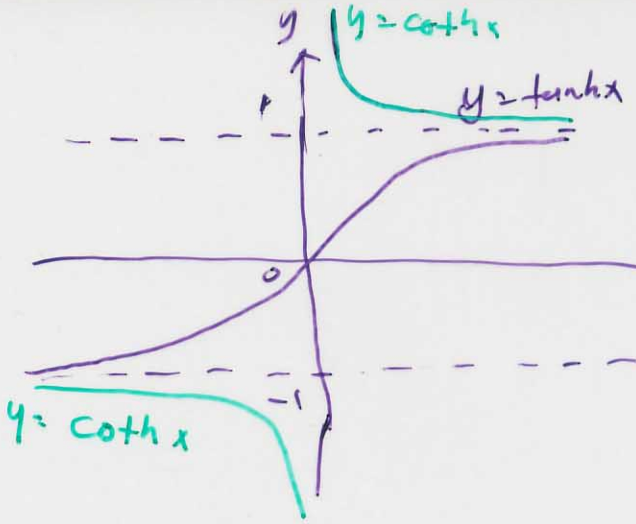
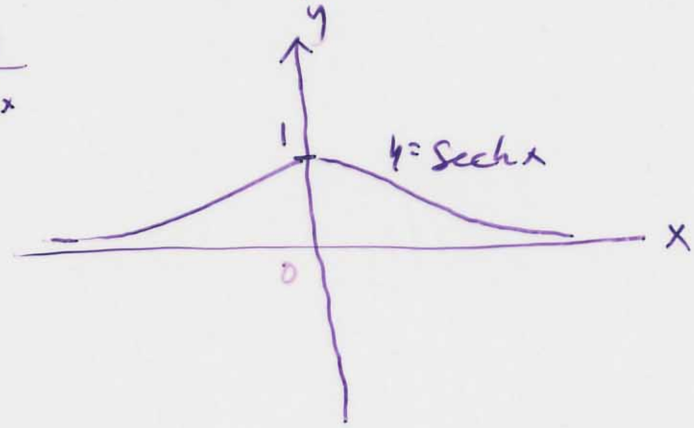
$$\tanh x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x - e^{-x}}{e^x - e^{-x}}$$

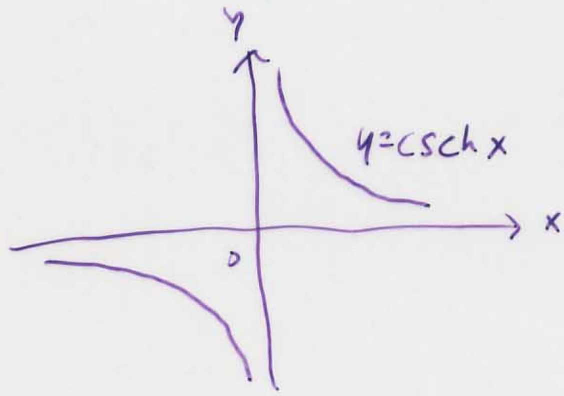


$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}} = \cancel{\frac{1}{\sinh x}} = \frac{1}{\cosh x}$$



~~secosh x =~~

$$\cancel{\cosh}\ \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$



Identities for hyperbolic f^n

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \cancel{\cos}\ \operatorname{csch}^2 x$$

cf these identities with

$$\cos^2 x + \sin^2 x = 1$$

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\sin^2 x = -\frac{\cos 2x + 1}{2}$$

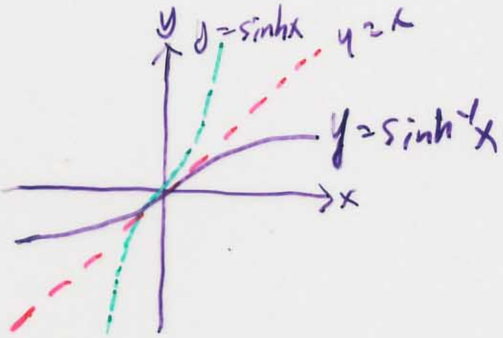
$$\cos^2 x = \frac{\cos 2x + 1}{2}$$

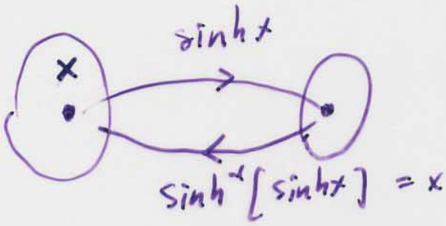
$$\tan^2 x = \sec^2 hx - 1$$

$$\cot^2 x = \csc^2 x - 1$$

Inverse hyperbolic functions

3/7.8

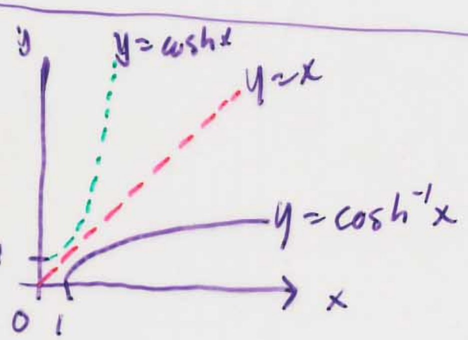




$y = \sinh x$ is 1-1 on $(-\infty, \infty)$

range of $\sinh x$ is $(-\infty, +\infty)$.

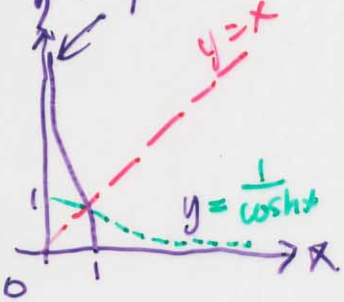
From the graph above, the reflection of $y = \sinh x$ on the straight line $y = x$ gives the curve of inverse of $y = \sinh x$, i.e $y = \sinh^{-1} x$. The range for $\sinh^{-1} x$ is $(-\infty, \infty)$, domain $(-\infty, \infty)$.



shown is the green curve of $y = \cosh x$, ~~y = cosh x~~ for $x \geq 0$, the domain in which the curve $\cosh x$ is 1-1 so that we can obtain it's inverse by reflecting it on $y = x$.

The domain of $y = \cosh^{-1} x$ is

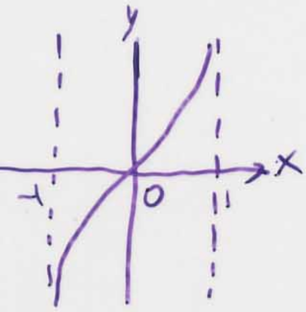
$x \geq 1$, range is $[0, +\infty)$



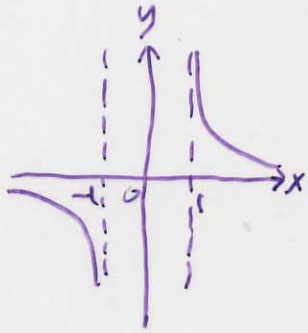
in the domain $[1, \infty]$, $y = \operatorname{sech}^{-1} x$ is 1-1.

The inverse $y = \operatorname{sech}^{-1} x$ has domain $(0, 1]$, range $[0, +\infty)$.

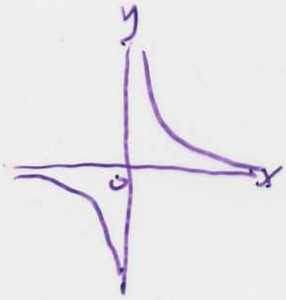
Similarly, 4/7.8



$y = \tanh^{-1} x$



$y = \coth^{-1} x$



$y = \cancel{\operatorname{csh}}\ \operatorname{csch}^{-1} x$

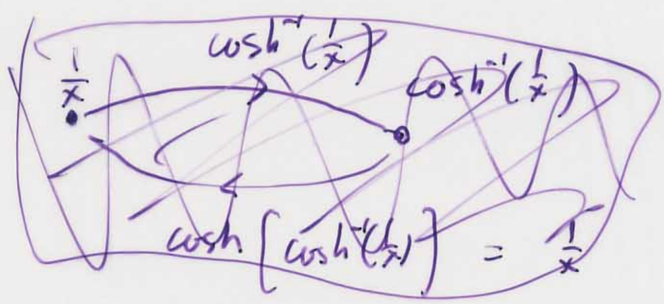
Identities for hyperbolic ~~cos~~ f^n s

$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$ — ①

$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$ — ②

$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$ ③

To prove ①:



$$\operatorname{sech} x = \frac{1}{\cosh(x)}$$

let $x \to \cosh^{-1}(\frac{1}{x})$,

$$\Rightarrow \operatorname{sech}\left[\cosh^{-1}(\tfrac{1}{x})\right] = \frac{1}{\cosh\left[\cosh^{-1}(\frac{1}{x})\right]} = \frac{1}{(\frac{1}{x})} = x$$

take sech^{-1} on both sides,

$$\operatorname{sech}^{-1}\left\{\operatorname{sech}\left[\cosh^{-1}(\tfrac{1}{x})\right]\right\} = \operatorname{sech}^{-1}(x)$$

$$\cosh^{-1}(\tfrac{1}{x}) = \operatorname{sech}^{-1}(x)$$

To prove ②:

$$\operatorname{csch}(x) = \frac{1}{\sinh x}$$

let $x \to \sinh^{-1}\left(\frac{1}{x}\right)$

$$\therefore \operatorname{csch}\left[\sinh^{-1}\left(\frac{1}{x}\right)\right] = \frac{1}{\sinh\left[\sinh^{-1}\left(\frac{1}{x}\right)\right]} = x$$

take csch^{-1} on both sides,

$$\sinh^{-1}\left(\frac{1}{x}\right) = \operatorname{csch}^{-1}(x)$$

Prove ③ yourself.

Derivatives of hyperbolic function.

1. $$\frac{d}{dx}(\sinh u) = \frac{d}{dx}\left[\frac{e^{u} - e^{-u}}{2}\right] = \frac{1}{2}\left\{\frac{de^{u}}{dx} - \frac{de^{-u}}{dx}\right\}$$

$$= \frac{1}{2}\left\{\frac{d}{du}e^{u}\cdot\frac{du}{dx} - \frac{d}{du}e^{-u}\cdot\frac{du}{dx}\right\}$$

$$= \frac{1}{2}\left\{e^{u}\frac{du}{dx} + e^{-u}\frac{du}{dx}\right\} = \frac{1}{2}\left[e^{u} + e^{-u}\right]\frac{du}{dx}$$

$$= \cosh u \cdot \frac{du}{dx}.$$

for special case of $u = x$,

$$\frac{d}{dx}\sinh x = \cosh x \qquad \left[\text{c.f. } \frac{d}{dx}\sin x = \cos x\right].$$

6/7-8

2. $\frac{d}{dx}\cosh u = \frac{d}{du}\cosh u \cdot \frac{du}{dx}$

$$= \frac{du}{dx}\left\{\frac{d}{du}\cdot\left[\frac{e^{u}+e^{-u}}{2}\right]\right\} = \frac{1}{2}\frac{du}{dx}\left\{e^{u}-e^{-u}\right\}$$

$$= \frac{du}{dx}\cdot \sinh u.$$

Special case $u=x$,

$\frac{d}{dx}\cosh x = \sinh x$ (c.f $\frac{d}{dx}\cos x = \sin x$).

---

3. $\frac{d}{dx}(\tanh u) = \frac{du}{dx}\cdot\frac{d}{du}(\tanh u)$

$$= \frac{du}{dx}\cdot\frac{d}{du}\left(\frac{\sinh u}{\cosh u}\right) = \frac{du}{dx}\cdot\left\{\frac{\cosh u\cdot\frac{d}{du}(\sinh u) - \sinh u\frac{d}{du}\cosh u}{\cosh^2 u}\right\}$$

$$= \frac{du}{dx}\cdot\left\{\frac{\cosh u\cdot\cosh u - \sinh u\cdot\sinh u}{\cosh^2 u}\right\}$$

$$= \frac{du}{dx}\cdot\left\{1-\tanh^2 u\right\} = \frac{du}{dx}\cdot\operatorname{sech}^2 u$$

Special case, $u=x$, $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$ (c.f $\frac{d}{dx}\tan x = \sec^2 x$

7/7-8

4. $\frac{d}{dx}(\coth u) = \frac{du}{dx} \cdot \frac{d}{du}\left[\frac{\cosh u}{\sinh u}\right]$

$= \frac{du}{dx} \cdot \frac{\sinh u \cdot \sinh u - \cosh u \cancel{\sin} \cosh u}{\sinh^2 u} = \frac{du}{dx} \cdot \left[1 - \coth^2 u\right]$

$= -\frac{du}{dx} \cdot \operatorname{csch}^2 x.$

$x = u,$

$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x$, c.f $\frac{d}{dx}\cot x = -\csc^2 x$

---

5. $\frac{d}{dx}(\operatorname{sech} u) = \frac{du}{dx} \cdot \frac{d}{du}(\operatorname{sech} u) = \frac{du}{dx}\frac{d}{dx}\left[\frac{1}{\cosh u}\right]$

$= \frac{du}{dx} \cdot \left\{\frac{-\cosh u \cdot 0 - \sinh u}{\cosh^2 u}\right]$

$= -\frac{du}{dx} \cdot \left[\cancel{\operatorname{sech} u}^{\,0} + \tanh u \cdot \operatorname{sech} u\right]$

$u = x,$

$\frac{d}{dx}\operatorname{sech} x = - \tanh x \operatorname{sech} x)$

c.f $\frac{d}{dx}(\sec x) = + \sec x \tan x$

---

6. $\frac{d}{dx}(\operatorname{csch} u) = \frac{du}{dx} \cdot \frac{d}{du}\left[\frac{1}{\sinh u}\right] = \frac{du}{dx} \cdot \left[\frac{0 - \cosh u}{\sinh^2 u}\right]$

$= -\frac{du}{dx}\left[\operatorname{csch} u \cdot \coth u\right]$

$u = x,$

$\frac{d}{dx}(\operatorname{csch} \cancel{x}) = -\operatorname{csch} x \coth x$, c.f $\frac{d}{dx}\csc x = -\csc x \cot$

From the derivatives, we ~~deduces that~~ 8/7.8

deduce the integral formula for the hyperbolic f2

1. $\frac{d}{dx}(\sinh x) = \cosh x \xrightarrow[\text{antiderivatives}]{\text{taking}} \int \cosh x \, dx = \sinh x + C$

2. $\frac{d}{dx}(\cosh x) = \sinh x \longrightarrow \int \sinh x \, dx = \cosh x + c$

3. $\frac{d}{dx}(\tanh x) = \cancel{\coth^2}\ \operatorname{sech}^2 x \rightarrow \int \operatorname{sec}^2 h x \, dx = \tanh x + c$

4. $\frac{d}{dx} \coth x = -\operatorname{csch}^2 x \longrightarrow \int \operatorname{csch}^2 x \, dx = -\coth x + c$

5. $\frac{d}{dx}(\operatorname{sech} x) = -\tanh x \operatorname{sech} x \rightarrow \int \tanh x \operatorname{sech} x \, dx = -\operatorname{sech} x + C$

6. $\frac{d}{dx}(\operatorname{csch} x) = -\coth x \operatorname{csch} x \rightarrow \int \coth x \operatorname{csch} x \, dx = -\operatorname{csch} x + C$

Derivatives of inverse hyperbolic f^n

e.g. 2 (pg 540) $\frac{d}{dx}(\cosh^{-1} u) = ?$; $u > 1$

Have to recall theorem 10.2 on differentiation formula for inverse functions; If $f(x_0)$ is differentiable & $f'(x_0) \neq 0$, then f^{-1} is differentiable on $y_0 = f(x_0)$, & $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$

$= \frac{1}{f'[f^{-1}(y_0)]}$

~~or $\frac{dx}{dy} = \frac{1}{dy/dx}$, $x = f^{-1}(y)$~~

$x_0 = f^{-1}(y_0)$

 9/7.8

$$(f^{-1})'(u) = \frac{1}{f'[f^{-1}(u)]}.$$

(1) we want to determine what $(\cosh^{-1}u)'$ is.

$\therefore$ identify $f(u) = \cosh(u)$; $f^{-1}(u) = \cosh^{-1}(u)$; $u > 1$

$f'(u) = \sinh(u)$

LHS:

$\therefore (f^{-1})'(u) \equiv \frac{d}{du}\cosh^{-1}(u)$

RHS: $\dfrac{1}{f'[f^{-1}(u)]} = \dfrac{1}{\sinh[f^{-1}(u)]} = \dfrac{1}{\sinh[\cosh^{-1}(u)]}$

but $\sinh[\cosh^{-1}(u)] = \sqrt{\cosh^2[\cosh^{-1}(u)] - 1}$

(recall $\cosh^2 x - \sinh^2 x = 1$)

$\therefore$ RHS $= \dfrac{1}{\sqrt{\cosh^2(\cosh^{-1}(u)) - 1}} = \dfrac{1}{\sqrt{u^2 - 1}}$

note: $\cosh(\cosh^{-1}(u)) = u$

$\therefore \cosh^2(\cosh^{-1}(u)) = u^2$

~~As~~ LHS = RHS, $\dfrac{d}{du}(\cosh^{-1}(u)) = \dfrac{1}{\sqrt{u^2-1}}$

$\therefore \dfrac{d}{dx}(\cosh^{-1}u) = \dfrac{du}{dx} \cdot \dfrac{1}{\sqrt{u^2-1}}$ $\forall$ $u > 1$

~~HHE~~.

e.g. $\frac{d}{dx}\sin^{-1}(u) = ?$ ~~This time, we use another tactics (previous method is also applicable too). Use implicit differentiation & chain rule.~~ 10/7

The theorem $(f^{-1})'(x) = \frac{1}{f'[f^{-1}(x)]}$ is also written as $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$. We will use the latter version to derive $\frac{d}{du}[\sinh^{-1}(x)]$.

② let $y = \sinh^{-1}(u) \Rightarrow \sinh y = u$; $-\infty < u < \infty$

take $\frac{d}{du}$: $\cosh y \frac{dy}{du} = 1$

$$\therefore \frac{dy}{du} = \frac{1}{\cosh y} = \frac{1}{\cosh(\sinh^{-1}(u))}$$

$\cosh^2 x - \sinh^2 x = 1$

$$= \frac{1}{\sqrt{1+\sinh^2(\sinh^{-1}(u))}} = \frac{1}{\sqrt{1+u^2}}$$

$$\therefore \frac{d}{du}(\sinh^{-1}(u)) = \frac{1}{\sqrt{1+u^2}}$$

$$\frac{d}{dx}(\sinh^{-1}(u)) = \frac{du}{dx}\frac{1}{\sqrt{1+u^2}} \quad ※.$$

$-\infty < u < \infty$

e.s. 11/7.8.

③ $\frac{d}{dx}(\tanh^{-1}u) = ?$

let $y = \tanh^{-1}u$; $-1 < u < 1$ or $|u| < 1$.

$x = \tanh y \rightarrow 1 = \operatorname{sech}^2 y \frac{dy}{dx}$

$$\frac{dy}{du} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1 - \tanh^2 y}$$

$$= \frac{1}{1 - \tanh^2(\tanh^{-1}u)} = \frac{1}{1-u^2}$$

$\therefore \frac{d}{du}[\tanh^{-1}u] = \frac{1}{1-u^2}$; $|u| < 1$

$\frac{d}{dx}\tanh^{-1}(u) = \frac{du}{dx} \cdot \frac{1}{1-u^2}$; $|u| < 1$.

D.I.Y :

④ $\frac{d}{dx}(\coth^{-1}u) = \frac{du}{dx}\frac{1}{1-u^2}$, $|u| > 1$

⑤ $\frac{d}{dx}(\operatorname{sech}^{-1}u) = \frac{-du/dx}{u\sqrt{1-u^2}}$, $0 < u < 1$

⑥ $\frac{d}{dx}(\operatorname{csch}^{-1}u) = \frac{-du/dx}{|u|\sqrt{1+u^2}}$ $u \neq 0$.

12/7

From these differentiation formula, we obtain the integral formula for hyperbolic f^{n}:

① $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}(u/a) + C, \quad u > a > 0$

② $\displaystyle\int \frac{du}{\sqrt{u^2+a^2}} = \sinh^{-1}(u/a) + C, \quad a > 0$

③ $\displaystyle\int \frac{du}{a^2-u^2} = \begin{cases} \tanh^{-1}(u/a) + C & \text{if } u^2 < a^2 \\ \frac{1}{a}\coth^{-1}(u/a) + C & \text{if } u^2 > a^2 \end{cases}$

④ $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = \frac{1}{a}\operatorname{sech}^{-1}(u/a) + C, \quad 0 < u < a$

⑤ $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C,$

$u \neq 0, \ a > 0.$

13/7.8.

e.g 3. evaluate $\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}$.

use eq. (2):

$$2\int_0^1 \frac{dx}{\sqrt{3+4x^2}} = \cancel{2\sinh^{-1}} \frac{\cancel{2}}{2}\int_0^1 \frac{dx}{\sqrt{x^2+3/4}}$$

$$= \int_0^1 \frac{dx}{\sqrt{x^2+\left(\sqrt{3/4}\right)^2}} = \left[\sinh^{-1}\left(\frac{x}{\sqrt{3/4}}\right)\right]_0^1$$

$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$

punch calculator

$$= 0.9866 - 0$$

$$= 0.9866$$

Tutorial questions.

Chap 26.

Supp 10(a), 10(b), 10(d), 10(g)

23, 24.