
UNIVERSITI SAINS MALAYSIA

First Semester Examination

Academic Session 2006/2007

October/November 2006

ZCA 110/4 - Calculus and Linear Algebra

[ZCA 110/4 - Kalkulus dan Aljabar Linear]

Duration: 3 hours

[Masa: 3 jam]

Please check that this examination paper consists of **XXX** printed pages before the examination begins.
*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **XXX** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Answer **FOUR** out of SIX questions in Section A. Please indicate the chosen questions clearly on the front page of each answer booklet. Also note that **only the first FOUR questions will be graded** if students answer more than FOUR questions.
Answer **BOTH (TWO) questions** in Section B.
[Jawab EMPAT daripada ENAM soalan yang diberikan dalam Seksyen A. Sila tunjukkan soalan-soalan pilihan anda dengan jelas di muka surat depan tiap-tiap buku jawapan. Juga diingatkan bahawa hanya EMPAT soalan pertama akan diperiksaan jika penuntut menjawab lebih dari EMPAT soalan.
Jawab KEDUA-DUA soalan dalan Seksyen B.]

1. (a) Given the following functions: *[Diberi fungsi berikut:]*

$$f(x) = \tanh(x) \text{ and } g(x) = \ln(x+1),$$

(i) Find the full domain and the corresponding range of $f(x)$ (2/100)
[Dapatkan domain penuh dan julat f(x) yang sepadan]

(ii) Find the full domain and the corresponding range of $g(x)$ (2/100)
[Dapatkan domain penuh dan julat g(x) yang sepadan]

(iii) Find the function $f \circ g$ (2/100)
[Dapatkan fungsi $f \circ g$]

(iv) Find the full domain and the corresponding range of $f \circ g$. (2/100)
[Dapatkan domain penuh dan julat $f \circ g$ yang sepadan]

(b) Find the following limits. *[Cari had-had berikut.]*

(i) $\lim\limits_{x \to 2} \dfrac{1-\sqrt{x}}{1-x}$

(ii) $\lim\limits_{x \to 1} \dfrac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$

(iii) $\lim\limits_{x \to 0} \dfrac{(x+h)^2 - x^2}{h}$

(iv) $\lim\limits_{x \to \infty} \dfrac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

(8/100)

Solution:

1(a)(i) $D_f = \mathbb{R}$, $R_f = (-1,1)$ [1 mark + 1 mark for each correct answer]

1(a)(ii) $D_g = (-1, \infty)$, $R_g = (-\infty, +\infty)$ [1 mark + 1 mark for each correct answer]

1(a)(iii) $f \circ g = f\left[g(x)\right] = \tanh\left[g(x)\right] = \tanh\left[\ln(x+1)\right]$ [2 marks]

1(a)(iv) $D_{f \circ g} = (-1, \infty)$, $R_{f \circ g} = (-1,1)$ [1 mark + 1 mark for each correct answer]

2. (a) Evaluate the following limits *[Dapatkan had-had berikut]*

(i) $\lim_{x\to\infty}\frac{\sin x}{\sqrt{x}}$ (2/100)

(ii) $\lim_{x\to 0^+}\frac{\sin x}{\sqrt{x}}$ (2/100)

(b) Determine the discontinuities (if exist) of the following functions. Determine whether they are removable or jump discontinuity.
[Tentukan ketidakselanjaran (jika wujud) fungi-fungsi berikut. Tentukan samada mereka adalah ketidakselajanran tersingkirkan atau ketidakselajanran lompatan.]

(i) $f(x) = |x| - x$ (2/100)

(ii) $f(x) = \begin{cases} x & \text{if } x=0 \\ x^2 & \text{if } 0<x<1 \\ 3-x & \text{if } x \ge 1 \end{cases}$ (2/100)

(c) Find the point and equation of the tangents on the curve $y = x - 1/(2x)$ where the gradient is 3.
[Cari titik dan persamaan bagi tangen-tangen pada lengkung $y = x - 1/(2x)$ *di mana gradiennya ialah 3]*;

(8/100)

Solution:
Q2(a) (i), Engineering Mathematics, Vol.2, CWL et al, pg. 215, Q13 (vi)

$$\lim_{x\to\infty}\frac{\sin x}{\sqrt{x}} = 0$$ [2 marks]

No intermediate steps required.

Solution:

Q2(a) (ii) $\lim_{x\to 0^+}\frac{\sin x}{\sqrt{x}} = \lim_{x\to 0^+}\left(\frac{\sqrt{x}}{\sqrt{x}}\cdot\frac{\sin x}{\sqrt{x}}\right) = \lim_{x\to 0^+}\left(\sqrt{x}\cdot\frac{\sin x}{x}\right) = \lim_{x\to 0^+}\sqrt{x}\cdot\lim_{x\to 0^+}\frac{\sin x}{x} = 0\cdot 1 = 0.$

[2 marks]

1 mark be given for showing intermediate steps.
1 mark be given for the correct answer 0.

Solution:
Q2(b) (i) Schaum's series, pg. 77, Supp. Prob. 4 (c).

No discontinuity.
2 marks given if the statement "No discontinuity" is given.

Solution:
Q2(b) (ii) Schaum's series, pg. 76, Solv.Prob.1(k). (Mistake in Schaum's series original question corrected.)

Jump discontinuity at *x*=1.
1 mark given if the statement "jump discontinuity" is given. 1 mark is given if the statement "*x*=1" is mentioned.

3. (a) (i) Given $x\sqrt{x^2+y} = 8\ln x - y^2 x \sin(xy)$, find $\frac{dy}{dx}$ in terms of *x* and *y*.

[Diberi $x\sqrt{x^2+y} = 8\ln x - y^2 x \sin(xy)$, dapatkan $\frac{dy}{dx}$ dalam sebutan x dan y.]

(4/100)

(ii) Differentiate $y = (x^2+4)^2 (2x^3-1)^3$

[Bezakan $y = (x^2+4)^2 (2x^3-1)^3$] (4/100)

(b) Find the local extreme values and inflection points of the function $y = \frac{x}{x^2+1}$. Plot its graph for the domain $[-5, 5]$ and identify these points on the graph.
[Cari nilai-nilai ekstreme tempatan dan titik-titk perubahan cekungan bagi fungsi $y = \frac{x}{x^2+1}$ .Lukis grafnya bagi domain $[-5, 5]$ dan tunjuk titik-titik ini di atas graf.]

(8/100)

Solution
Q3(a) (i), Engineering Mathematics, Vol.2, CWL et al, pg. 217, Q26 (ii)

Taking d/d*x* on both sides, we obtain:

$$\frac{d}{dx}\left(x\sqrt{x^2+y}\right)=\frac{d}{dx}\left(8\ln x-y^2x\sin\left(xy\right)\right)$$

$$LHS=x^2\left(x^2+y\right)^{-1/2}+\frac{x}{2}\left(x^2+y\right)^{-1/2}\frac{dy}{dx}+\left(x^2+y\right)^{1/2}$$

$$RHS=\frac{8}{x}-y^2x^2\cos\left(xy\right)\frac{dy}{dx}-y^3x\cos\left(xy\right)-y^2\sin\left(xy\right)-2xy\sin\left(xy\right)\frac{dy}{dx}$$

$$\left[\frac{x}{2}\left(x^2+y\right)^{-1/2}+2xy\sin\left(xy\right)+y^2x^2\cos\left(xy\right)\right]\frac{dy}{dx}=$$
$$\frac{8}{x}-y^3x\cos\left(xy\right)-y^2\sin\left(xy\right)-x^2\left(x^2+y\right)^{-1/2}-\left(x^2+y\right)^{1/2}$$

Simplifying and collecting d*y*/d*x* to the L
HS, we obtain

$$\frac{dy}{dx}=\frac{\frac{8}{x}-y^3x\cos\left(xy\right)-y^2\sin\left(xy\right)-x^2\left(x^2+y\right)^{-1/2}-\left(x^2+y\right)^{1/2}}{\frac{x}{2}\left(x^2+y\right)^{-1/2}+y^2x^2\cos\left(xy\right)+2xy\sin\left(xy\right)}$$

1 mark for correctly showing the LHS, 1 mark for showing the RHS, 2 marks be given for correct final answer.

Q3(a) (ii)

$$y=\left(x^2+4\right)^2\left(2x^3-1\right)^3$$

$$\frac{dy}{dx}=\frac{d}{dx}\left[\left(x^2+4\right)^2\left(2x^3-1\right)^3\right]=\left(2x^3-1\right)^3\frac{d\left(x^2+4\right)^2}{dx}+\left(x^2+4\right)^2\frac{d\left(2x^3-1\right)^3}{dx}$$

$$=2\left(2x^3-1\right)^3\left(x^2+4\right)\left(2x\right)+3\left(x^2+4\right)^2\left(2x^3-1\right)^2\left(6x^2\right)$$

$$=2\left(x^2+4\right)\left(2x^3-1\right)^2\left[2x\left(2x^3-1\right)+9x^2\left(x^2+4\right)\right]$$

$$=2x\left(x^2+4\right)\left(2x^3-1\right)^2\left[13x^3+36x-2\right]$$

$$=$$
$$-16x+288x^2-4x^3+240x^4-1152x^5+42x^6-768x^7$$
$$+1152x^8-120x^9+704x^{10}+104x^{12}$$

3 marks be given if intermediate steps are explicitly and correctly shown. 1 mark be given for correct final answer.

4. (a) Show that $f\left(x\right)=\sqrt[3]{\left(x-2\right)^{2}}$ is continuous at x=2 but not differentiable at x=2.

[Tunjukkan bahawa $f\left(x\right)=\sqrt[3]{\left(x-2\right)^{2}}$ *adalah selanjar pada x=2 tapi tak terbezakan pada x=2.]*

(8/100)

(b) A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?
[Suatu bidang tanah bersegiempat bujur adalah dibatasi pada satu sisi oleh sebatang sungai dan pada tiga sisi lagi oleh suatu pagar elektrik berdawai tunggal. Dengan 800 m dawai untuk kegunaan anda, apakah luas terbesar di mana anda boleh liputi, dan apakah dimensinya?]

(8/100)

Solution

Q4(a) Engineering Mathematics, Vol.2, CWL et al, pg. 217, Q23.

Given that $f\left(x\right)=(x\text{-}2)^{3/2}$,

$f(2)$ is defined with $f(2) = 0$.

Also, $\lim_{x\to 2} f\left(x\right)$ exists, with $\lim_{x\to 2} f\left(x\right)=0$.

Hence $f\left(x\right)$ is continuous at $x = 2$.

[4 marks if steps showing continuity of $f\left(x\right)$ at $x = 2$ is provided.]

Next, consider the derivative of $f\left(x\right)$ at $x = 2$:

$$\lim_{x\to 2}\frac{f\left(x\right)-f\left(2\right)}{x-2}=\lim_{x\to 2}\frac{\left(x-2\right)^{2/3}-0}{x-2}=\lim_{x\to 2}\frac{1}{\left(x-2\right)^{1/3}}$$

Limit does not exist. Hence, $f\left(x\right)$ not differentiable $x = 2$.

[4 marks if steps showing non-differentiability of $f\left(x\right)$ at $x = 2$ is provided.]

5. (a)(i) Prove *[Buktikan]*

$$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{x\sqrt{x^{2}-1}},\ |x|<1.$$

(4/100)

(ii) Prove *[Buktikan]*

$$\frac{d}{dx}\left(\operatorname{csch}^{-1}x\right)=\frac{-1}{|x|\sqrt{1+x^{2}}},\ x\neq 0.$$

(4/100)

(b) Evaluate the following integrals:
[Nilaikan kamiran-kamiran berikut:]

(i) $\int \frac{1}{\sqrt{25+y^2}}\, dy$

(ii) $\int x^2 \sin(1-x)\, dx$

(8/100)

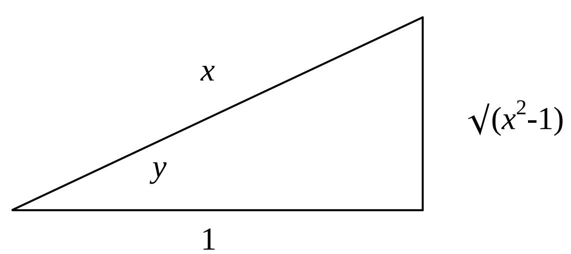
Solution

Q5(a) (i) Schaum's series, pg. 170, Solv. Prob. 1.

Let $y = \sec^{-1} x$, then $x = \sec y$.

$\frac{d}{dx} x = \frac{d}{dx} \sec y$

$RHS = \frac{d}{dx} \sec y = \frac{dy}{dx} \cdot \frac{d}{dy} \sec y = \frac{dy}{dx} \cdot \tan y \sec y$



LHS = 1

RHS = LHS: $\frac{dy}{dx} = 1/\tan y \sec y =$

tan y = $\sqrt{(x^2-1)}$; sec y = x;

tan y sec y = $\sqrt{(x^2-1)}/x$

$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$

Full intermediate steps must be explicitly and correctly shown in order to be given full 4 marks. Deduce marks accordingly if intermediate steps are inconsistent or erroneous.

Solution: Q5(a) (ii)

Let $y = \operatorname{csch}^{-1} x$, then $x = \operatorname{csch} y = \dfrac{1}{\sinh y}$.

$$\frac{d}{dx} x = \frac{d}{dx} \frac{1}{\sinh y}$$

$$RHS = \frac{d}{dx} \frac{1}{\sinh y} = \frac{dy}{dx} \cdot \frac{d}{dy} \left(\frac{1}{\sinh y} \right) = \frac{dy}{dx} \cdot \left(\frac{0 \text{ - } \cosh y}{\sinh^2 y} \right) = -\frac{dy}{dx} \cdot \left(\frac{\cosh y}{\sinh^2 y} \right)$$

LSH = RHS

$$1 = -\frac{dy}{dx} \cdot \left(\frac{\cosh y}{\sinh^2 y} \right)$$

$$\frac{dy}{dx} = \frac{\text{-}\sinh^2 y}{\cosh y} = \frac{\text{-}\sinh^2 y}{\sqrt{1+\sinh^2 y}} = \frac{-(1/x)^2}{\sqrt{1+(1/x)^2}} = -\frac{1}{x^2} \frac{|x|}{\sqrt{x^2+1}} = -\frac{1}{|x|} \frac{1}{\sqrt{x^2+1}} =$$

Full intermediate steps must be explicitly and correctly shown in order to be given full 4 marks. Deduce marks accordingly if intermediate steps are inconsistent or erroneous.

6. (a) (i) Evaluate the indefinite integration $\int \tan^4 x \, dx$.

[Nilaikan kamiran tak tentu $\int \tan^4 x \, dx$ *]* (4/100)

(ii) Find the arc length L of the curve $y = x^{3/2}$ from $x = 0$ to $x = 5$.

[Dapatkan L, panjang lengkuk bagi lengkung $y = x^{3/2}$ dari x=0 ke x=5.]

(4/100)

(b) Evaluate the following integrals: *[Nilaikan kamiran-kamiran berikut:]*

(i) $\int \frac{2 - \cos x + \sin x}{\sin^2 x} dx$

(ii) $\int \frac{x^3}{x^2 - 2x + 1} dx$

(8/100)

Solution

Q6(a) (i) Schaum's series, pg. 297, Solv. Prob. 16.

$$\begin{aligned}\int \tan^4 x\, dx &= \int \tan^2 x \tan^2 x\, dx = \int \tan^2 x(\sec^2 x - 1)\, dx \\ &= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx \\ &= \tfrac{1}{3}\tan^3 x - \int (\sec^2 x - 1)\, dx \\ &= \tfrac{1}{3}\tan^3 x - (\tan x - x) + C \\ &= \tfrac{1}{3}\tan^3 x - \tan x + x + C\end{aligned}$$

Full intermediate steps must be explicitly and correctly shown in order to be given full 4 marks. Deduce marks accordingly if intermediate steps are inconsistent or erroneous.

Solution

Q6(a) (ii) Schaum's series, pg. 261, e.g. 3.

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b \sqrt{1 + (y')^2}\, dx$$

1 mark for stating the formula

Find the arc length L of the curve $y = x^{3/2}$ from $x = 0$ to $x = 5$.
since $y' = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$,

$$L = \int_0^5 \sqrt{1 + (y')^2}\, dx = \int_0^5 \sqrt{1 + \tfrac{9}{4}x}\, dx$$

$$= \tfrac{4}{9}\int_0^5 (1 + \tfrac{9}{4}x)^{1/2}(\tfrac{9}{4})\, dx = \tfrac{4}{9}\tfrac{2}{3}(1 + \tfrac{9}{4}x)^{3/2}\Big]_0^5$$ (by Quick Formula I and the Fundamental Theorem of Calculus)

$$= \tfrac{8}{27}((\tfrac{49}{4})^{3/2} - 1^{3/2}) = \tfrac{8}{27}(\tfrac{343}{8} - 1) = 335/27 = 12.4$$

2 marks for showing correct intermediate steps.
1 mark for correct answer.

SECTION B

Instruction: Answer ALL questions in this Section. Each question carries 18 marks.
[Arahan: Jawab semua soalan dalam Bahagian ini. Setiap soalan membawa 18 markah.]

7. (a) Evaluate *[Nilaikan]*

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots \qquad (5/100)$$

(b) Determine whether the sequence $\langle s_n \rangle$ converges or diverges. If it converges, find the limit.
[Tentukan samada jujukan $\langle s_n \rangle$ menumpu atau mencapah. Jika menumpu, dapatkan hadnya.]

(i) $s_n = \dfrac{n}{2n+1}$ (2/100)

(ii) $s_n = (8 - 2n)$ (2/100)

(c) Find *[Cari]*
(i) the Fourier cosine series and *[siri cosinus Fourier dan]*
(ii) the Fourier sine series *[siri sinus Fourier]*

of f on the given interval. *[f atas selang yang diberikan]*

$$f(x) = \cos x, \quad 0 < x < \pi/2.$$

(9/100)

Solution
Q7(a) Schaum's series, pg. 397, Solv. Prob. 6.

This is a geometric series with
ratio r = - ½ [1 mark]
and first term
a = 1. [1 mark]
Since $|r|$ = ½ < 1 the series converges and its sum is

$S = \dfrac{a}{1-r}$ [2 marks]

$= \dfrac{1}{1-(1/2)} = \dfrac{2}{3}$ [1 mark]

Solution
Q7(b) Anton et. al, 8th edition, pg. 633, e.g. 3 (a), (d)

Q7(b)(i) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{(2+1/n)} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}(2+1/n)} = \frac{1}{2}$ [1 mark]

The sequence, $\langle s_n \rangle$, converges. [1 mark]

Q7(b)(ii) $\lim_{n\to\infty} s_n = \lim_{n\to\infty}(8-2n) = -\infty$. [1 mark]

The sequence, $\langle s_n \rangle$, diverges. [1 mark]

8. (a) Write down the following system of linear equations in matrix notation and find the solutions of this system using Cramer's Rule.
[Tulis sistem persamaan linear berikut dengan menggunakan nyataan matriks dan cari penyelesaian sistem ini dengan menggunakan kaedah Cramer.]

$$\begin{aligned} -4x_1 + 2x_2 + x_3 &= 7 \\ x_1 - 2x_3 &= 3 \\ 2x_1 + 5x_4 &= 2 \\ 3x_1 + 2x_2 - x_3 + x_4 &= 1 \end{aligned}$$

(9/100)

(b)

(i) Reduce, using elementary row operations, the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ to row reduced echelon form.
[Tutunkan, dengan menggunakan operasi-operasi baris permulaan, matriks $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ kepada bentuk echelon terturun.] (4/100)

(ii) Hence, or otherwise, determine the inverse of *A*.
[Oleh yang demikian, atau dengan cara lain, tentukan songsangan bagi A]

(5/100)

Solution
8b(i)

$$\begin{pmatrix}1&1&0\\0&1&1\\1&1&1\end{pmatrix}\xrightarrow{R_3-R_1}\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\xrightarrow{R_2-R_3}\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{R_1-R_2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

procedural steps = 3 marks
correct answer = 1 marks

8b(ii)

By Gauss-Jordan elimination, the inverse of A is obtained by performing the consecutive operations of

R_3-R_1, R_2-R_3 and then R_1-R_2 on I_3:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{R_3-R_1}\begin{pmatrix}1&0&0\\0&1&0\\-1&0&1\end{pmatrix}\xrightarrow{R_2-R_3}\begin{pmatrix}1&0&0\\1&1&-1\\-1&0&1\end{pmatrix}\xrightarrow{R_1-R_2}\begin{pmatrix}0&-1&1\\1&1&-1\\-1&0&1\end{pmatrix}$$

Hence, $A^{-1}=\begin{pmatrix}0&-1&1\\1&1&-1\\-1&0&1\end{pmatrix}$.

Alternatively, also accept A^{-1} obtained via the following procedure:

$$|A|=\begin{pmatrix}1&1&0\\0&1&1\\1&1&1\end{pmatrix}=1\cdot\begin{vmatrix}1&1\\1&1\end{vmatrix}-0\cdot\begin{vmatrix}1&0\\1&1\end{vmatrix}+1\cdot\begin{vmatrix}1&0\\1&1\end{vmatrix}=0-0+1=1$$

$$Adj(A)=\begin{pmatrix}+\begin{vmatrix}1&1\\1&1\end{vmatrix}&-\begin{vmatrix}0&1\\1&1\end{vmatrix}&+\begin{vmatrix}0&1\\1&1\end{vmatrix}\\-\begin{vmatrix}1&0\\1&1\end{vmatrix}&+\begin{vmatrix}1&0\\1&1\end{vmatrix}&-\begin{vmatrix}1&1\\1&1\end{vmatrix}\\+\begin{vmatrix}1&0\\1&1\end{vmatrix}&-\begin{vmatrix}1&0\\0&1\end{vmatrix}&+\begin{vmatrix}1&1\\0&1\end{vmatrix}\end{pmatrix}^T=\begin{pmatrix}0&1&-1\\-1&1&0\\1&-1&1\end{pmatrix}^T=\begin{pmatrix}0&-1&1\\1&1&-1\\-1&0&1\end{pmatrix}$$

$$A^{-1}=\frac{Adj(A)}{|A|}=\begin{pmatrix}0&-1&1\\1&1&-1\\-1&0&1\end{pmatrix}.$$

procedural steps = 3 marks if all correct
correct answer = 2 marks