Chapter 8

Techniques of Integration

8.1

Basic Integration Formulas

(2nd lecture of week 17/09/07-22/09/07)

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C \qquad \text{(any number } k)$
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \frac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$
21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad (a > 0)$
22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \qquad (u > a > 0)$

Example 1 Making a simplifying substitution

$$\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \int \frac{d(\overbrace{x^2-9x}^{u})}{\sqrt{x^2-9x+1}}$$

$$= \int \frac{du}{\sqrt{u+1}} = \int \frac{d(u+1)}{\sqrt{u+1}} = \int \frac{dv}{\sqrt{v}} = 2v^{1/2} + C$$

$$= 2(u+1)^{1/2} + C = 2\left(x^2-9x+1\right)^{1/2} + C$$

Example 2 Completing the square

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x-4)^2}} =$$

$$\int \frac{d(x-4)}{\sqrt{16 - (x-4)^2}} = \int \frac{du}{\sqrt{4^2 - u^2}}$$

$$= \sin^{-1}\frac{u}{4} + C = \sin^{-1}\left(\frac{x-4}{4}\right) + C$$

Example 3 Expanding a power and using a trigonometric identity

$$\int (\sec x + \tan x)^2 dx$$

$$= \int (\sec^2 x + \tan^2 x + 2\sec x \tan x) dx.$$

$$\text{Racall:} \tan^2 x = \sec^2 x - 1; \frac{d}{dx}\tan x = \sec^2 x; \frac{d}{dx}\sec x = \tan x \sec x;$$

$$= \int (2\sec^2 x - 1 + 2\sec x \tan x) dx$$

$$= 2\tan x + -x + 2\sec x + C$$

Example 4 Eliminating a square root

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx =$$

$$\cos 4x = \cos 2(2x) = 2\cos^2(2x) - 1$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\,dx = \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, x$$

$$= \sqrt{2}\int_0^{\pi/4} \cos 2x\,dx = \ldots$$

Example 5 Reducing an improper fraction

$$\int \frac{3x^2 - 7x}{3x+2} dx$$

$$= \int x - 3 + \frac{6}{3x+2} dx$$

$$= \int x - 3 + \frac{2}{x+2/3} dx$$

$$= \frac{1}{2}x^2 - 3x + 2\ln\left| x + \frac{2}{3} \right| + C$$

Example 6 Separating a fraction

$$\int \frac{3x+2}{\sqrt{1-x^2}}dx$$

$$= 3\int \frac{x}{\sqrt{1-x^2}}dx + \int \frac{2}{\sqrt{1-x^2}}dx$$

$$= 3\int \frac{\frac{1}{2}d(x^2)}{\sqrt{1-x^2}} + 2\int \frac{1}{\sqrt{1-x^2}}dx$$

$$= \frac{3}{2}\int \frac{du}{\sqrt{1-u}} + 2\sin^{-1}x + C$$

$$\int \frac{du}{(1-u)^{1/2}} = -2(1-u)^{1/2} + C'$$

$$= \frac{3}{2}[-2(1-u)^{1/2}] + 2\sin^{-1}x + C''$$

$$= -3\sqrt{(1-x^2)} + 2\sin^{-1}x + C''$$

Example 7 Integral of $y = \sec x$

$$\int \sec x dx = ?$$

$$d \sec x = \sec x \tan x dx$$

$$d \tan x = \sec^2 x dx = \sec x \sec x dx$$

$$d(\sec x + \tan x) = \sec x(\sec x + \tan x) dx$$

$$\sec x dx = \frac{d(\sec x + \tan x)}{\sec x + \tan x}$$

$$\int \sec x dx = \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} = \ln|\sec x + \tan x| + C$$

TABLE 8.2 The secant and cosecant integrals

1. $$\int \sec u \, du = \ln |\sec u + \tan u| + C$$

2. $$\int \csc u \, du = -\ln |\csc u + \cot u| + C$$

Procedures for Matching Integrals to Basic Formulas

Procedure	Example
Making a simplifying substitution	$\frac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \frac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$
Using a trigonometric identity	$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x\tan x + \tan^2 x$ $= \sec^2 x + 2\sec x \tan x + (\sec^2 x - 1)$ $= 2\sec^2 x + 2\sec x\tan x - 1$
Eliminating a square root	$\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$
Reducing an improper fraction	$\frac{3x^2-7x}{3x+2} = x - 3 + \frac{6}{3x+2}$
Separating a fraction	$\frac{3x+2}{\sqrt{1-x^2}} = \frac{3x}{\sqrt{1-x^2}} + \frac{2}{\sqrt{1-x^2}}$
Multiplying by a form of 1	$\sec x = \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}$ $= \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}$

8.2

Integration by Parts

(2nd lecture of week 17/09/07-22/09/07)

Product rule in integral form

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$$

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int g(x)\frac{d}{dx}[f(x)]dx + \int f(x)\frac{d}{dx}[g(x)]dx$$

$$f(x)g(x) = \int g(x)f'(x)dx + \int f(x)g'(x)dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Integration by parts formula

Alternative form of the integration by parts formula

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$$

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int g(x)\frac{d}{dx}[f(x)]dx + \int f(x)\frac{d}{dx}[g(x)]dx$$

$$f(x)g(x) = \int g(x)df(x) + \int f(x)dg(x)$$

Let $u = f(x); v = g(x)$. The above formular is recast into the form

$$uv = \int vdu + \int udv$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

Example 4 Repeated use of integration by parts

$$\int x^2 e^x dx = ?$$

Example 5 Solving for the unknown integral

$$\int e^x \cos x dx = ?$$

Evaluating by parts for definite integrals

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \tag{3}$$

Example 6 Finding area

- Find the area of the region in Figure 8.1

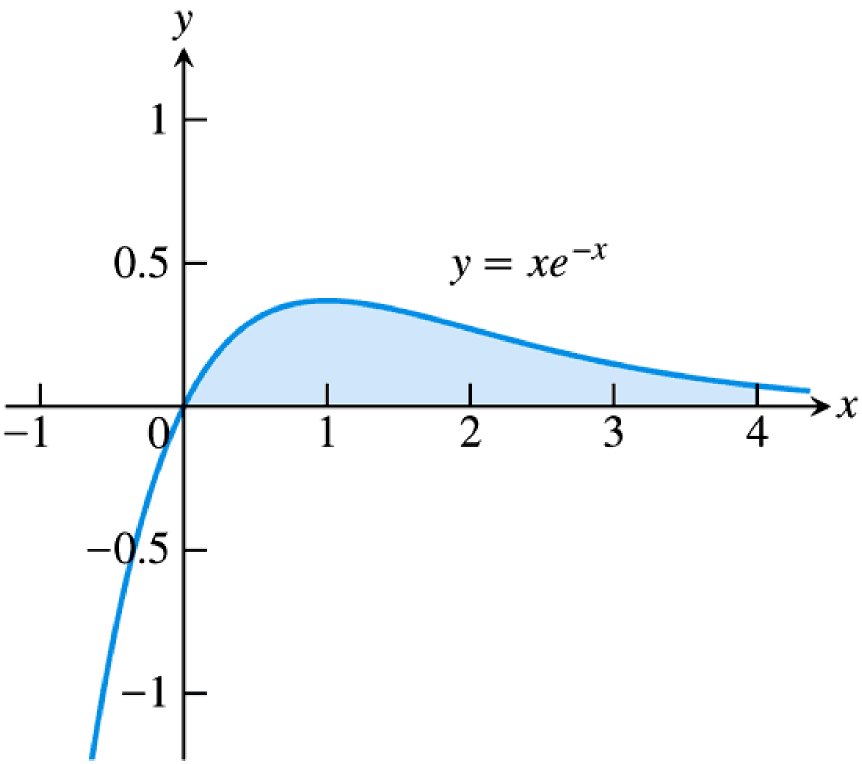


FIGURE 8.1 The region in Example 6.

Solution

$$\int_0^4 xe^{-x}dx = \ldots$$

Example 9 Using a reduction formula

- Evaluate $\int \cos^3 x dx$

8.3

Integration of Rational Functions by Partial Fractions

(3rd lecture of week 17/09/07-22/09/07)

General description of the method

- A rational function $f(x)/g(x)$ can be written as a sum of partial fractions. To do so:
- (a) The degree of $f(x)$ must be less than the degree of $g(x)$. That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term.
- We must know the factors of $g(x)$. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors.

Reducibility of a polynomial

- A polynomial is said to be **reducible** if it is the product of two polynomials of lower degree.
- A polynomial is **irreducible** if it is not the product of two polynomials of lower degree.

- THEOREM (Ayers, Schaum's series, pg. 305)
- Consider a polynomial $g(x)$ of order $n \geq 2$ (with leading coefficient 1). Two possibilities.

1. $g(x) = (x-r)h(x)$, where $h_1(x)$ is a polynomial of degree n-1, or
2. $g(x) = (x^2+px+q)\ h_2(x)$, where $h_2(x)$ is a polynomial of degree n-2, with the irreducible quadratic factor (x^2+px+q).

Example

$$g(x) = x^3 - 4x = \underbrace{(x-2)}_{\text{linear factor}} \cdot \underbrace{x(x+2)}_{\text{poly. of degree 2}}$$

$$g(x) = x^3 + 4x = \underbrace{(x^2+4)}_{\text{irreducible quadratic factor}} \cdot \underbrace{x}_{\text{poly. of degree 1}}$$

$$g(x) = x^4 - 9 = \underbrace{(x^2+3)}_{\text{irreducible quadratic factor}} \cdot \underbrace{(x+\sqrt{3})(x-\sqrt{3})}_{\text{poly. or degree 2}}$$

$$g(x) = x^3 - 3x^2 - x + 3 = \underbrace{(x+1)}_{\text{linear factor}} \underbrace{(x-2)^2}_{\text{poly. or degree 2}}$$

Quadratic polynomial

- A quadratic polynomial (polynomial or order $n = 2$) is either reducible or not reducible.
- Consider: $g(x) = x^2 + px + q$.
- If $(p^2 - 4q) \geq 0$, $g(x)$ is reducible, i.e. $g(x) = (x + r_1)(x + r_2)$.
- If $(p^2 - 4q) < 0$, $g(x)$ is irreducible.

- In general, a polynomial of degree *n* can always be expressed as the product of linear factors and irreducible quadratic factors:

$$P_n(x) = (x - r_1)^{n_1}(x - r_2)^{n_2} \ldots (x - r_l)^{n_l} \times$$

$$(x^2 + p_1 x + q_1)^{m_1}(x^2 + p_2 x + q_2)^{m_2} \ldots (x^2 + p_k x + q_k)^{m_k}$$

$$n = (n_1 + n_2 + \ldots + n_l) + 2(m_1 + m_2 + \ldots + m_l)$$

Integration of rational functions by partial fractions

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1 x + C_1}{x^2 + px + q} + \frac{B_2 x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_n x + C_n}{(x^2 + px + q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example 1 Distinct linear factors

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \ldots$$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)} = \ldots$$

Example 2 A repeated linear factor

$$\int \frac{6x+7}{(x+2)^2}\,dx = \ldots$$

$$\frac{6x+7}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

Example 3 Integrating an improper fraction

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx = \ldots$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{5x - 3}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)} = \ldots$$

Example 4 Integrating with an irreducible quadratic factor in the denominator

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \ldots$$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2} = \ldots$$

Example 5 A repeated irreducible quadratic factor

$$\int \frac{1}{x(x^2+1)^2}\,dx = ?$$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2} = \ldots$$

Other ways to determine the coefficients

- Example 8 Using differentiation
- Find A, B and C in the equation

$$\frac{x-1}{(x+1)^3}=\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

$$\frac{A(x+1)^2+B(x+1)+C}{(x+1)^3}=\frac{x-1}{(x+1)^3}$$

$$\Rightarrow A(x+1)^2+B(x+1)+C=x-1$$

$$x=-1\rightarrow C=-2$$

$$\Rightarrow A(x+1)^2+B(x+1)=x+1$$

$$\Rightarrow A(x+1)+B=1$$

$$\frac{d}{dx}[A(x+1)+B]=\frac{d}{dx}(1)=0$$

$$A=0$$

$$B=1$$

Example 9 Assigning numerical values to x

- Find A, B and C in

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}$$

$$=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$$

$$A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)\equiv f(x)$$

$$=x^2+1$$

$$f(1)=2A+=1^2+1=2\Rightarrow A=1$$

$$f(2)=-B=2^2+1=5;\Rightarrow B=-5$$

$$f(3)=2C=3^2+1=10;\Rightarrow C=5$$

8.4

Trigonometric Integrals

(3rd lecture of week 17/09/07-22/09/07)

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the work into three cases.

Case 1 If m is odd, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Example 1 *m* is odd

$$\int \sin^3 x \cos^2 x \; dx = ?$$

$$\int \sin^3 x \cos^2 x \; dx = -\int \sin^2 x \cos^2 x \; d\left(\cos x\right)$$

$$= \int (\cos^2 x - 1) \cos^2 x \; d\left(\cos x\right)$$

$$= \int (u^2 - 1) u^2 du = \ldots$$

Example 2 *m* is even and *n* is odd

$$\int \cos^5 x \quad dx = ?$$

$$\int \cos^3 x \cos^2 x \; dx = \int \cos^2 x \cos^2 x \; d\sin x$$

$$= \int (1\text{-}\sin^2 x)(1\text{-}\sin^2 x) \; d\sin x$$

$$= \int (1\text{-}u^2)(1\text{-}u^2) \; du = \ldots$$

Example 3 *m* and *n* are both even

$$\int \cos^2 x \sin^4 x \quad dx = ?$$

$$\int \cos^2 x \sin^4 x \, dx =$$

$$\int \left(\frac{1\text{-}\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 \, dx$$

$$= \frac{1}{4}\int (1\text{-}\cos 2x)(1+\cos 2x)^2 \, dx$$

$$= \frac{1}{4}\int \left(1 + \cos 2x - \cos^2 2x - \cos \;\; 2x\right) dx = \ldots$$

Example 4 Eliminating square roots

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx = ?$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx$$

$$= \int_0^{\pi/4} \sqrt{2\cos^2 2x}\,dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\,dx = \ldots$$

Example 6 Integrals of powers of tan x and sec x

$$\int \sec^3 x dx = ?$$

Use integration by parts.

$$\int \sec^3 x dx = \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x dx}_{dv};$$

$$dv = \sec^2 x dx \rightarrow v = \int \sec^2 x dx = \tan x$$

$$u = \sec \rightarrow du = \sec x \tan x dx$$

$$\int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x dx}_{dv}$$

$$= \sec x \tan x - \int \tan x \cdot \underbrace{\sec x \tan x dx}_{du}$$

$$= \sec x \tan x - \int \tan x^2 \sec x dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx$$

$$\int \sec^3 x dx = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx ...$$

$$\int \sec x dx = \int \sec x \frac{(\tan x + \sec x)}{\tan x + \sec x} dx$$

$$= \int \frac{(\sec x \tan x + \sec^2 x)}{\tan x + \sec x} dx$$

$$= \int \frac{d(\sec x + \tan x)}{\tan x + \sec x}$$

$$= \ln|\sec x + \tan x| + C$$

Example 7 Products of sines and cosines

$$\int \cos 5x \sin 3x dx = ?$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x];$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x];$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

$$\int \cos 5x \sin 3x dx$$

$$= \frac{1}{2}\int [\sin(-2x) + \sin 8x] dx$$

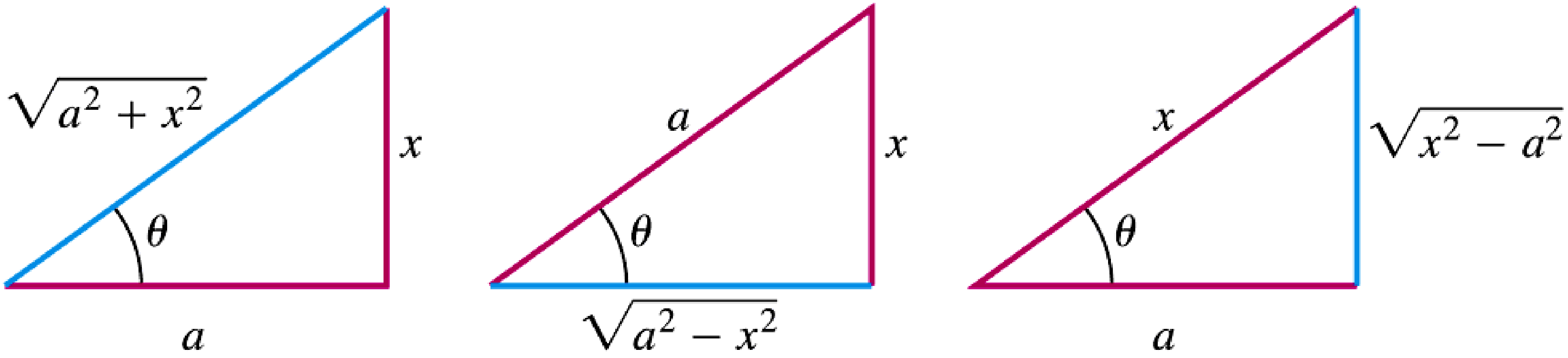
$$= \dots$$

8.5

Trigonometric Substitutions

(1st lecture of week 24/09/07-29/09/07)

Three basic substitutions



$x = a \tan \theta$ $\quad$ $x = a \sin \theta$ $\quad$ $x = a \sec \theta$

$\sqrt{a^2 + x^2} = a|\sec \theta|$ $\quad$ $\sqrt{a^2 - x^2} = a|\cos \theta|$ $\quad$ $\sqrt{x^2 - a^2} = a|\tan \theta|$

FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

Useful for integrals involving $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$

Example 1 Using the substitution $x = a\tan\theta$

$$\int \frac{dx}{\sqrt{4+x^2}} = ?$$

$$x = 2\tan y \rightarrow dx = 2\sec^2 y dy = 2(\tan^2 y + 1)dy$$

$$\int \frac{dx}{\sqrt{4+4\tan^2 y}} = \int \frac{2(\tan^2 y + 1)}{\sqrt{4+4\tan^2 y}} dy$$

$$= \int \frac{(\tan^2 y + 1)}{\sqrt{1+\tan^2 y}} dy = \int \sqrt{\sec^2 y} dy = \int |\sec y| dy$$

$$= \ln|\sec y + \tan y| + C$$

Example 2 Using the substitution $x = a\sin\theta$

$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = ?$$

$$x = 3\sin y \rightarrow dx = 3\cos y \; dy$$

$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2 y \cdot 3\cos y \; dy}{\sqrt{9-9\sin^2 y}} =$$

$$= 9\int \frac{\sin^2 y \cdot \cos y \; dy}{\sqrt{1-\sin^2 y}}$$

$$= 9\int \sin^2 y dy = \ldots$$

Example 3 Using the substitution $x = a\sec\theta$

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = ?$$

$$x = \frac{2}{5}\sec y \rightarrow dx = \frac{2}{5}\sec y \tan y \; dy$$

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \frac{2}{5}\int \frac{\sec y \tan y \; dy}{\sqrt{4\sec^2 y - 4}} = \frac{1}{5}\int \frac{\sec y \tan y \; dy}{\sqrt{\sec^2 y - 1}}$$

$$= \frac{1}{5}\int \frac{\sec y \tan y \; dy}{\sqrt{\sec^2 y - 1}} = \frac{1}{5}\int \sec y \; dy$$

$$= \frac{1}{5}\ln|\sec y + \tan y| + C = \ldots$$

Example 4 Finding the volume of a solid of revolution

$$V = 16\pi \int_0^2 \frac{dx}{\left(x^2+4\right)^2} = ?$$

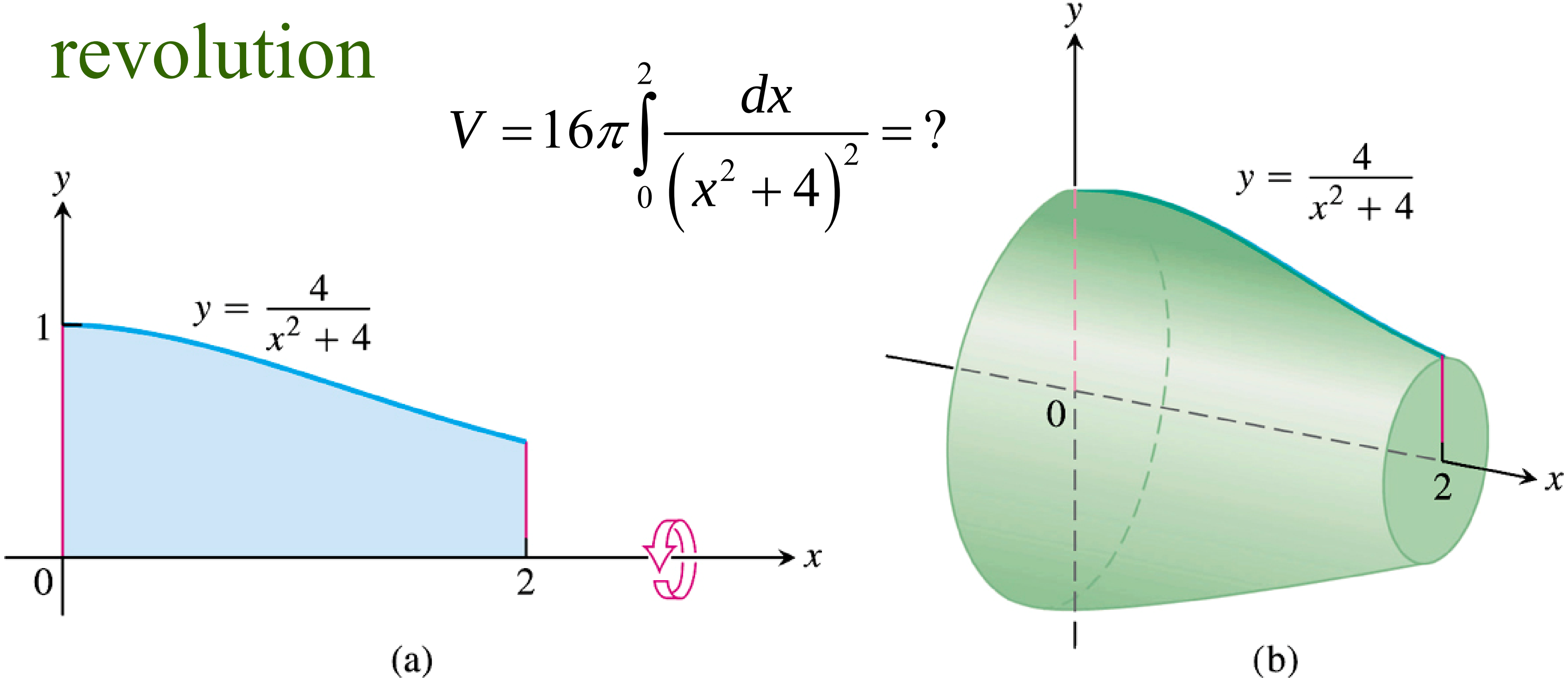


FIGURE 8.7 The region (a) and solid (b) in Example 4.

Solution

$$V = 16\pi \int_0^2 \frac{dx}{\left(x^2+4\right)^2} = ?$$

Let $x = 2\tan y \to dx = 2\sec^2 y dy$

$$V = \pi \int_0^{\pi/4} \frac{2\sec^2 y dy}{\left(\tan^2 y + 1\right)^2} = \pi \int_0^{\pi/4} \frac{2\sec^2 y dy}{\left(\sec^2 y\right)^2}$$

$$= 2\pi \int_0^{\pi/4} \cos^2 y dy = \ldots$$

8.6

Integral Tables

(1st lecture of week 24/09/07-29/09/07)

Integral tables is provided at the back of Thomas'

- T-4 A brief tables of integrals
- Integration can be evaluated using the tables of integral.

EXAMPLE 1 Find

$$\int x(2x + 5)^{-1}\,dx.$$

Solution We use Formula 8 (not 7, which requires $n \neq -1$):

$$\int x(ax + b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax + b| + C.$$

With $a = 2$ and $b = 5$, we have

$$\int x(2x + 5)^{-1}\,dx = \frac{x}{2} - \frac{5}{4}\ln|2x + 5| + C.$$

EXAMPLE 2 Find

$$\int \frac{dx}{x\sqrt{2x+4}}.$$

Solution We use Formula 13(b):

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{ax+b} - \sqrt{b}}{\sqrt{ax+b} + \sqrt{b}} \right| + C, \qquad \text{if } b > 0.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x+4}} = \frac{1}{\sqrt{4}} \ln \left| \frac{\sqrt{2x+4} - \sqrt{4}}{\sqrt{2x+4} + \sqrt{4}} \right| + C$$

$$= \frac{1}{2} \ln \left| \frac{\sqrt{2x+4} - 2}{\sqrt{2x+4} + 2} \right| + C.$$

EXAMPLE 3 Find

$$\int \frac{dx}{x\sqrt{2x - 4}}.$$

Solution We use Formula 13(a):

$$\int \frac{dx}{x\sqrt{ax - b}} = \frac{2}{\sqrt{b}} \tan^{-1} \sqrt{\frac{ax - b}{b}} + C.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x - 4}} = \frac{2}{\sqrt{4}} \tan^{-1} \sqrt{\frac{2x - 4}{4}} + C = \tan^{-1} \sqrt{\frac{x - 2}{2}} + C.$$

EXAMPLE 4 Find

$$\int \frac{dx}{x^2\sqrt{2x - 4}}.$$

Solution We begin with Formula 15:

$$\int \frac{dx}{x^2\sqrt{ax + b}} = -\frac{\sqrt{ax + b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax + b}} + C.$$

With $a = 2$ and $b = -4$, we have

$$\int \frac{dx}{x^2\sqrt{2x - 4}} = -\frac{\sqrt{2x - 4}}{-4x} + \frac{2}{2 \cdot 4}\int \frac{dx}{x\sqrt{2x - 4}} + C.$$

We then use Formula 13(a) to evaluate the integral on the right (Example 3) to obtain

$$\int \frac{dx}{x^2\sqrt{2x - 4}} = \frac{\sqrt{2x - 4}}{4x} + \frac{1}{4}\tan^{-1}\sqrt{\frac{x - 2}{2}} + C.$$

EXAMPLE 5 Find

$$\int x \sin^{-1} x \, dx.$$

Solution We use Formula 99:

$$\int x^n \sin^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \sin^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1} \, dx}{\sqrt{1 - a^2x^2}}, \qquad n \neq -1.$$

With $n = 1$ and $a = 1$, we have

$$\int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2 \, dx}{\sqrt{1 - x^2}}.$$

The integral on the right is found in the table as Formula 33:

$$\int \frac{x^2}{\sqrt{a^2 - x^2}} \, dx = \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) - \frac{1}{2} x \sqrt{a^2 - x^2} + C.$$

With $a = 1$,

$$\int \frac{x^2 \, dx}{\sqrt{1 - x^2}} = \frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1 - x^2} + C.$$

The combined result is

$$\int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \left(\frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1 - x^2} + C \right)$$

$$= \left(\frac{x^2}{2} - \frac{1}{4} \right) \sin^{-1} x + \frac{1}{4} x \sqrt{1 - x^2} + C'.$$

8.8

Improper Integrals

(2nd lecture of week 24/09/07-29/09/07)

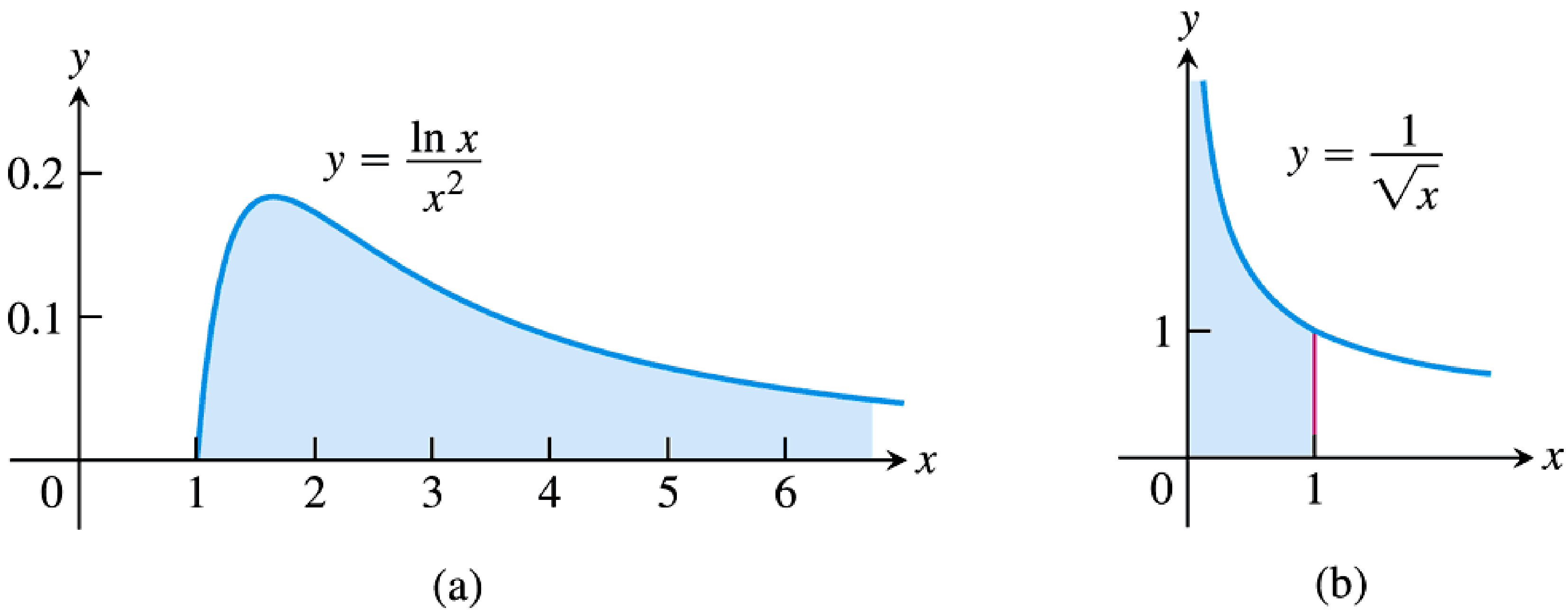


FIGURE 8.17 Are the areas under these infinite curves finite?

Infinite limits of integration

$$A(a) = \lim_{b\to\infty} A(b) = \lim_{b\to\infty} 2 - 2e^{-b/2} = 2$$

$$A(b) = \int_0^b e^{-x/2}dx = \ldots = 2 - 2e^{-b/2}$$

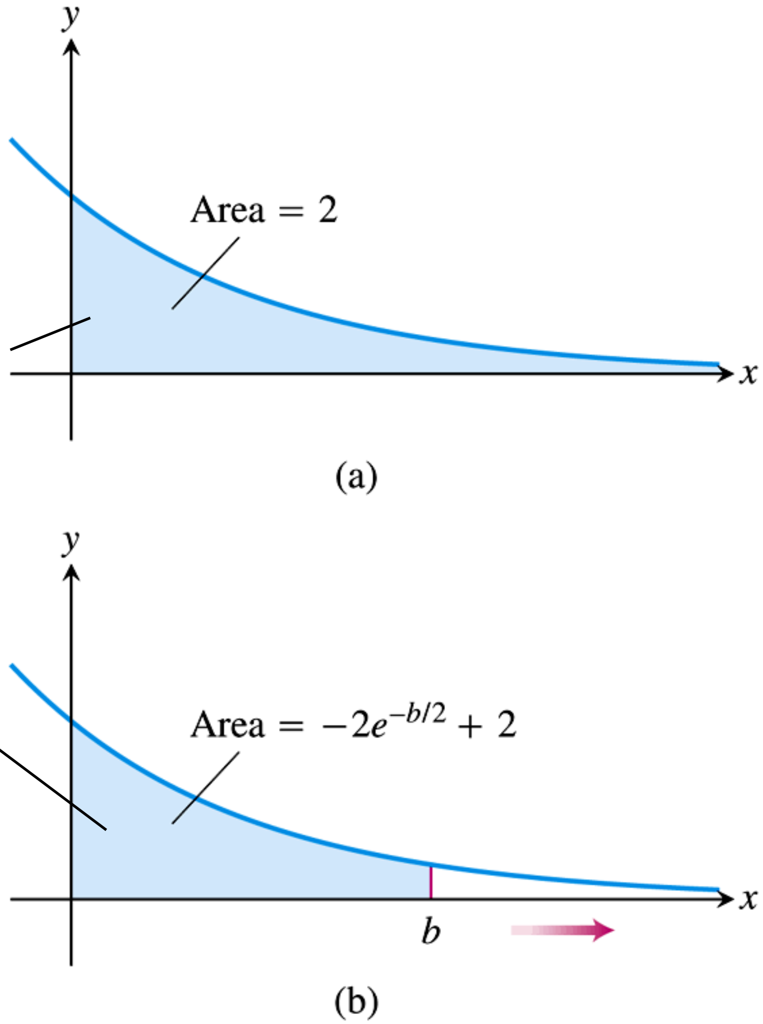


FIGURE 8.18 (a) The area in the first quadrant under the curve $y = e^{-x/2}$ is (b) an improper integral of the first type.

DEFINITION Type I Improper Integrals

Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

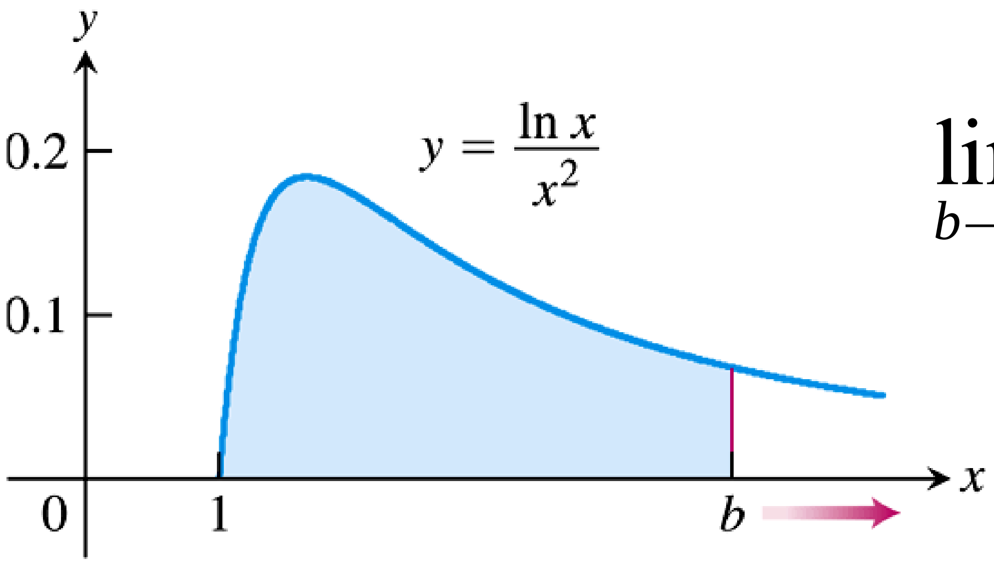
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Example 1 Evaluating an improper integral on [1,∞]

- Is the area under the curve $y=(\ln x)/x^2$ from 1 to ∞ finite? If so, what is it?



$$\lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = ?$$

FIGURE 8.19 The area under this curve is an improper integral (Example 1).

Solution

$$\int_1^b \frac{\ln x}{x}\frac{dx}{x} = \int_1^b \frac{\ln x}{x} d(\ln x) = \int_{\ln 1}^{\ln b} \frac{u}{e^u} du \qquad ;u = \ln x, x = e^u$$

$$\int_0^{\ln b} u \underbrace{e^{-u} du}_{dw} = u \underbrace{(-e^{-u})}_{w}\Big|_0^{\ln b} - \int_0^{\ln b} \underbrace{(-e^{-u})}_{w} du$$

$$= ue^{-u}\Big|_{\ln b}^{0} + \int_0^{\ln b} e^{-u} du = ue^{-u}\Big|_{\ln b}^{0} - e^{-u}\Big|_0^{\ln b}$$

$$= -\ln b \cdot e^{-\ln b} - (e^{-\ln b} - 1) = -\frac{1}{b}\ln b - \frac{1}{b} + 1$$

$$\lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2} dx = \lim_{b\to\infty}\left[-\frac{1}{b}\ln b - \frac{1}{b} + 1\right] = 1$$

Example 2 Evaluating an integral on [-∞,∞]

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = ?$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_{-b}^{0} \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2}$$

$$= 2\lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2}$$

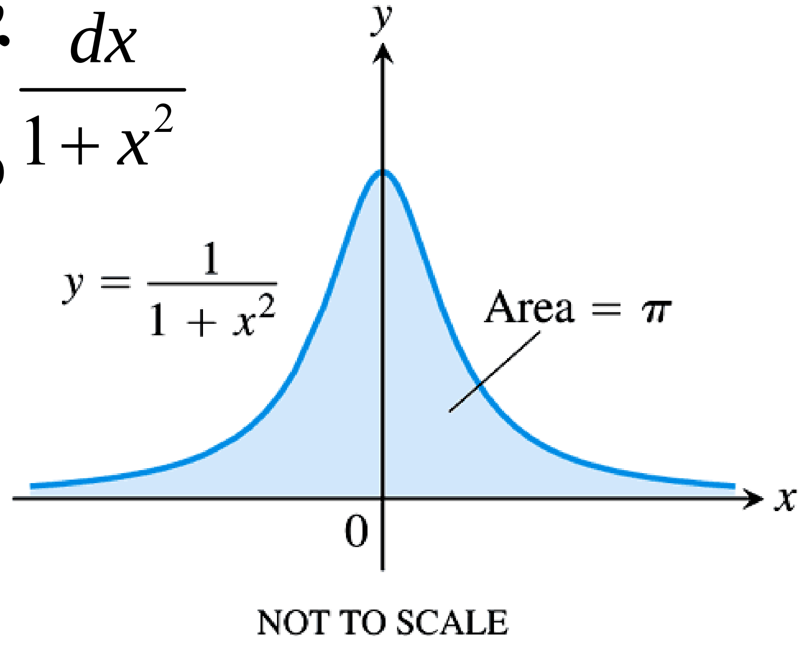


FIGURE 8.20 The area under this curve is finite (Example 2).

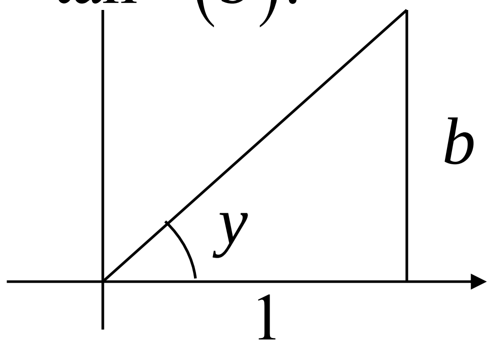
Solution

Using the integral table (Eq. 16)

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int_0^b \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^b = \tan^{-1}(b) - \tan^{-1} 0 = \tan^{-1}(b).$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\lim_{b\to\infty}\tan^{-1} b = 2\cdot\frac{\pi}{2} = \pi$$



$$y = \tan^{-1} b \Rightarrow b = \tan y$$

$$\lim_{b\to\infty}\tan^{-1} b = \frac{\pi}{2}$$

DEFINITION **Type II Improper Integrals**

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

Example 3 Integrands with vertical asymptotes

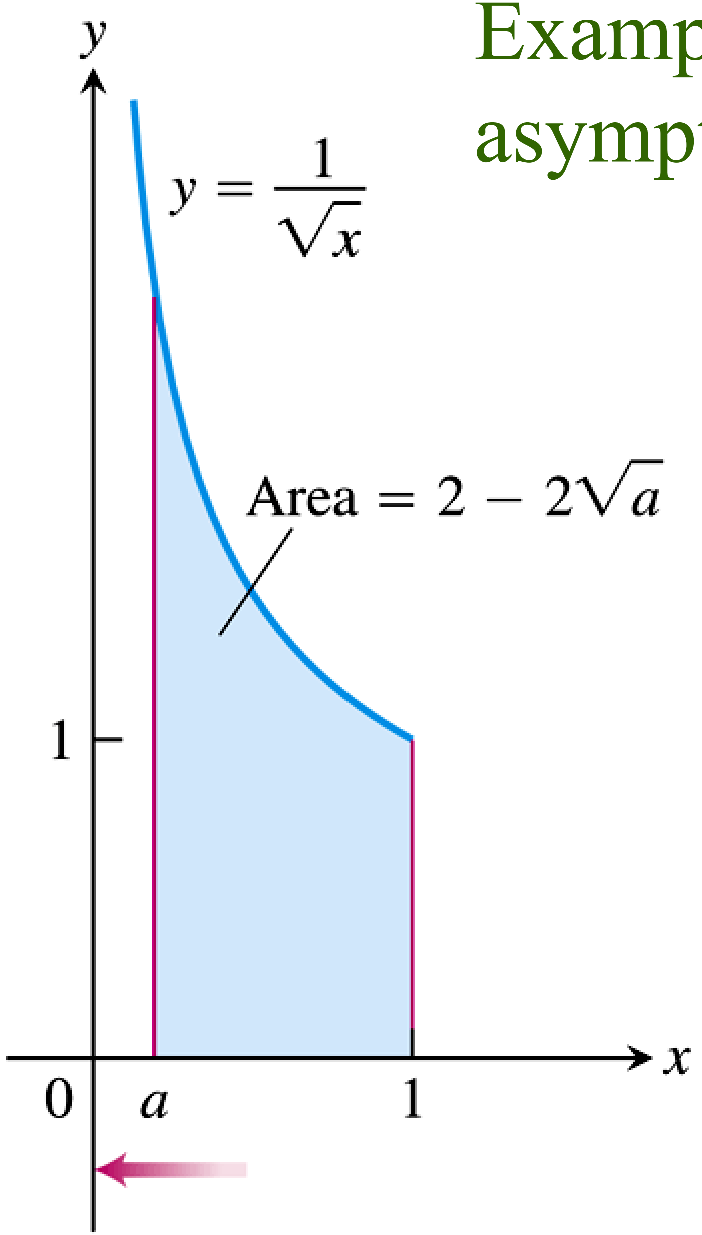
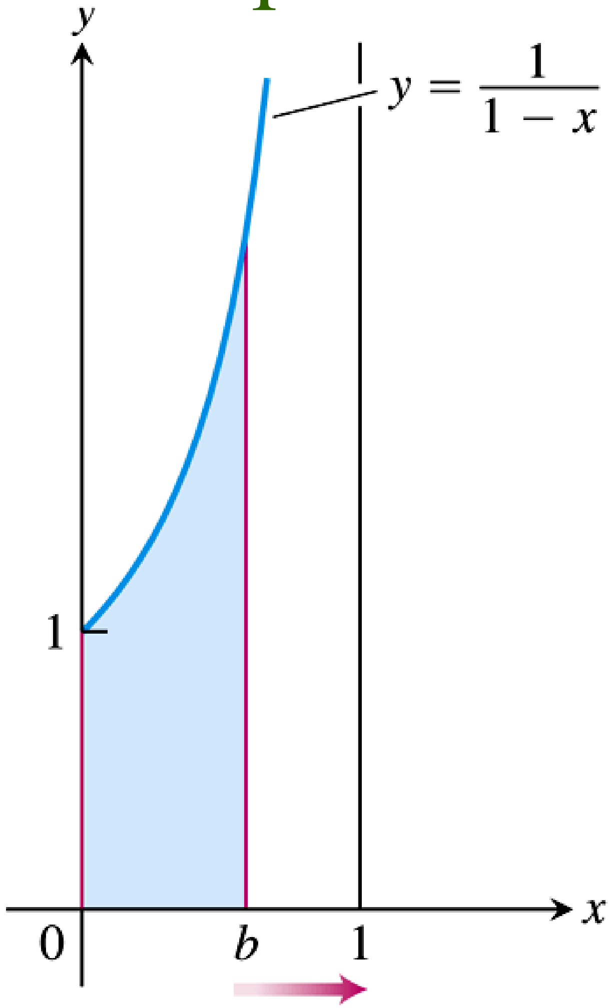


FIGURE 8.21 The area under this curve is

$$\lim_{a \to 0^+} \int_a^1 \left(\frac{1}{\sqrt{x}}\right) dx = 2,$$

an improper integral of the second kind.

Example 4 A divergent improper integral



- Investigate the convergence of $\int_0^1 \frac{dx}{1-x}$

FIGURE 8.22 The limit does not exist:

$$\int_0^1 \left(\frac{1}{1-x}\right) dx = \lim_{b \to 1^-} \int_0^b \frac{1}{1-x} dx = \infty$$

The area beneath the curve and above the x-axis for [0, 1) is not a real number (Example 4).

Solution

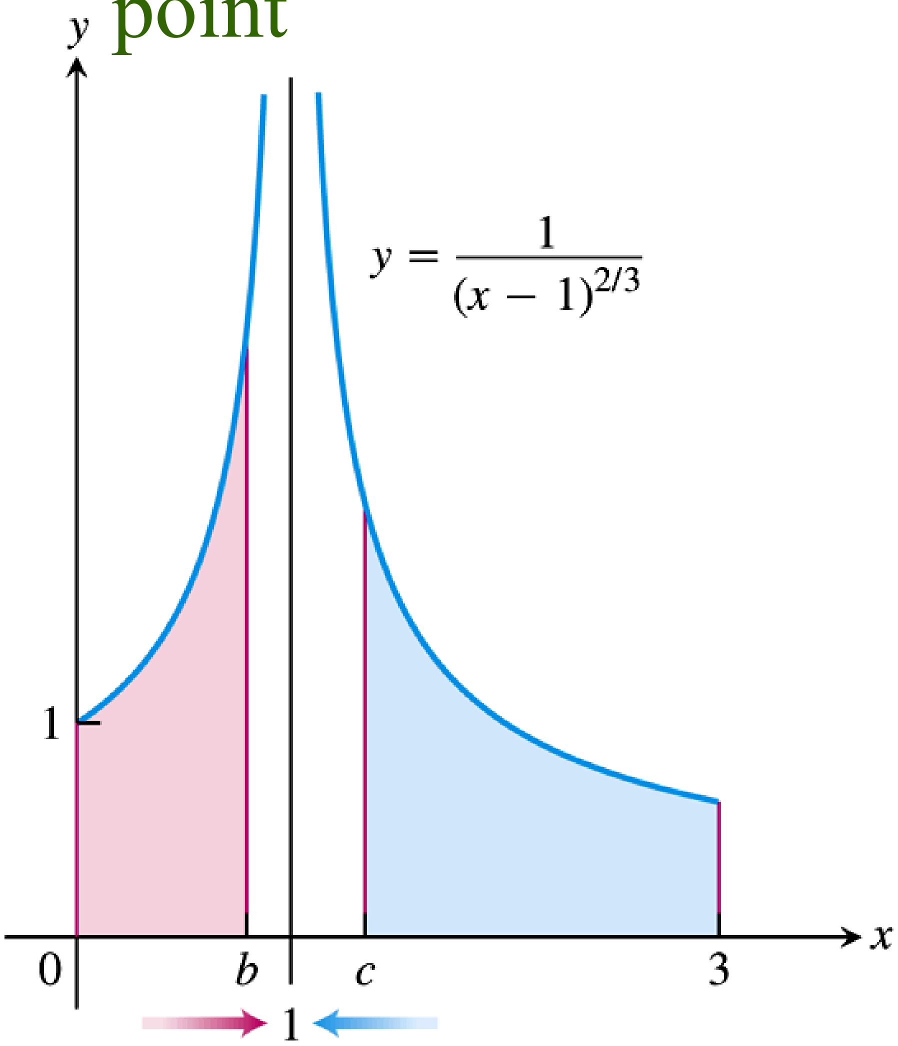
$$\int_0^1 \frac{dx}{1-x} = \lim_{b\to 1^-} \int_0^b \frac{dx}{1-x} = -\lim_{b\to 1^-} \left[\ln|x-1|\right]_0^b$$

$$= -\lim_{b\to 1^-} \left[\ln|b-1| - \ln|0-1|\right]$$

$$= -\lim_{b\to 1^-} \left[\ln|b-1| - \ln|0-1|\right] = \lim_{b\to 1^-} \left[\ln|b-1|^{-1}\right]$$

$$= \lim_{\varepsilon\to 0} \left[\ln\frac{1}{\varepsilon}\right] = \infty$$

Example 5 Vertical asymptote at an interior point



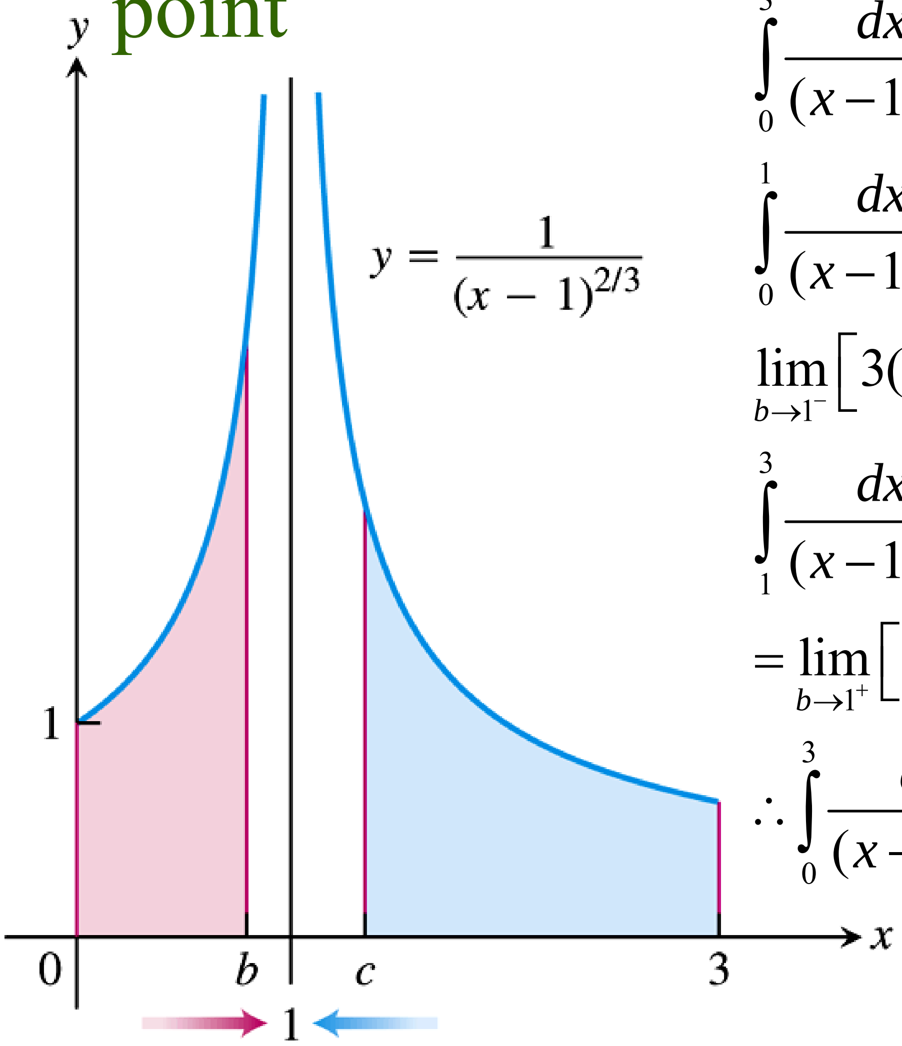
$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = ?$$

FIGURE 8.23 Example 5 shows the convergence of

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = 3 + 3\sqrt[3]{2},$$

so the area under the curve exists (so it is a real number).

Example 5 Vertical asymptote at an interior point



$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \left[3(x-1)^{1/3}\right]_0^b =$$

$$\lim_{b\to 1^-} \left[3(b-1)^{1/3} - 3(-1)^{1/3}\right] = \lim_{b\to 1^-} [0+3] = 3;$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1+} \int_b^3 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^+} \left[3(x-1)^{1/3}\right]_b^3$$

$$= \lim_{b\to 1^+} \left[3(3-1)^{1/3} - 3(b-1)^{1/3}\right] = 3\cdot 2^{2/3}$$

$$\therefore \int_0^3 \frac{dx}{(x-1)^{2/3}} = 3(1+2^{2/3})$$

Example 7 Finding the volume of an infinite solid

- The cross section of the solid in Figure 8.24 perpendicular to the x-axis are circular disks with diameters reaching from the x-axis to the curve $y = e^x$, $-\infty < x < \ln 2$. Find the volume of the horn.

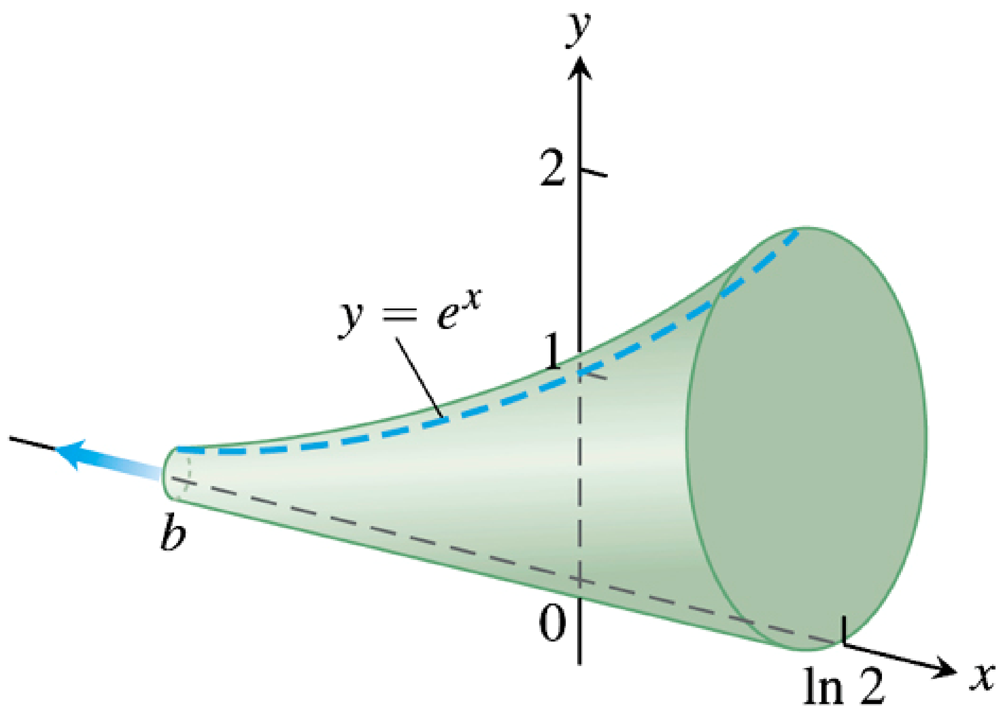


FIGURE 8.24 The calculation in Example 7 shows that this infinite horn has a finite volume.

Example 7 Finding the volume of an infinite solid

volume of a slice of disk of thickness dx,diameter y

$$V=\int_0^V dV=\frac{1}{4}\lim_{b\to-\infty}\int_b^{\ln 2}\pi y(x)^2\,dx$$

$$=\frac{1}{4}\lim_{b\to-\infty}\int_b^{\ln 2}\pi e^{2x}dx$$

$$=\frac{1}{8}\lim_{b\to-\infty}\left[\pi e^{2x}\right]_b^{\ln 2}$$

$$=\frac{1}{8}\lim_{b\to-\infty}\left[4\pi-\pi e^{2b}\right]$$

$$=\frac{1}{8}\pi\lim_{b\to-\infty}(4-e^{2b})=\frac{\pi}{2}$$

$$dV=\pi(y/2)^2\,dx$$

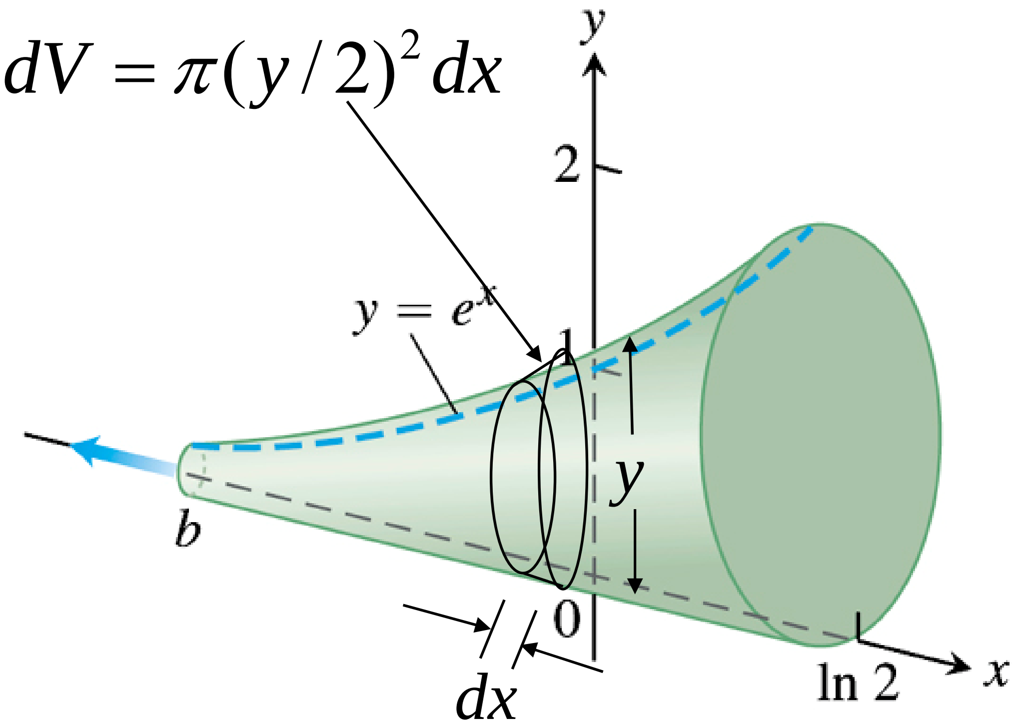


FIGURE 8.24 The calculation in Example 7 shows that this infinite horn has a finite volume.