Linear transformation and the change of basis

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Abstract

This short note supplements the set of ZCA 110 lecture notes on linear algebra. It expounds in understandable language the idea of linear transformation involving different bases.

1 Going from one basis to another

Consider two bases, $W = (W_1, W_2, \cdots, W_n)$ and $Z = (Z_1, Z_2, \cdots, Z_n)$ in the vector space $\mathrm{V}_n(R)$. Note that here Z_i, W_i are colume vectors of $n-$components. The connection between the two bases can be worked out via the following consideration:

Consider a general n-vector in V_n(R), call it Q. Q can be represented in the W-basis or in the Z-basis.

In the W-basis: $Q = W \cdot Q_W$,where Q_W is the coordinate vector of Q relative to the W-basis. In the Z-basis: $Q = Z \cdot Q_Z$,where Q_Z is the coordinate vector of Q relative to the Z-basis. The vector Q is the same vector irrespective of its basis representation, hence

$$\begin{aligned} Q &= W \cdot Q_W = Z \cdot Q_Z \\ \Rightarrow Q_Z &= (Z^{-1} \cdot W) Q_W \equiv P Q_W. \end{aligned} \tag{1}$$

Eq. (1) relates the coordinate vector of an general vector Q in the Z-basis to that in the W-basis. In other words, if the basis Z and W is given, we can form a matrix

$$P = Z^{-1} \cdot X \tag{2}$$

which allows the coodinate vector in one basis to be determined if the coordinate vector in the other is given. Eq. (1) can also equivalently be stated as

$$Q_W = P^{-1} Q_Z.$$

2 Linear Transformation

A transformation can be carried out in any basis. Consider a vector X is tranformed into vector Y. Such a transformatino can be represented in both the W-basis and the Z-basis. In each basis representationm the transformation takes on different forms. Say A is the transformation matrix in the X-basis representation, whereas B is the correspoinding transformation in the Z-basis representation. The linear transformations in both bases are given by:

W-basis	Z-basis
$W = (W_1, W_2, \cdots, W_n)$	$Z = (Z_1, Z_2, \cdots, Z_n)$
$X \xrightarrow{A} Y$	$X \xrightarrow{B} Y$
In component form	In component form
$X_W \xrightarrow{A} Y_W$	$X_Z \xrightarrow{B} Y_Z$
In matrix form	In matrix form
$Y_W = AX_W$	$Y_Z = BX_Z$

Now, we shall prove that: If

$$\begin{aligned} X_W &\xrightarrow{A} Y_W \text{ in the W-basis} \\ X_Z &\xrightarrow{B} Y_Z \text{ in the Z-basis,} \end{aligned}$$

then the two transformation A, B are similar, i.e.,

$$B = Q^{-1}AQ,$$

where $Q = P^{-1} = (Z^{-1}W)^{-1} = W^{-1}Z$.

The proof is as followed: We begin with

$$Y_Z = BX_Z. \tag{3}$$

In Eq. (3), the LHS, i.e. Y_Z is related to Y_W via $Y_Z = PY_W$, whereas X_Z in the RHS is related to X_W via $X_Z = PX_W$. Hence, Eq. (3) can be written as

$$\begin{aligned} & PY_W = B(PX_W) \\ \Rightarrow \quad & Y_W = (P^{-1}BP)X_W. \end{aligned} \tag{4}$$

Eq. (4) is just the tranformation of X_W into Y_W by A (in the W-basis), i.e.

$$Y_W = (P^{-1}BP)X_W = AX_W. \tag{5}$$

Hence, we can identify

$$A = P^{-1}BP,$$

or

$$B = PAP^{-1} = Q^{-1}AQ,$$

where P is given by Eq. (2).