Chapter 1

Preliminaries

1.3

Functions and Their Graphs

(1st lecture of week 06/08/07 - 11/08/07)

Function

- $y = f(x)$
- f represents function (a rule that tell us how to calculate the value of y from the variable x
- x : independent variable (input of f)
- y : dependent variable (the correspoinding output value of f at x)

DEFINITION Function

A **function** from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

Definition Domain of the function

The set of D of all possible input values

Definition Range of the function

The set of all values of $f(x)$ as x varies throughout D

FIGURE 1.22 A diagram showing a function as a kind of machine.

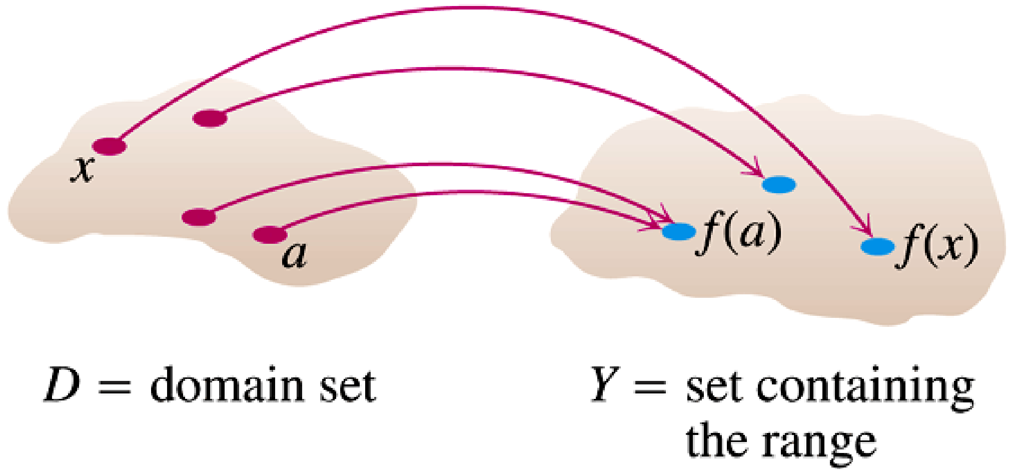

FIGURE 1.23 A function from a set D to a set Y assigns a unique element of Y to each element in D.

Natural Domain

- When a function $y = f(x)$is defined and the domain is not stated explicitly, the domain is assumed to be the largest set of real x-values for the formula gives real y-values.
- e.g. compare "$y = x^2$" c.f. "$y = x^2, x \geqslant 0$"
- Domain may be open, closed, half open, finite, infinite.

Verify the domains and ranges of these functions

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Graphs of functions

- Graphs provide another way to visualise a function
- In set notation, a graph is

$$\{(x, f(x)) \mid x \in D\}$$

- The graph of a function is a useful picture of its behaviour.

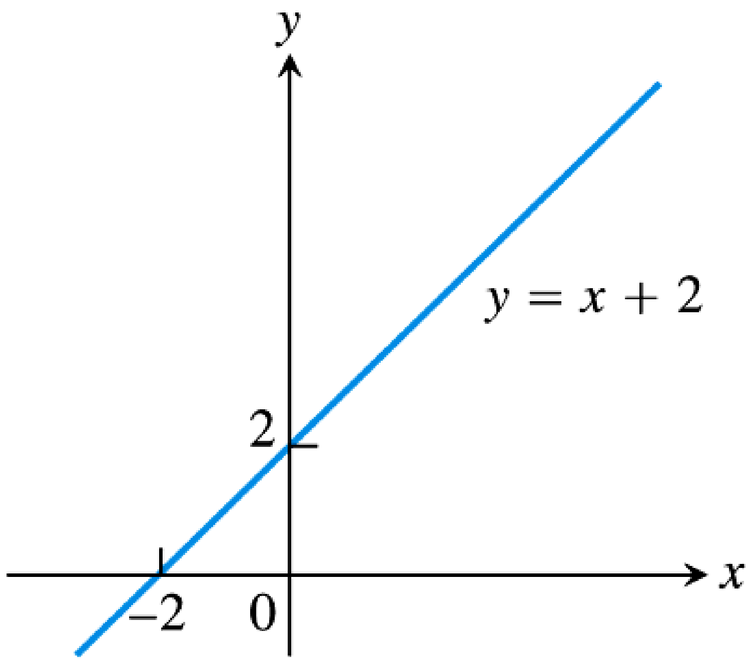

FIGURE 1.24 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

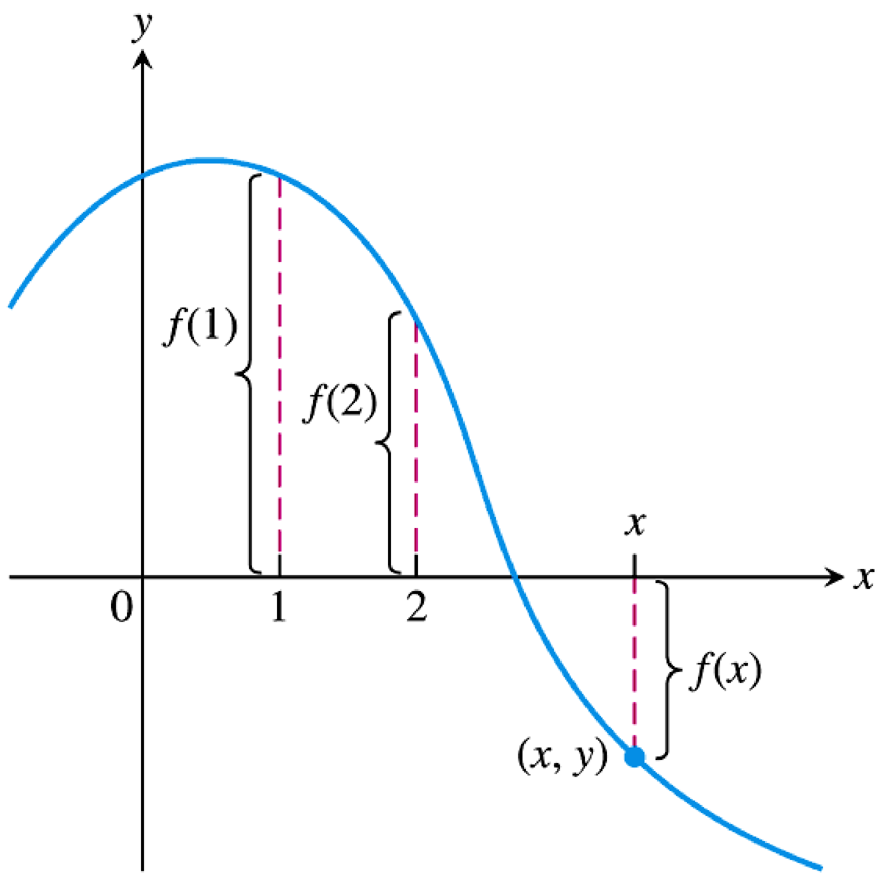

FIGURE 1.25 If (x, y) lies on the graph of f, then the value $y = f(x)$ is the height of the graph above the point x (or below x if $f(x)$ is negative).

Example 2 Sketching a graph

- Graph the function $y = x^2$ over the interval [-2,2]

The vertical line test

- Since a function must be single valued over its domain, no vertical line can intersect the graph of a function more than once.
- If a is a point in the domain of a function f, the vertical line $x=a$ can intersect the graph of f in a single point $(a, f(a))$.

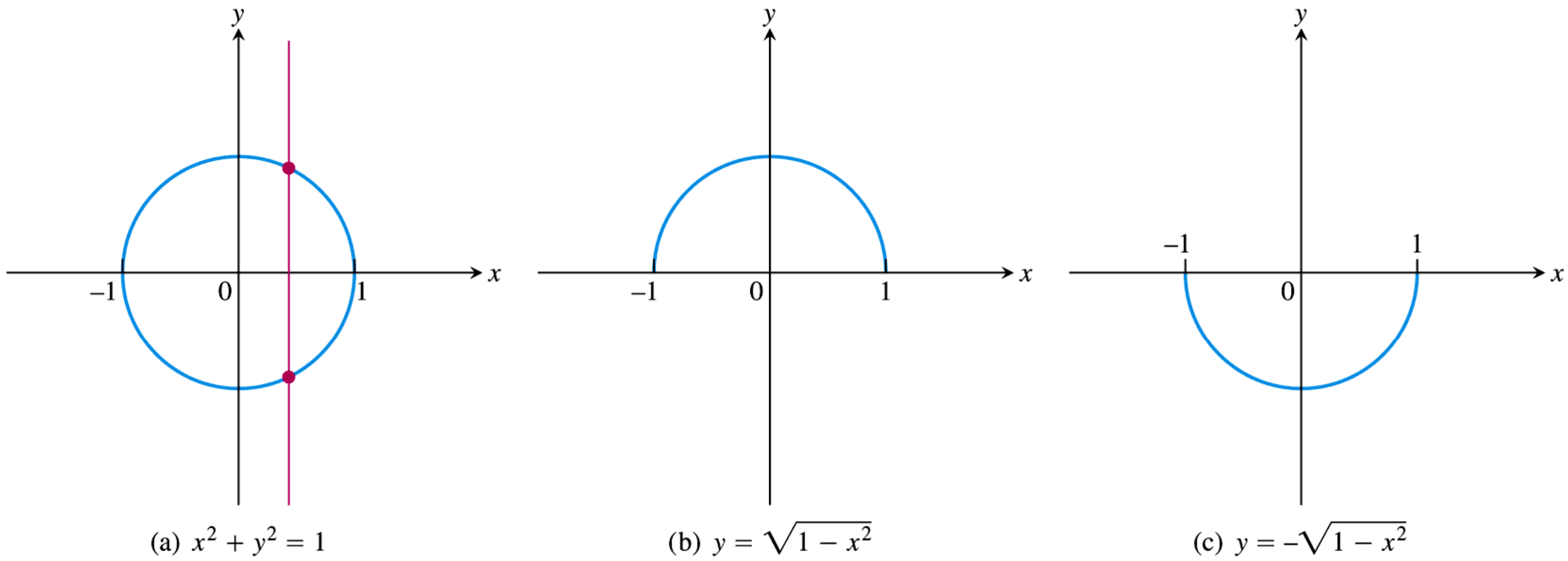

FIGURE 1.28 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of a function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of a function $g(x) = -\sqrt{1 - x^2}$.

Piecewise-defined functions

- The absolute value function

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

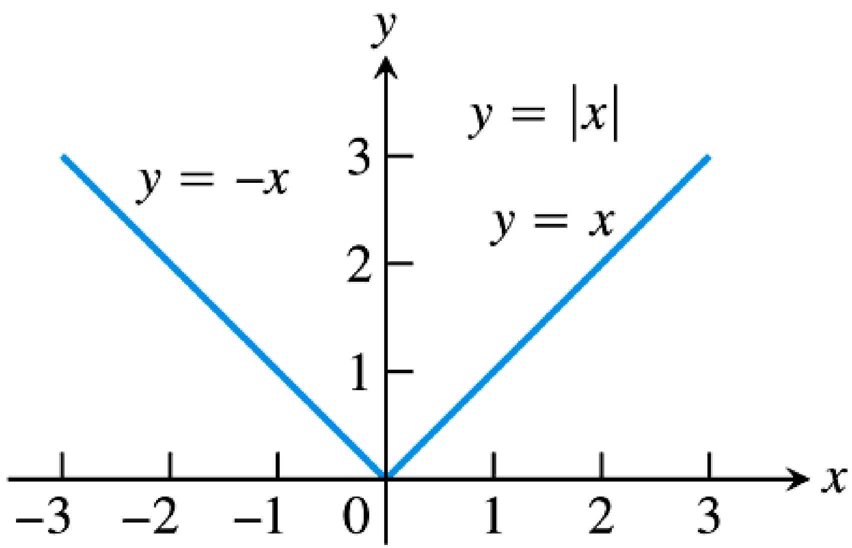

FIGURE 1.29 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

Graphing piecewise-defined functions

- Note: this is *just one function* with a domain covering all real number

$$f(x)=\begin{cases} -x & x<0 \\ x^2 & 0\le x\le 1 \\ 1 & x>1 \end{cases}$$

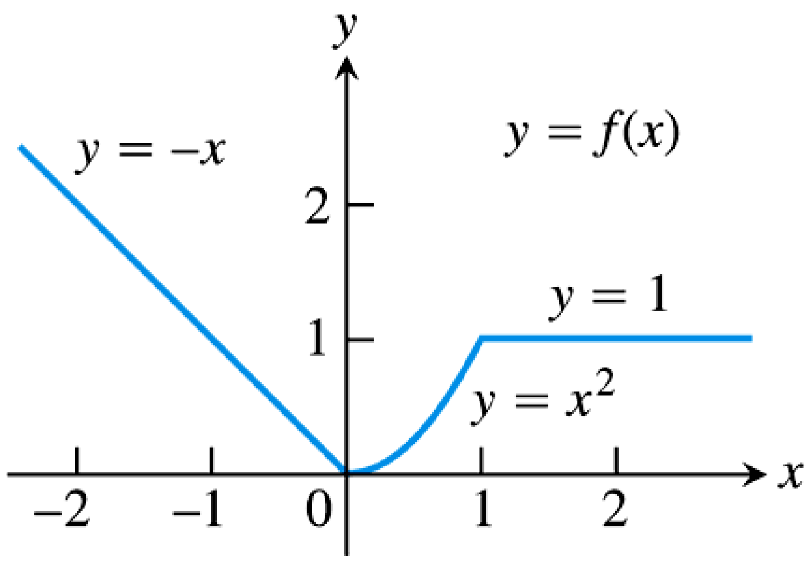

FIGURE 1.30 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 5).

The greatest integer function

- Also called integer floor function
- $f = [x]$, defined as greatest integer less than or equal to x.
- e.g.
- $[2.4] = 2$
- $[2]=2$
- $[-2] = -2$, etc.

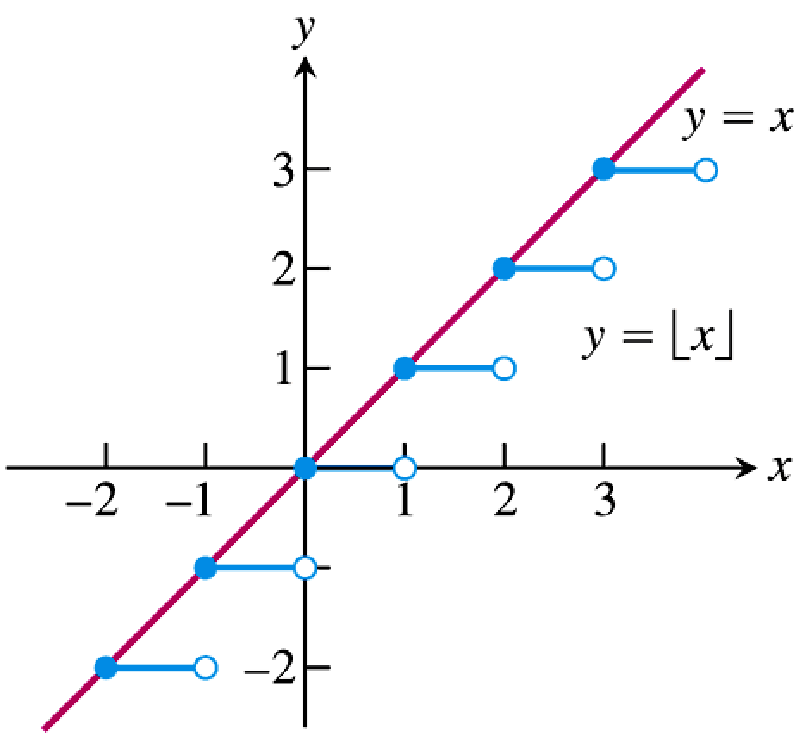

FIGURE 1.31 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x (Example 6).

Note: the graph is the blue colour lines, not the one in red

Writing formulas for piecewise-defined functions

- Write a formula for the function $y=f(x)$ in Figure 1.33

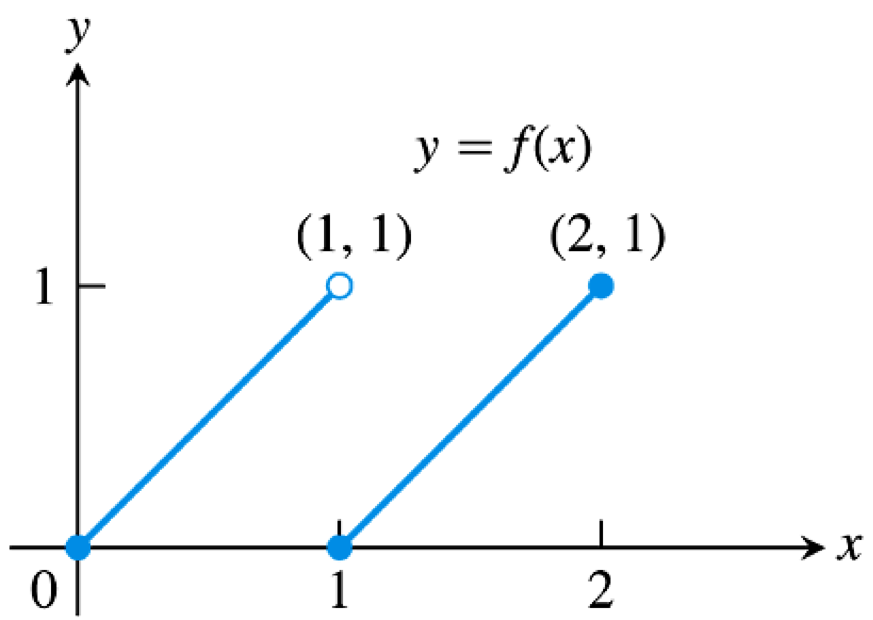

FIGURE 1.33 The segment on the left contains (0, 0) but not (1, 1). The segment on the right contains both of its endpoints (Example 8).

1.4

Identifying Functions;
Mathematical Models
(1st lecture of week 06/08/07 - 11/08/07)

Linear functions

- Linear function takes the form of
- $y=mx + b$
- m, b constants
- m slope of the graph
- b intersection with the y-axis
- The linear function reduces to a constant function $f = c$ when $m = 0$,

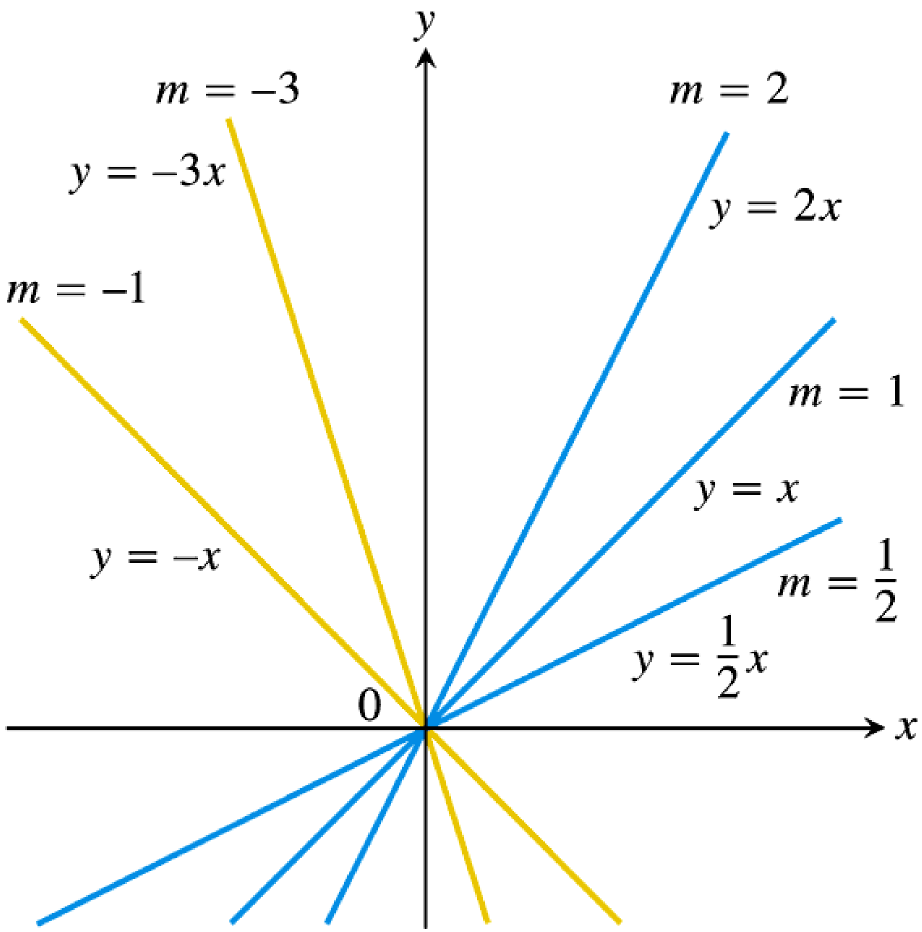

FIGURE 1.34 The collection of lines $y = mx$ has slope m and all lines pass through the origin.

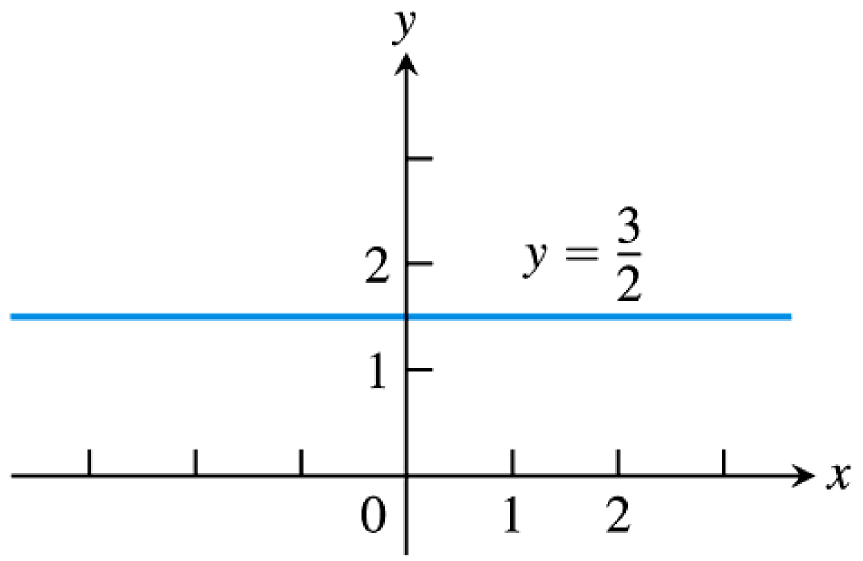

FIGURE 1.35 A constant function has slope $m = 0$.

Power functions

- $f(x) = x^a$
- a constant
- Case (a): $a = n$, a positive integer

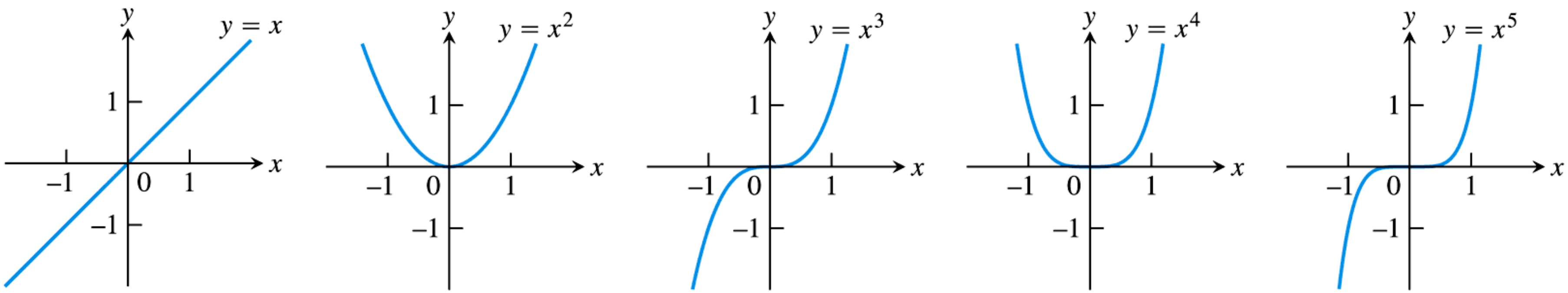

FIGURE 1.36 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$ defined for $-\infty < x < \infty$.

go back

Power functions

- Case (b):
- a = -1 (hyperbola)
- or a=-2

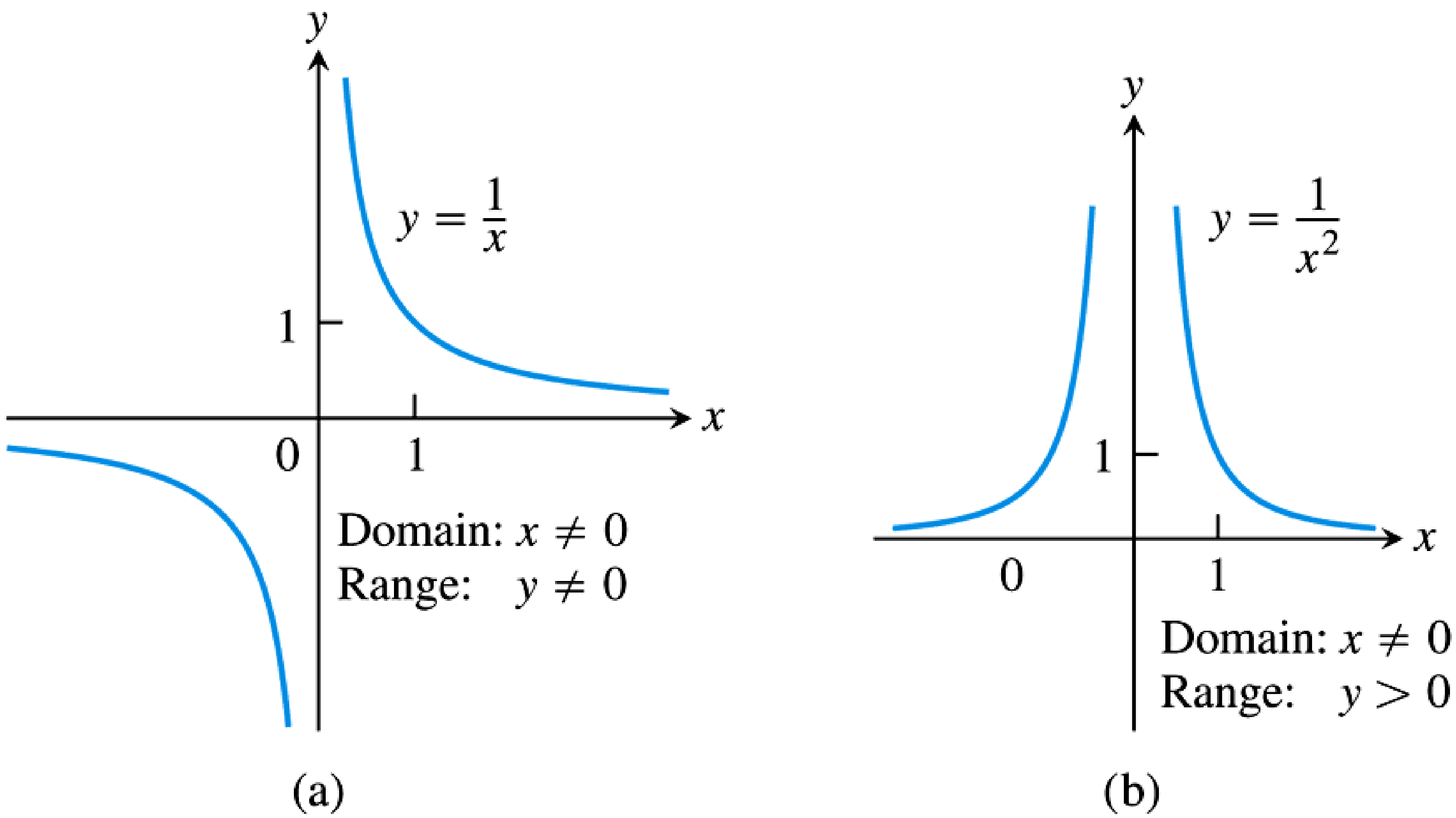

FIGURE 1.37 Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

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Power functions

- Case (c):
- a = ½, 1/3, 3/2, and 2/3
- $f(x) = x^{1/2} = \sqrt{x}$ (square root) , domain = $[0 \leqslant x < \infty)$
- $g(x) = x^{1/3} = \sqrt[3]{x}$(cube root), domain = $(-\infty < x < \infty)$

- $p(x) = x^{2/3} = (x^{1/3})^{2}$, domain = ?
- $q(x) = x^{3/2} = (x^{3})^{1/2}$ domain = ?

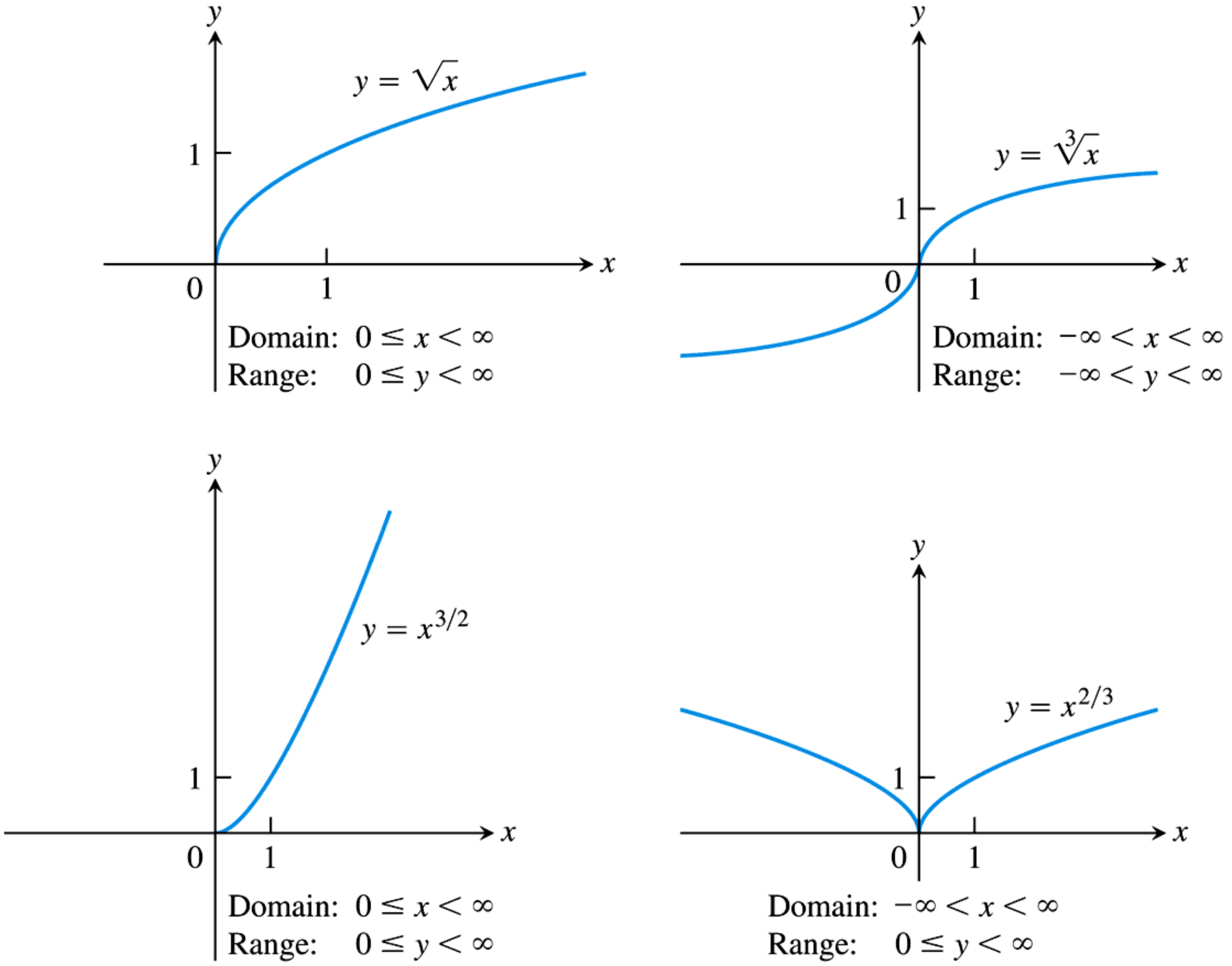


FIGURE 1.38 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

go back

Polynomials

- $p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots a_1 x + a_0$
- n nonnegative integer (1,2,3…)
- a's coefficients (real constants)
- If $a_n \neq 0$, n is called the degree of the polynomial

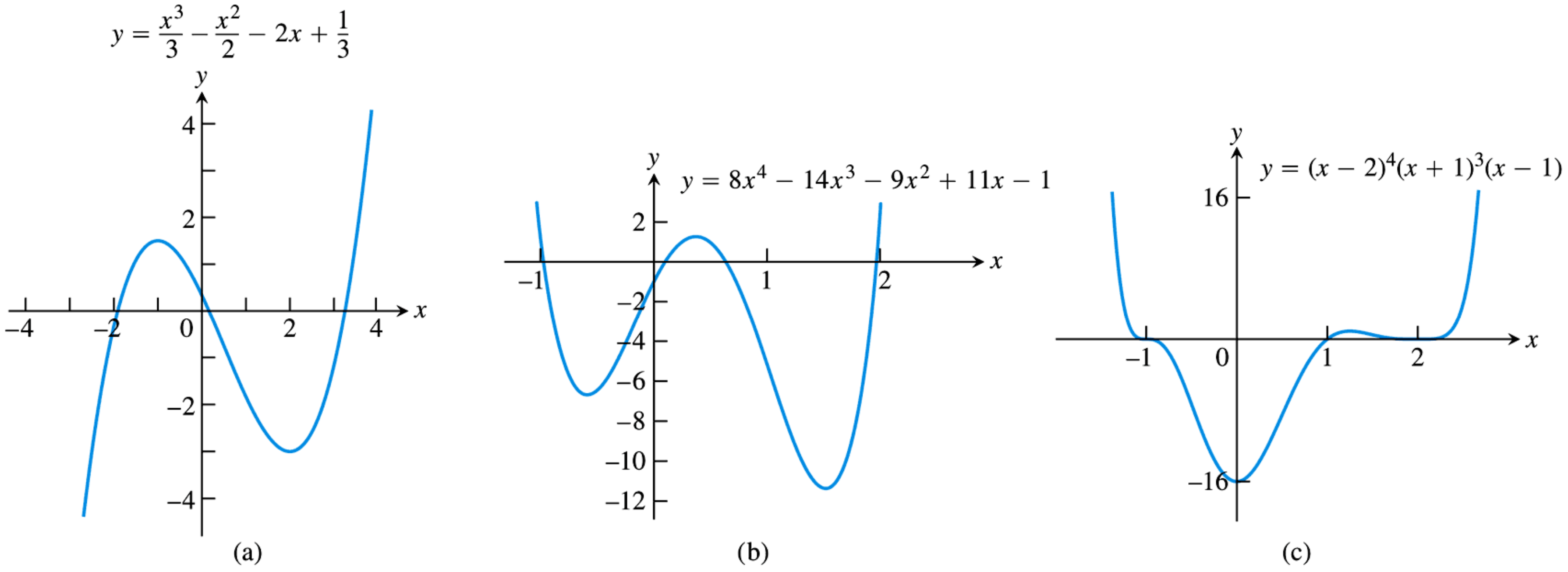


FIGURE 1.39 Graphs of three polynomial functions.

Rational functions

- A rational function is a quotient of two polynomials:
- $f(x) = p(x) / q(x)$
- p,q are polynomials.
- Domain of $f(x)$ is the set of all real number x for which $q(x) \neq 0$.

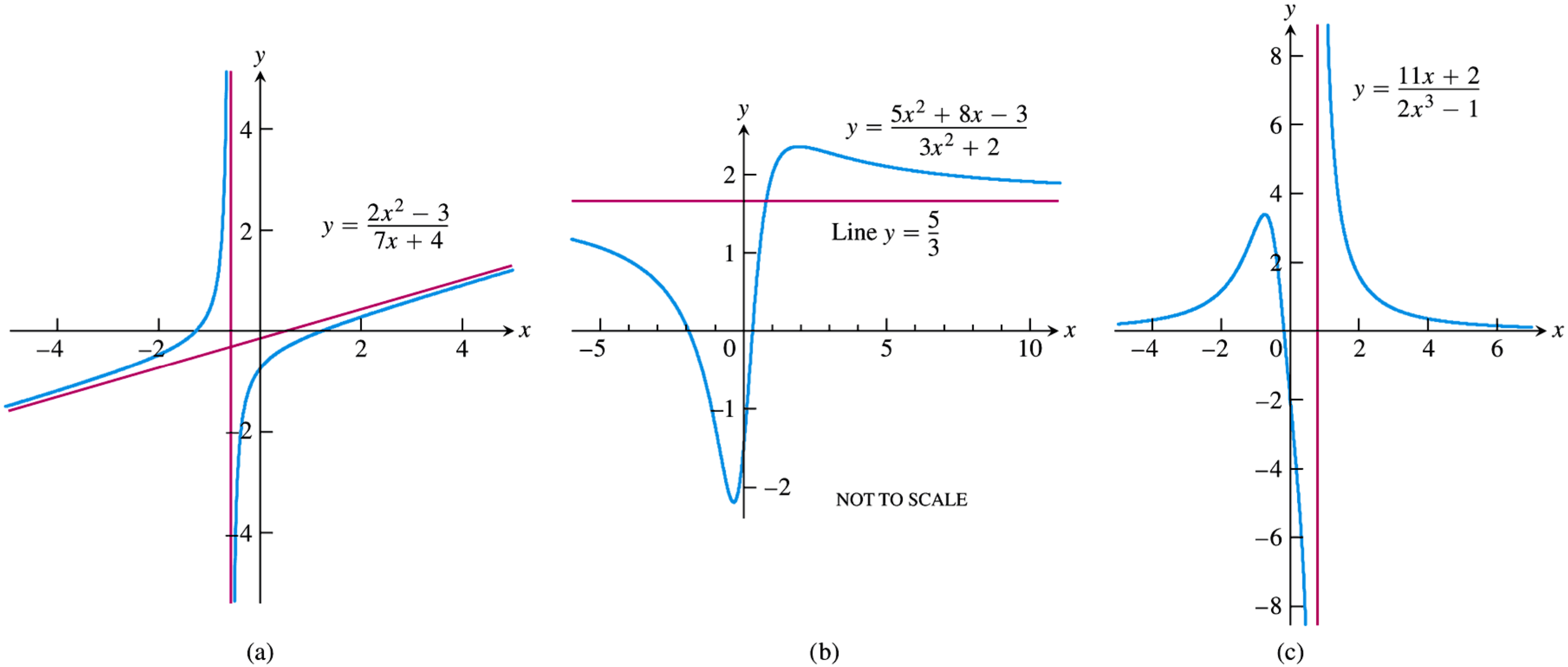


FIGURE 1.40 Graphs of three rational functions.

Algebraic functions

- Functions constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots)

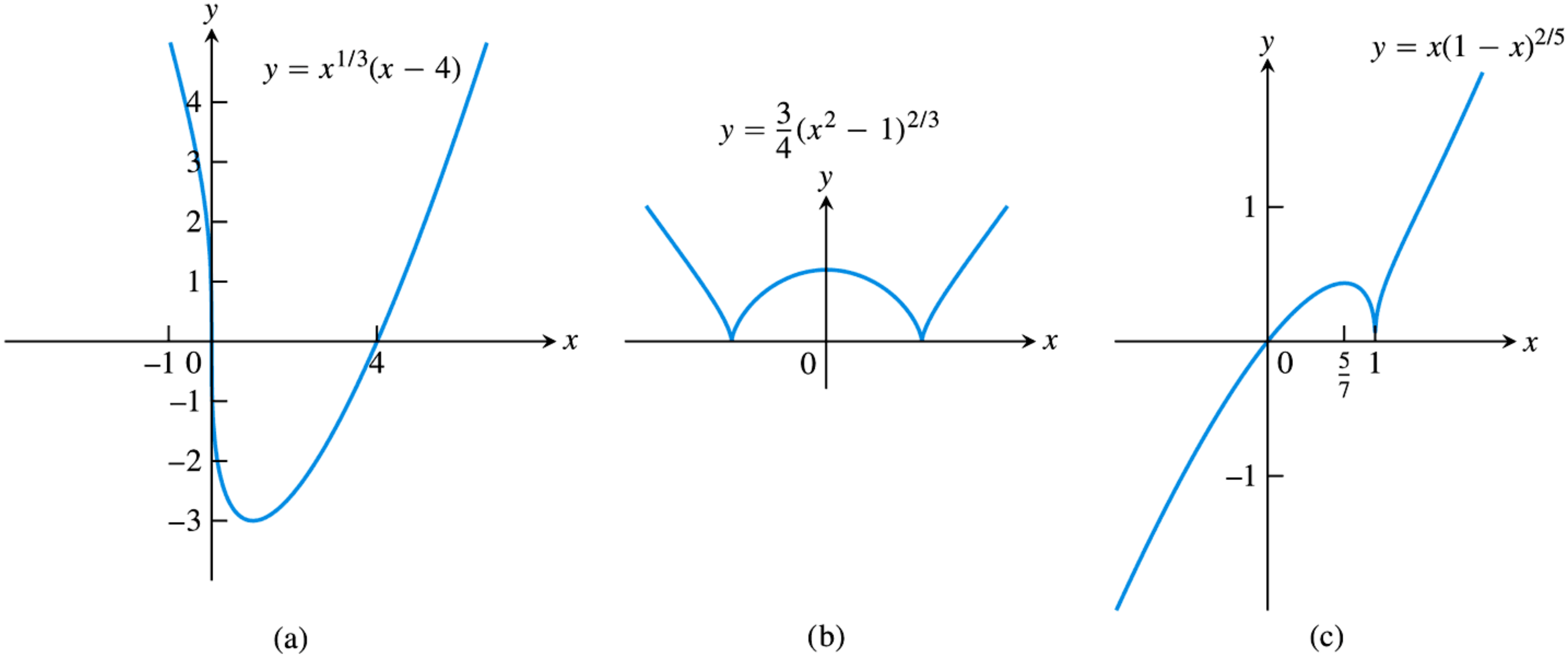


FIGURE 1.41 Graphs of three algebraic functions.

Trigonometric functions

- More details in later chapter

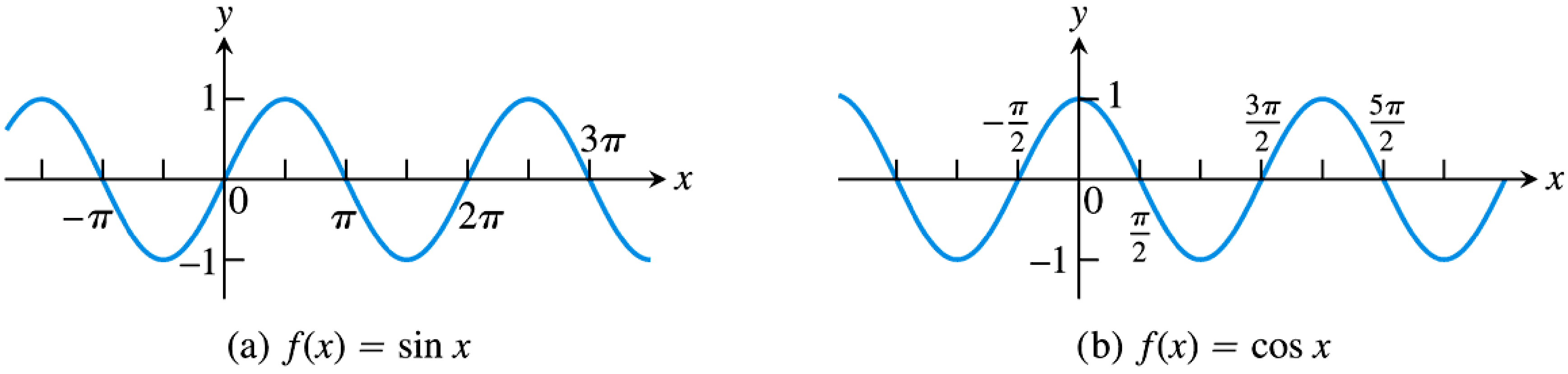


FIGURE 1.42 Graphs of the sine and cosine functions.

Exponential functions

- $f(x) = a^x$
- Where $a > 0$ and $a \neq 0$. a is called the 'base'.
- Domain $(-\infty, \infty)$
- Range $(0, \infty)$
- Hence, $f(x) > 0$
- More in later chapter

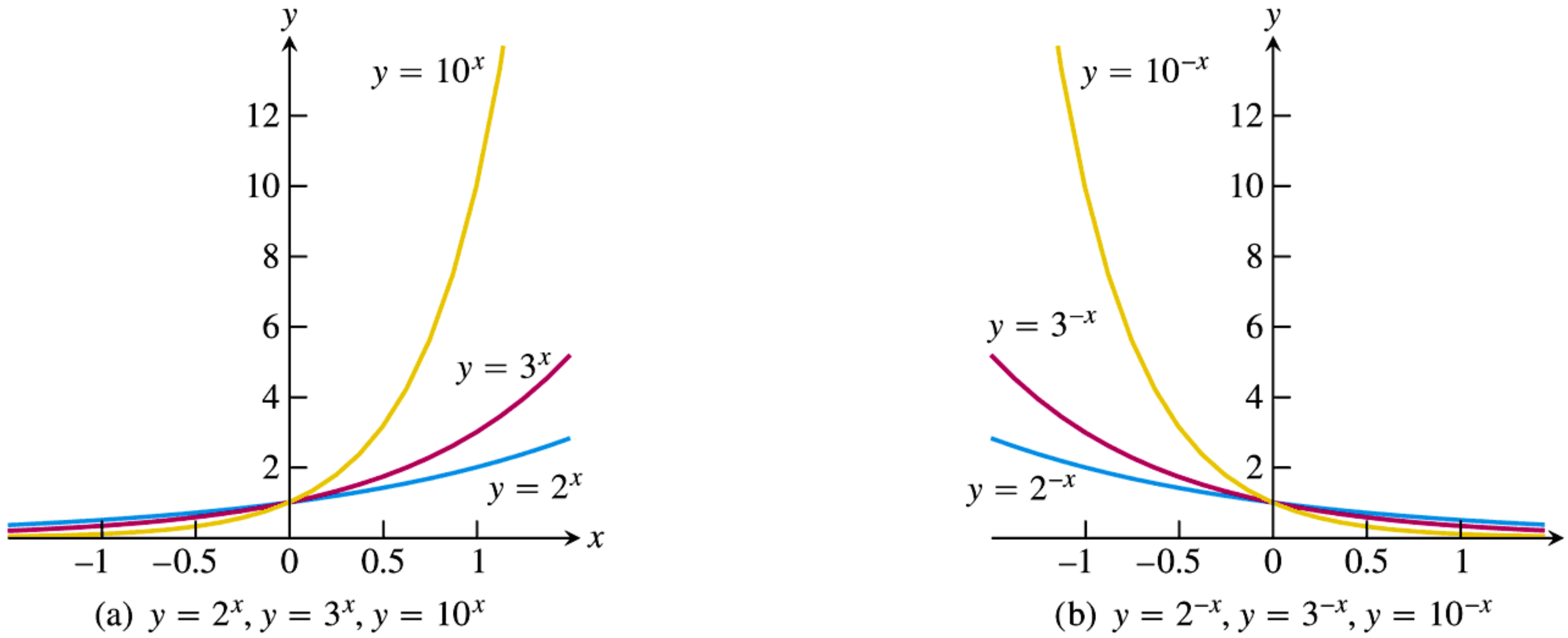


FIGURE 1.43 Graphs of exponential functions.

Note: graphs in (a) are reflections of the corresponding curves in (b) about the *y*-axis. This amount to the symmetry operation of $x \leftrightarrow -x$.

Logarithmic functions

- $f(x) = \log_a x$
- a is the base
- $a \neq 1$, $a > 0$
- Domain $(0, \infty)$
- Range $(-\infty, \infty)$
- They are the *inverse functions* of the exponential functions (more in later chapter)

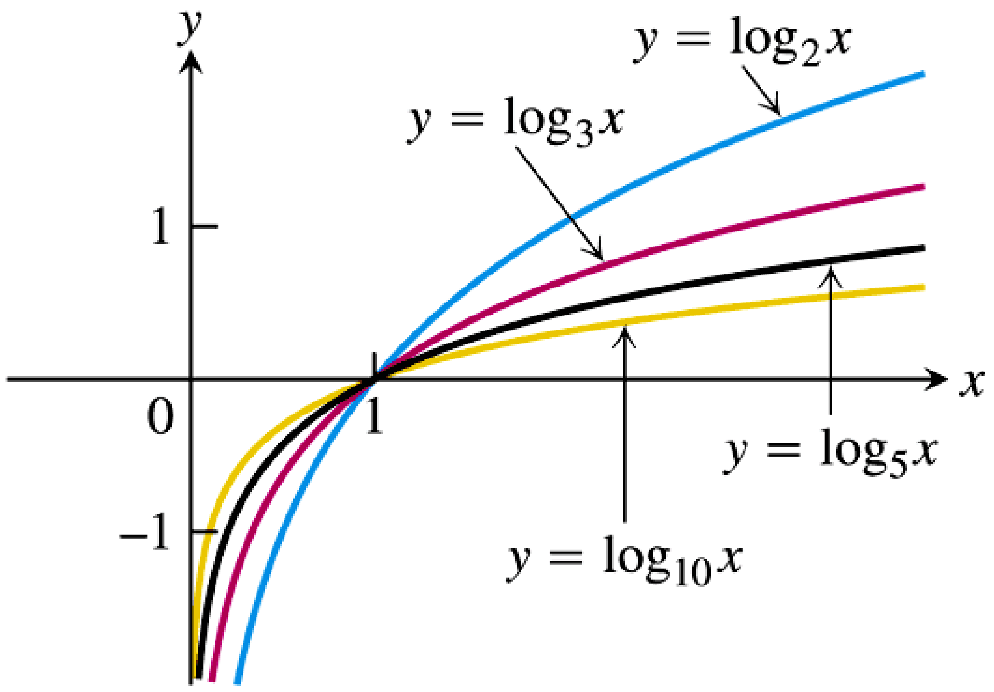


FIGURE 1.44 Graphs of four logarithmic functions.

Transcendental functions

- Functions that are not algebraic
- Include: trigonometric, inverse trigonometric, exponential, logarithmic, hyperbolic and many other functions

Example 1

- Recognizing Functions
- (a) $f(x) = 1 + \mathrm{x} - \frac{1}{2}x^5$
- (b) $g(x) = 7^x$
- (c) $h(z) = z^7$
- (d) $y(t) = \sin(t - \pi/4)$

Increasing versus decreasing functions

- A function is said to be increasing if it rises as you move from left to right
- A function is said to be decreasing if it falls as you move from left to right

Function	Where increasing	Where decreasing
$y = x^2$	$0 \le x < \infty$	$-\infty < x \le 0$
$y = x^3$	$-\infty < x < \infty$	Nowhere
$y = 1/x$	Nowhere	$-\infty < x < 0$ and $0 < x < \infty$
$y = 1/x^2$	$-\infty < x < 0$	$0 < x < \infty$
$y = \sqrt{x}$	$0 \le x < \infty$	Nowhere
$y = x^{2/3}$	$0 \le x < \infty$	$-\infty < x \le 0$

$y=x^2, y=x^3$; $y=1/x, y=1/x^2$; $y=x^{1/2}, y=x^{2/3}$

DEFINITIONS Even Function, Odd Function

A function $y = f(x)$ is an

$$\textbf{even function of } x \quad \text{if } f(-x) = f(x),$$
$$\textbf{odd function of } x \quad \text{if } f(-x) = -f(x),$$

for every x in the function's domain.

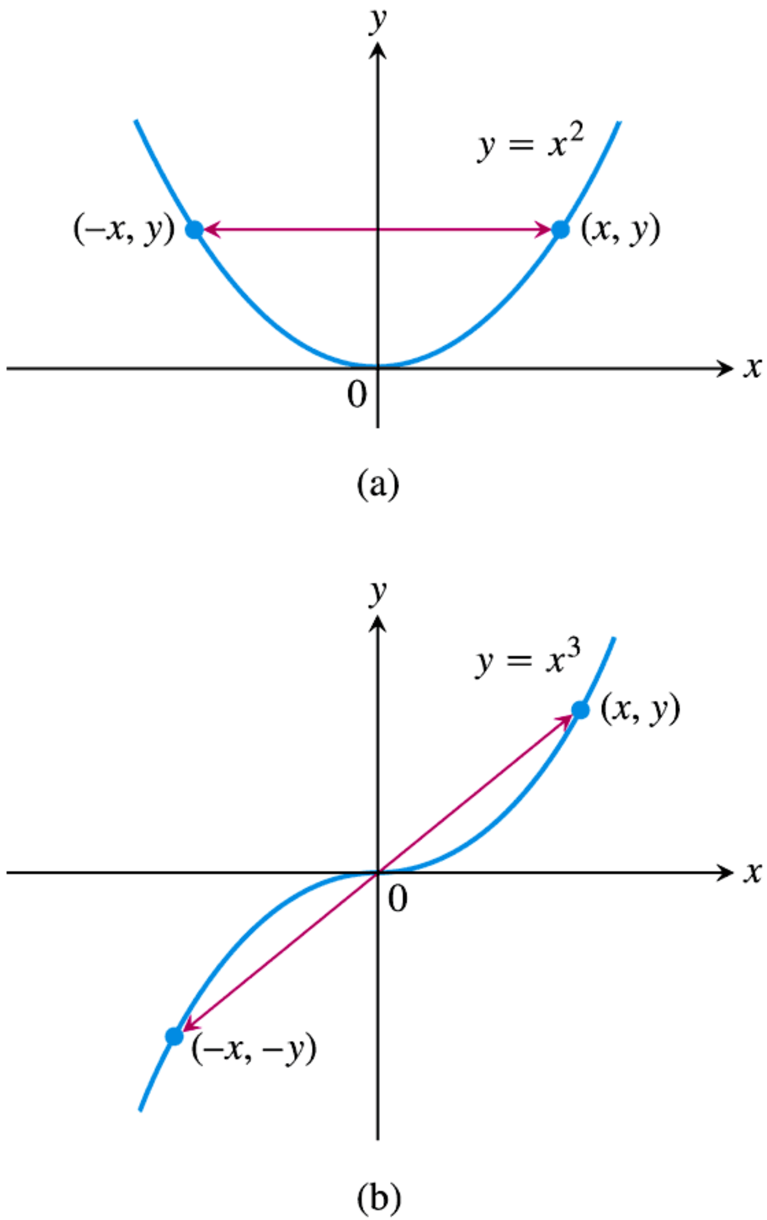


FIGURE 1.46 In part (a) the graph of $y = x^2$ (an even function) is symmetric about the y-axis. The graph of $y = x^3$ (an odd function) in part (b) is symmetric about the origin.

Recognising even and odd functions

- $f(x) = x^2$ Even function as $(-x)^2 = x^2$ for all x, symmetric about the all x, symmetric about the y-axis.
- $f(x) = x^2 + 1$ Even function as $(-x)^2 + 1 = x^2 + 1$ for all x, symmetric about the all x, symmetric about the y-axis.

Recognising even and odd functions

- $f(x) = x$. Odd function as $(-x) = -x$ for all x, symmetric about origin.
- $f(x) = x+1$. Odd function ?

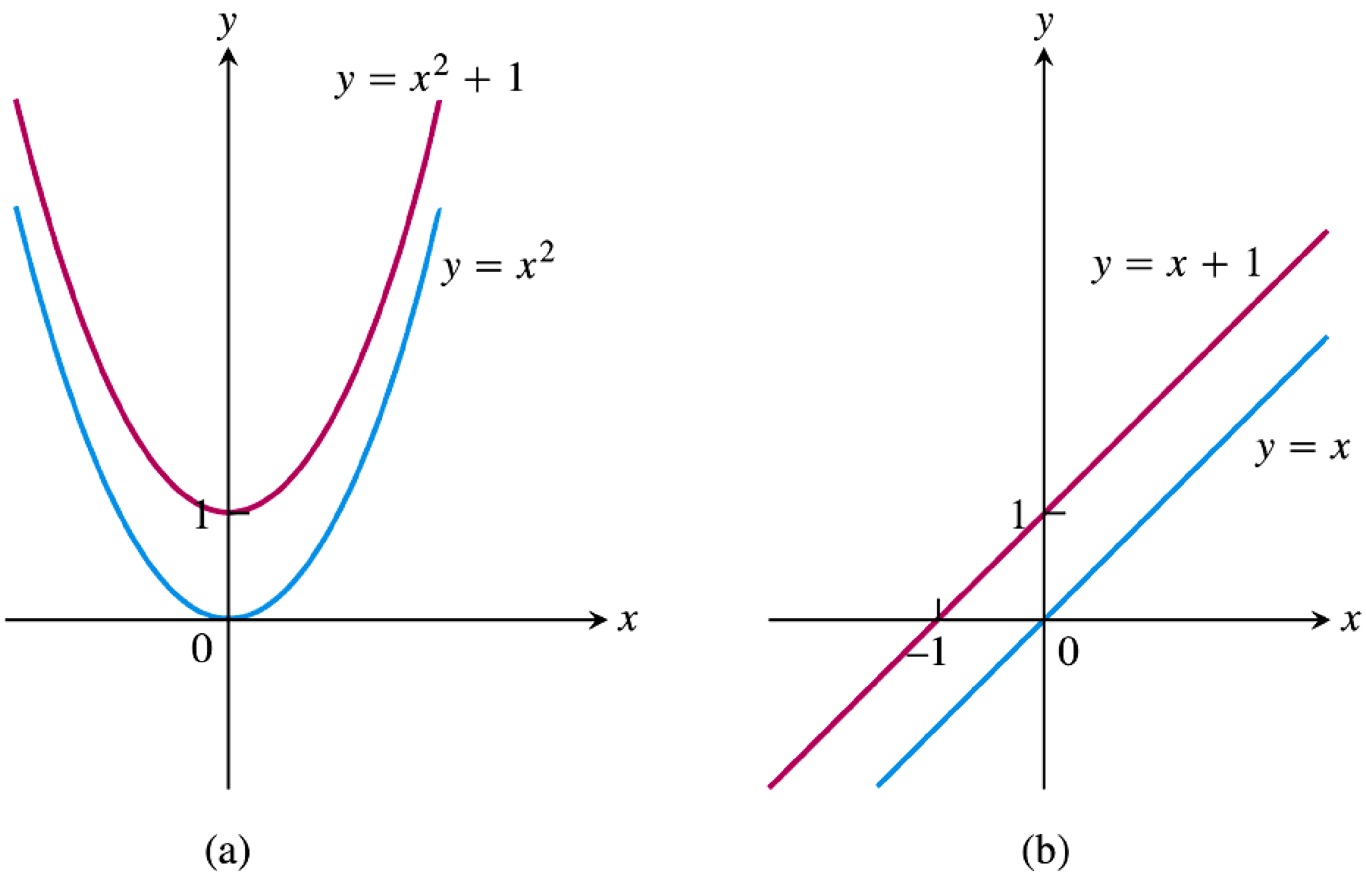


FIGURE 1.47 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd. The symmetry about the origin is lost (Example 2).

1.5

Combining Functions;
Shifting and Scaling Graphs
(2nd lecture of week 06/08/07 - 11/08/07)

Sums, differences, products and quotients

- f, g are functions
- For $x \in D(f) \cap D(g)$, we can define the functions of
- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$,
- $(cf)(x) = cf(x)$, c a real number
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

Example 1

- $f(x) = \sqrt{x}$, $g(x) = \sqrt{(1-x)}$,
- The domain common to both *f*,*g* is
- $D(f) \cap D(g) = [0,1]$ (work it out)

Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1 - x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1 - x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

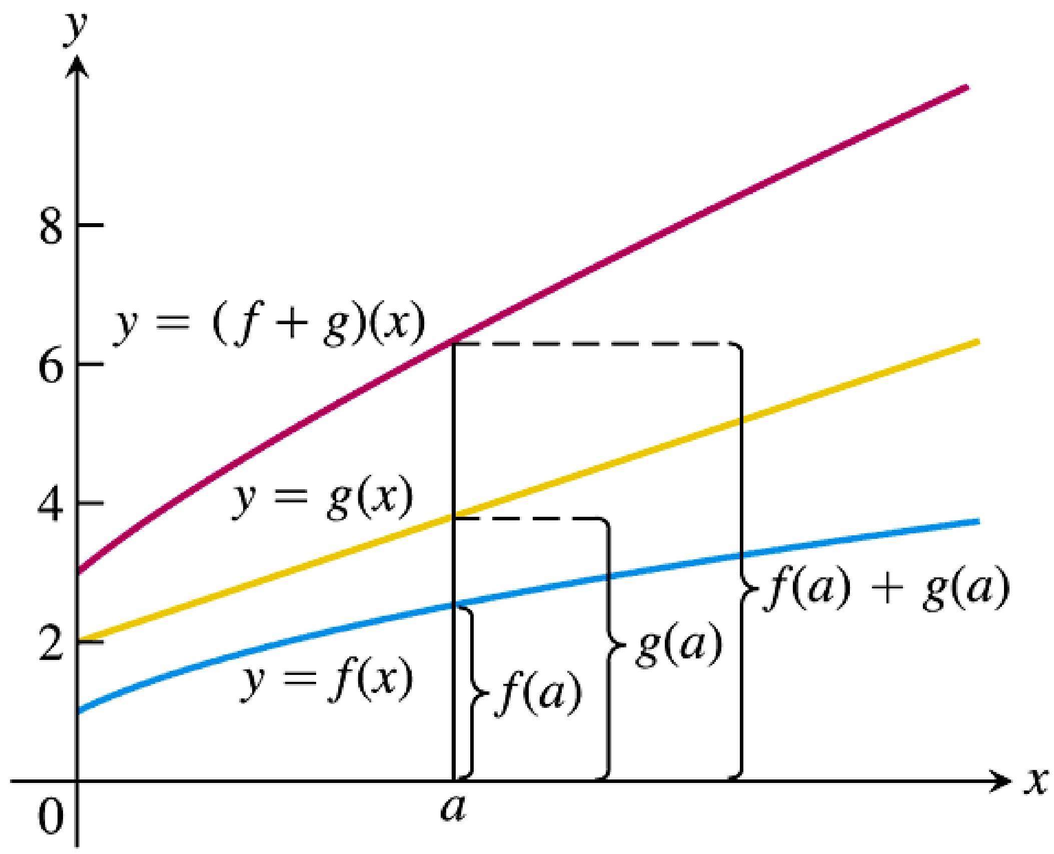


FIGURE 1.50 Graphical addition of two functions.

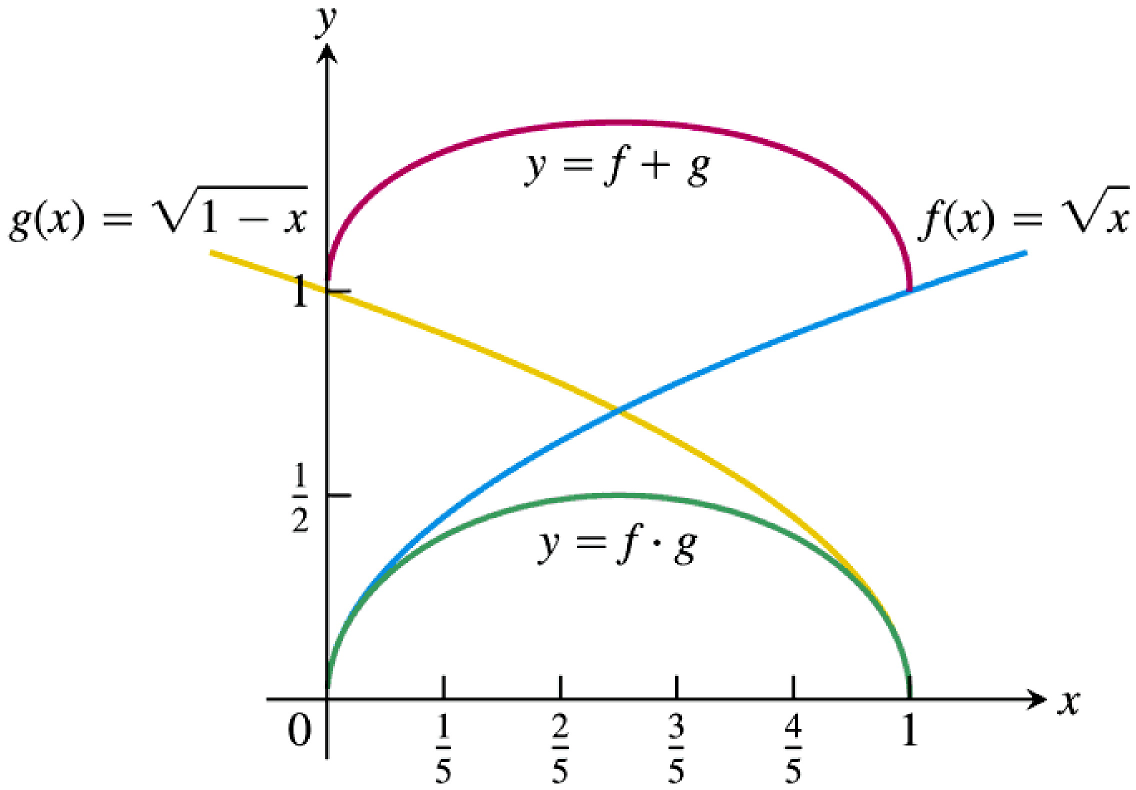


FIGURE 1.51 The domain of the function $f + g$ is the intersection of the domains of f and g, the interval [0, 1] on the x-axis where these domains overlap. This interval is also the domain of the function $f \cdot g$ (Example 1).

Composite functions

- Another way of combining functions

DEFINITION Composition of Functions

If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

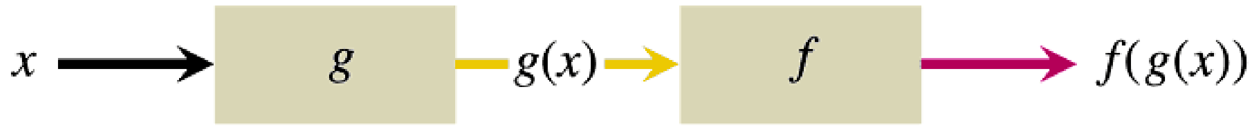


FIGURE 1.52 Two functions can be composed at x whenever the value of one function at x lies in the domain of the other. The composite is denoted by $f \circ g$.

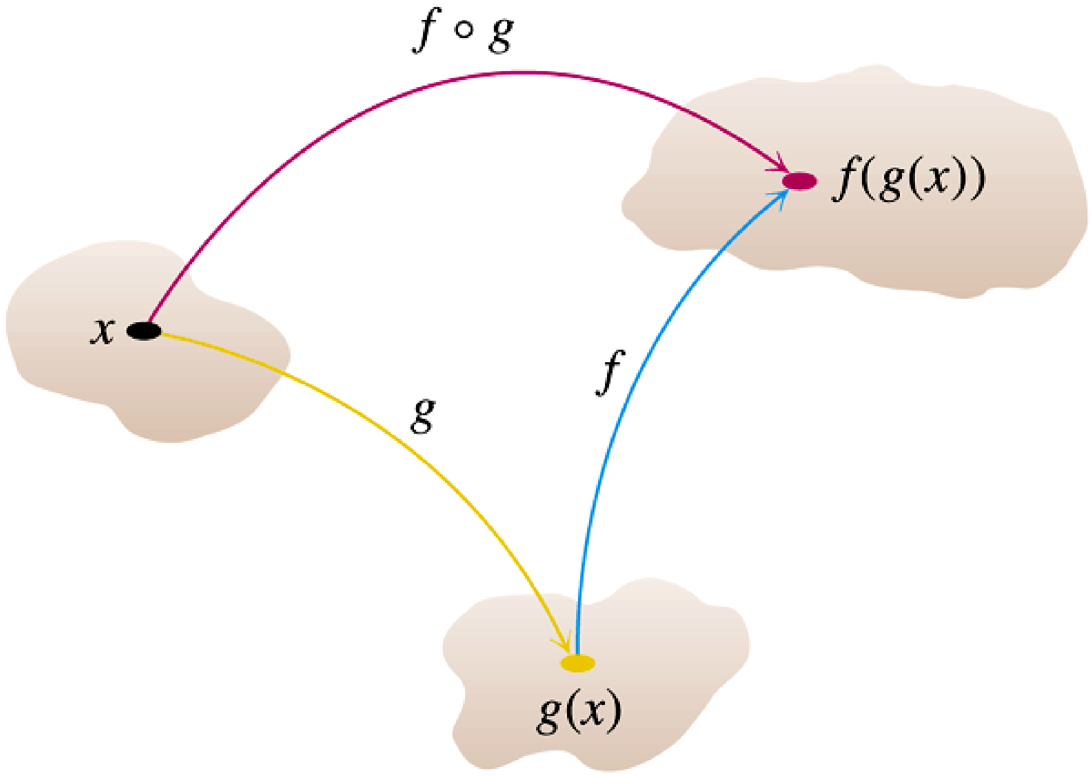


FIGURE 1.53 Arrow diagram for $f \circ g$.

Example 2

- Viewing a function as a composite
- $y(x) = \sqrt{(1 - x^2)}$ is a composite of
- $g(x) = 1 - x^2$ and $f(x) = \sqrt{x}$
- i.e. $y(x) = f\,[g(x)] = \sqrt{(1 - x^2)}$
- Domain of the composite function is $|x| \leqslant 1$, or [-1,1]
- Is $f\,[g(x)] = g\,[f(x)]$?

Example 3

- Read it yourself
- Make sure that you know how to work out the domains and ranges of each composite functions listed

Shifting a graph of a function

Shift Formulas

Vertical Shifts

$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$
	Shifts it *down* $\lvert k \rvert$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$	Shifts the graph of f *left* h units if $h > 0$
	Shifts it *right* $\lvert h \rvert$ units if $h < 0$

Example 4

- (a) $y = x^2$, $y = x^2 + 1$
- (b) $y = x^2$, $y = x^2 - 2$

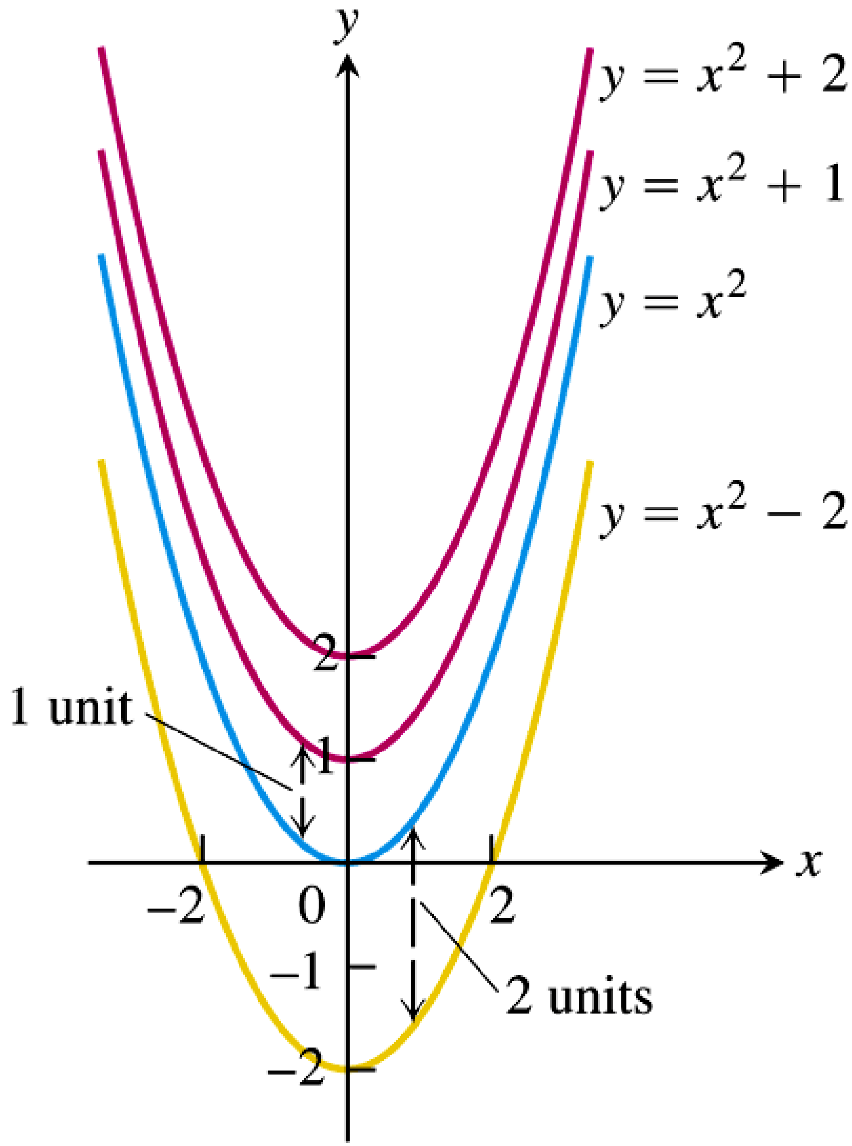


FIGURE 1.54 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Example 4a and b).

Example 4

- (c) $y = x^2, y = (x + 3)^2, \ y = (x - 3)^2$

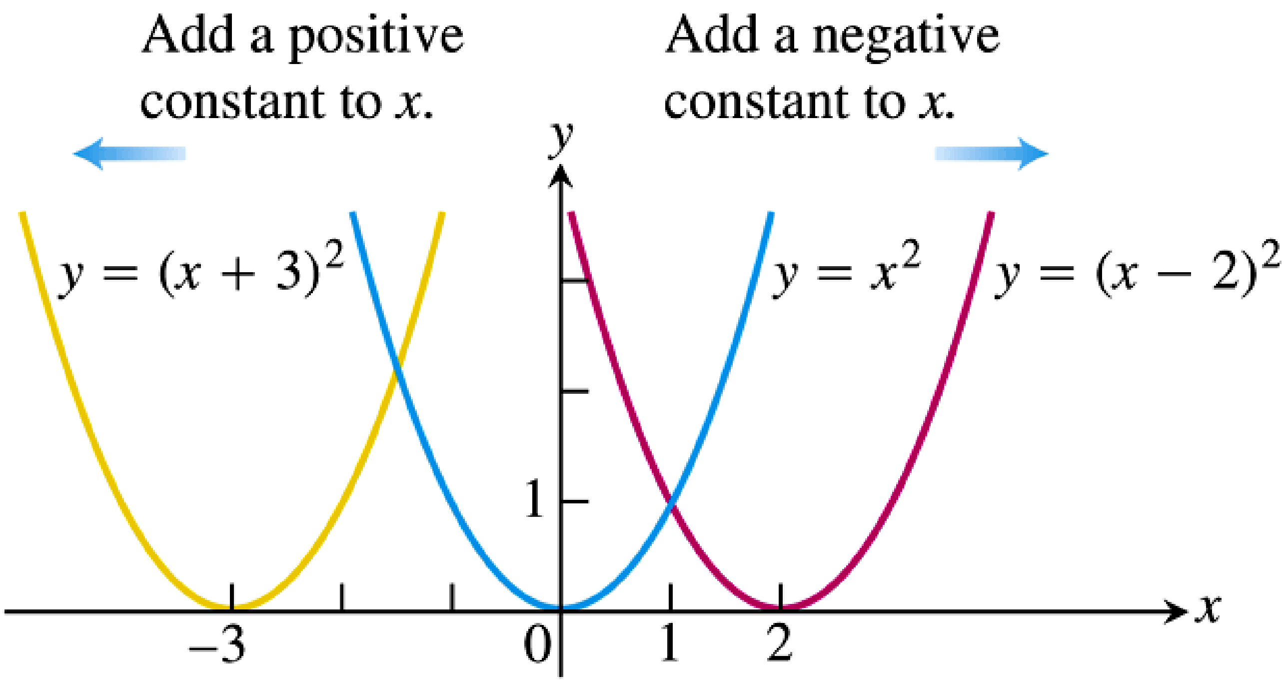


FIGURE 1.55 To shift the graph of $y = x^2$ to the left, we add a positive constant to x. To shift the graph to the right, we add a negative constant to x (Example 4c).

Example 4

- (d) $y = |x|$, $y = |x - 2| - 1$

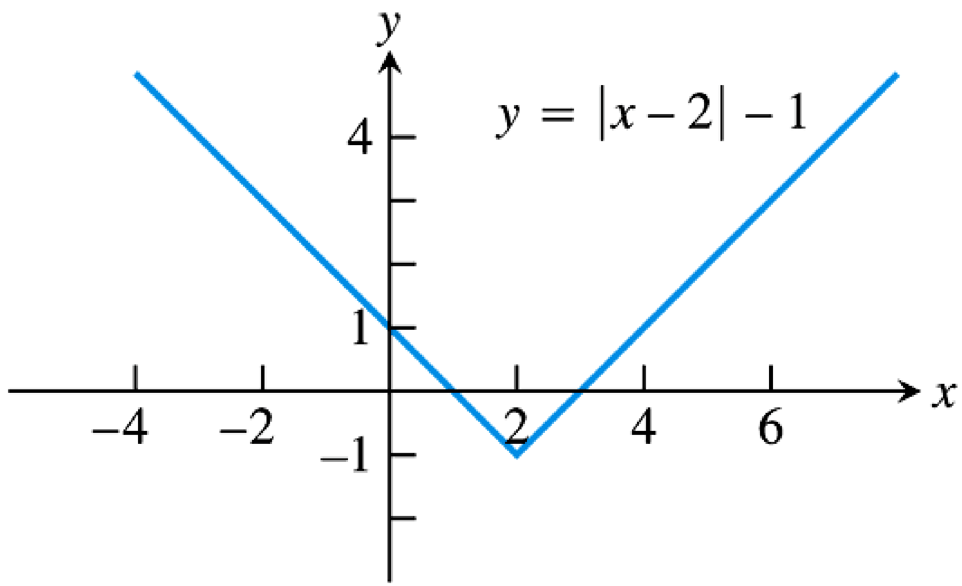


FIGURE 1.56 Shifting the graph of $y = |x|$ 2 units to the right and 1 unit down (Example 4d).

Scaling and reflecting a graph of a function

- To scale a graph of a function is to stretch or compress it, vertically or horizontally.
- This is done by multiplying a constant c to the function or the independent variable

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$,

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$,

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

Example 5(a)

- Vertical stretching and compression of the graph $y = \sqrt{x}$ by a factor or 3

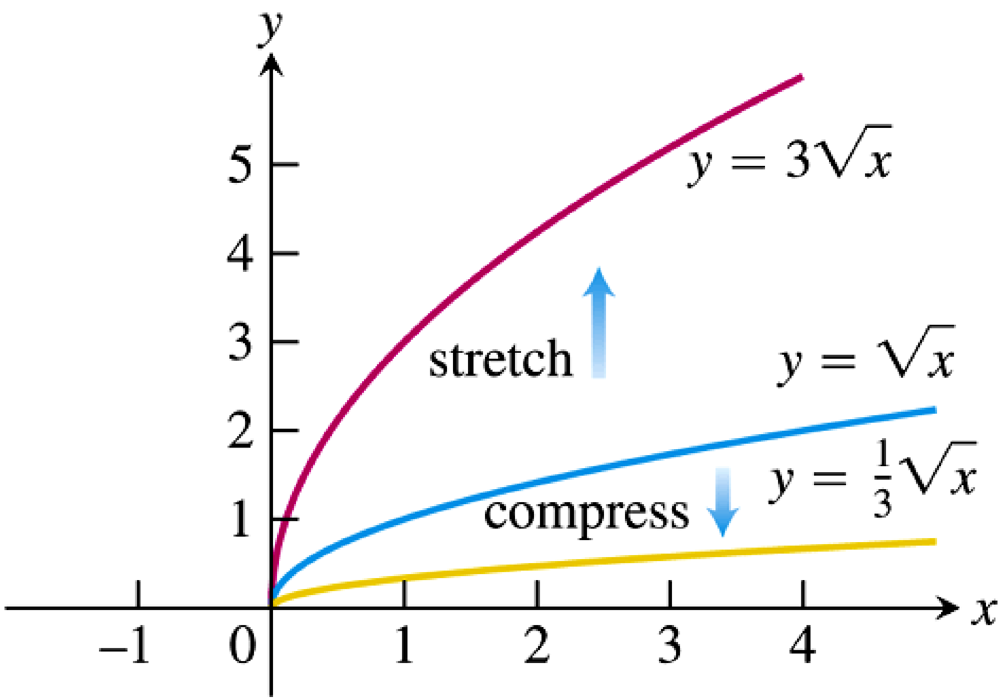


FIGURE 1.57 Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 5a).

Example 5(b)

- Horizontal stretching and compression of the graph $y = \sqrt{x}$ by a factor of 3

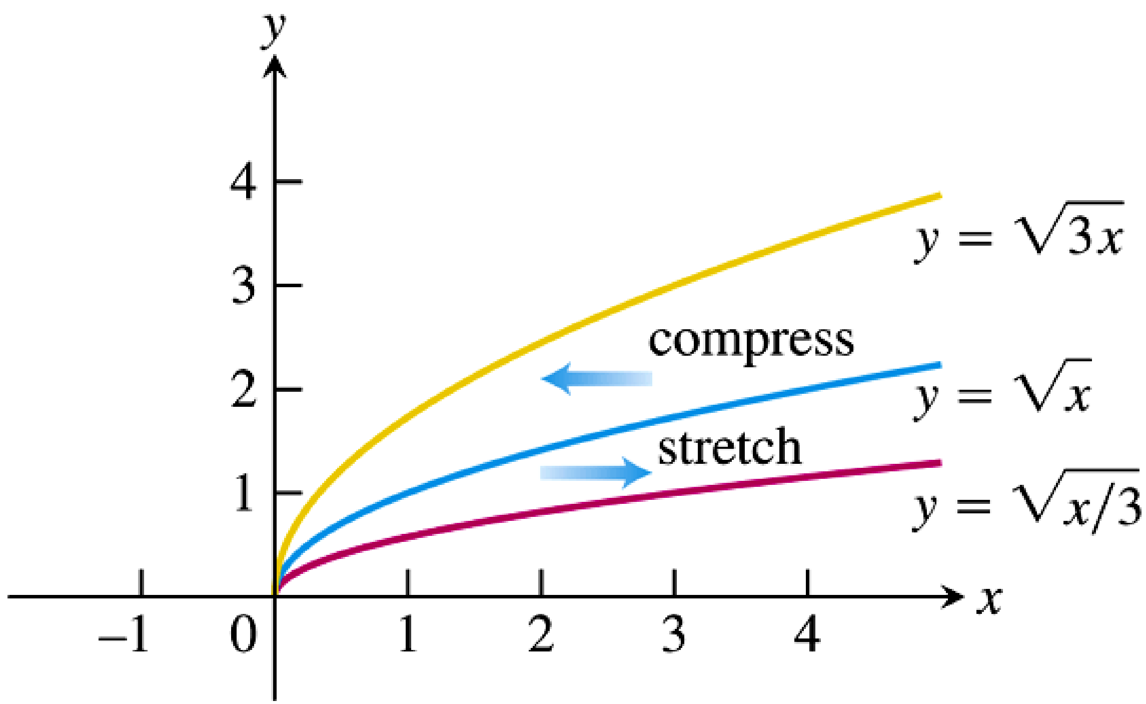


FIGURE 1.58 Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 5b).

Example 5(c)

- Reflection across the *x*- and *y*- axes
- c = -1

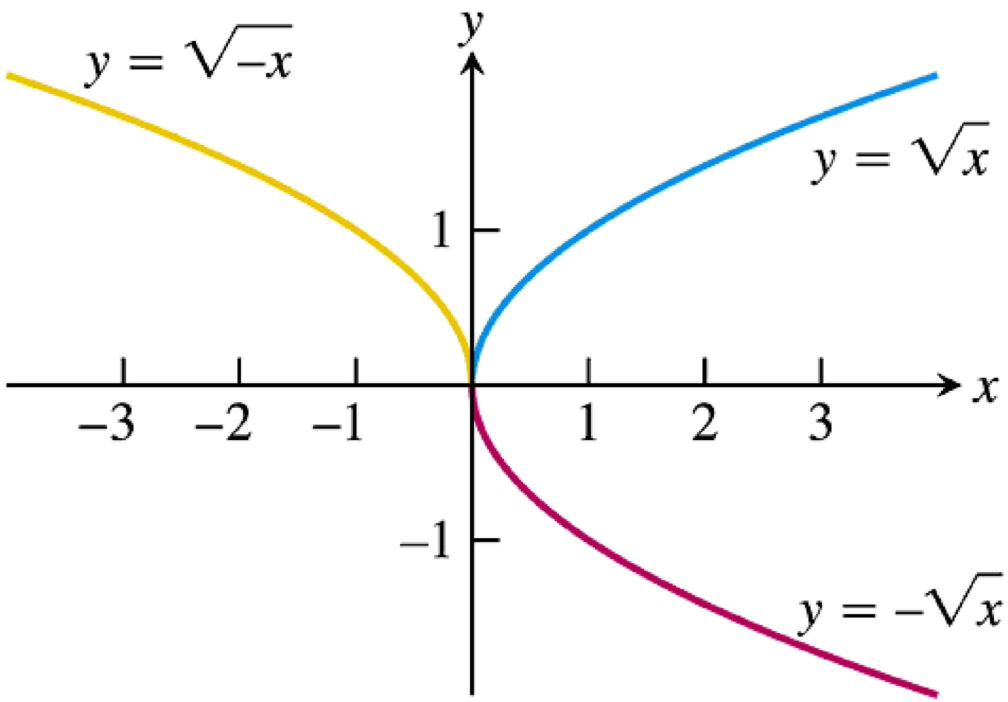


FIGURE 1.59 Reflections of the graph $y = \sqrt{x}$ across the coordinate axes (Example 5c).

Example 6

- Read it yourself

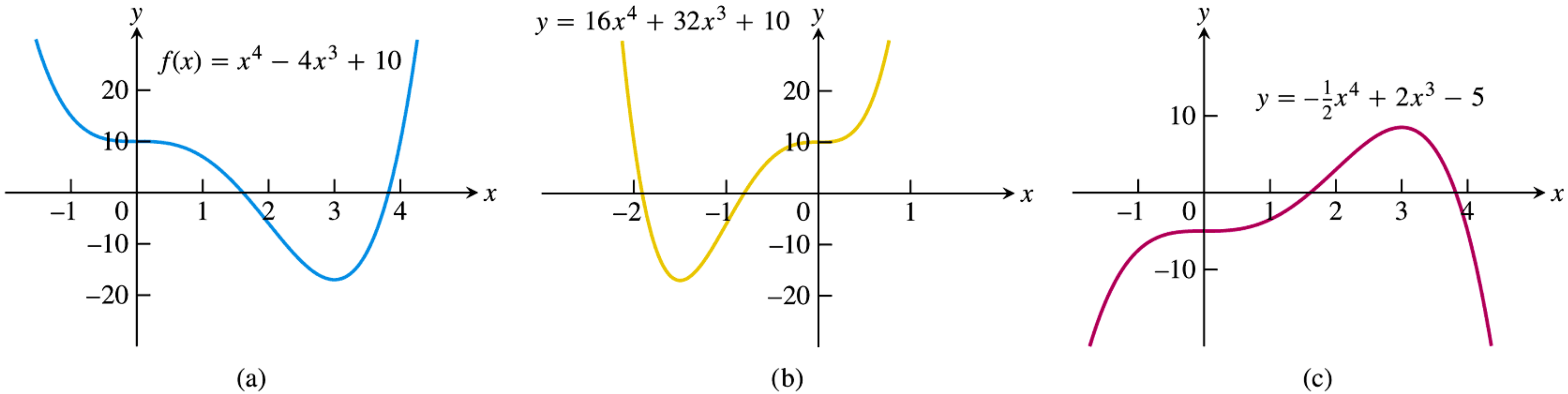


FIGURE 1.60 (a) The original graph of f. (b) The horizontal compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the y-axis. (c) The vertical compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the x-axis (Example 6).

1.6

Trigonometric Functions

(2nd lecture of week 06/08/07 - 11/08/07)

Radian measure

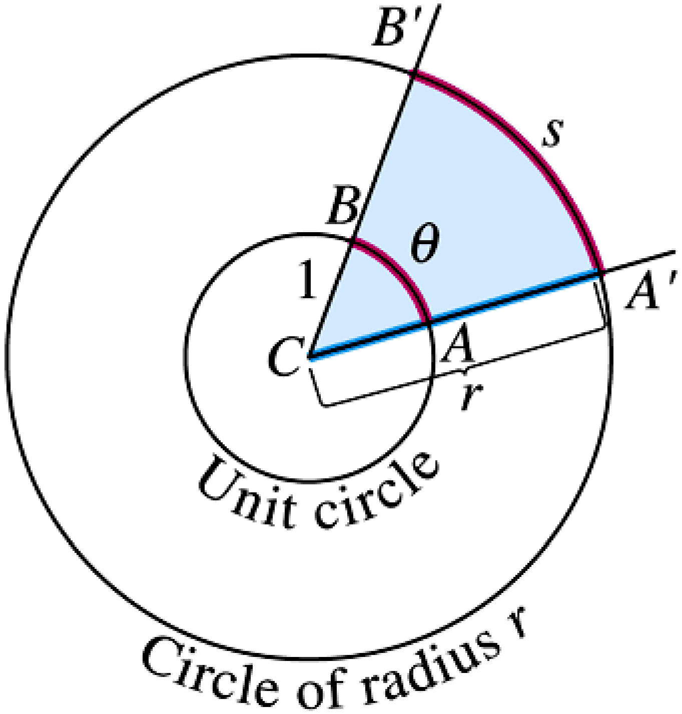


FIGURE 1.63 The radian measure of angle ACB is the length θ of arc AB on the unit circle centered at C. The value of θ can be found from any other circle, however, as the ratio s/r. Thus $s = r\theta$ is the length of arc on a circle of radius r when θ is measured in radians.

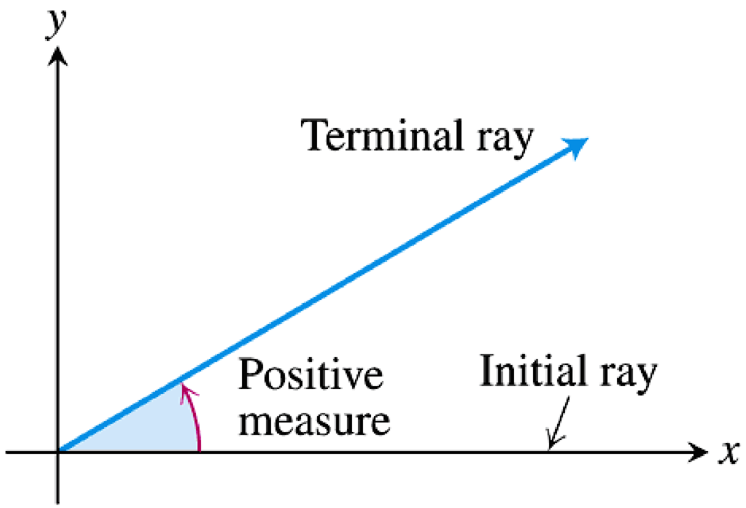
Conversion Formulas

$$1 \text{ degree} = \frac{\pi}{180} (\approx 0.02) \text{ radians}$$

$$\text{Degrees to radians: multiply by } \frac{\pi}{180}$$

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57) \text{ degrees}$$

$$\text{Radians to degrees: multiply by } \frac{180}{\pi}$$



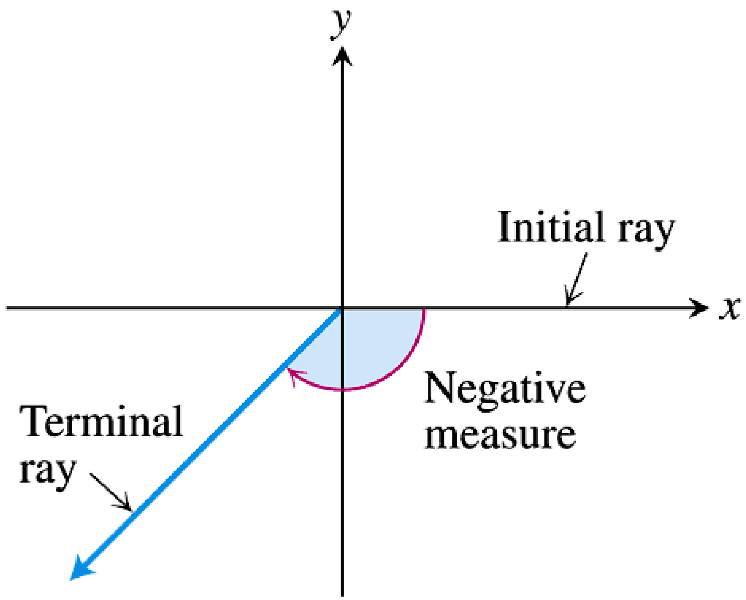


FIGURE 1.65 Angles in standard position in the *xy*-plane.

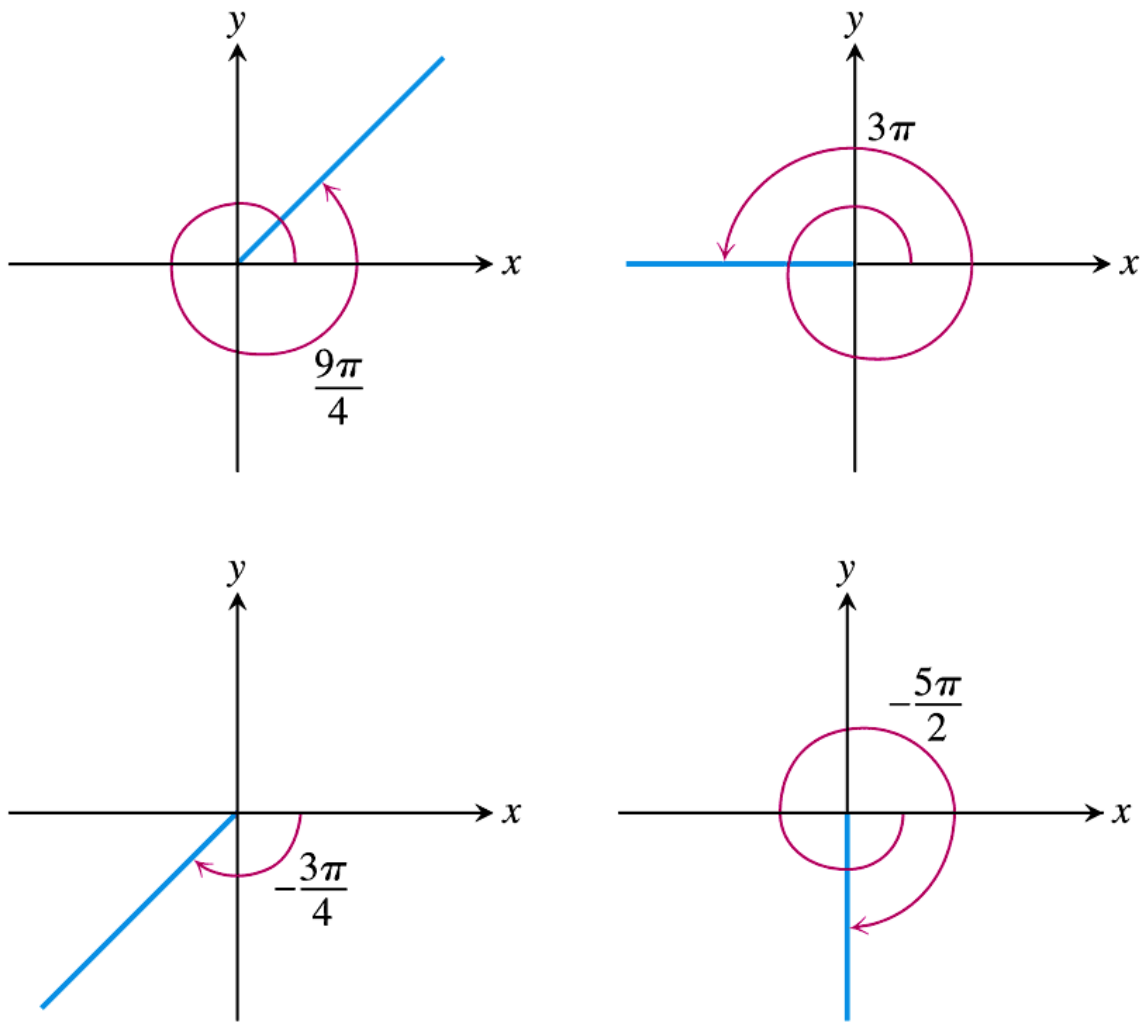
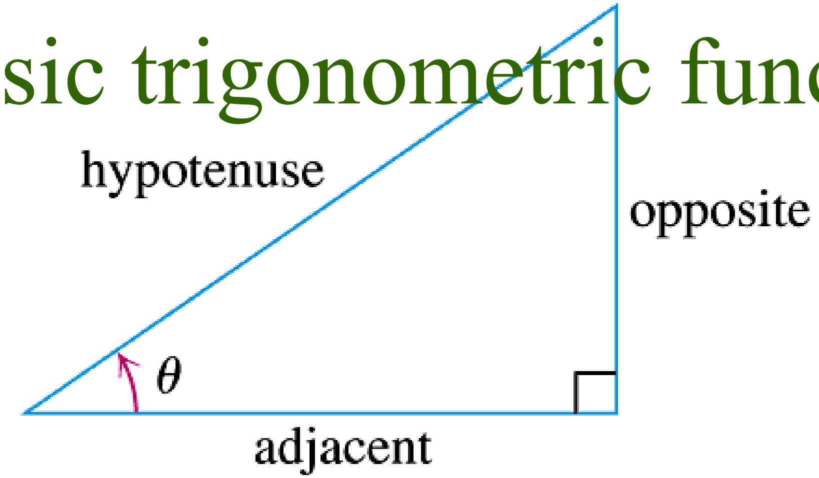


FIGURE 1.66 Nonzero radian measures can be positive or negative and can go beyond 2π.

Angle convention

- Be noted that angle will be expressed in terms of radian unless otherwise specified.
- Get used to the change of the unit

The six basic trigonometric functions



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

FIGURE 1.67 Trigonometric ratios of an acute angle.

Generalised definition of the six trigo functions

- Define the trigo functios in terms of the coordinats of the point $P(x,y)$ on a circle of radius r

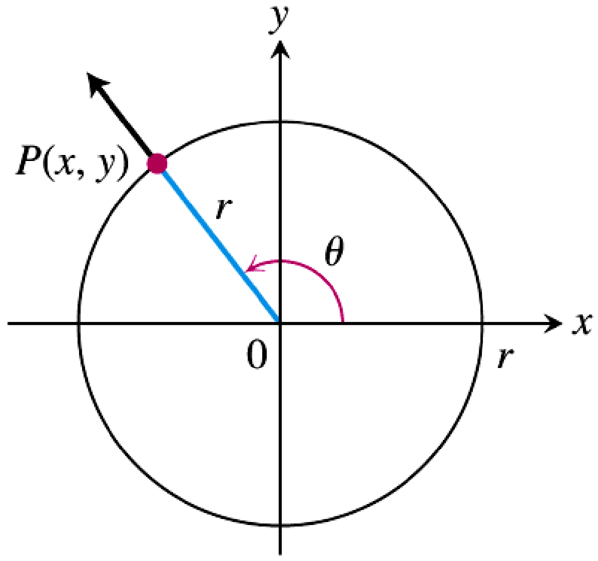


FIGURE 1.68 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.

- **sine:** $\sin\theta = y/r$
- **cosine:** $\cos\theta = x/r$
- **tangent:** $\tan\theta = y/x$
- **cosecant:** $\csc\theta = r/y$
- **secant:** $\sec\theta = r/x$
- **cotangent:** $\cot\theta = x/y$

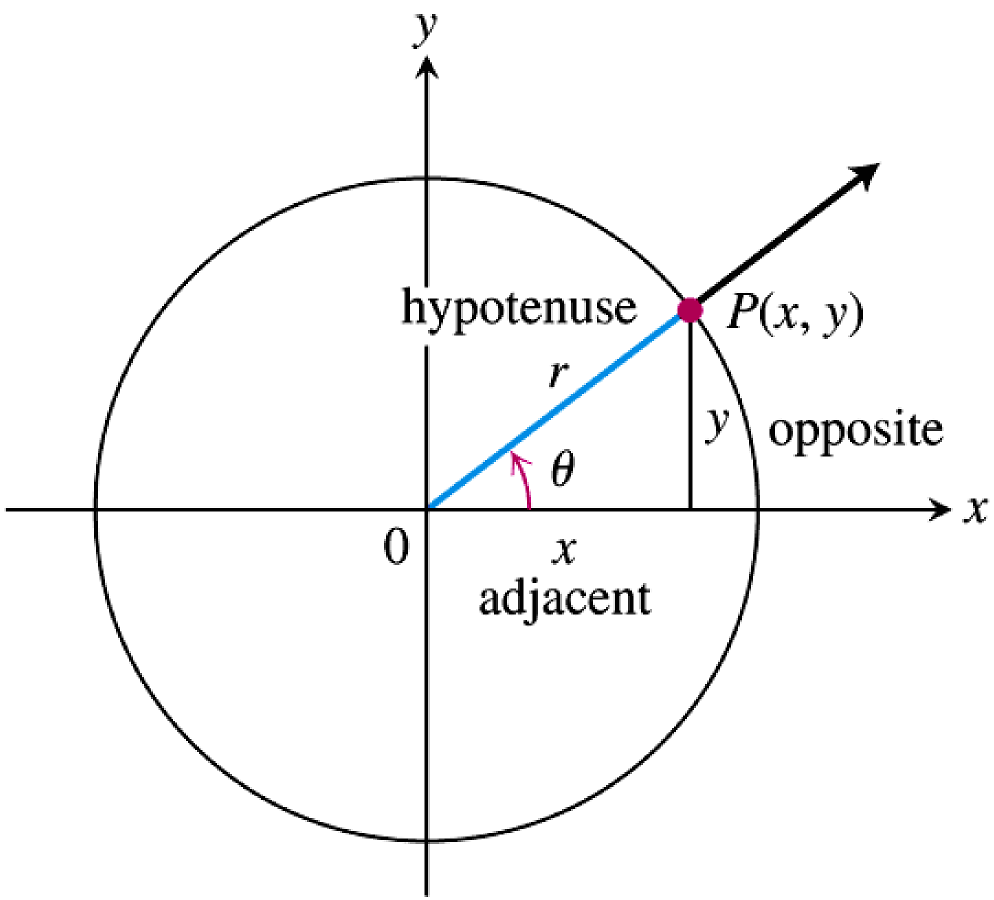


FIGURE 1.69 The new and old definitions agree for acute angles.

Mnemonic to remember when the basic trigo functions are positive or negative

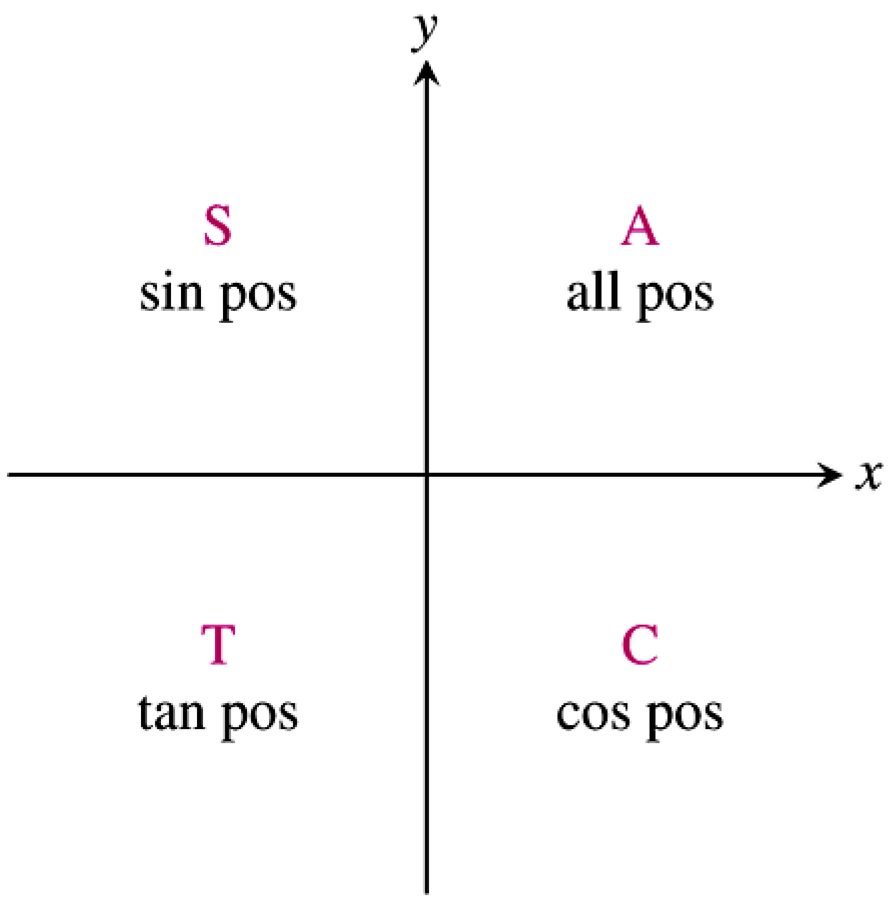


FIGURE 1.70 The CAST rule, remembered by the statement "All Students Take Calculus," tells which trigonometric functions are positive in each quadrant.

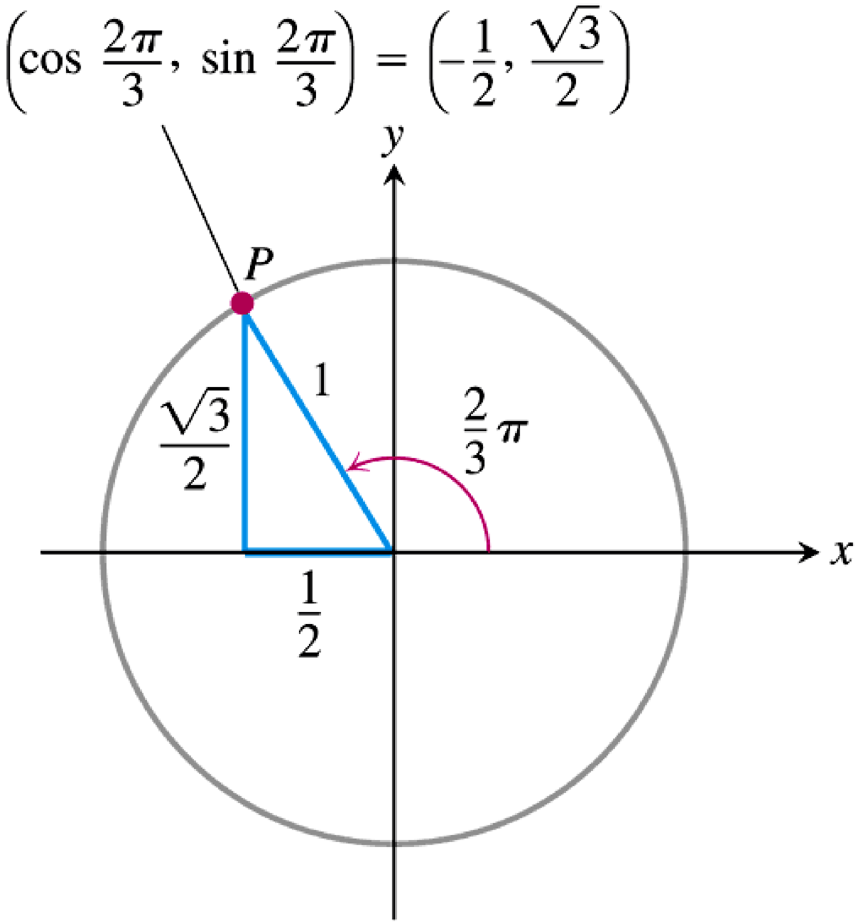


FIGURE 1.71 The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	**−180**	**−135**	**−90**	**−45**	**0**	**30**	**45**	**60**	**90**	**120**	**135**	**150**	**180**	**270**	**360**
θ **(radians)**	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1	0	1
$\tan\theta$	0	1		-1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0		0

Periodicity and graphs of the trigo functions

Trigo functions are also periodic.

DEFINITION **Periodic Function**

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

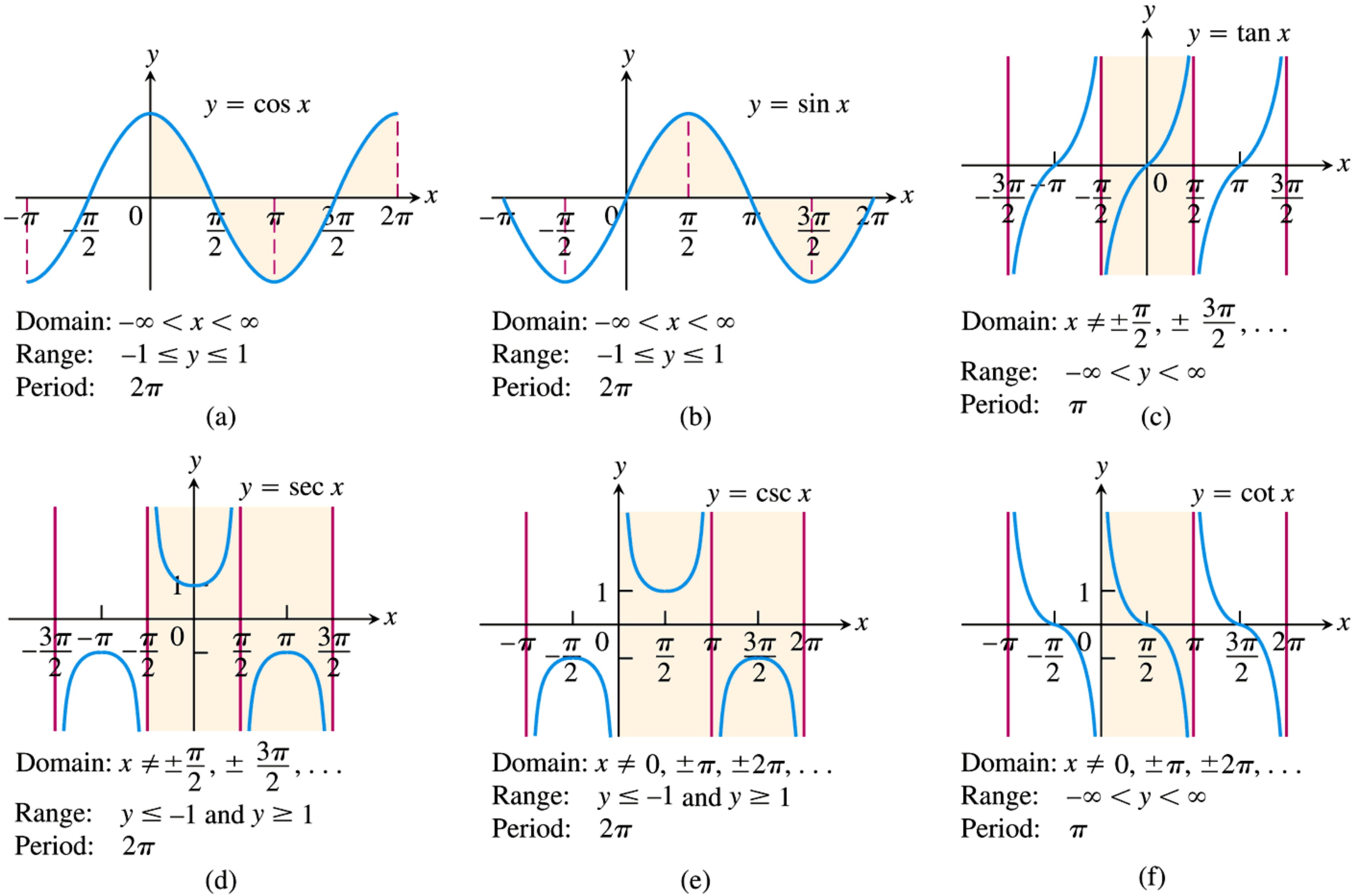


FIGURE 1.73 Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Parity of the trigo functions

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

The parity is easily deduced from the graphs.

Identities

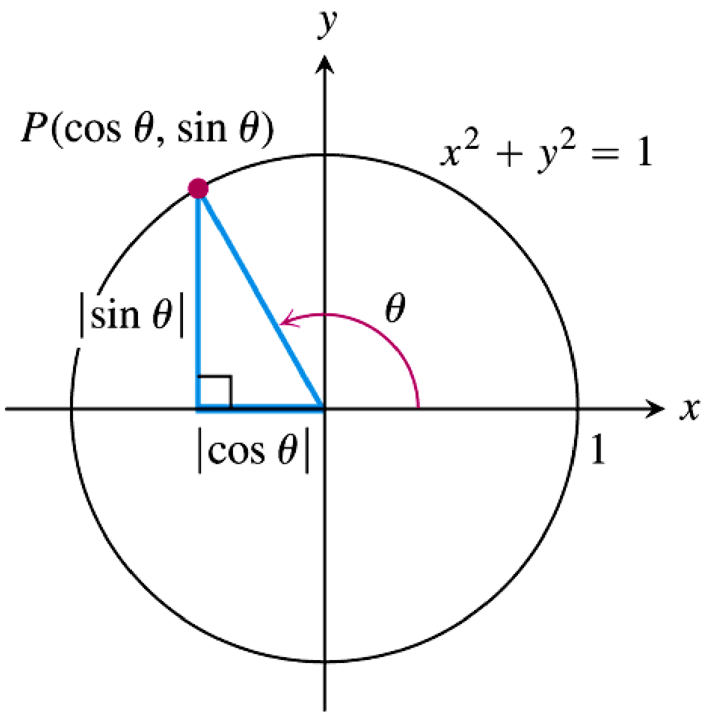


FIGURE 1.74 The reference triangle for a general angle θ.

Applying Pythagorean theorem to the right triangle leads to the identity

$$\cos^2\theta + \sin^2\theta = 1. \quad (1)$$

Dividing identity (1) by $\cos^2\theta$ and $\sin^2\theta$ in turn gives the next two identities

$$1 + \tan^2\theta = \sec^2\theta.$$
$$1 + \cot^2\theta = \csc^2\theta.$$

Addition Formulas

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A + B) = \sin A \cos B + \cos A \sin B \qquad (2)$$

There are also similar formulas for cos (A-B) and sin (A-B). Do you know how to deduce them?

Double-Angle Formulas

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta \end{aligned} \qquad (3)$$

Identity (3) is derived by setting $A = B$ in (2)

Half-Angle Formulas

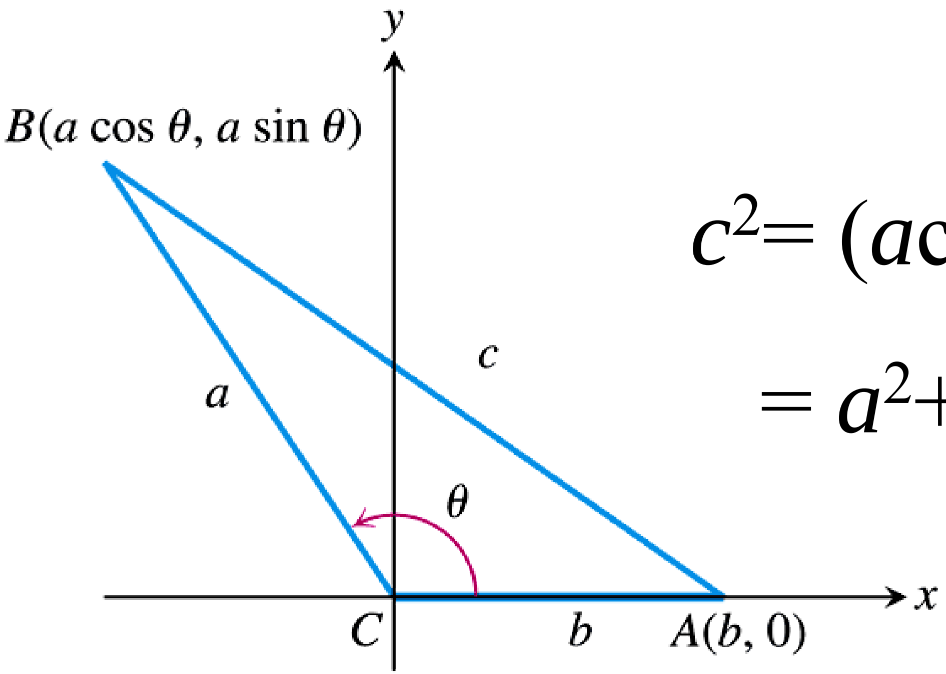
$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad (4)$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \qquad (5)$$

Identities (4,5) are derived by combining (1) and (3(i))

Law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \qquad (6)$$



$$c^2 = (a\cos\theta - b)^2 + (a\sin\theta)^2$$

$$= a^2 + b^2 - 2ab\cos\theta$$

FIGURE 1.75 The square of the distance between A and B gives the law of cosines.

Chapter 2

Limits and Continuity

2.1

Rates of Change and Limits

(3rd lecture of week 06/08/07 - 11/08/07)

Average Rates of change and Secant Lines

- Given an arbitrary function $y=f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y, Δy, by the length Δx

DEFINITION **Average Rate of Change over an Interval**

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

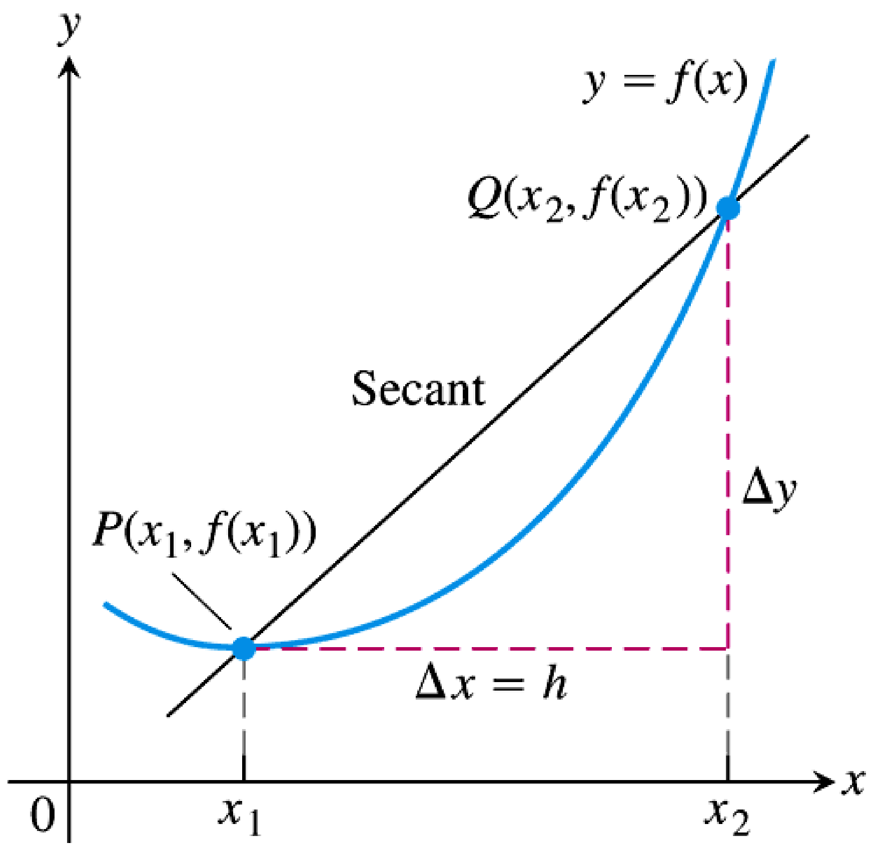


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Example 4

- Figure 2.2 shows how a population of fruit flies grew in a 50-day experiment.
- (a) Find the average growth rate from day 23 to day 45.
- (b) How fast was the number of the flies growing on day 23?

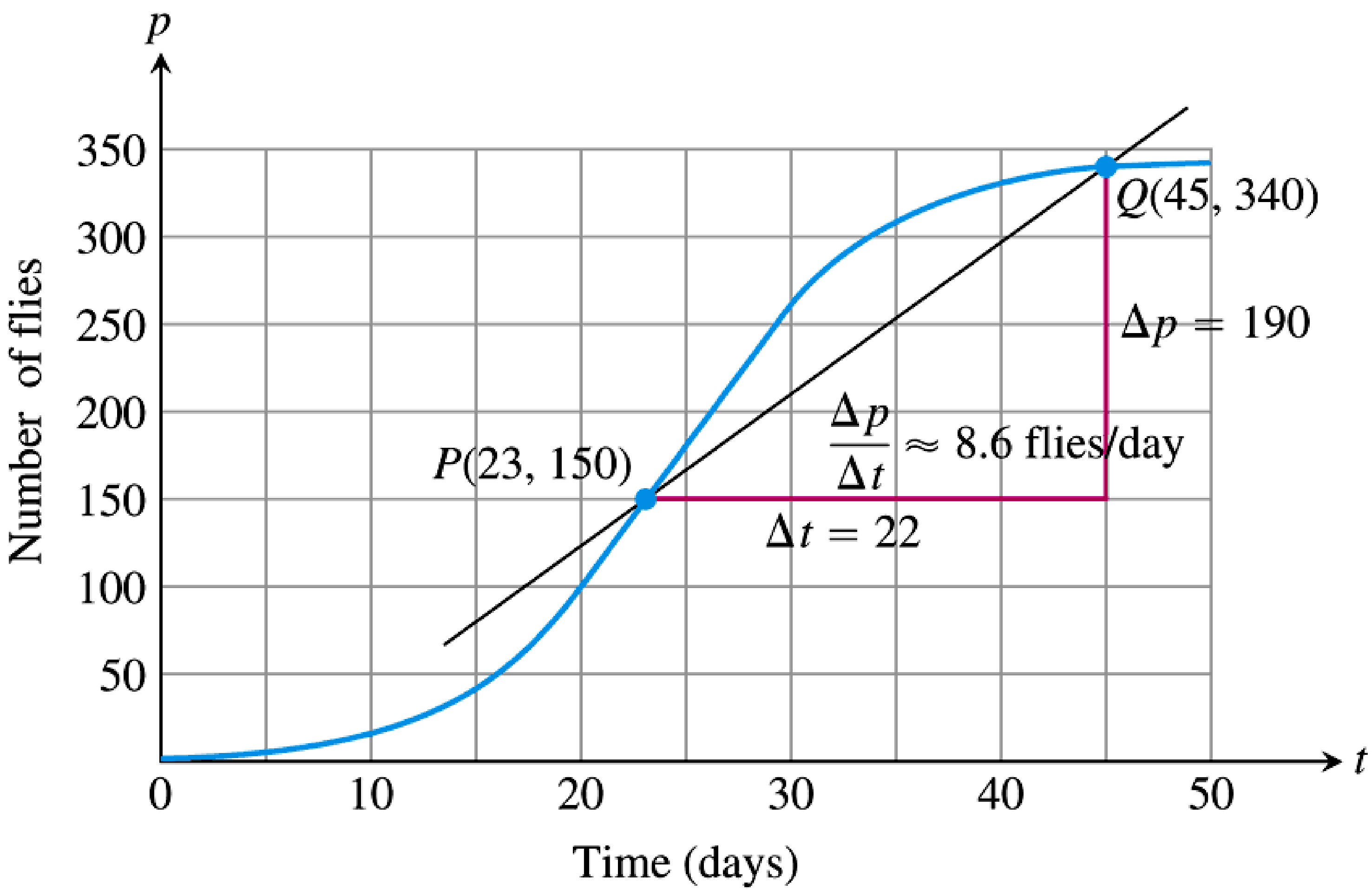


FIGURE 2.2 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line.

Slop at $P \approx$ (250 - 0)/(35-14) = 16.7 flies/day

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\frac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\frac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\frac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\frac{265 - 150}{30 - 23} \approx 16.4$

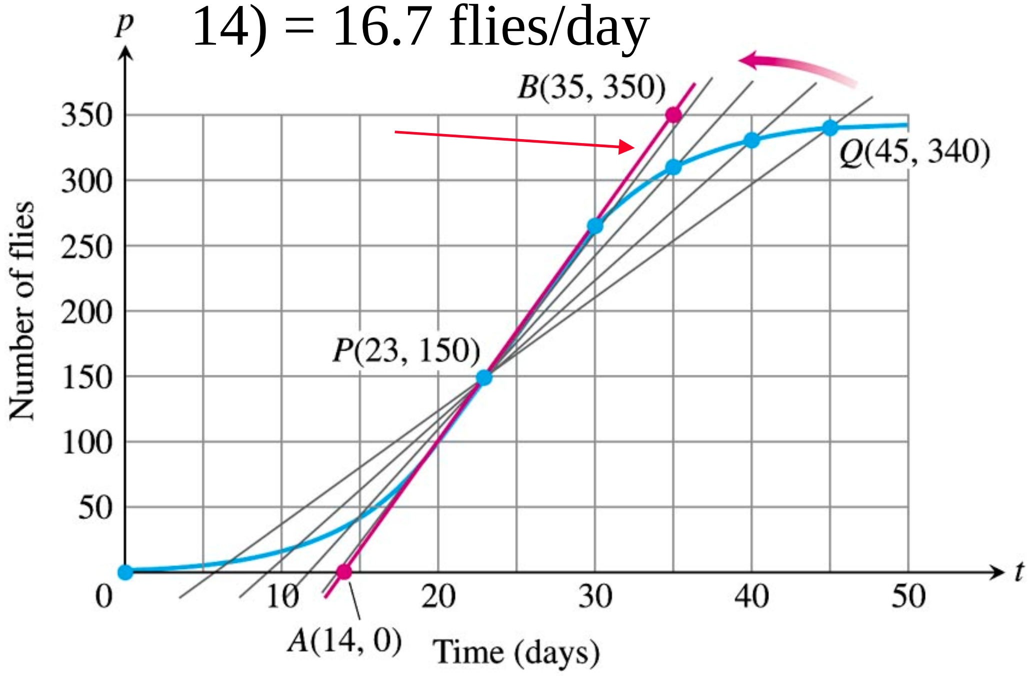


FIGURE 2.3 The positions and slopes of four secants through the point P on the fruit fly graph (Example 4).

The grow rate at day 23 is calculated by examining the average rates of change over increasingly short time intervals starting at day 23. Geometrically, this is equivalent to evaluating the slopes of secants from P to Q with Q approaching P.

Limits of function values

- Informal definition of limit:
- Let f be a function defined on an open interval about x_0, except possibly at x_0 itself.
- If f gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0

$$\lim_{x \to x_0} f(x) = L$$

- "Arbitrarily close" is not yet defined here (hence the definition is informal).

Example 5

- How does the function behave near $x=1$?

$$f(x)=\frac{x^2-1}{x-1}$$

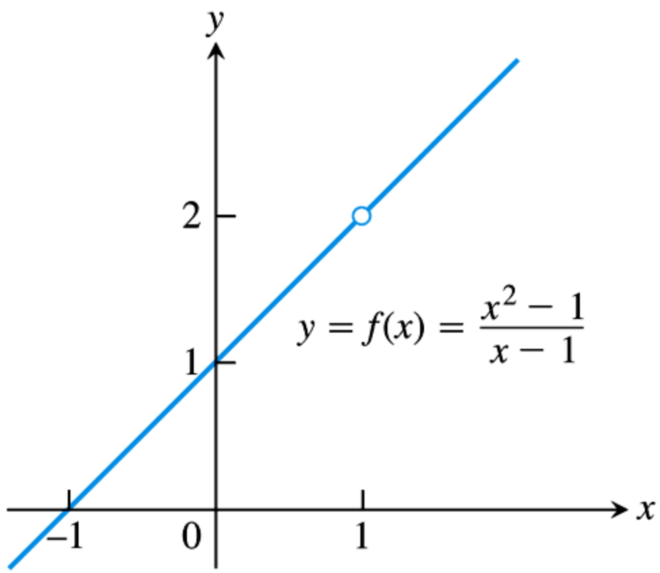
- Solution:

$$f(x)=\frac{(x-1)(x+1)}{x-1}=x+1 \quad \text{for } x\neq 1$$

TABLE 2.2 The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

Values of x below and above 1	$f(x) = \frac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

We say that $f(x)$ approaches the limit 2 as x approaches 1, $\lim_{x \to 1} f(x) = 2$ or $\lim_{x \to 1} = \frac{x^2 - 1}{x - 1} = 2$



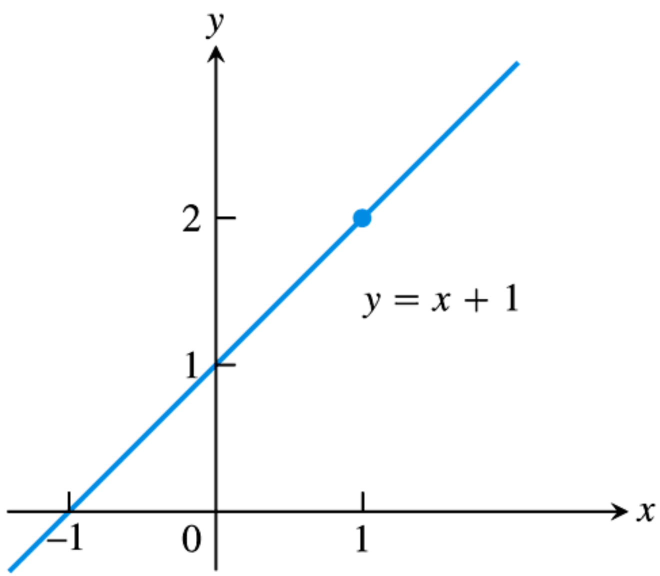


FIGURE 2.4 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 5).

Example 6

- The limit value does not depend on how the function is defined at x_0.

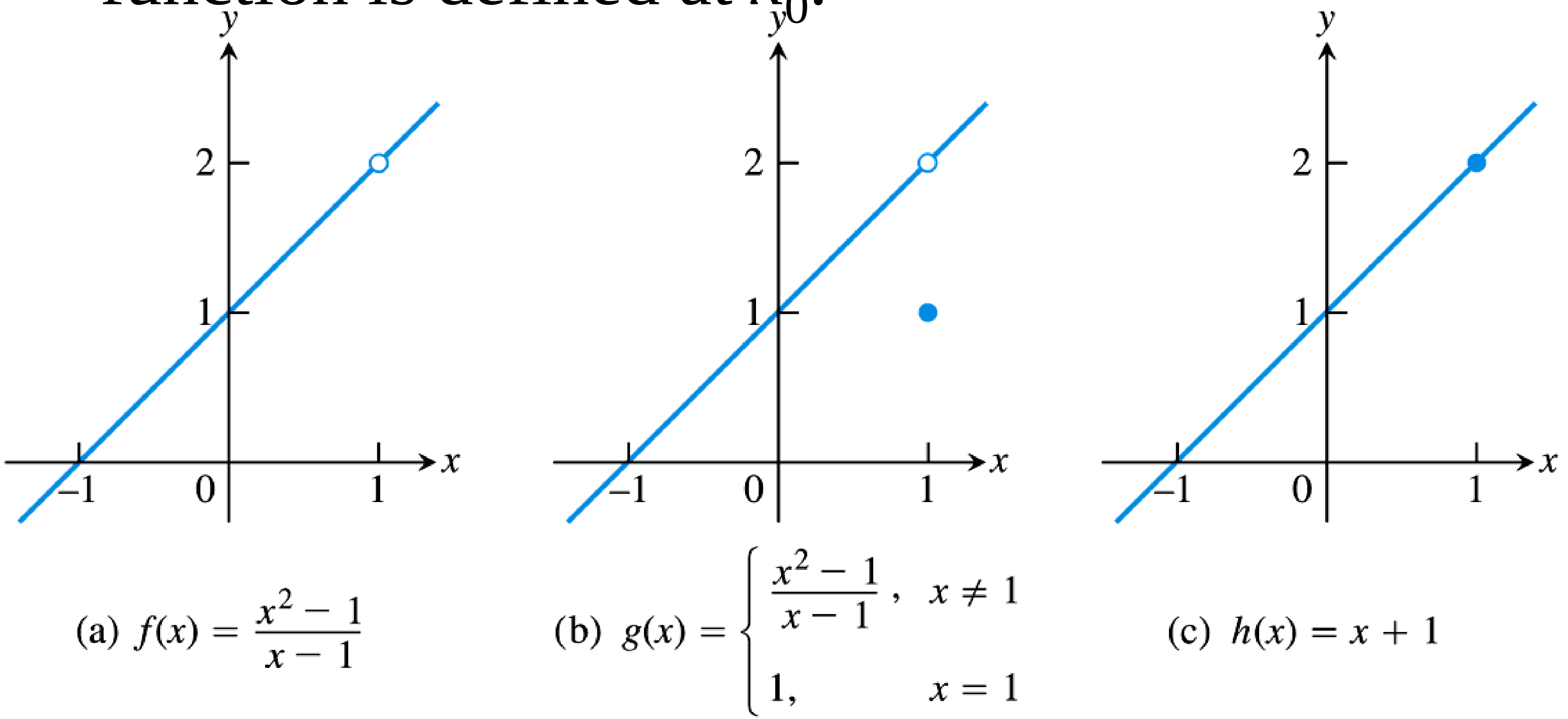


FIGURE 2.5 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 6).

Example 7

- In some special cases $\lim_{x \to x0} f(x)$ can be evaluated by calculating $f(x_0)$. For example, constant function, rational function and identity function for which $x=x_0$ is defined
- (a) $\lim_{x \to 2} (4) = 4$ (constant function)
- (b) $\lim_{x \to -13} (4) = 4$ (constant function)
- (c) $\lim_{x \to 3} x = 3$ (identity function)
- (d) $\lim_{x \to 2} (5x-3) = 10 – 3 =7$ (polynomial function of degree 1)
- (e) $\lim_{x \to -2} (3x+4)/(x+5) = (-6+4)/(-2+5) =-2/3$ (rational function)

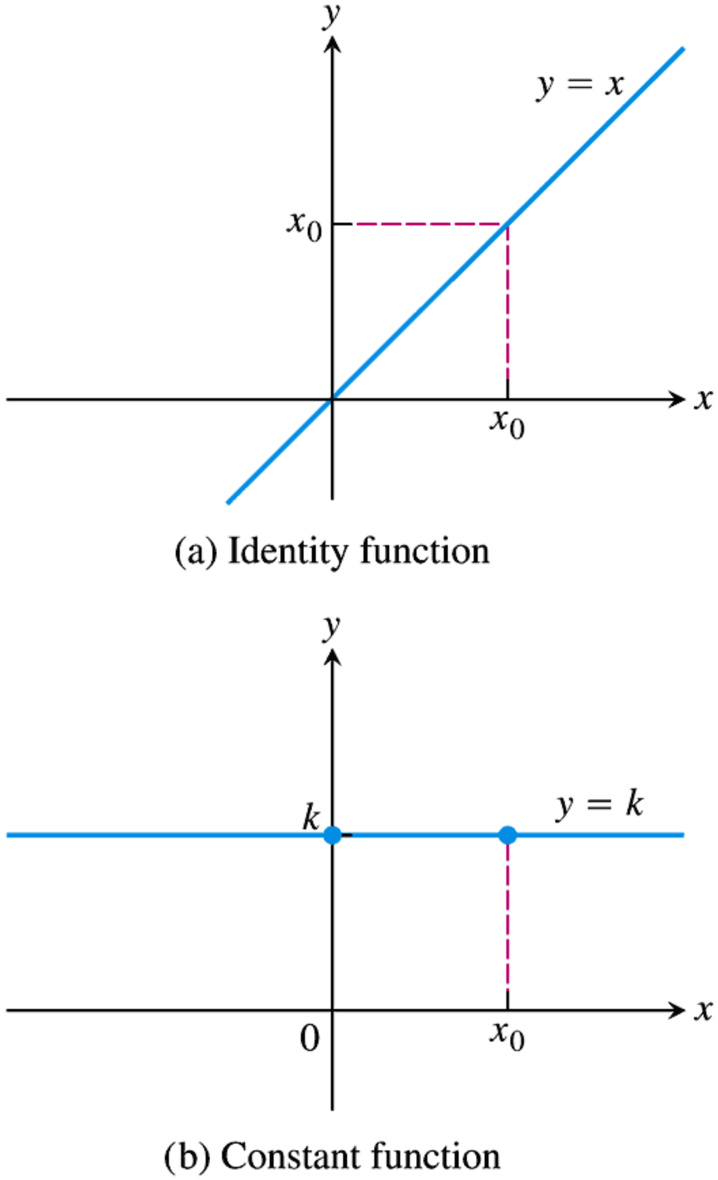


FIGURE 2.6 The functions in Example 8.

Example 9

- A function may fail to have a limit exist at a point in its domain.

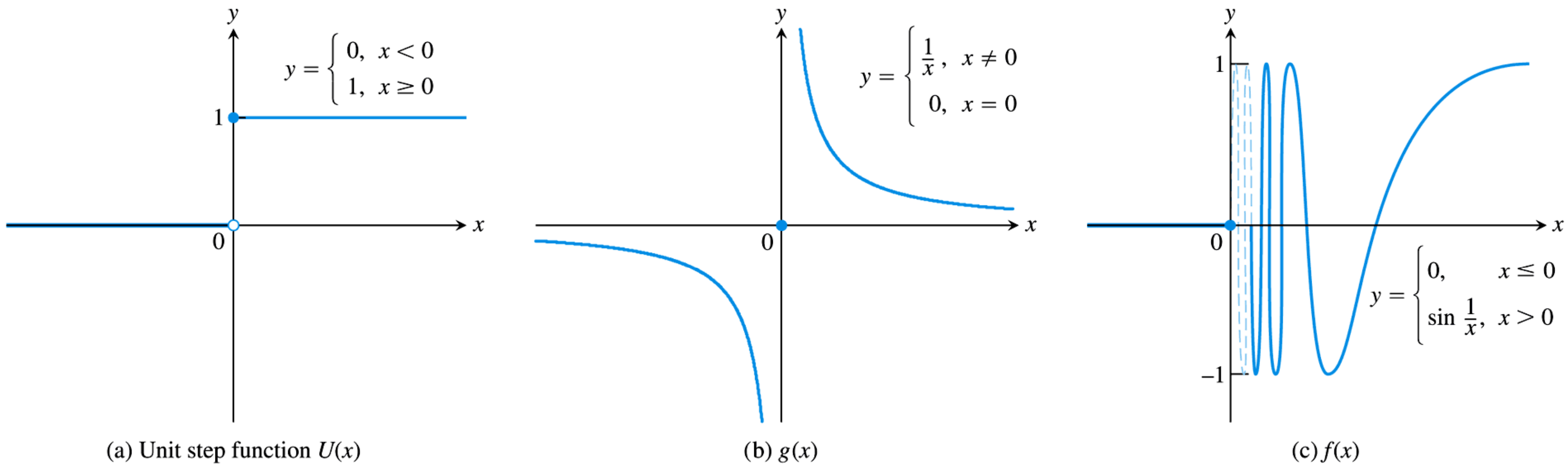


FIGURE 2.7 None of these functions has a limit as x approaches 0 (Example 9).

Jump

Grow to infinities

Oscillate

2.2

Calculating limits using the limits laws

(3rd lecture of week 06/08/07 - 11/08/07)

The limit laws

- Theorem 1 tells how to calculate limits of functions that are arithmetic combinations of functions whose limit are already known.

THEOREM 1 Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $$\lim_{x\to c}(f(x) + g(x)) = L + M$$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $$\lim_{x\to c}(f(x) - g(x)) = L - M$$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $$\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $$\lim_{x\to c}(k\cdot f(x)) = k\cdot L$$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Example 1 Using the limit laws

- (a) $\lim_{x \to c} (x^3+4x^2-3)$

  $= \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$

  (sum and difference rule)

  $= c^3 + 4c^2 - 3$

  (product and multiple rules)

Example 1

- (b) $\lim_{x\to c} (x^4+x^2-1)/(x^2+5)$

  $= \lim_{x\to c} (x^4+x^2-1) / \lim_{x\to c} (x^2+5)$

$=(\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1)/(\lim_{x\to c} x^2 + \lim_{x\to c} 5)$

$= (c^4 + c^2 - 1)/(c^2 + 5)$

Example 1

- (c) $\lim_{x \to -2} \sqrt{(4x^2-3)} = \sqrt{\lim_{x \to -2} (4x^2-3)}$

  Power rule with $r/s = ½$

  $= \sqrt{[\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3]}$

  $= \sqrt{[4(-2)^2 - 3]} = \sqrt{13}$

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example 2

- Limit of a rational function

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

Eliminating zero denominators algebraically

Identifying Common Factors

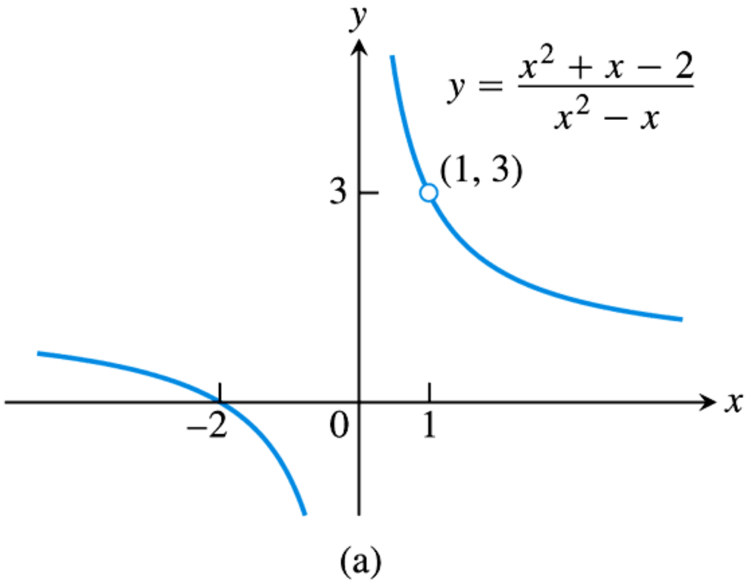
It can be shown that if $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.

Example 3 Canceling a common factor

- Evaluate $\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}$
- Solution: We can't substitute x=1 since

  $f(x = 1)$ is not defined. Since $x \neq 1$, we can cancel the common factor of x-1:

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{(x-1)(x+2)}{x(x-1)} = \lim_{x\to 1} \frac{(x+2)}{x} = 3$$



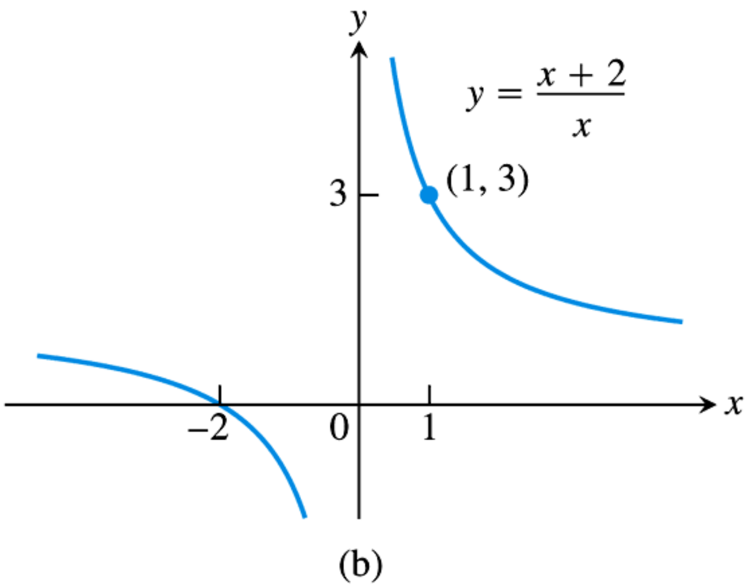


FIGURE 2.8 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$ (Example 3).

The Sandwich theorem

THEOREM 4 The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

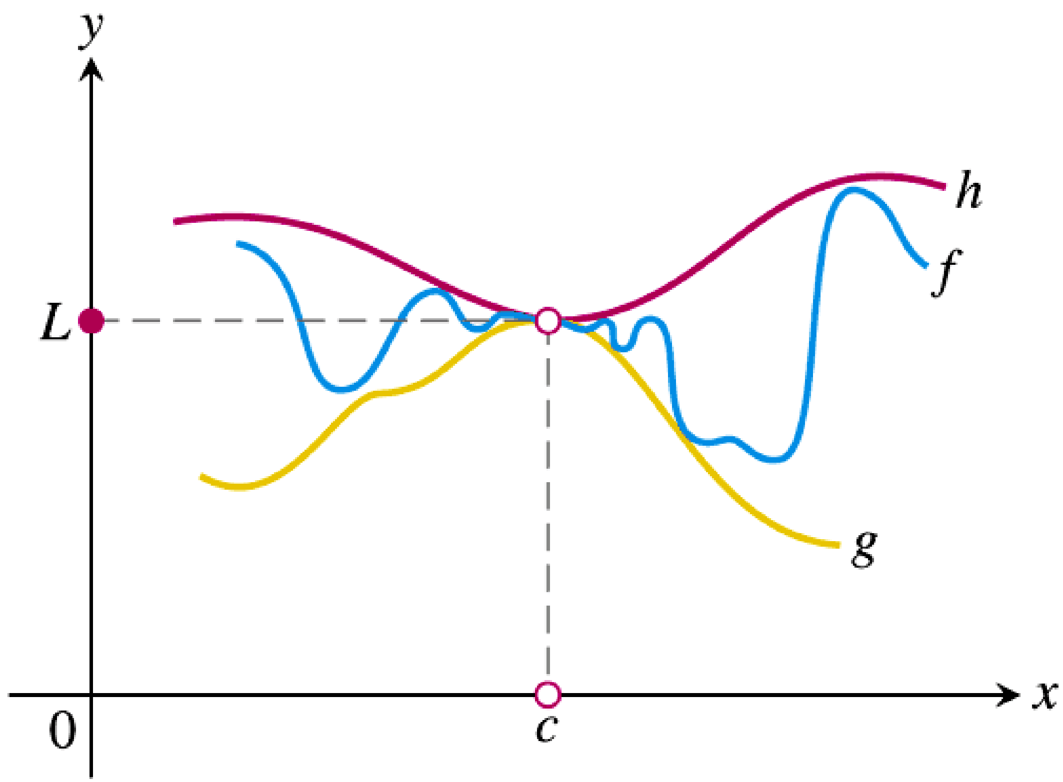


FIGURE 2.9 The graph of f is sandwiched between the graphs of g and h.

Example 6

- (a)
- The function $y = \sin\theta$ is sandwiched between $y = |\theta|$ and $y = -|\theta|$ for all values of θ. Since $\lim_{\theta\to 0} (-|\theta|) = \lim_{\theta\to 0} (|\theta|) = 0$, we have $\lim_{\theta\to 0} \sin\theta = 0$.
- (b)
- From the definition of $\cos\theta$,

$0 \leqslant 1 - \cos\theta \leqslant |\theta|$ for all θ, and we have the limit $\lim_{x\to 0} \cos\theta = 1$

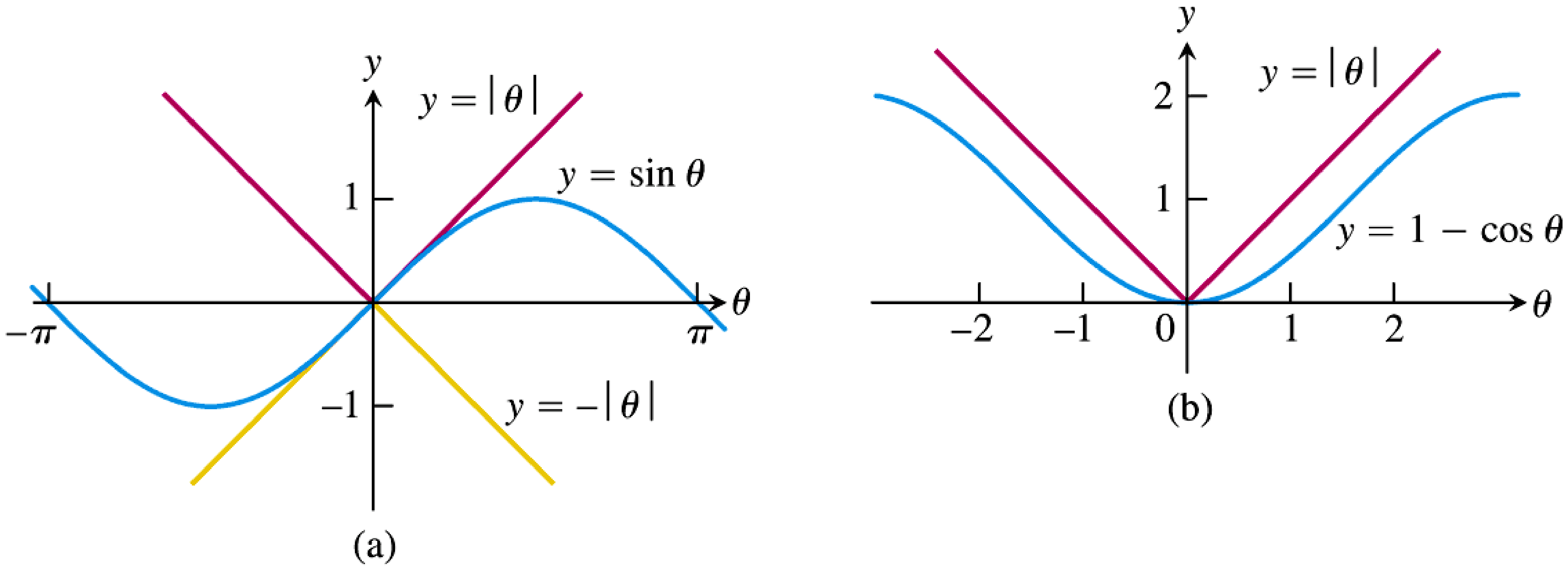


FIGURE 2.11 The Sandwich Theorem confirms that (a) $\lim_{\theta \to 0} \sin \theta = 0$ and (b) $\lim_{\theta \to 0} (1 - \cos \theta) = 0$ (Example 6).

Example 6(c)

- For any function $f(x)$, if $\lim_{x\to 0} (|f(x)|) = 0$, then $\lim_{x\to 0} f(x) = 0$ due to the sandwich theorem.
- Proof: $-|f(x)| \leqslant f(x) \leqslant |f(x)|$.
- Since $\lim_{x\to 0} (|f(x)|) = \lim_{x\to 0} (-|f(x)|) = 0$
- $\Rightarrow \lim_{x\to 0} f(x) = 0$

2.3

The Precise Definition of a Limit

(1st lecture of week 13/08/07-18/08/07)

Example 1 A linear function

- Consider the linear function $y = 2x - 1$ near $x_0 = 4$. Intuitively it is close to 7 when x is close to 4, so $\lim_{x \to 0} (2x\text{-}1)=7$. How close does x have to be so that $y = 2x -1$ differs from 7 by less than 2 units?

Solution

- For what value of x is $|y-7| < 2$?
- First, find $|y-7|<2$ in terms of x:

$|y-7|<2 \equiv |2x-8|<2$

$\equiv -2 < 2x-8 < 2$

$\equiv 3 < x < 5$

$\equiv -1 < x - 4 < 1$

Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0=7$.

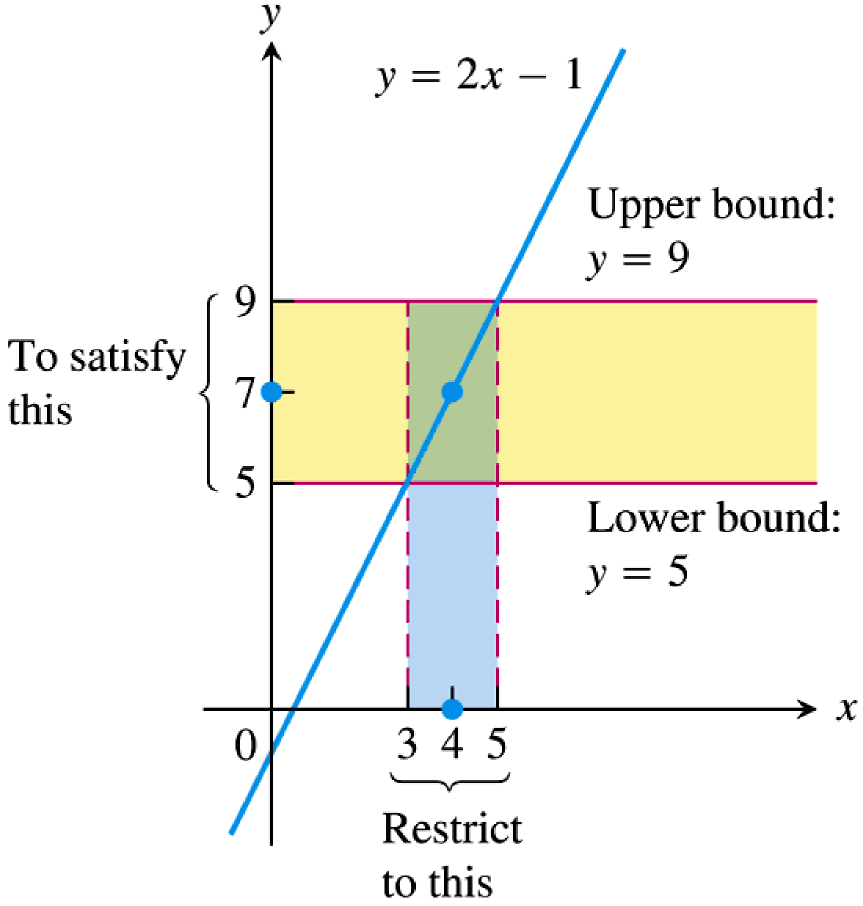


FIGURE 2.12 Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Example 1).

Definition of limit

DEFINITION Limit of a Function

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Definition of limit

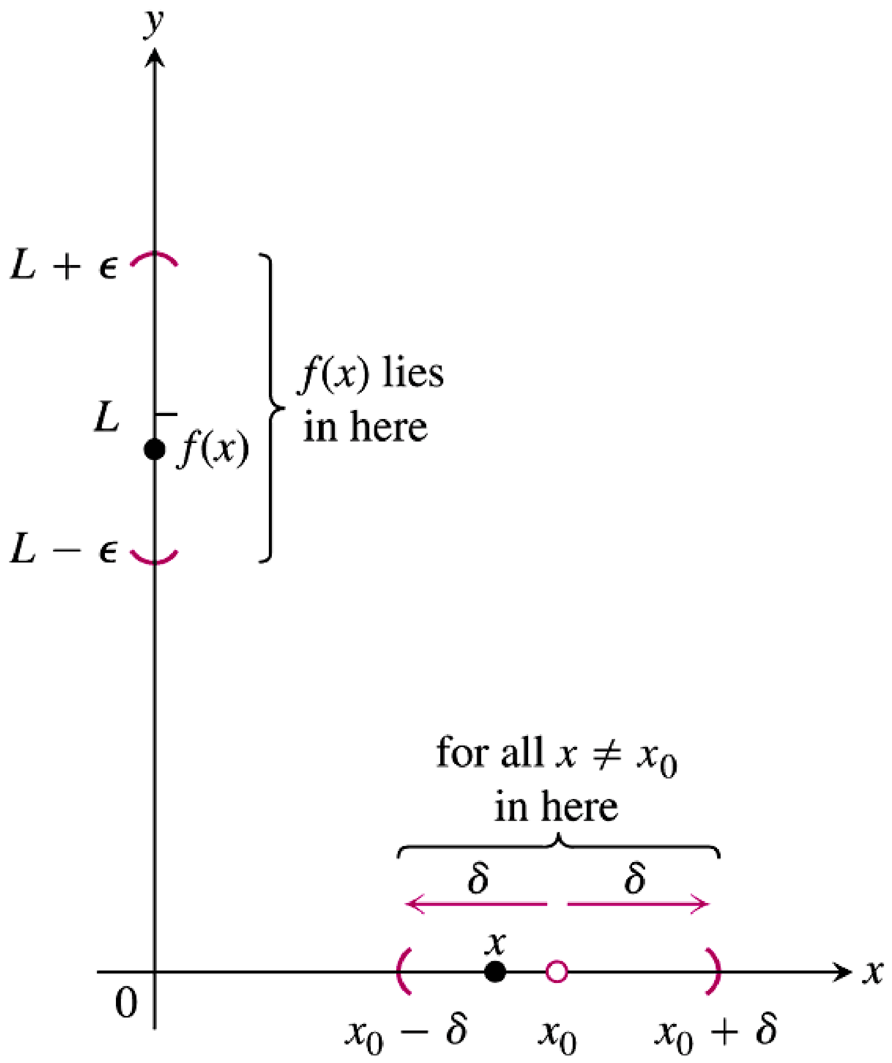


FIGURE 2.14 The relation of δ and ϵ in the definition of limit.

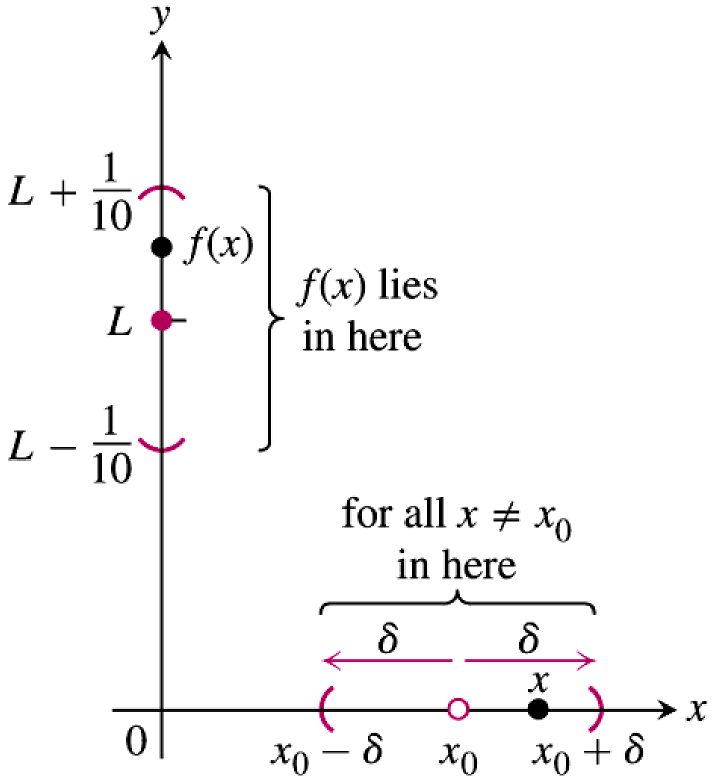
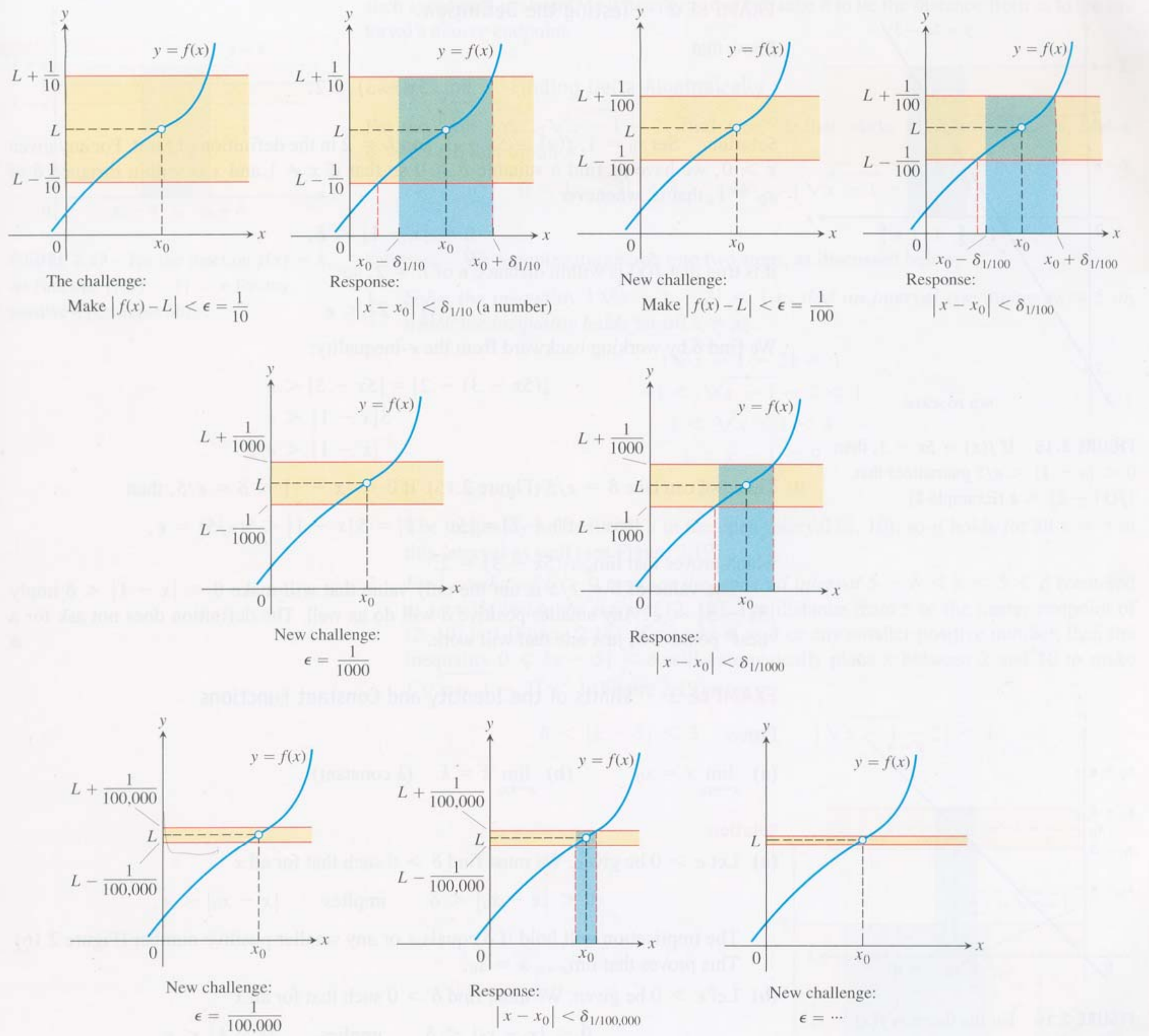


FIGURE 2.13 How should we define $\delta > 0$ so that keeping x within the interval $(x_0 - \delta, x_0 + \delta)$ will keep $f(x)$ within the interval $\left(L - \frac{1}{10}, L + \frac{1}{10}\right)$?

- The problem of proving L as the limit of $f(x)$ as x approaches x_0 is a problem of proving the existence of δ, such that whenever
- $x_0 - \delta < x < x_0 + \delta$,
- $L+\varepsilon < f(x) < L-\varepsilon$ for any arbitrarily small value of ε.
- As an example in Figure 2.13, given $\varepsilon = 1/10$, can we find a corresponding value of δ?
- How about if $\varepsilon = 1/100$? $\varepsilon = 1/1234$?
- If for any arbitrarily small value of ε we can always find a corresponding value of δ, then we has successfully proven that L is the limit of f as x approaches x_0



y = f(x)
L + 1/10
L
L − 1/10
0
x₀
The challenge:
Make |f(x) − L| < ε = 1/10
x₀ − δ1/10
x₀ + δ1/10
Response:
|x − x₀| < δ1/10 (a number)
L + 1/100
L − 1/100
New challenge:
Make |f(x) − L| < ε = 1/100
x₀ − δ1/100
x₀ + δ1/100
|x − x₀| < δ1/100
L + 1/1000
L − 1/1000
New challenge:
ε = 1/1000
|x − x₀| < δ1/1000
L + 1/100,000
L − 1/100,000
ε = 1/100,000
|x − x₀| < δ1/100,000
ε = ···

Example 2 Testing the definition

- Show that

$$\lim_{x \to 1}(5x-3)=2$$

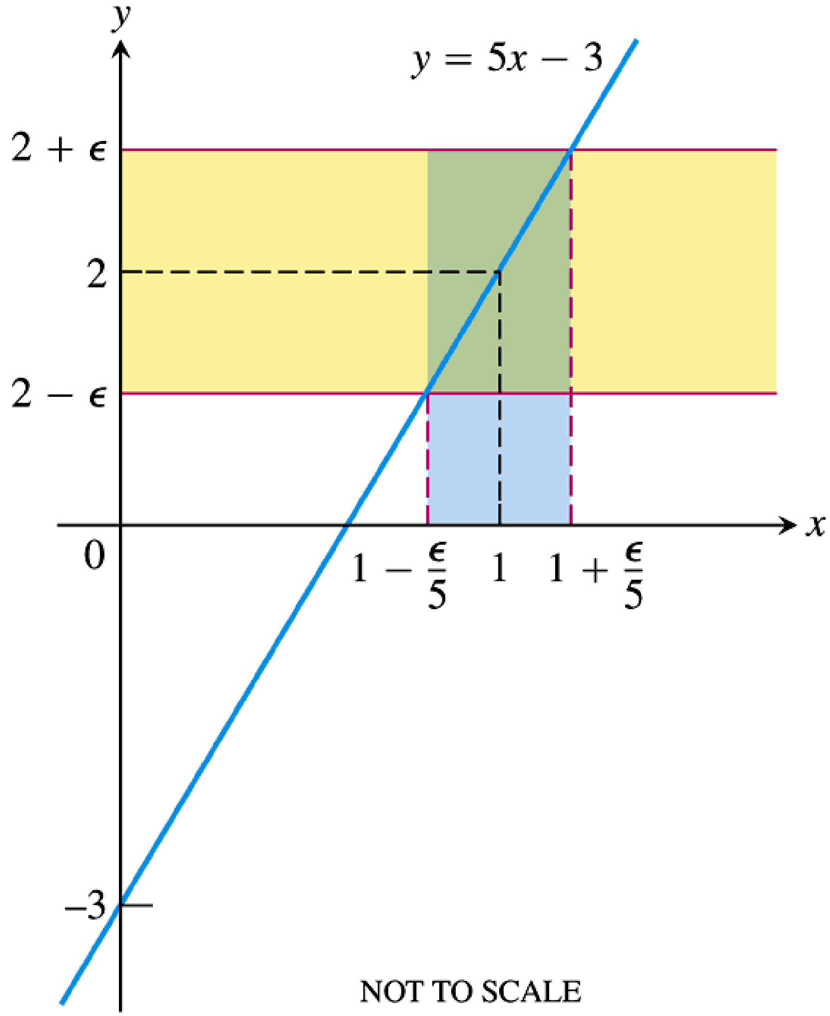


FIGURE 2.15 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

Solution

- Set $x_0=1$, $f(x)=5x-3$, $L=2$.
- For any given ε, we have to find a suitable $\delta > 0$ so that whenever

  $0<| x-1|< \delta$, $x\neq1$,

  it is true that $f(x)$ is within distance ε of $L=2$, i.e.

  $|f(x)-2|< \varepsilon$.

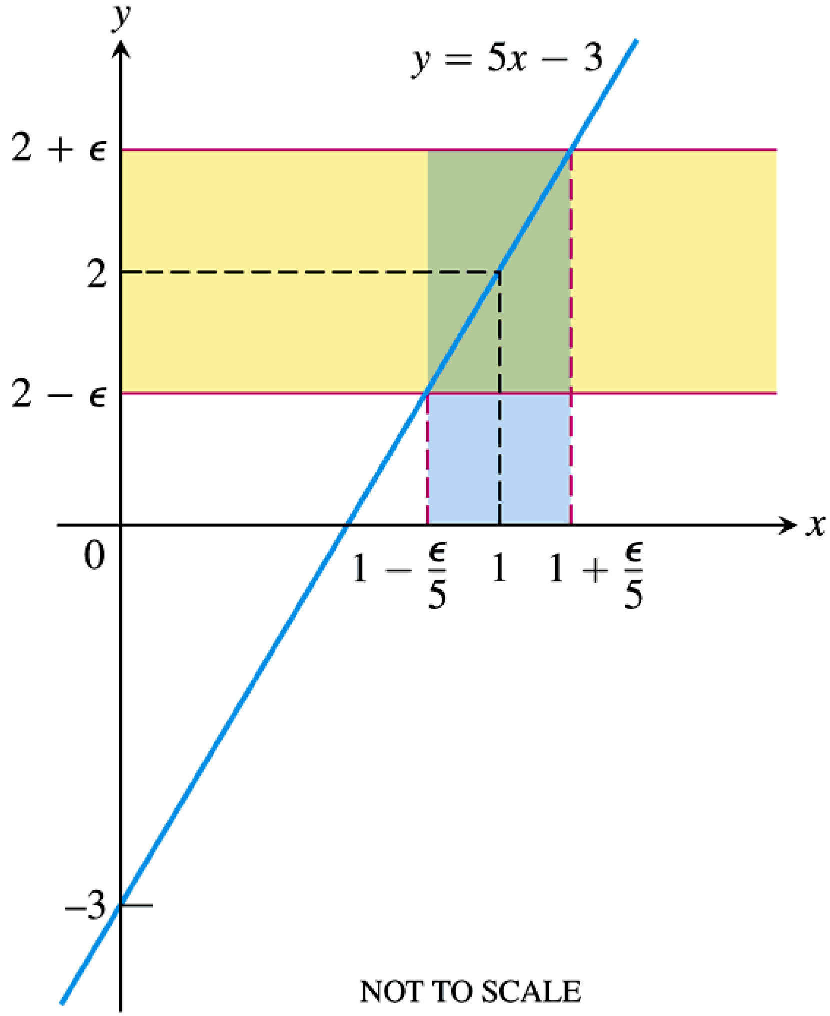
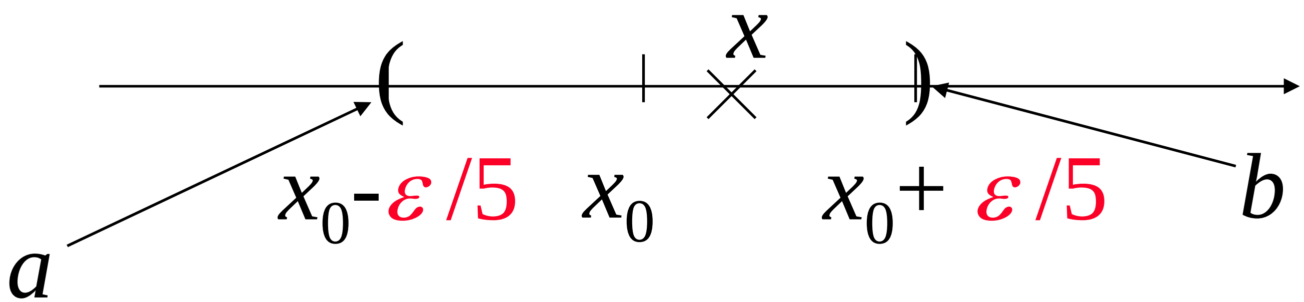


FIGURE 2.15 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

- First, obtain an open interval (a,b) in which

  $|f(x) - 2| < \varepsilon \equiv |5x - 5| < \varepsilon \equiv$

  $-\varepsilon/5 < x - 1 < \varepsilon/5 \equiv -\varepsilon/5 < x - x_0 < \varepsilon/5$



- choose $\delta < \varepsilon / 5$. This choice will guarantee that

  $|f(x) - L| < \varepsilon$ whenever $x_0 - \delta < x < x_0 + \delta$.

  We have shown that for any value of ε given, we can always find an corresponding value of δ that meets the "challenge" posed by an ever diminishing ε. This is an proof of existence.

  Thus we have proven that the limit for $f(x)$=5x-3 is L=2 when $x \rightarrow x_0$=1.

Example 3(a)

- Limits of the identity functions
- Prove

$$\lim_{x \to x_0} x = x_0$$

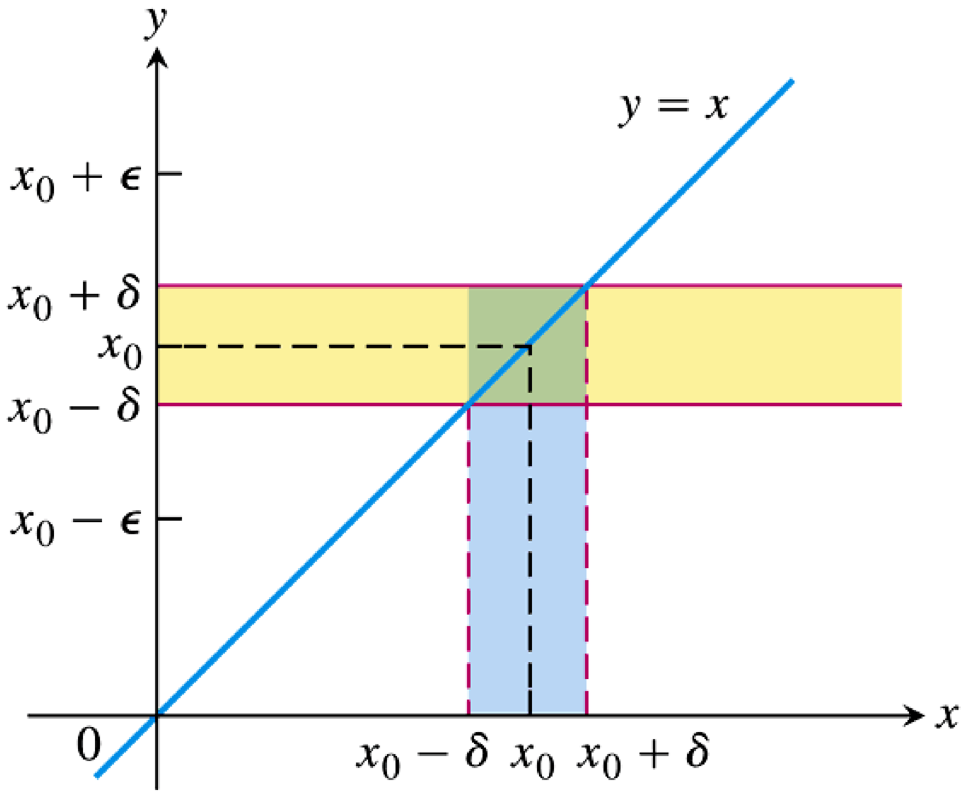


FIGURE 2.16 For the function $f(x) = x$, we find that $0 < |x - x_0| < \delta$ will guarantee $|f(x) - x_0| < \epsilon$ whenever $\delta \leq \epsilon$ (Example 3a).

Solution

- Let $\varepsilon > 0$. We must find $\delta > 0$ such that for all x, $0 < |x-x_0| < \delta$ implies $|f(x)-x_0| < \varepsilon$., here, $f(x)=x$, the identity function.
- Choose $\delta < \varepsilon$ will do the job.
- The proof of the existence of δ proves $\lim_{x \to x_0} x = x_0$

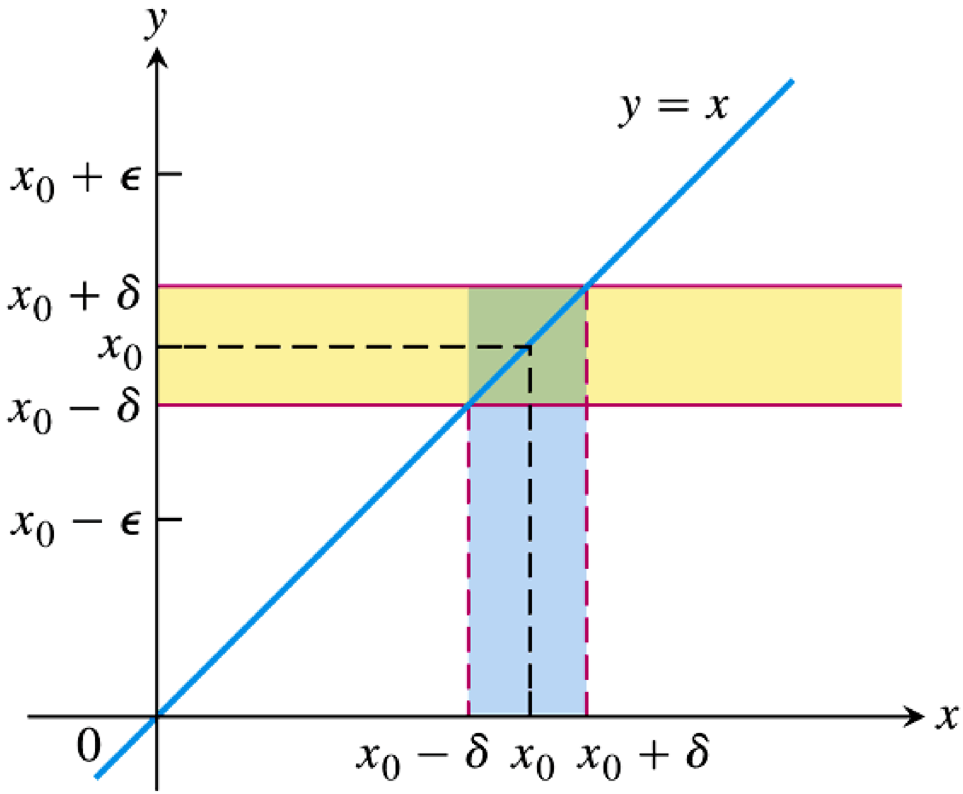


FIGURE 2.16 For the function $f(x) = x$, we find that $0 < |x - x_0| < \delta$ will guarantee $|f(x) - x_0| < \epsilon$ whenever $\delta \leq \epsilon$ (Example 3a).

Example 3(b)

- Limits constant functions
- Prove

  $$\lim_{x \to x_0} k = k \quad (k \text{ constant})$$

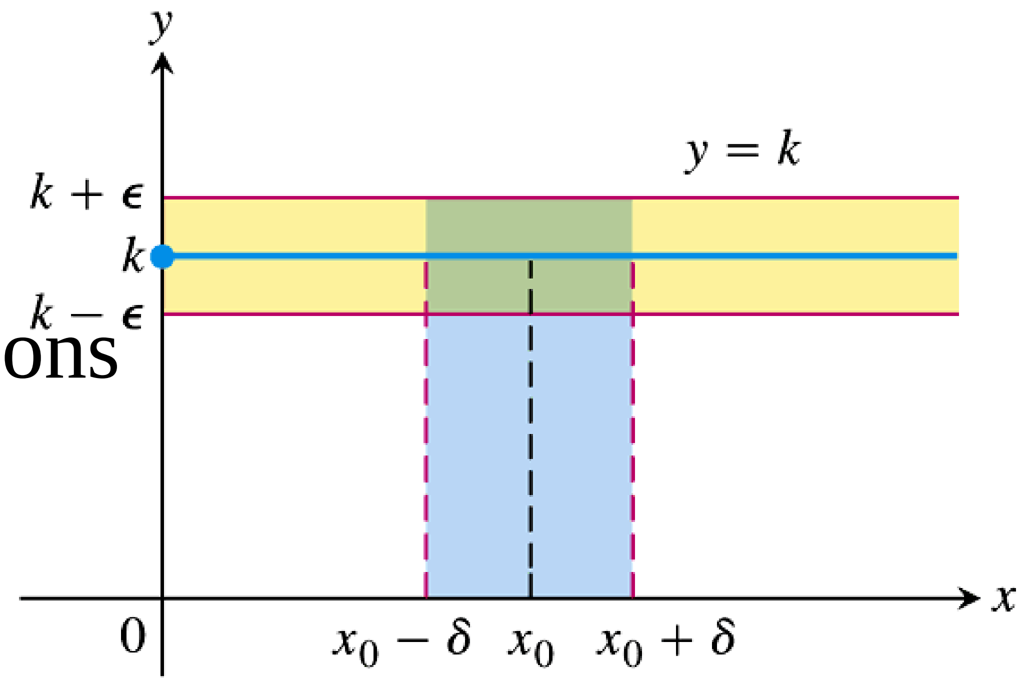


FIGURE 2.17 For the function $f(x) = k$, we find that $|f(x) - k| < \epsilon$ for any positive δ (Example 3b).

Solution

- Let $\varepsilon > 0$. We must find $\delta > 0$ such that for all x, $0 < |x-x_0| < \delta$ implies $|f(x) - k| < \varepsilon$., here, $f(x)=k$, the constant function.
- Choose any δ will do the job.
- The proof of the existence of δ proves $\lim_{x \to x_0} k = k$

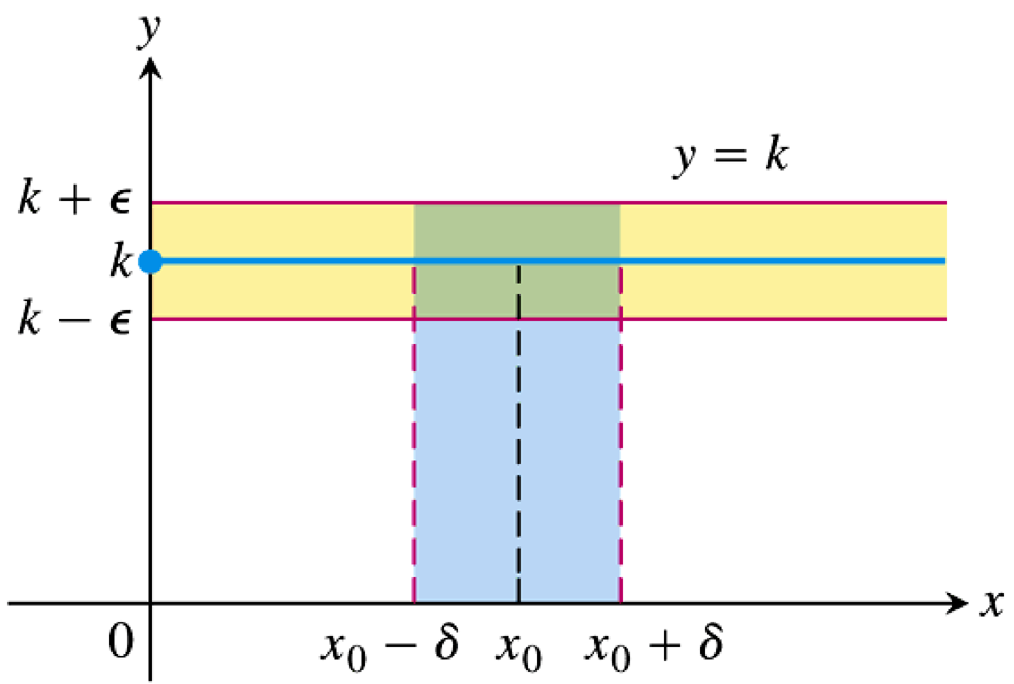


FIGURE 2.17 For the function $f(x) = k$, we find that $|f(x) - k| < \epsilon$ for any positive δ (Example 3b).

Finding delta algebraically for given epsilons

- Example 4: Finding delta algebraically
- For the limit $\lim_{x \to 5} \sqrt{x-1} = 2$
  find a $\delta > 0$ that works for $\varepsilon = 1$. That is,
  find a $\delta > 0$ such that for all x,

$$0 < |x-5| < \delta \Rightarrow 0 < \left|\sqrt{x-1} - 2\right| < 1$$

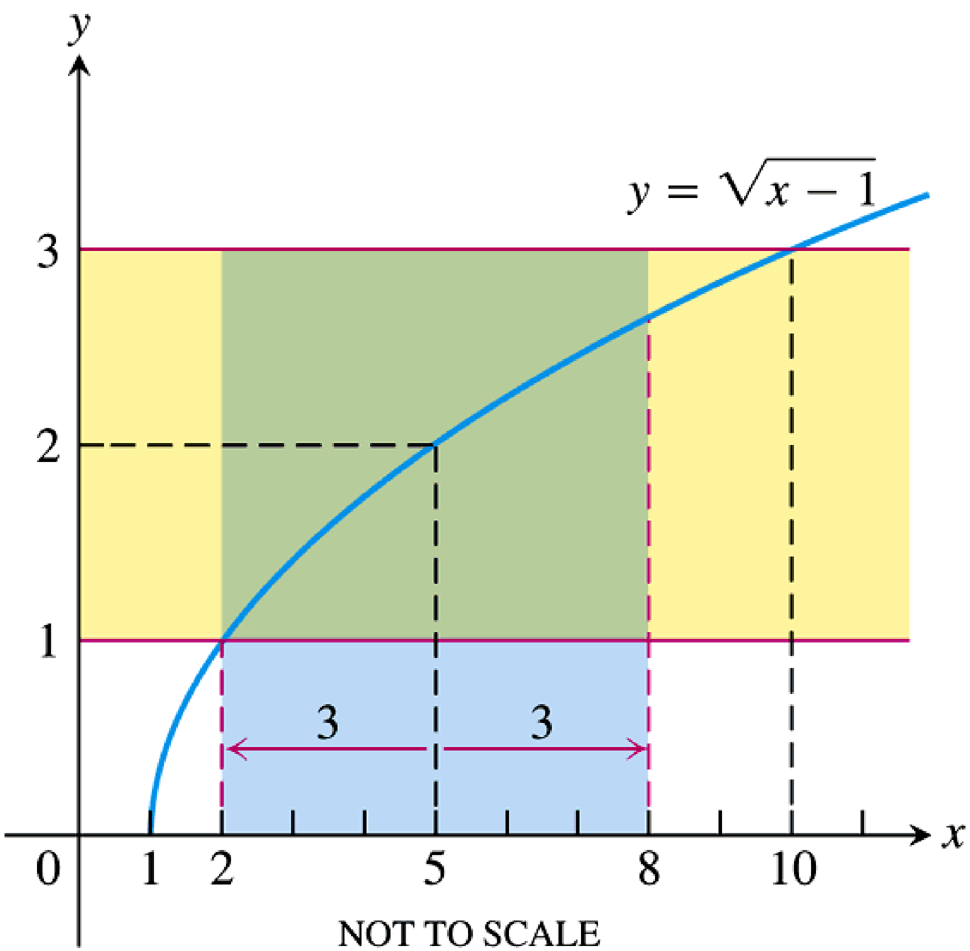


FIGURE 2.19 The function and intervals in Example 4.

Solution

- δ is found by working backward:

How to Find Algebraically a δ for a Given f, L, x_0, and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.
2. *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

Solution

- Step one: Solve the inequality $|f(x)-L|<\varepsilon$

$$0<\left|\sqrt{x-1}-2\right|<1 \Rightarrow 2<x<10$$

- Step two: Find a value of $\delta > 0$ that places the open interval $(x_0-\delta, x_0+\delta)$ centered at x_0 inside the open interval found in step one. Hence, we choose $\delta = 3$ or a smaller number

By doing so, the inequality $0<|x - 5| < \delta$ will automatically place x between 2 and 10 to make $0<\left|f(x)-2\right|<1$

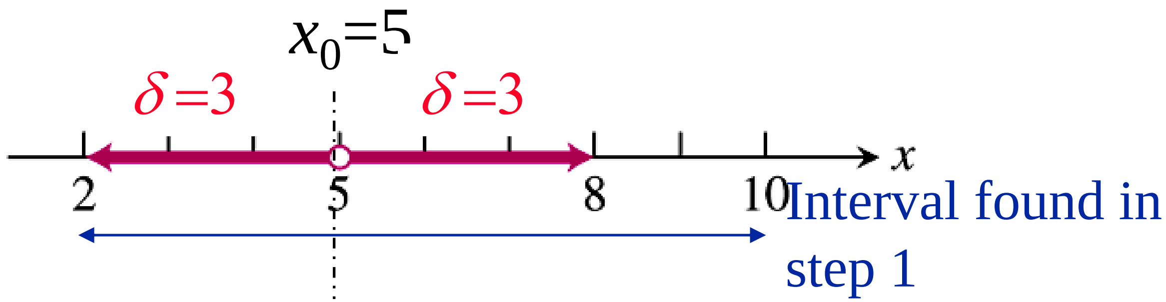


FIGURE 2.18 An open interval of radius 3 about $x_0 = 5$ will lie inside the open interval (2, 10).

Example 5

- Prove that

$$\lim_{x \to 2} f(x) = 4 \text{ if}$$

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$$

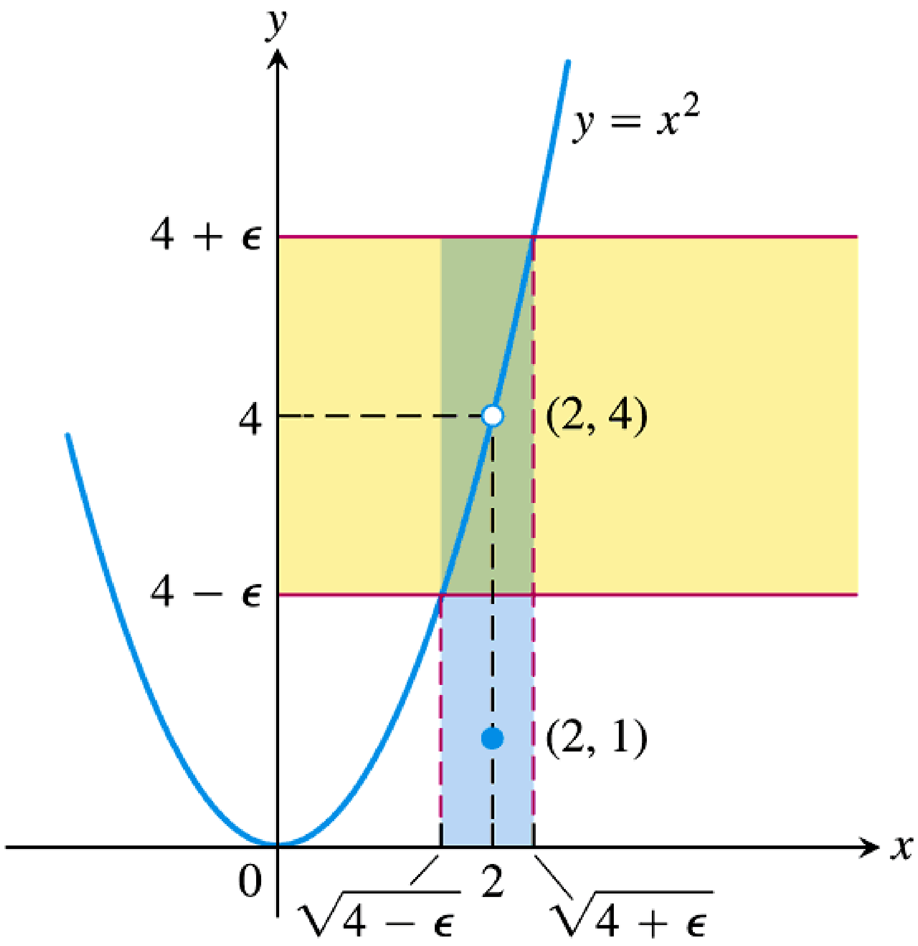


FIGURE 2.20 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \epsilon$.

Solution

- Step one: Solve the inequality $|f(x)-L|<\varepsilon$:

$$0<\left|x^2-2\right|<\varepsilon \Rightarrow \sqrt{4-\varepsilon}<x<\sqrt{4+\varepsilon}, x \neq 2$$

- Step two: Choose
- $\delta <$ min [2-√(4-ε)2, √(4+ε) – 2]

- For all x,
- $0 < |x - 2| < \delta \Rightarrow |f(x)-4|<\varepsilon$
- This completes the proof.

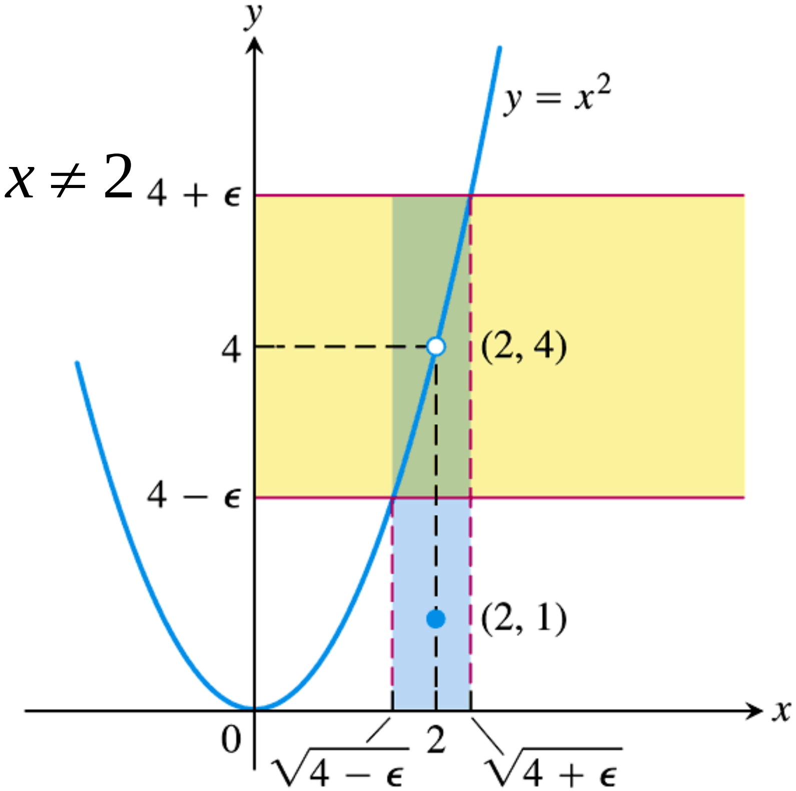


FIGURE 2.20 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \epsilon$.

2.4

One-Sided Limits and Limits at Infinity

(1st lecture of week 13/08/07-18/08/07)

Two sided limit does not exist for *y*;

But

y does has two one-sided limits

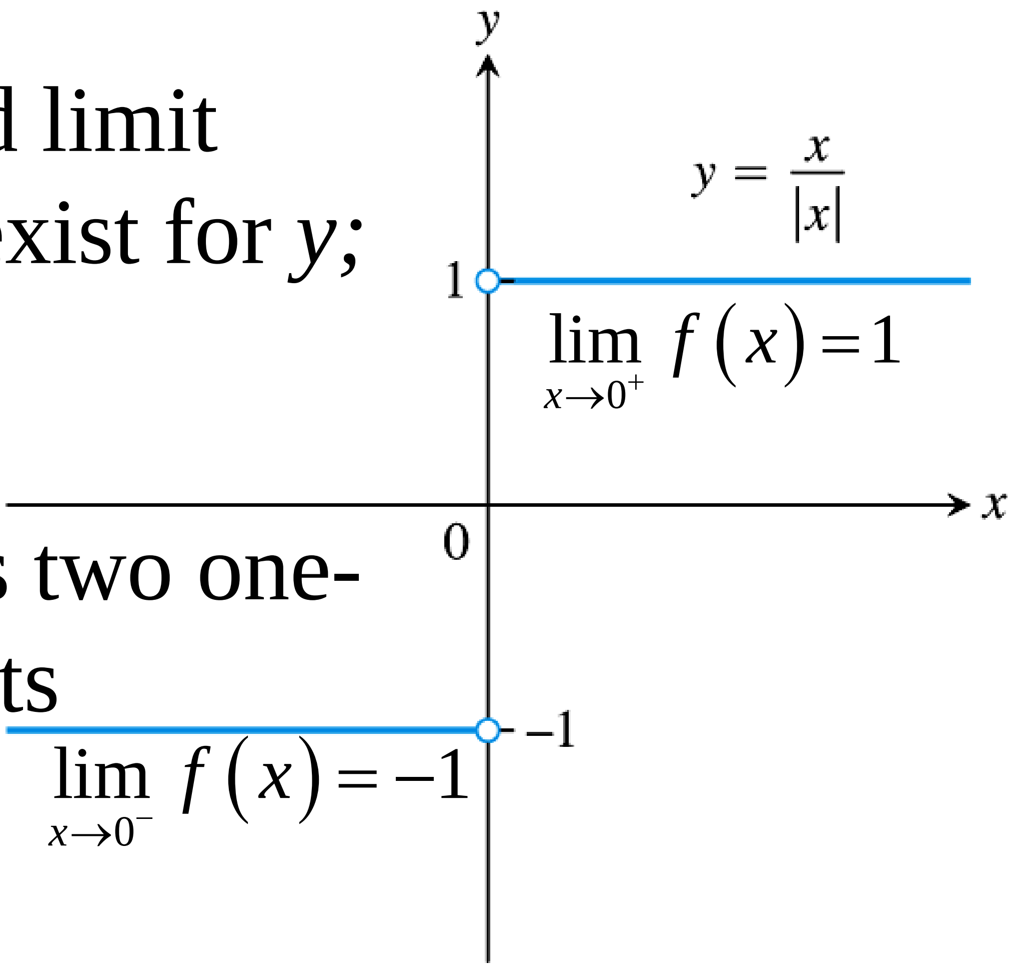


FIGURE 2.21 Different right-hand and left-hand limits at the origin.

One-sided limits

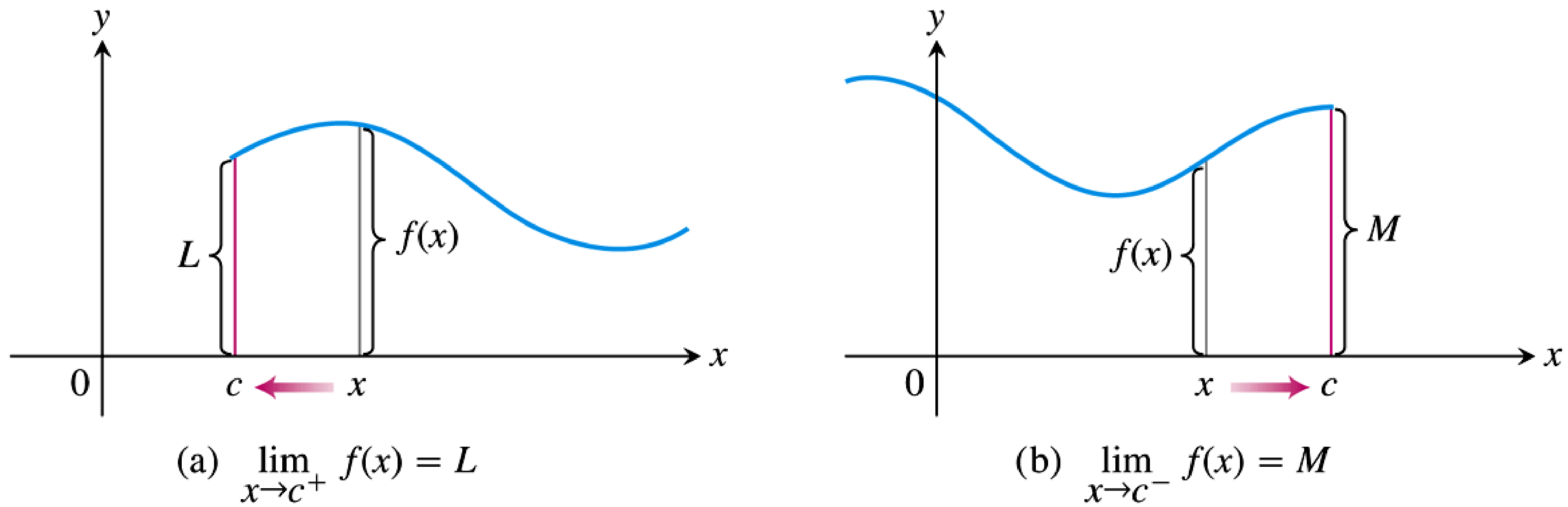


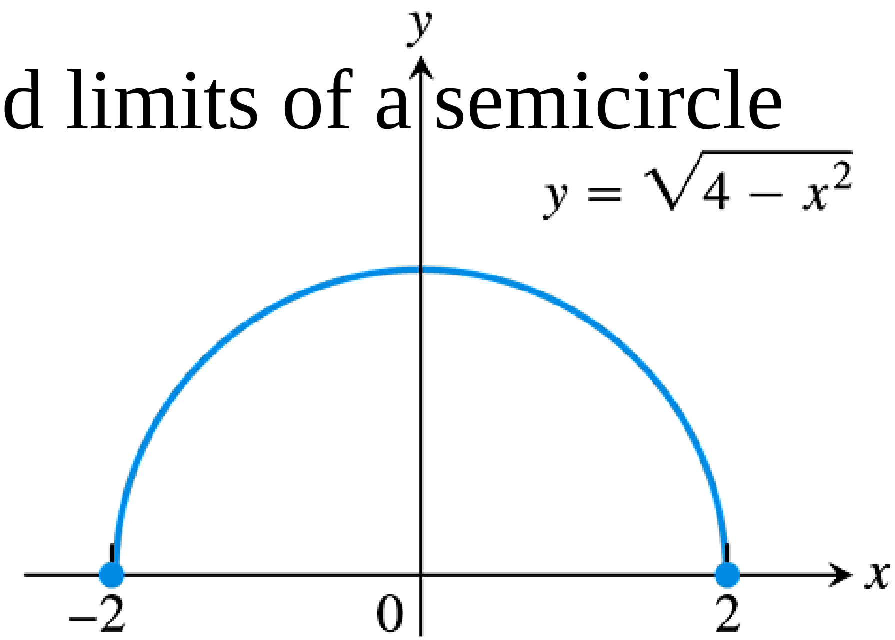
FIGURE 2.22 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

Right-hand limit

Left-hand limit

Example 1

- One sided limits of a semicircle



No right hand limit at *x*=2;

No two sided limit at *x*= 2;

No left hand limit at *x*= -2;

No two sided limit at *x*= -2;

FIGURE 2.23 $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$ and $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$ (Example 1).

THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Example 2

- Limits of the function graphed in Figure 2.24
- Can you write down all the limits at $x=0$, $x=1$, $x=2$, $x=3$, $x=4$?
- What is the limit at other values of x?

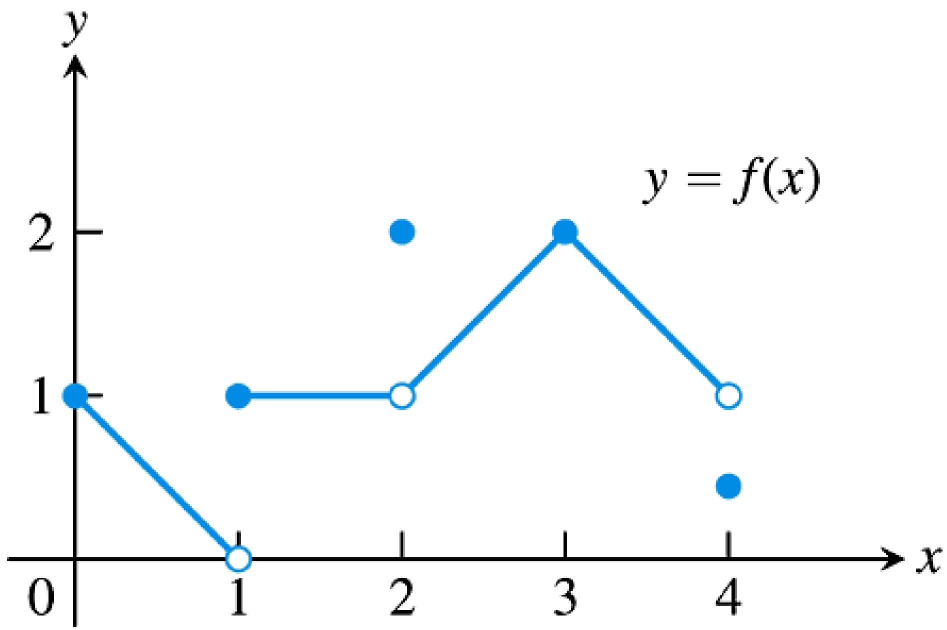


FIGURE 2.24 Graph of the function in Example 2.

Precise definition of one-sided limits

DEFINITIONS Right-Hand, Left-Hand Limits

We say that $f(x)$ has **right-hand limit L at x_0**, and write

$$\lim_{x \to x_0^+} f(x) = L \qquad \text{(See Figure 2.25)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit L at x_0**, and write

$$\lim_{x \to x_0^-} f(x) = L \qquad \text{(See Figure 2.26)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

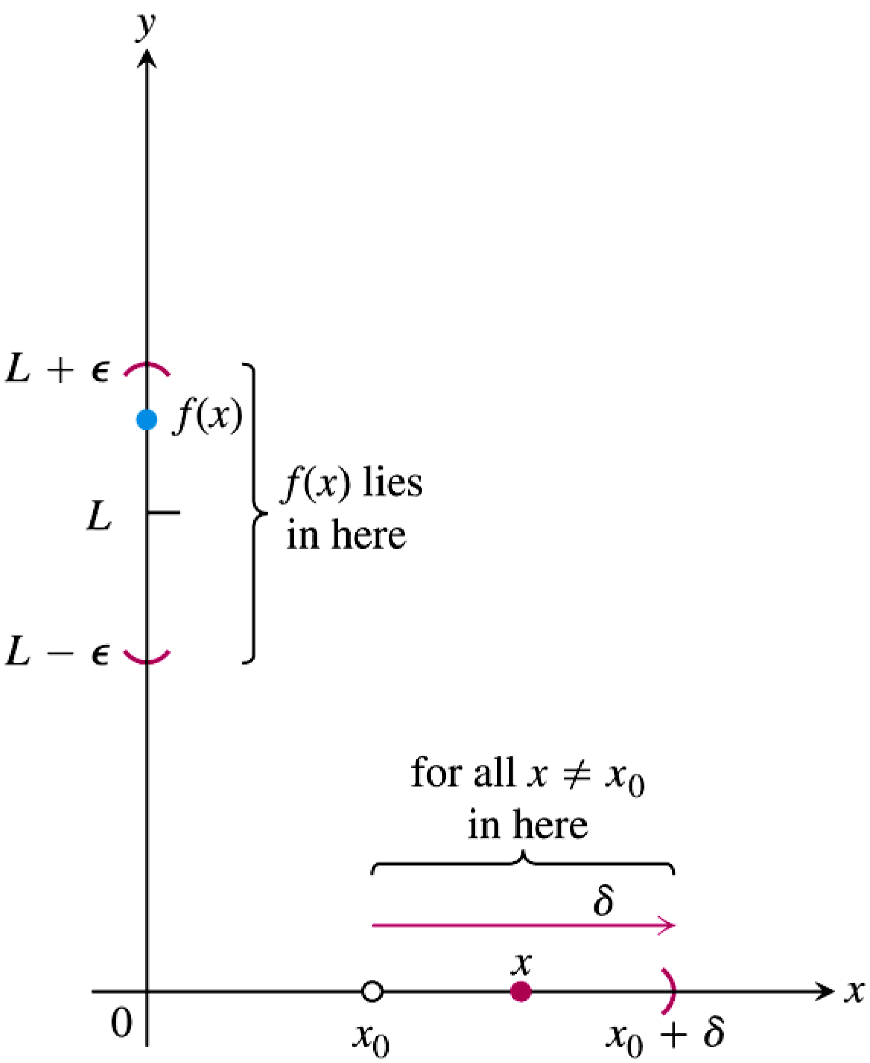


FIGURE 2.25 Intervals associated with the definition of right-hand limit.

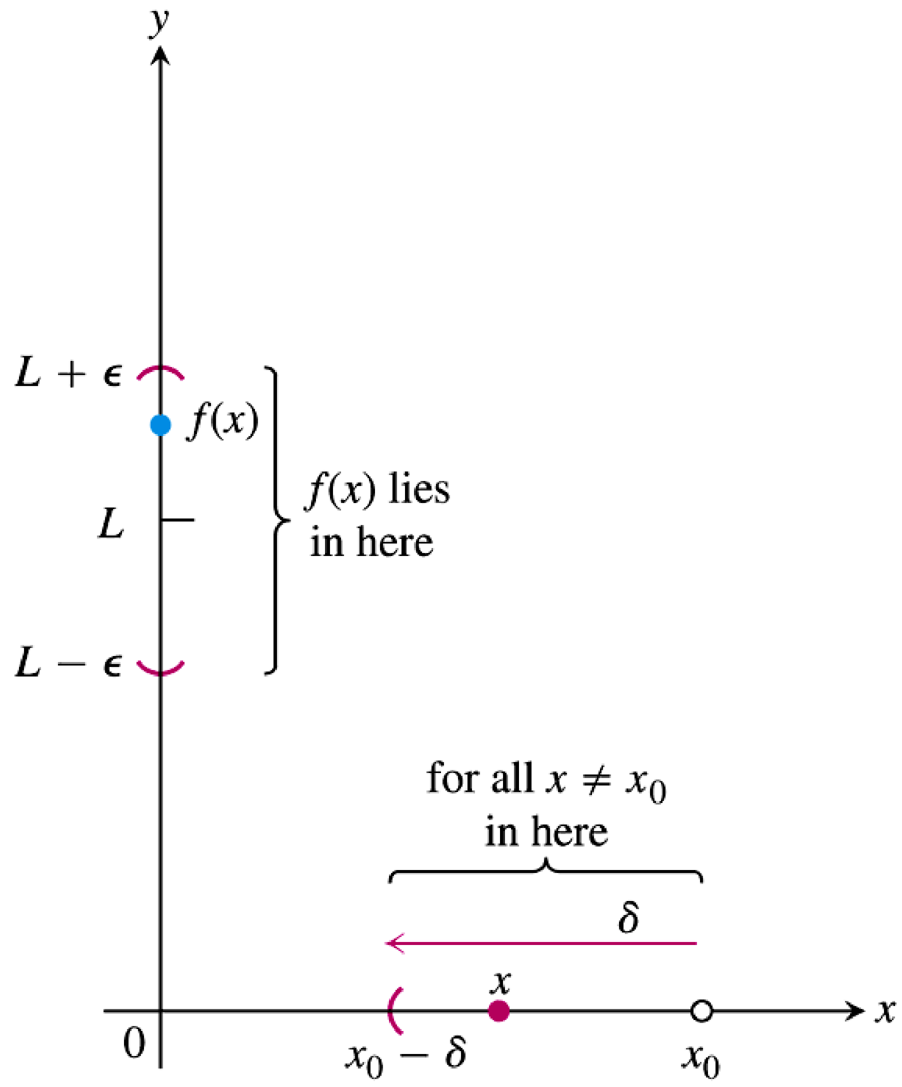


FIGURE 2.26 Intervals associated with the definition of left-hand limit.

Limits involving $(\sin\theta)/\theta$

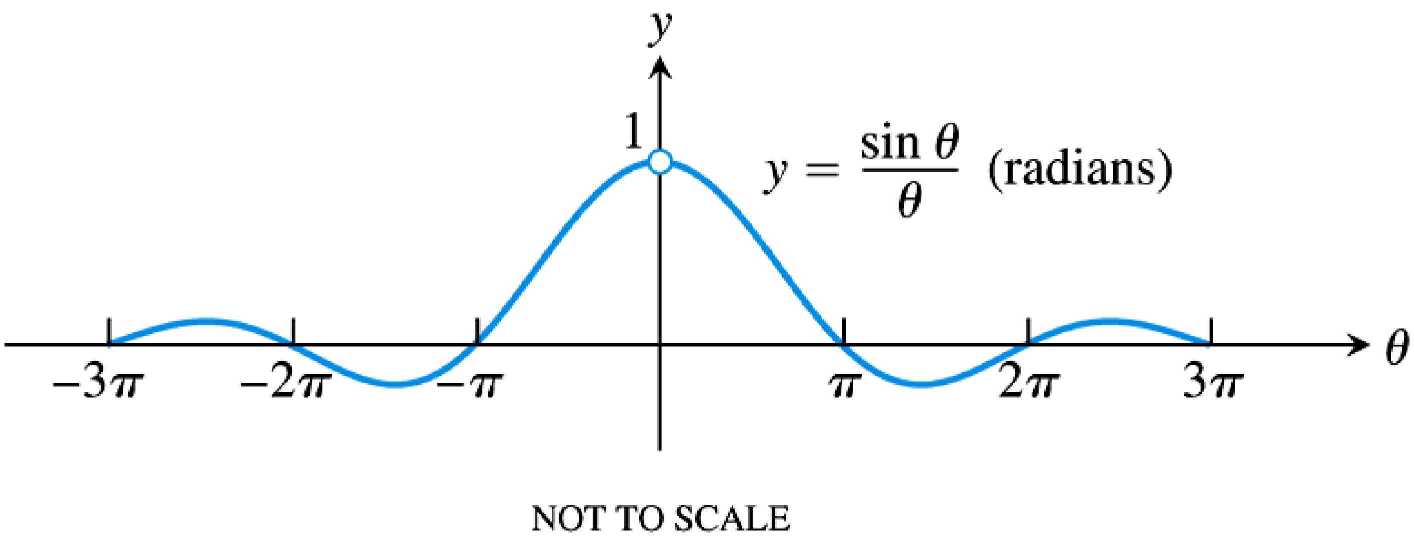


FIGURE 2.29 The graph of $f(\theta) = (\sin\theta)/\theta$.

THEOREM 7

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{1}$$

Proof

Area ΔOAP = ½ $\sin\theta$

Area sector OAP = $\theta/2$

Area ΔOAT = ½ $\tan\theta$

½ $\sin\theta < \theta/2 <$ ½ $\tan\theta$

$1 < \theta/\sin\theta < 1/\cos\theta$

$1 > \sin\theta/\theta > \cos\theta$

Taking limit $\theta \rightarrow 0^{\pm}$,

$$\lim_{\theta\to 0^{\pm}} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0} \frac{\sin\theta}{\theta}$$

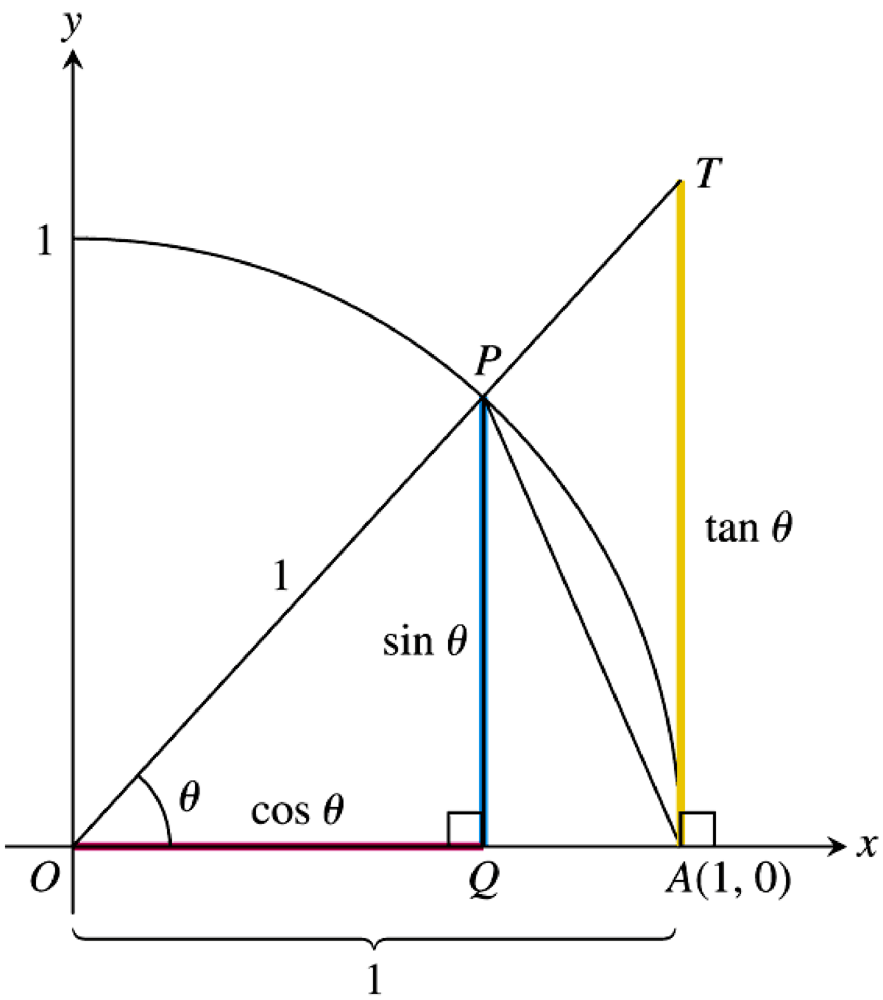


FIGURE 2.30 The figure for the proof of Theorem 7. $TA/OA = \tan\theta$, but $OA = 1$, so $TA = \tan\theta$.

Example 5(a)

- Using theorem 7, show that $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

Example 5(b)

- Using theorem 7, show that $\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$

Finite limits as $x \to \pm\infty$

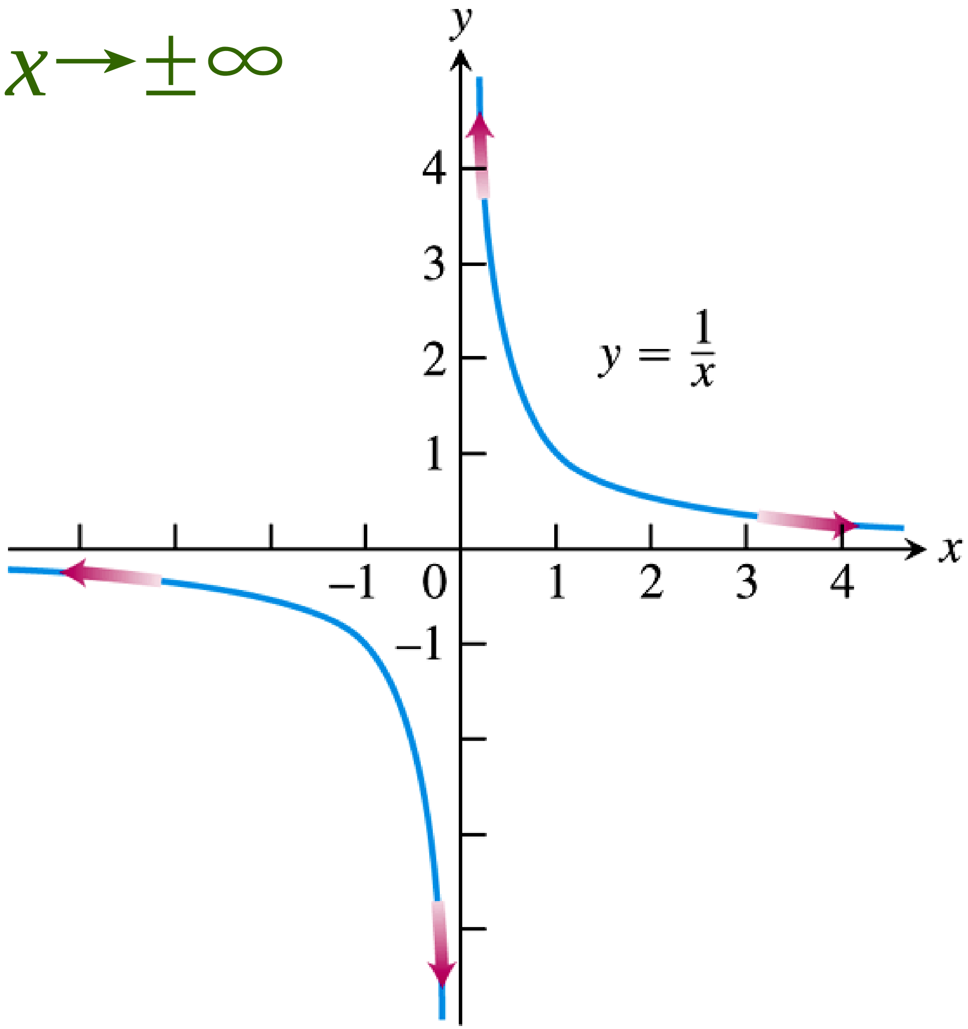


FIGURE 2.31 The graph of $y = 1/x$.

Precise definition

DEFINITIONS **Limit as x approaches ∞ or $-\infty$**

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon .$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon .$$

Example 6

- Limit at infinity for $f(x) = \dfrac{1}{x}$
- (a) Show that $\lim_{x \to \infty} \dfrac{1}{x} = 0$
- (b) Show that $\lim_{x \to -\infty} \dfrac{1}{x} = 0$

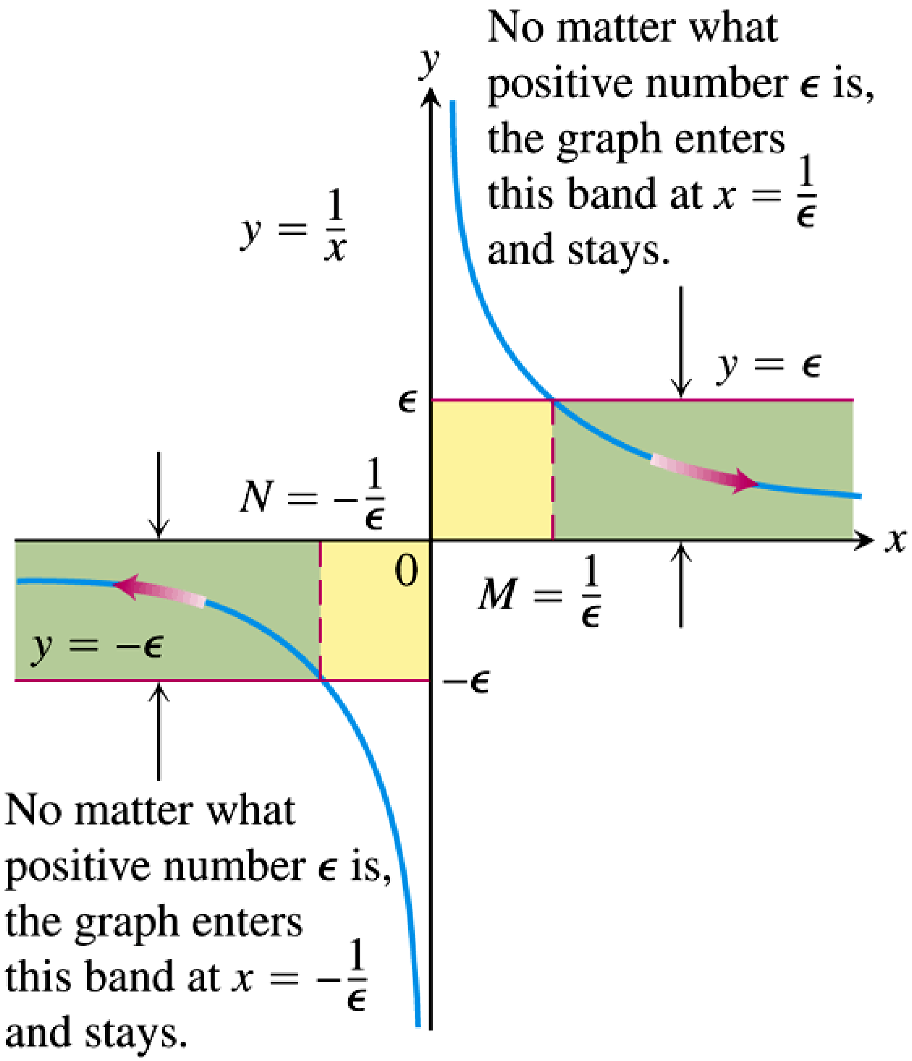


FIGURE 2.32 The geometry behind the argument in Example 6.

THEOREM 8 Limit Laws as $x \to \pm\infty$

If L, M, and k, are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L \quad \text{and} \quad \lim_{x \to \pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to \pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to \pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x \to \pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x \to \pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

Example 7(a)

- Using Theorem 8

$$\lim_{x\to\infty}\left(5+\frac{1}{x}\right)=\lim_{x\to\infty}5+\lim_{x\to\infty}\frac{1}{x}=5+0=5$$

Example 7(b)

$$\lim_{x\to\infty}\frac{\pi\sqrt{3}}{x^2} = \pi\sqrt{3}\lim_{x\to\infty}\frac{1}{x^2}$$

$$= \pi\sqrt{3}\lim_{x\to\infty}\frac{1}{x}\cdot\lim_{x\to\infty}\frac{1}{x}$$

$$= \pi\sqrt{3}\cdot 0\cdot 0 = 0$$

Limits at infinity of rational functions

- Example 8

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to\infty}\frac{5+(8/x)-(3/x^2)}{3+(2/x^2)}=$$

$$=\frac{5+\lim_{x\to\infty}(8/x)-\lim_{x\to\infty}(3/x^2)}{3+\lim_{x\to\infty}(2/x^2)}=\frac{5+0-0}{3+0}=\frac{5}{3}$$

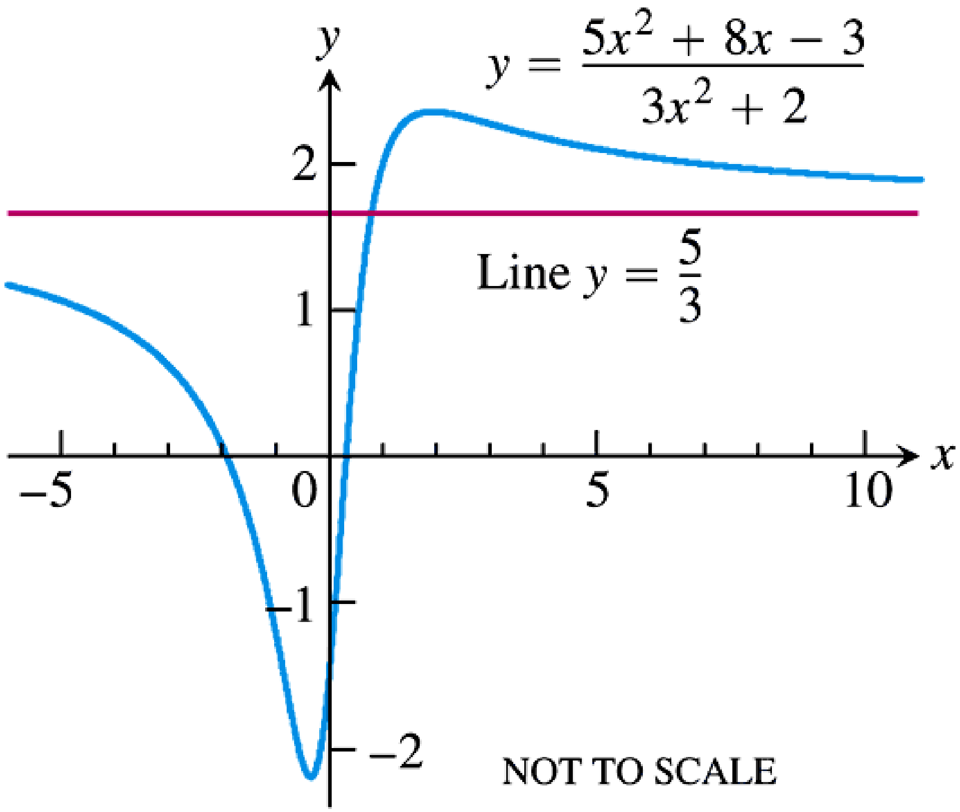


FIGURE 2.33 The graph of the function in Example 8. The graph approaches the line $y = 5/3$ as $|x|$ increases.

go back

Example 9

- Degree of numerator less than degree of denominator

$$\lim_{x\to\infty}\frac{11x+2}{2x^3-1}=\lim_{x\to\infty}\frac{(11/x)+(2/x^2)}{2-(1/x^2)}=\frac{0+0}{2-0}=0$$

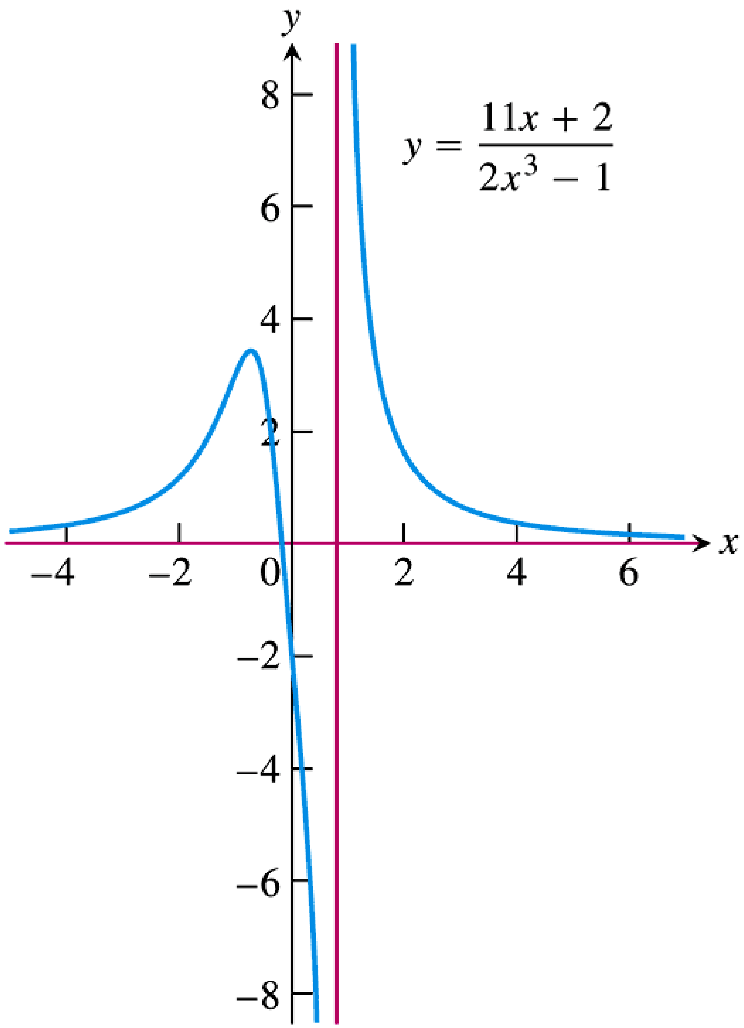
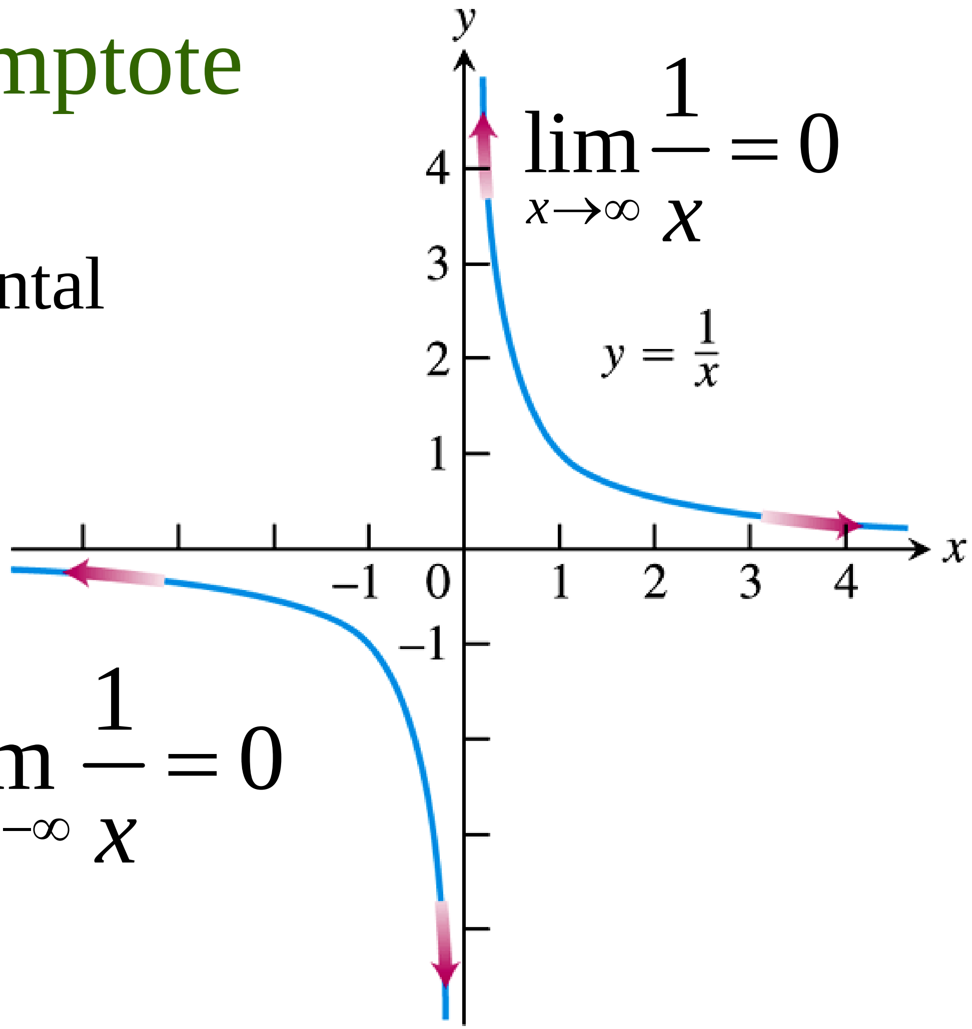


FIGURE 2.34 The graph of the function in Example 9. The graph approaches the x-axis as $|x|$ increases.

Horizontal asymptote

- x-axis is a horizontal asymptote



$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

FIGURE 2.31 The graph of $y = 1/x$.

DEFINITION Horizontal Asymptote

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

Figure 2.33 has the line $y=5/3$ as a horizontal asymptote on both the right and left because

$$\lim_{x \to \infty} f(x) = \frac{5}{3} \qquad \lim_{x \to -\infty} f(x) = \frac{5}{3}$$

Oblique asymptote

- Happen when the degree of the numerator polynomial is one greater than the degree of the denominator
- By long division, recast $f(x)$ into a linear function plus a remainder. The remainder shall $\to 0$ as $x \to \pm\infty$. The linear function is the asymptote of the graph.

Example 12

- Find the oblique asymptote

$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

- Solution

$$f(x) = \frac{2x^2 - 3}{7x + 4} = \overbrace{\left(\frac{2}{7}x - \frac{8}{49}\right)}^{\text{linear function}} + \frac{-115}{49(7x+4)}$$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(\frac{2}{7}x - \frac{8}{49}\right) + \lim_{x\to\pm\infty}\frac{-115}{49(7x+4)}$$

$$= \lim_{x\to\pm\infty}\left(\frac{2}{7}x - \frac{8}{49}\right) + 0 = \lim_{x\to\pm\infty}\left(\frac{2}{7}x - \frac{8}{49}\right)$$

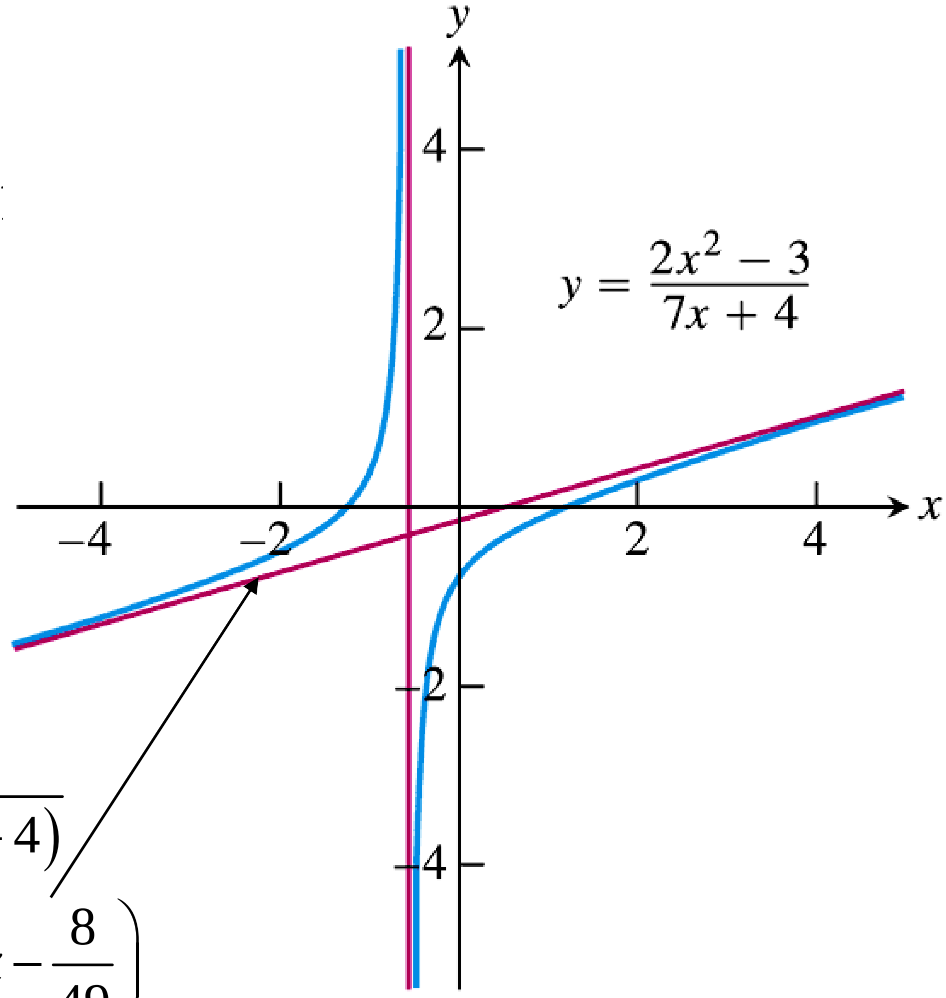


FIGURE 2.36 The function in Example 12 has an oblique asymptote.

2.5

Infinite Limits and Vertical Asymptotes

(2nd lecture of week 13/08/07-18/08/07)

Infinite limit

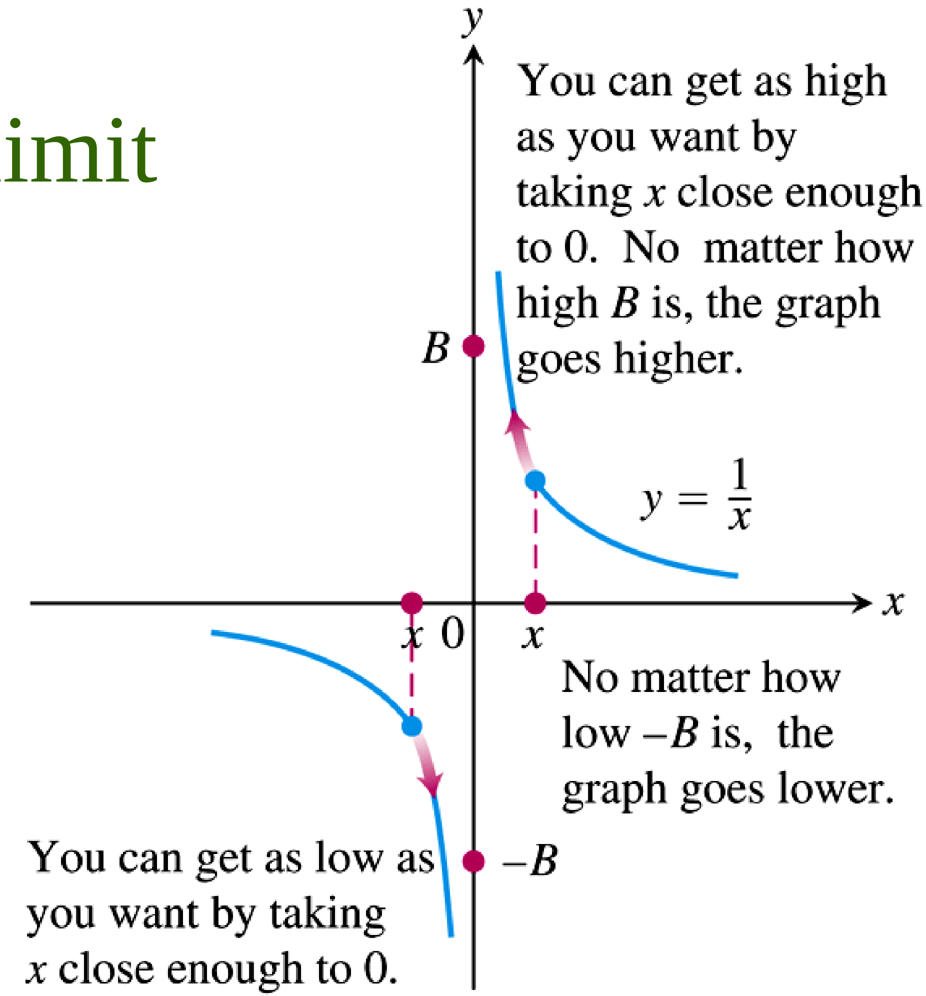


FIGURE 2.37 One-sided infinite limits:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Example 1

- Find $\lim_{x\to 1^+} \frac{1}{x-1}$ and $\lim_{x\to 1^-} \frac{1}{x-1}$

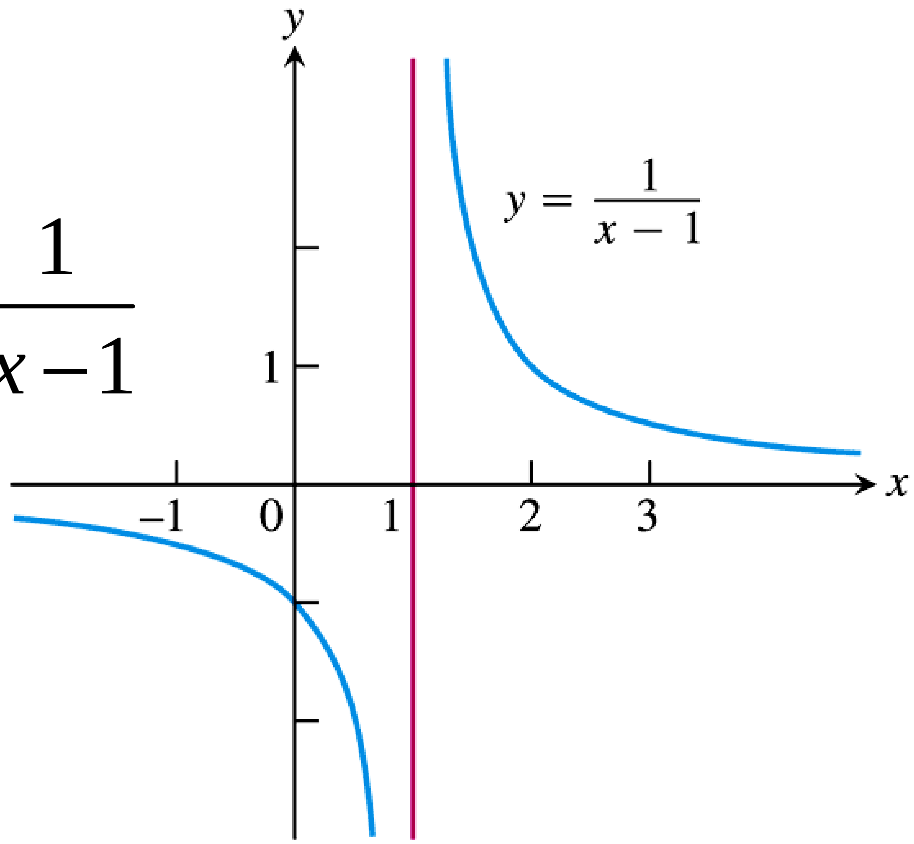


FIGURE 2.38 Near $x = 1$, the function $y = 1/(x - 1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 1).

Example 2 Two-sided infinite limit

- Discuss the behavior of

$$(a) \qquad f(x) = \frac{1}{x^2} \text{ near } x = 0$$

$$(b) \qquad g(x) = \frac{1}{(x+3)^2} \text{ near } x = -3$$

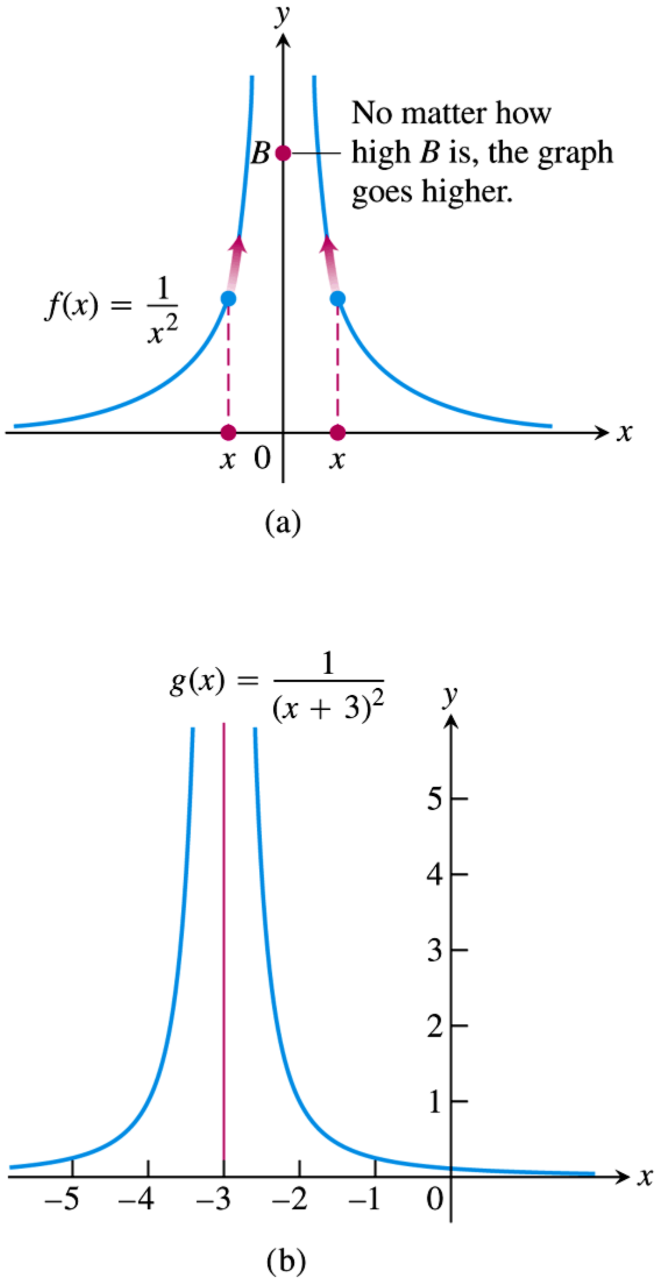


FIGURE 2.39 The graphs of the functions in Example 2. (a) $f(x)$ approaches infinity as $x \to 0$. (b) $g(x)$ approaches infinity as $x \to -3$.

Example 3

- Rational functions can behave in various ways near zeros of their denominators

$$(a)\lim_{x\to 2}\frac{(x-2)^2}{x^2-4}=\lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{(x-2)}{(x+2)}=0$$

$$(b)\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{(x+2)}=\frac{1}{4}$$

$$(c)\lim_{x\to 2^+}\frac{x-3}{x^2-4}=\lim_{x\to 2}\frac{x-3}{(x-2)(x+2)}=-\infty \quad \text{(note: } x>2\text{)}$$

$$(d)\lim_{x\to 2^-}\frac{x-3}{x^2-4}=\lim_{x\to 2}\frac{x-3}{(x-2)(x+2)}=+\infty \quad \text{(note: } 0<x<2\text{)}$$

Example 3

$$(e)\lim_{x\to 2}\frac{x-3}{x^2-4}=\lim_{x\to 2}\frac{x-3}{(x-2)(x+2)} \quad \text{limit does not exist}$$

$$(f)\lim_{x\to 2}\frac{2-x}{(x-2)^3}=-\lim_{x\to 2}\frac{x-2}{(x-2)(x-2)^2}=-\lim_{x\to 2}\frac{1}{(x-2)^2}=-\infty$$

Precise definition of infinite limits

DEFINITIONS **Infinity, Negative Infinity as Limits**

1. We say that ***f*(*x*) approaches infinity as *x* approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) > B.$$

2. We say that ***f*(*x*) approaches negative infinity as *x* approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) < -B.$$

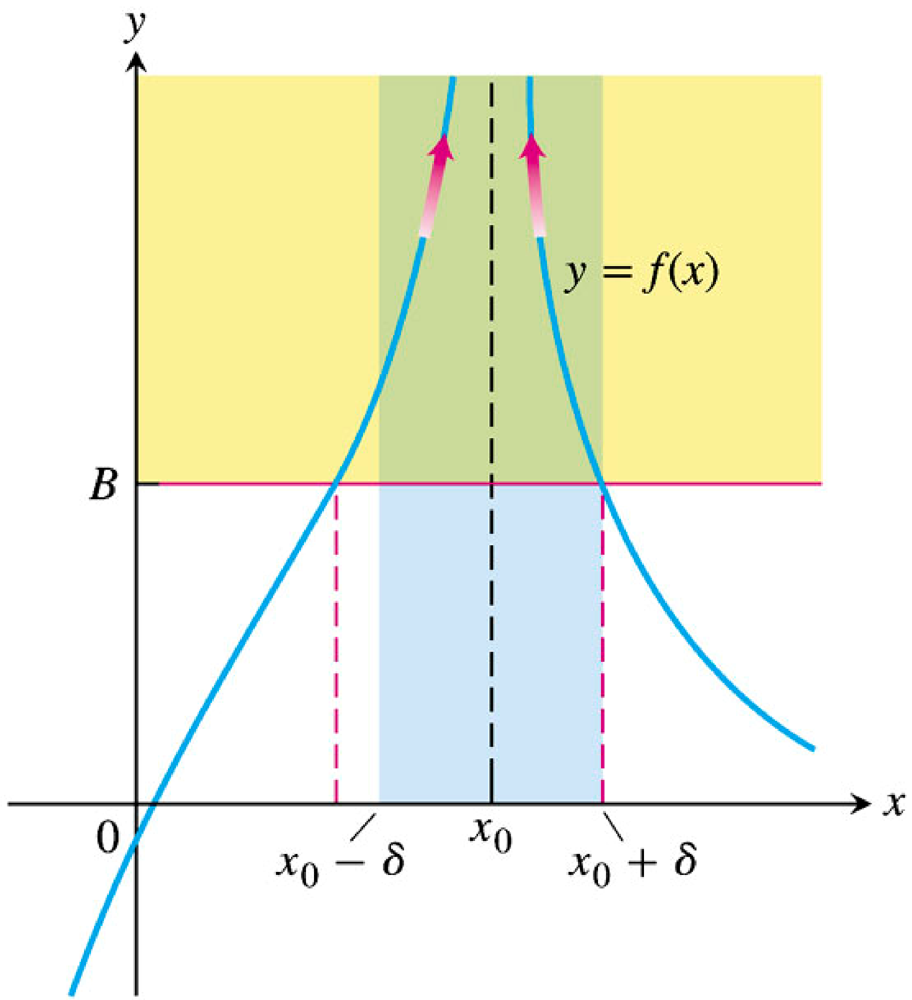


FIGURE 2.40 For $x_0 - \delta < x < x_0 + \delta$, the graph of $f(x)$ lies above the line $y = B$.

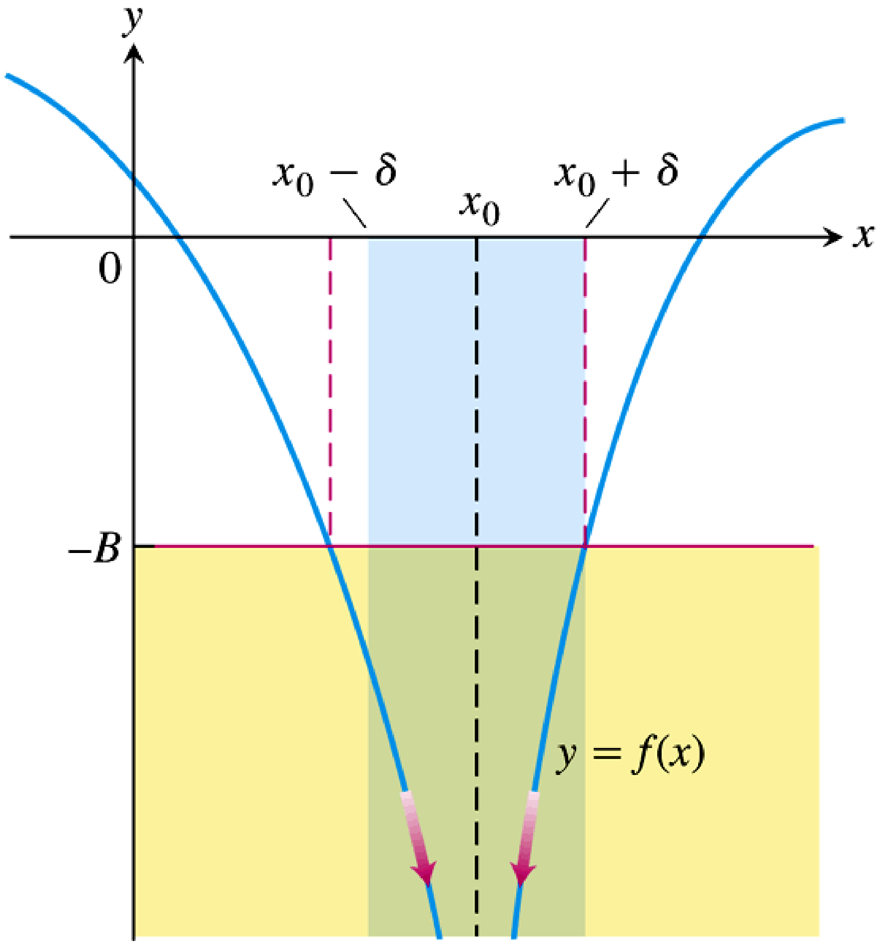


FIGURE 2.41 For $x_0 - \delta < x < x_0 + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

Example 4

- Using definition of infinite limit
- Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Given B>0, we want to find δ>0 such that

$$0 < |x - 0| < \delta \qquad \text{implies} \qquad \frac{1}{x^2} > B$$

Example 4

Now

$$\frac{1}{x^2} > B \text{ if and only if } x^2 < 1/B \equiv |x| < 1/\sqrt{B}$$

By choosing $\delta = 1/\sqrt{B}$

(or any smaller positive number), we see that

$$|x| < \delta \text{ implies } \frac{1}{x^2} > \frac{1}{\delta^2} \geq B$$

Vertical asymptotes

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

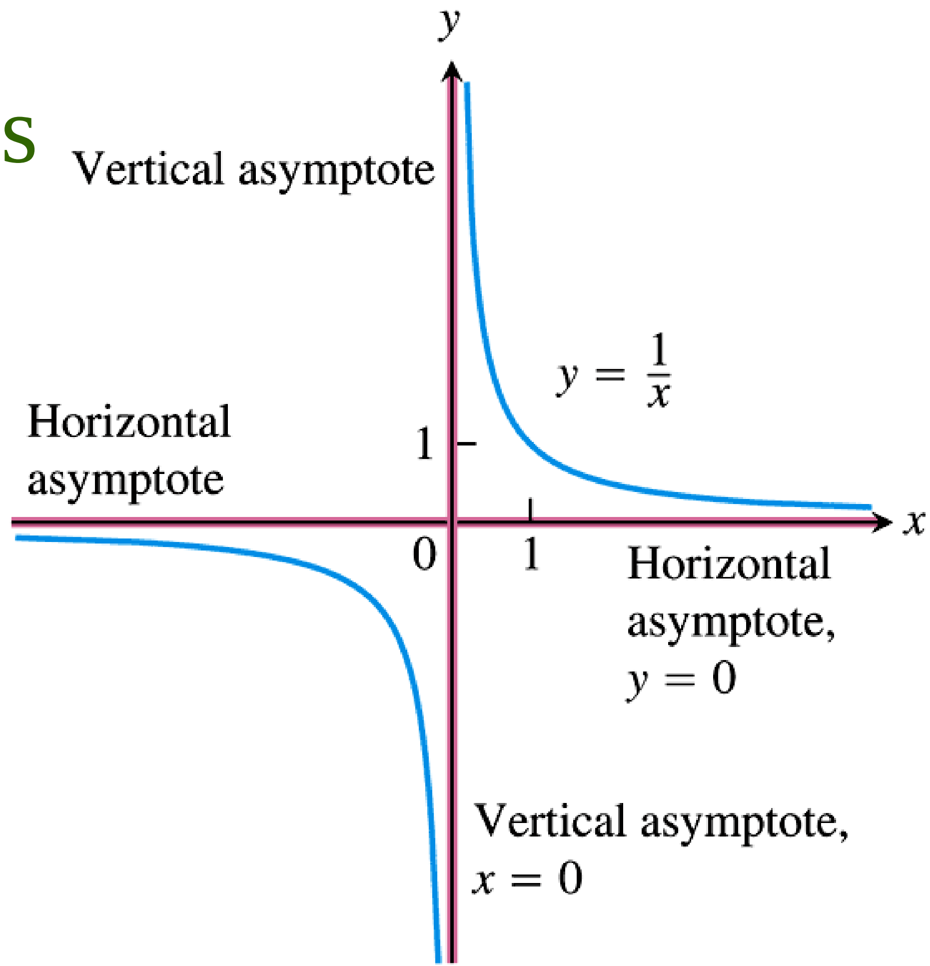


FIGURE 2.42 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

DEFINITION **Vertical Asymptote**

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty .$$

Example 5 Looking for asymptote

- Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x+3}{x+2}$$

- Solution:

$$y = 1 + \frac{1}{x+2}$$

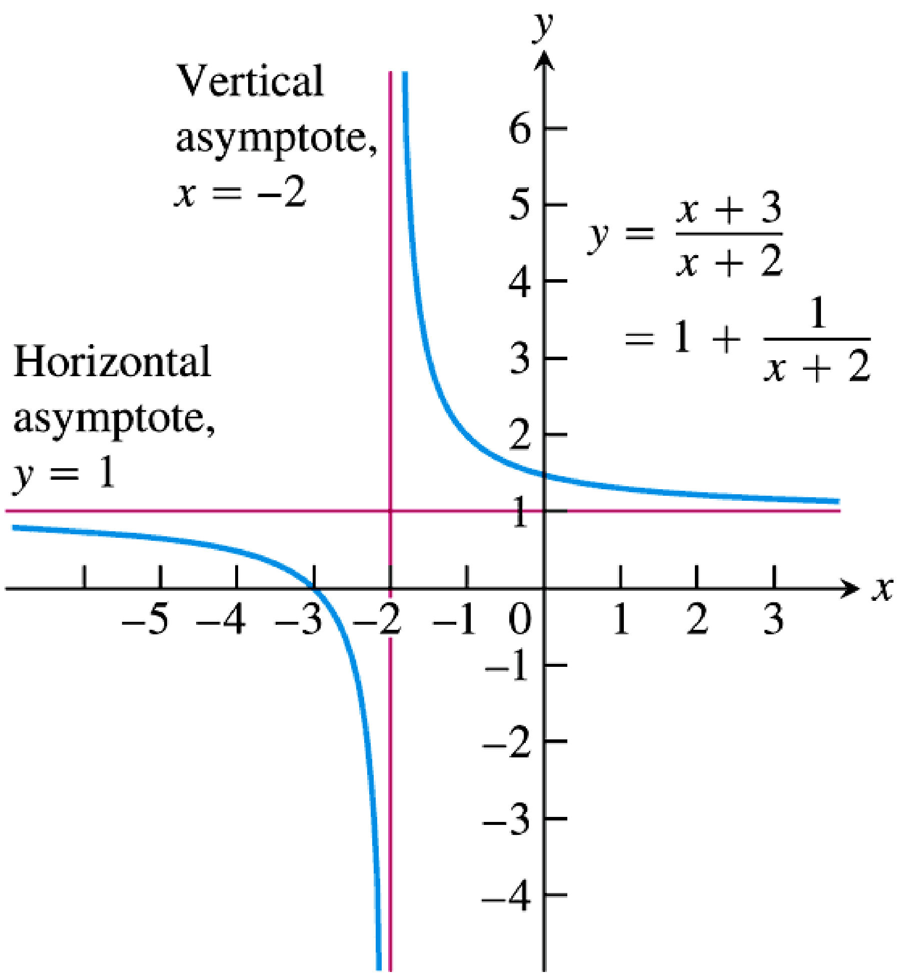


FIGURE 2.43 The lines $y = 1$ and $x = -2$ are asymptotes of the curve $y = (x + 3)/(x + 2)$ (Example 5).

Asymptote need not be two-sided

- Example 6

$$f(x) = -\frac{8}{x^2 - 2}$$

- Solution:

$$f(x) = -\frac{8}{x^2 - 2} = -\frac{8}{(x-2)(x+2)}$$

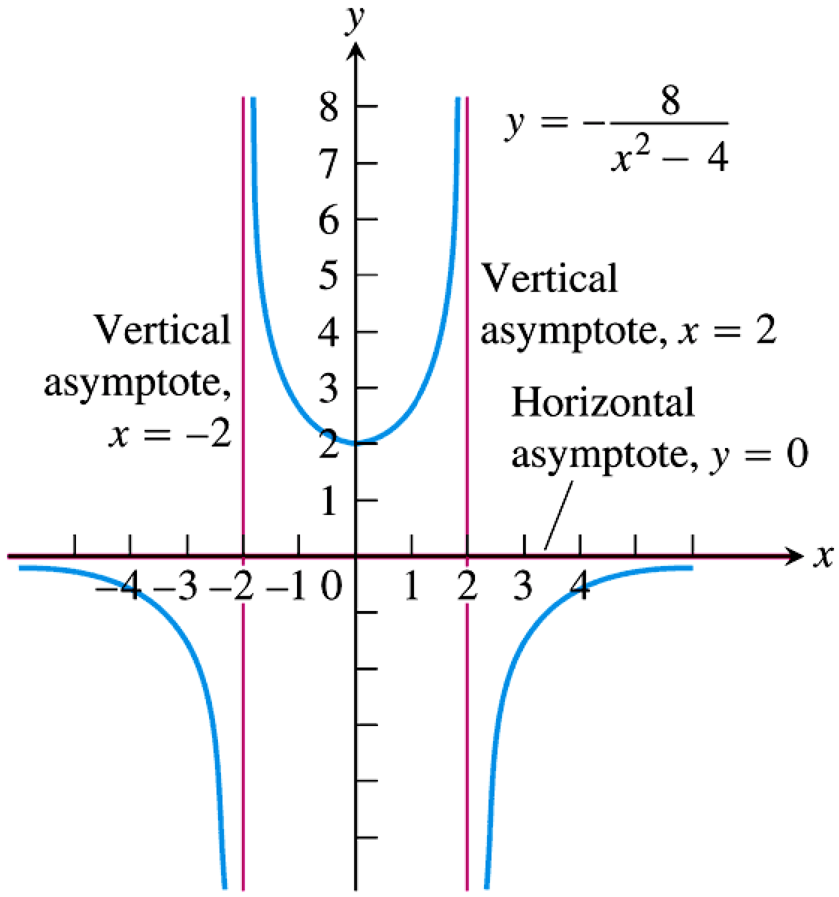


FIGURE 2.44 Graph of $y = -8/(x^2 - 4)$. Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided (Example 6).

Example 8

- A rational function with degree of freedom of numerator greater than degree of denominator

$$f(x)=\frac{x^2-3}{2x-4}$$

- Solution:

linear remainder

$$f(x)=\frac{x^2-3}{2x-4}=\boxed{\frac{x}{2}+1}+\boxed{\frac{1}{2x-4}}$$

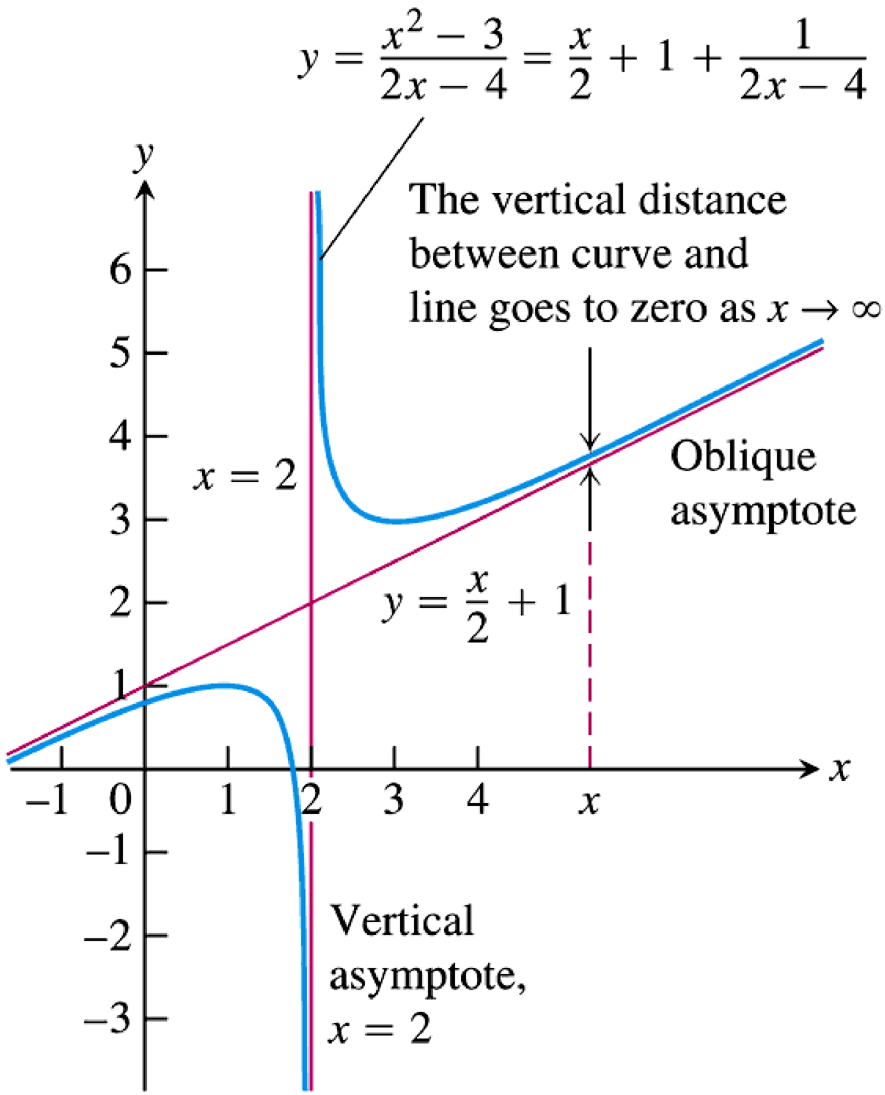


FIGURE 2.47 The graph of $f(x) = (x^2 - 3)/(2x - 4)$ has a vertical asymptote and an oblique asymptote (Example 8).

2.6

Continuity

(2nd lecture of week 13/08/07-18/08/07)

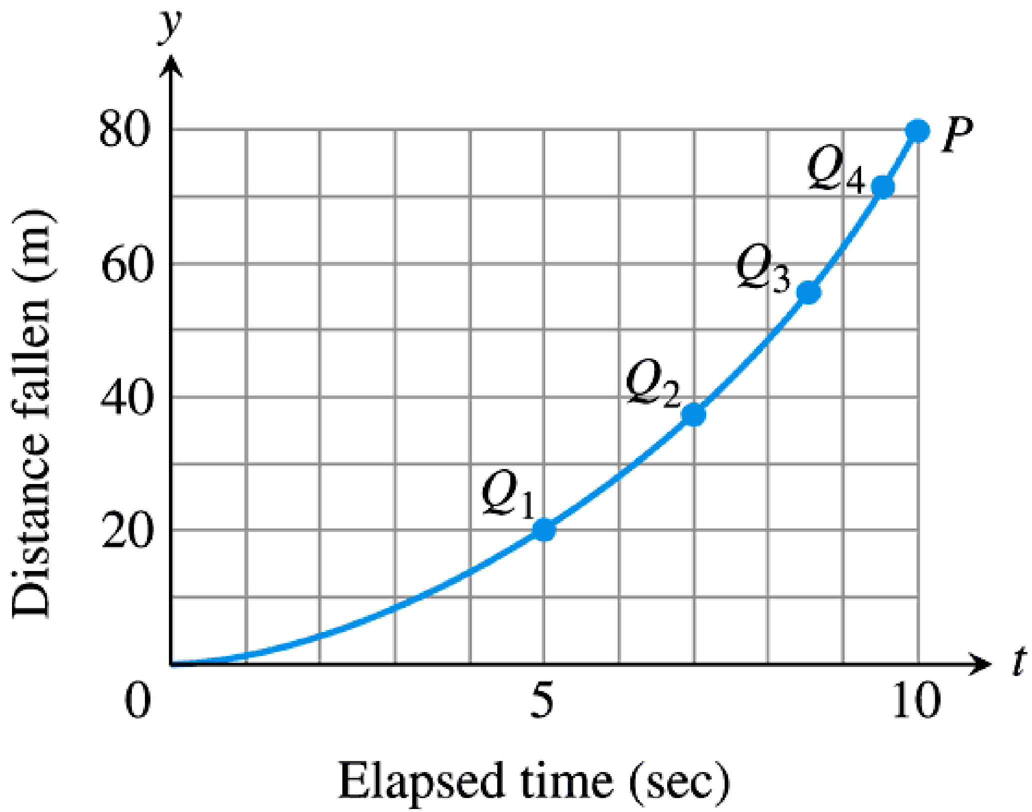


FIGURE 2.49 Connecting plotted points by an unbroken curve from experimental data $Q_1, Q_2, Q_3, \ldots$ for a falling object.

Continuity at a point

- Example 1
- Find the points at which the function f in Figure 2.50 is continuous and the points at which f is discontinuous.

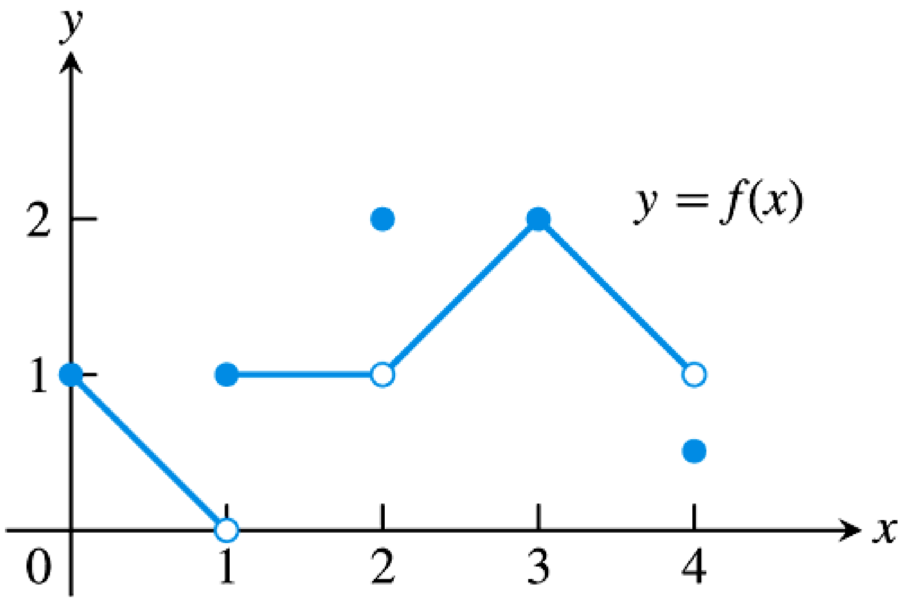


FIGURE 2.50 The function is continuous on [0, 4] except at $x = 1, x = 2$, and $x = 4$ (Example 1).

- f continuous:
- At $x = 0$
- At $x = 3$
- At $0 < c < 4$, $c \neq 1,2$

- f discontinuous:
- At $x = 1$
- At $x = 2$
- At $x = 4$
- $0 > c$, $c > 4$
- Why?

- To define the continuity at a point in a function’s domain, we need to
- define continuity at an interior point
- define continuity at an endpoint

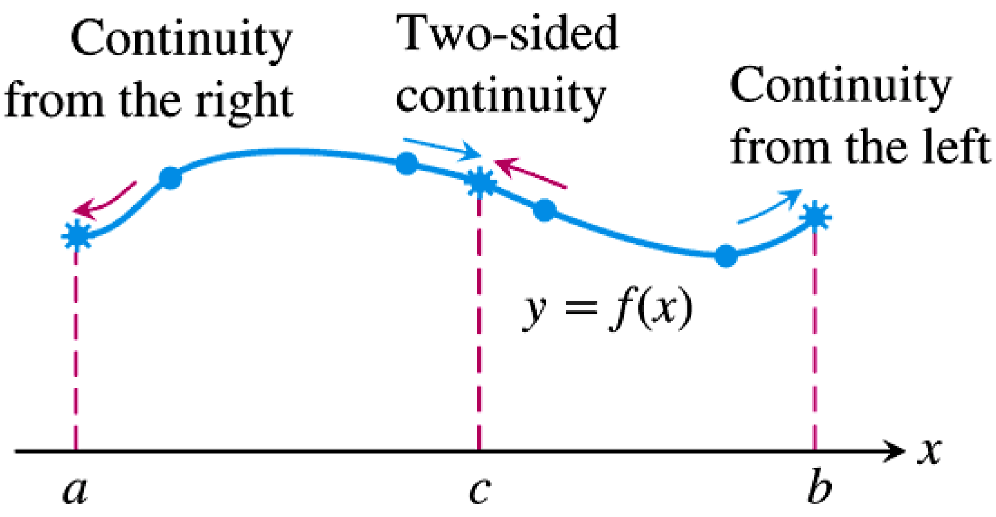


FIGURE 2.51 Continuity at points a, b, and c.

DEFINITION Continuous at a Point

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

Example 2

- A function continuous throughout its domain

$$f(x) = \sqrt{4 - x^2}$$

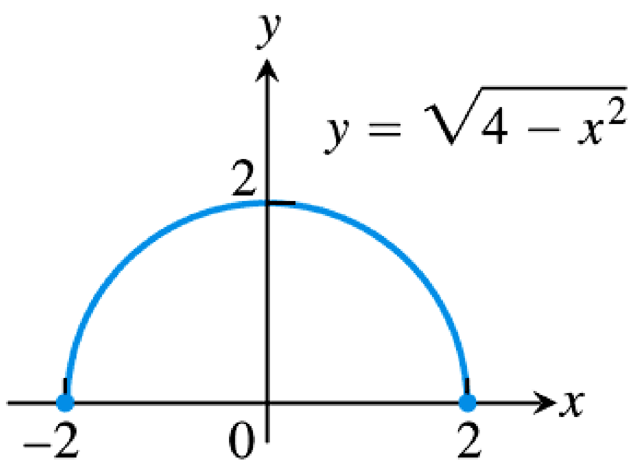


FIGURE 2.52 A function that is continuous at every domain point (Example 2).

Example 3

- The unit step function has a jump discontinuity

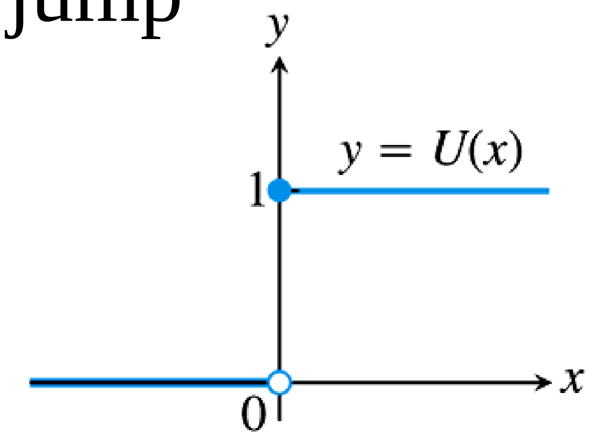


FIGURE 2.53 A function that is right-continuous, but not left-continuous, at the origin. It has a jump discontinuity there (Example 3).

Summarize continuity at a point in the form of a test

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1.	$f(c)$ exists	(c lies in the domain of f)
2.	$\lim_{x \to c} f(x)$ exists	(f has a limit as $x \to c$)
3.	$\lim_{x \to c} f(x) = f(c)$	(the limit equals the function value)

For one-sided continuity and continuity at an endpoint, the limits in parts 2 and parts 3 of the test should be replaced by the appropriate one-sided limits.

Example 4

- The greatest integer function,
- $y=[x]$
- The function is not continuous at the integer points since limit does not exist there (left and right limits not agree)

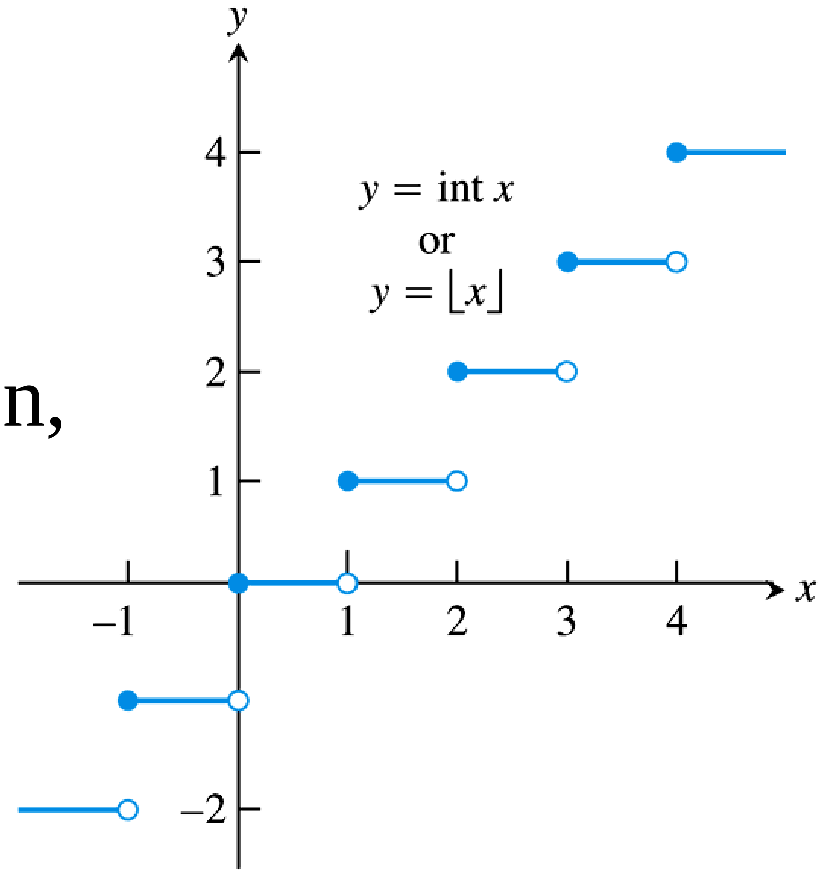


FIGURE 2.54 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

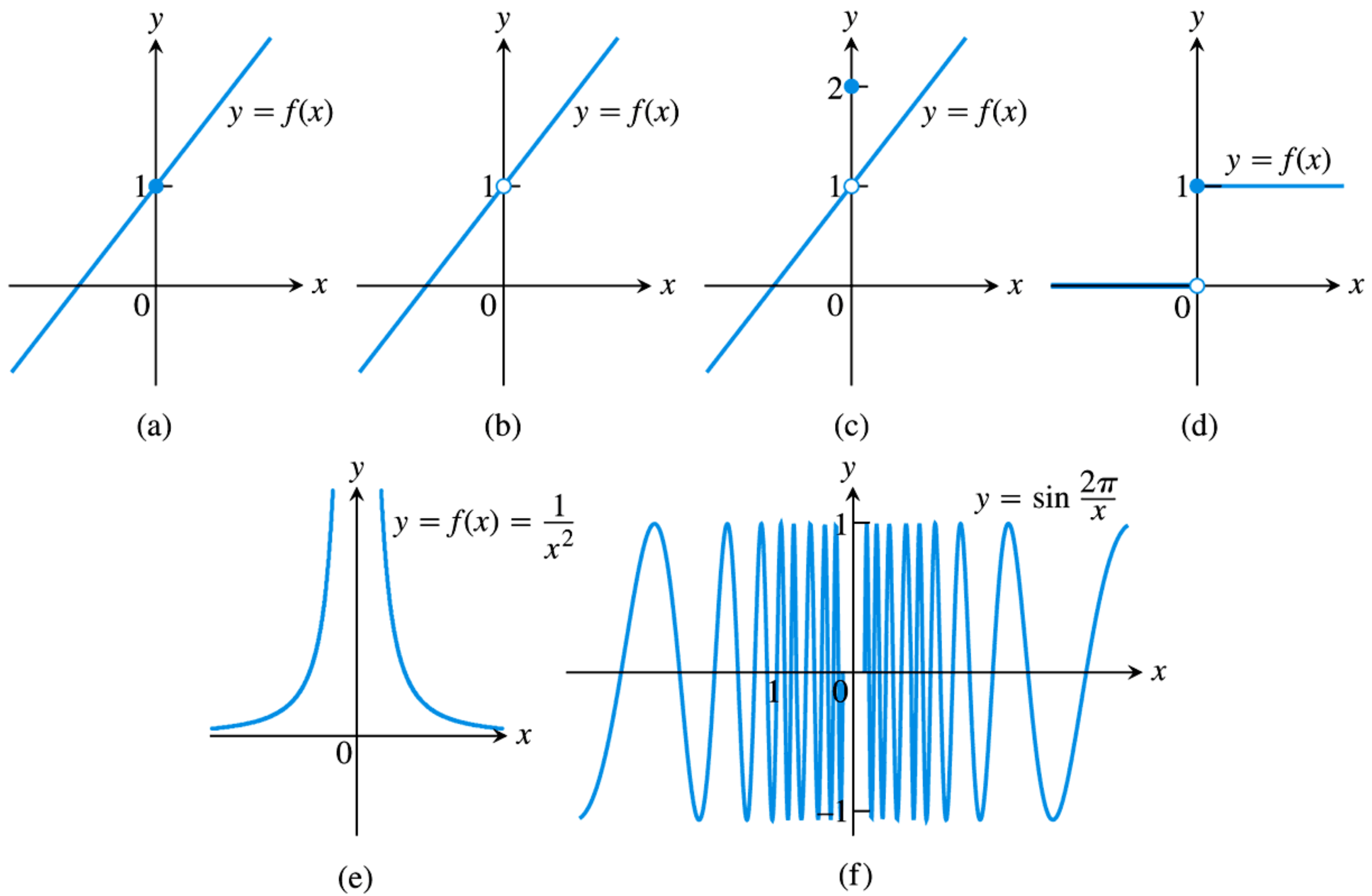


FIGURE 2.55 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

Discontinuity types

- (b), (c) removable discontinuity
- (d) jump discontinuity
- (e) infinite discontinuity
- (f) oscillating discontinuity

Continuous functions

- A function is continuous on an interval if and only if it is continuous at every point of the interval.
- Example: Figure 2.56
- $1/x$ not continuous on [-1,1] but continuous over $(-\infty,0) \cup (0, \infty)$

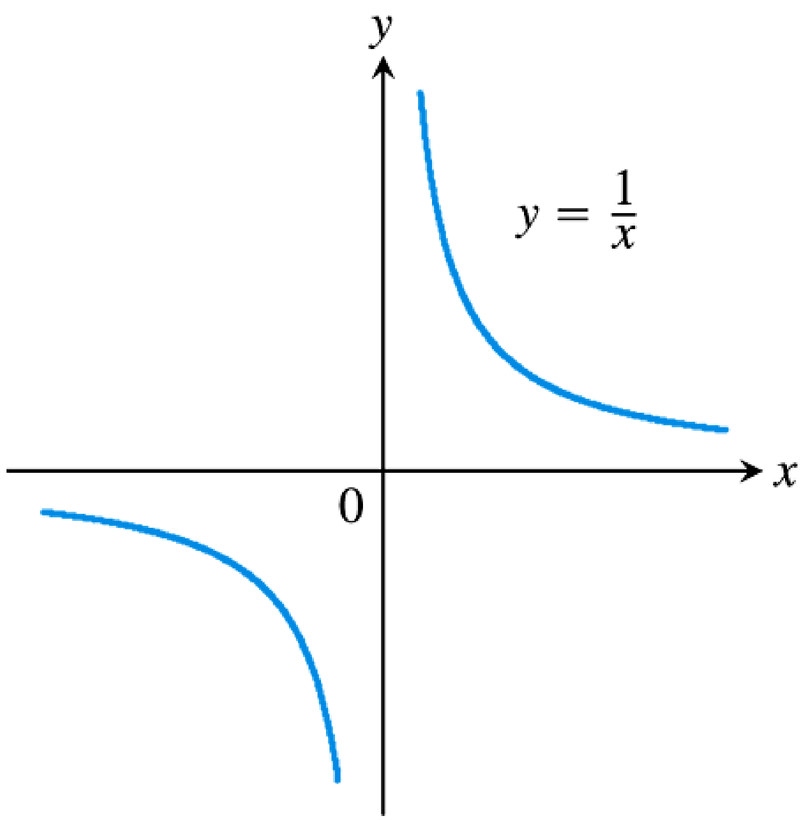


FIGURE 2.56 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$ (Example 5).

Example 5

- Identifying continuous function
- (a) $f(x)=1/x$
- (b) $f(x)= x$
- Ask: is $1/x$ continuous over its domain?

THEOREM 9 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

Example 6

- Polynomial and rational functions are continuous
- (a) Every polynomial is continuous by
- (i) $\lim_{x \to c} P(x) = P(c)$
- (ii) Theorem 9
- (b) If $P(x)$ and $Q(x)$ are polynomial, the rational function $P(x)/Q(x)$ is continuous whenever it is defined.

Example 7

- Continuity of the absolute function
- $f(x) = |x|$ is everywhere continuous

- Continuity of the sinus and cosinus function
- $f(x) = \cos x$ and $\sin x$ is everywhere continuous

Composites

- All composites of continuous functions are continuous

THEOREM 10 Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

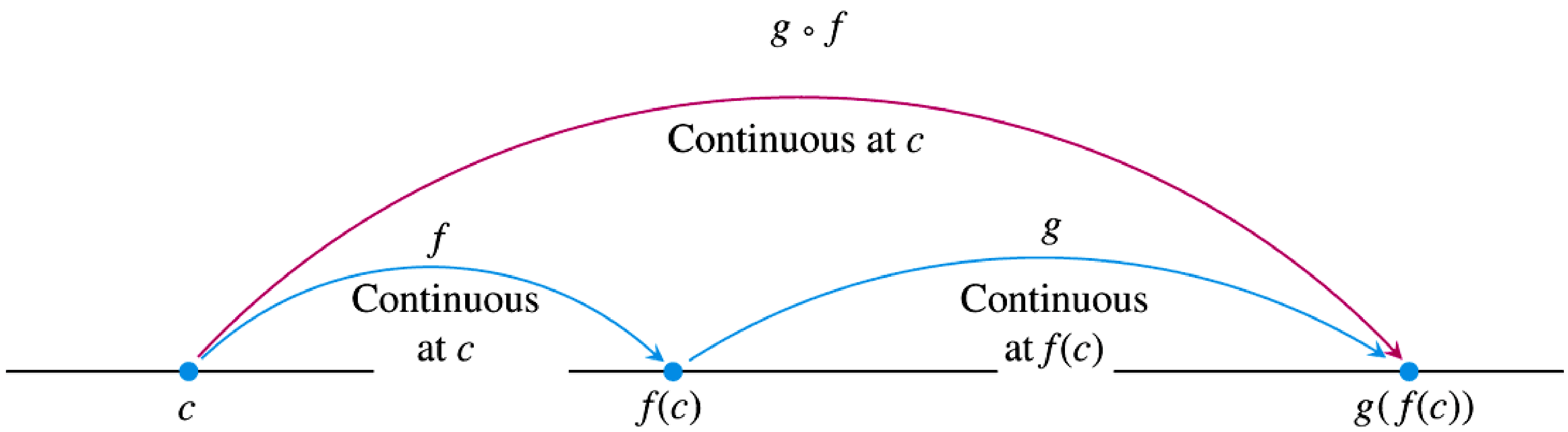


FIGURE 2.57 Composites of continuous functions are continuous.

Example 8

- Applying Theorems 9 and 10
- Show that the following functions are continuous everywhere on their respective domains.

$$(a)\ y = \sqrt{x^2 - 2x - 5} \qquad (b)\ y = \frac{x^{2/3}}{1 + x^4}$$

$$(c)\ y = \left|\frac{x - 2}{x^2 - 2}\right| \qquad (d)\ y = \left|\frac{x \sin x}{x^2 + 2}\right|$$

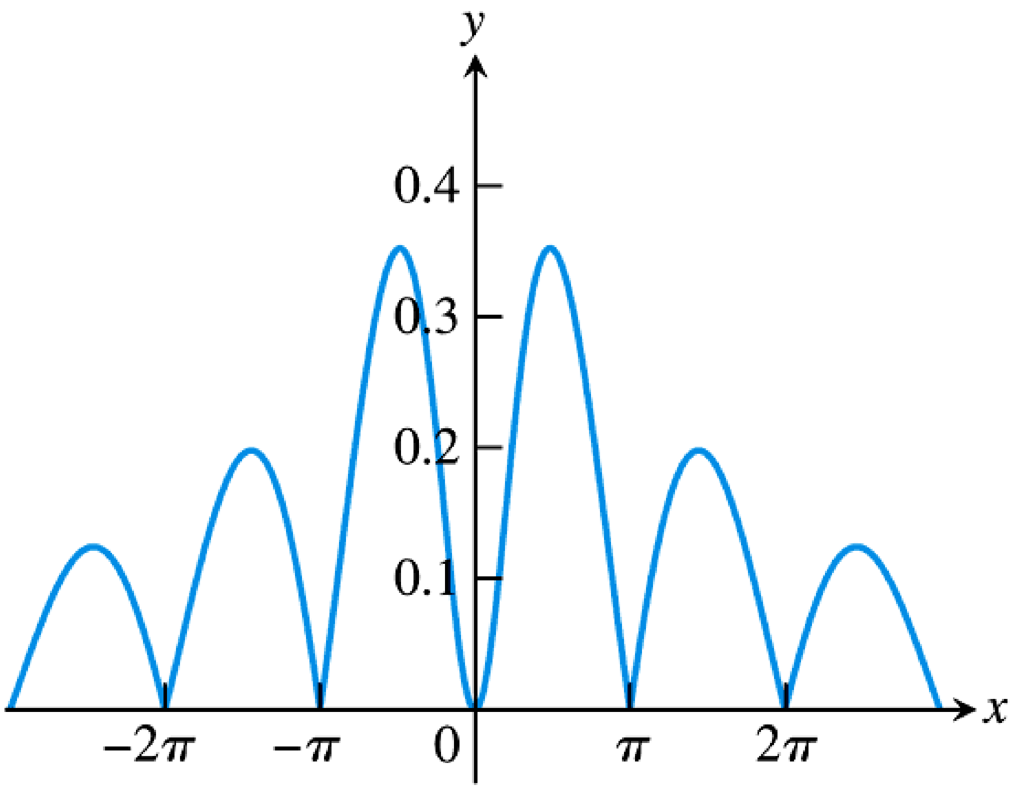
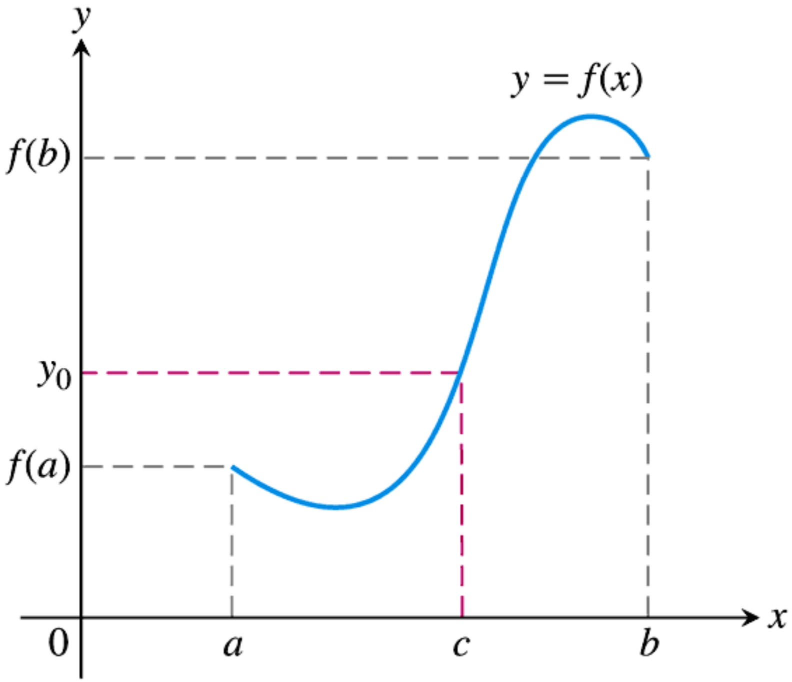


FIGURE 2.58 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous (Example 8d).

THEOREM 11 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



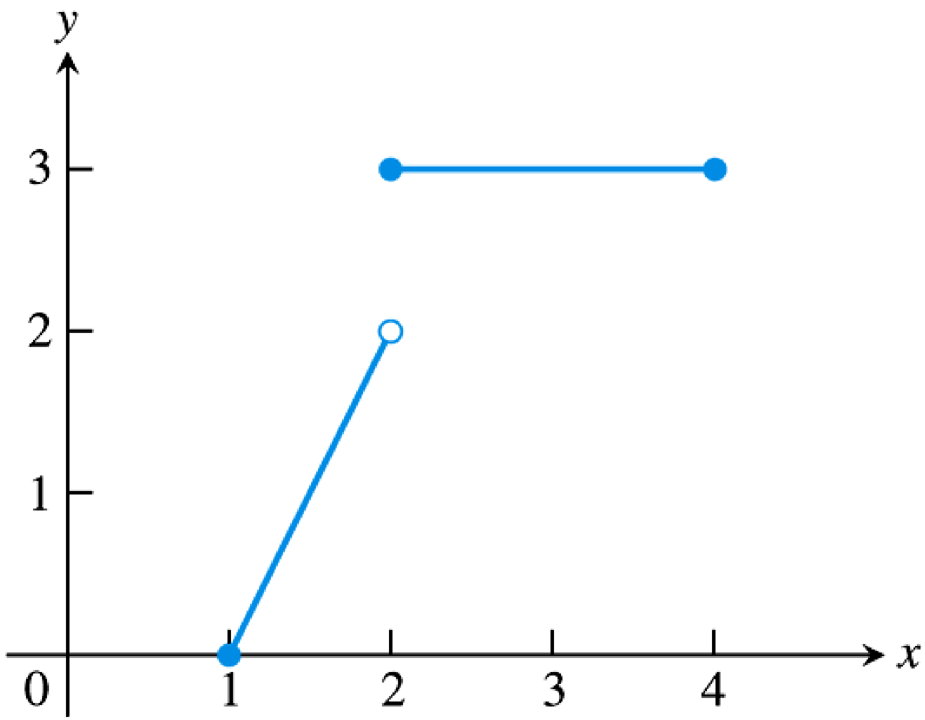


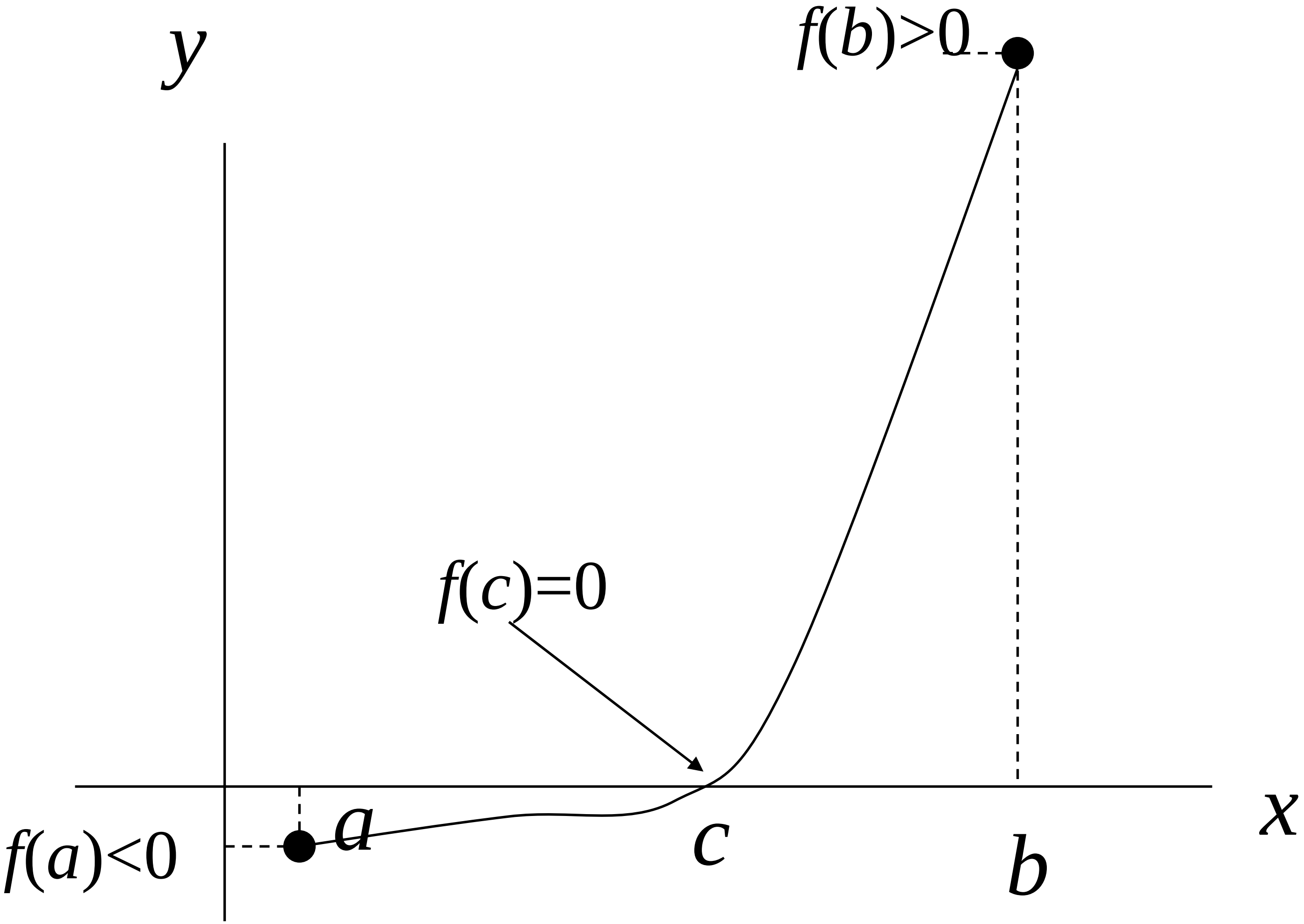
FIGURE 2.61 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \leq x < 2 \\ 3, & 2 \leq x \leq 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

Consequence of root finding

- A solution of the equation $f(x)=0$ is called a root.
- For example, $f(x)= x^2 + x - 6$, the roots are $x=2$, $x=-3$ since $f(-3)=f(2)=0$.
- Say f is continuous over some interval.
- Say a, b (with $a < b$) are in the domain of f, such that $f(a)$ and $f(b)$ have opposite signs.
- This means either $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$
- Then, as a consequence of theorem 11, there must exist at least a point c between a and b, i.e. $a < c < b$ such that $f(c)= 0$. $x=c$ is the root.


y
f(b)>0
f(c)=0
x
f(a)<0
a
c
b

Example

- Consider the function $f(x) = x - \cos x$
- Prove that there is at least one root for $f(x)$ in the interval $[0, \pi/2]$.

- **Solution**
- $f(x)$ is continuous on $(-\infty, \infty)$.
- Say $a = 0$, $b = \pi/2$.
- $f(x=0) = -1$; $f(x = \pi/2) = \pi/2$
- $f(a)$ and $f(b)$ have opposite signs
- Then, as a consequence of theorem 11, there must exist at least a point c between a and b, i.e. $a=0 < c < b= \pi/2$ such that $f(c)= 0$. $x=c$ is the root.

2.7

Tangents and Derivatives

(3rd lecture of week 13/08/07-18/08/07)

What is a tangent to a curve?

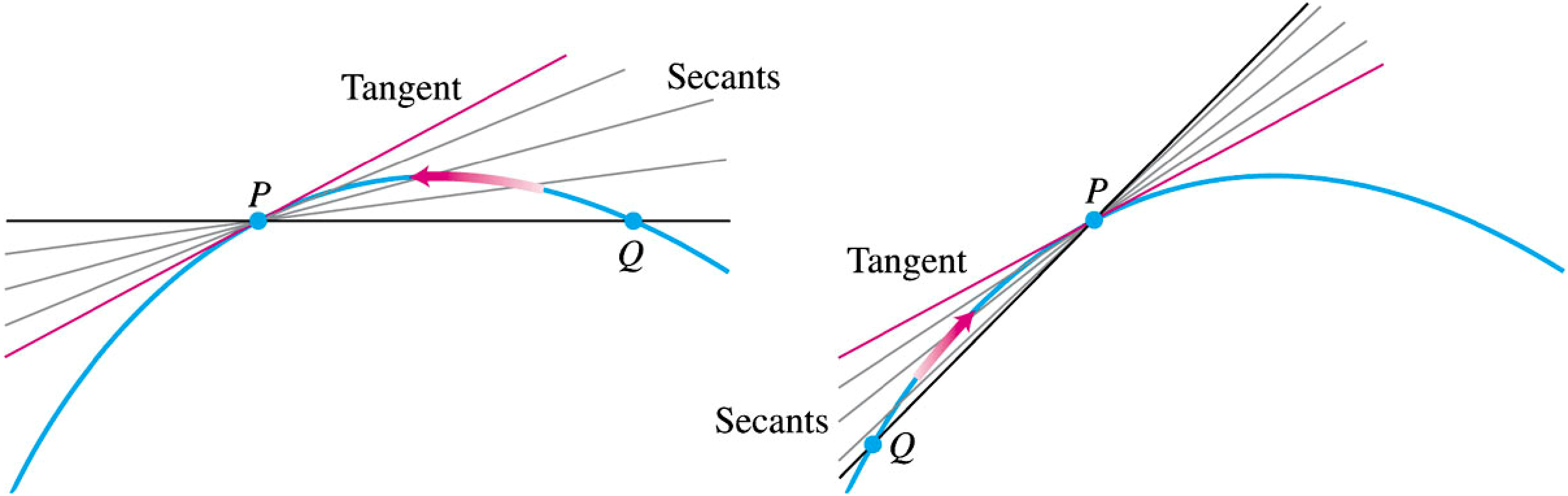


FIGURE 2.65 The dynamic approach to tangency. The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

DEFINITIONS **Slope, Tangent Line**

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

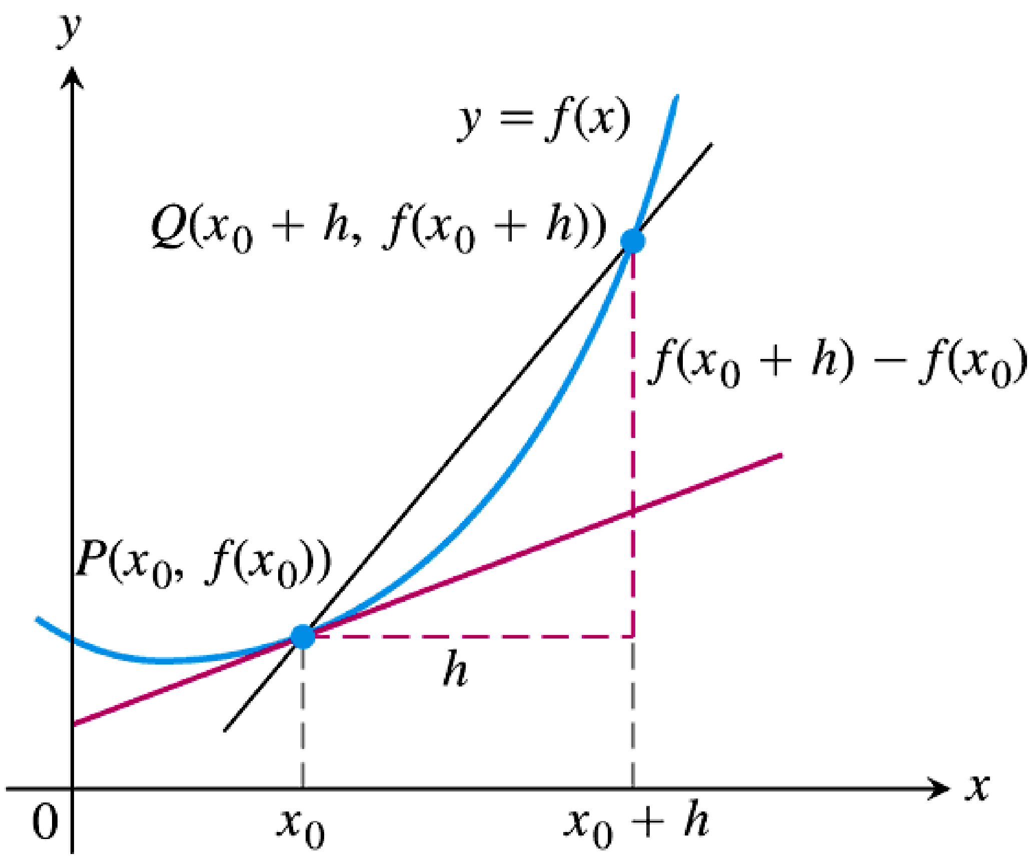


FIGURE 2.67 The slope of the tangent line at P is $\lim_{h\to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$.

Example 1: Tangent to a parabola

- Find the slope of the parabola $y=x^2$ at the point $P(2,4)$. Write an equation for the tangent to the parabola at this point.

Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

3. If the limit exists, find the tangent line as

$$y = y_0 + m(x - x_0).$$

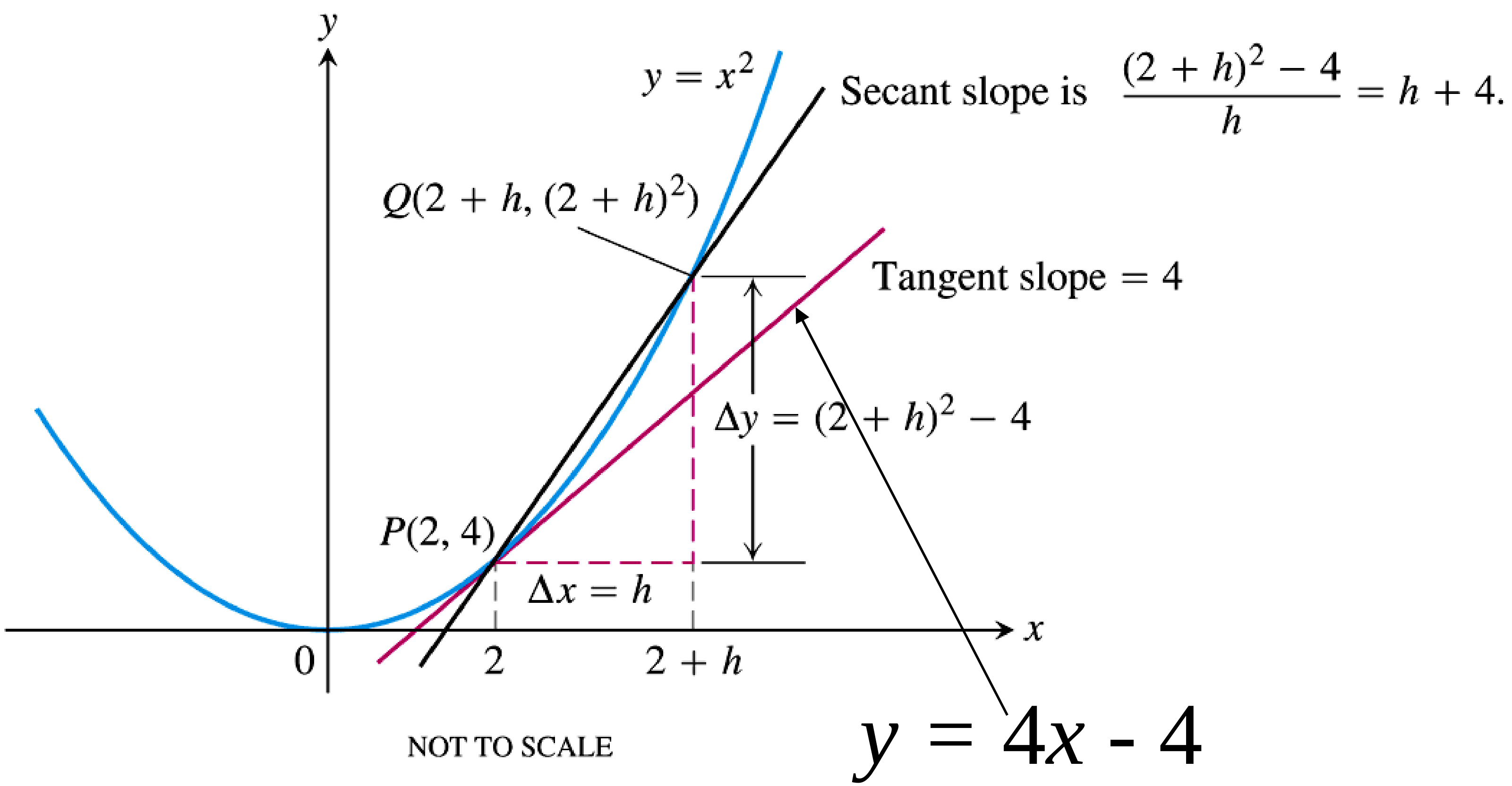


FIGURE 2.66 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ (Example 1).

Example 3

- Slope and tangent to $y=1/x$, $x\neq 0$
- (a) Find the slope of $y=1/x$ at $x = a \neq 0$
- (b) Where does the slope equal -1/4?
- (c) What happens to the tangent of the curve at the point $(a, 1/a)$ as a changes?

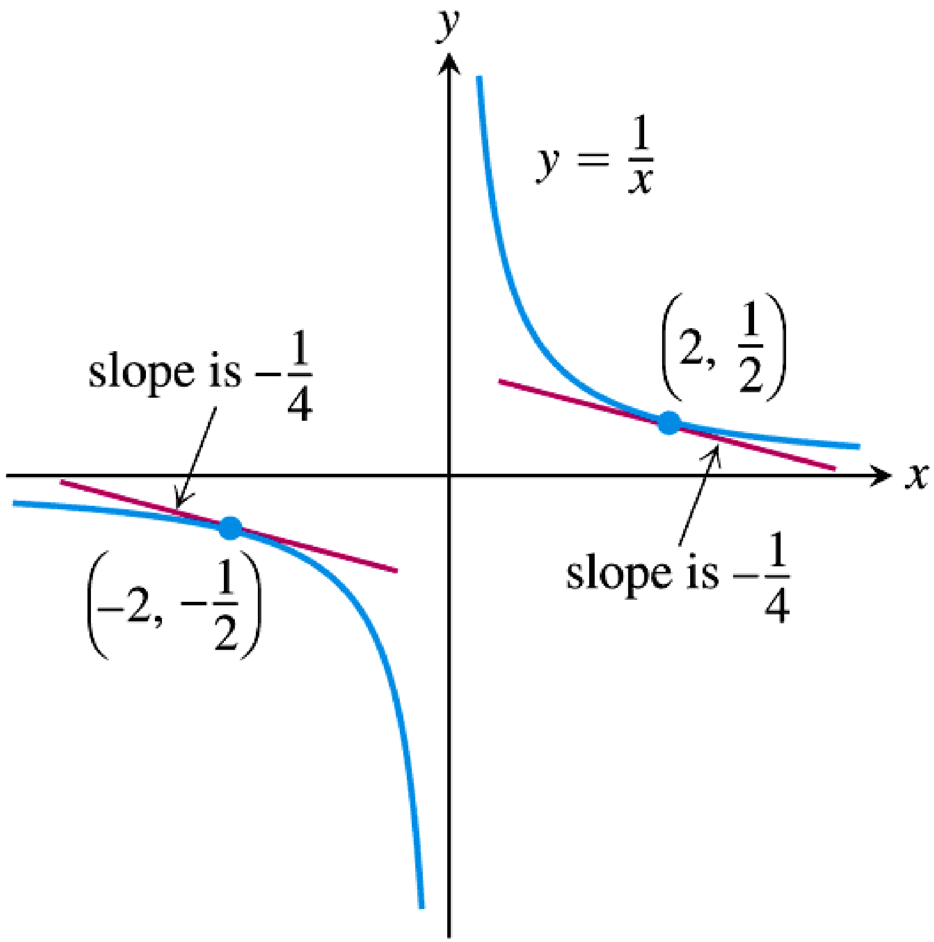


FIGURE 2.68 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 3).

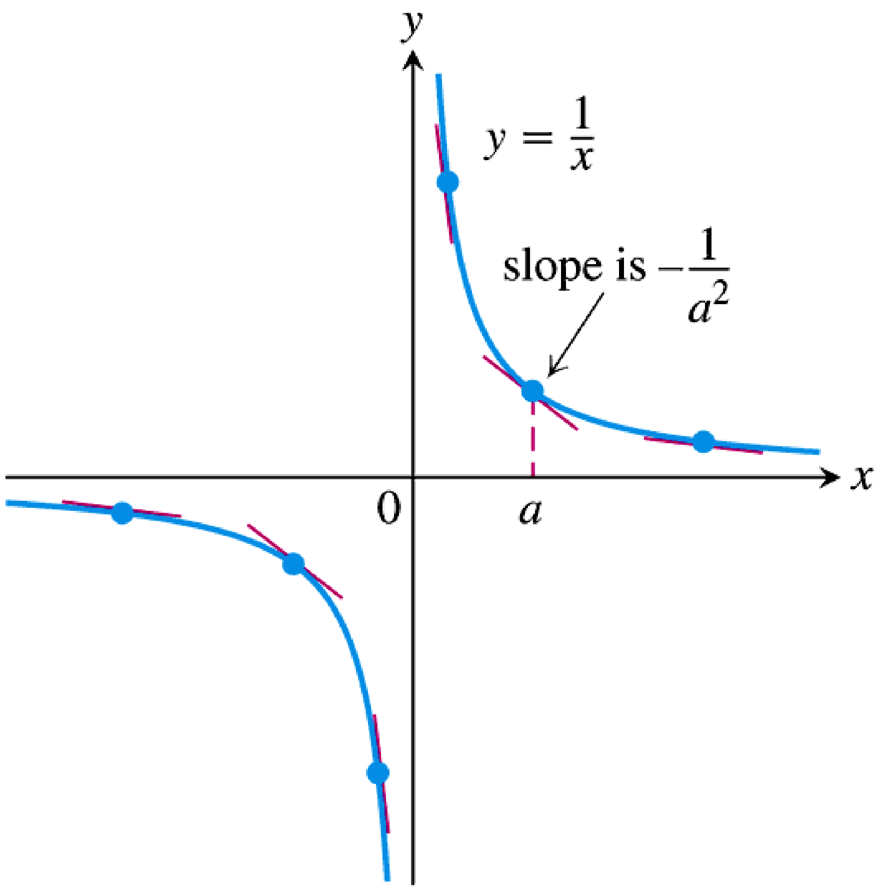


FIGURE 2.69 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away.

Chapter 3

Differentiation

3.1

The Derivative as a Function

(3rd lecture of week 13/08/07-18/08/07)

DEFINITION Derivative Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

- The limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

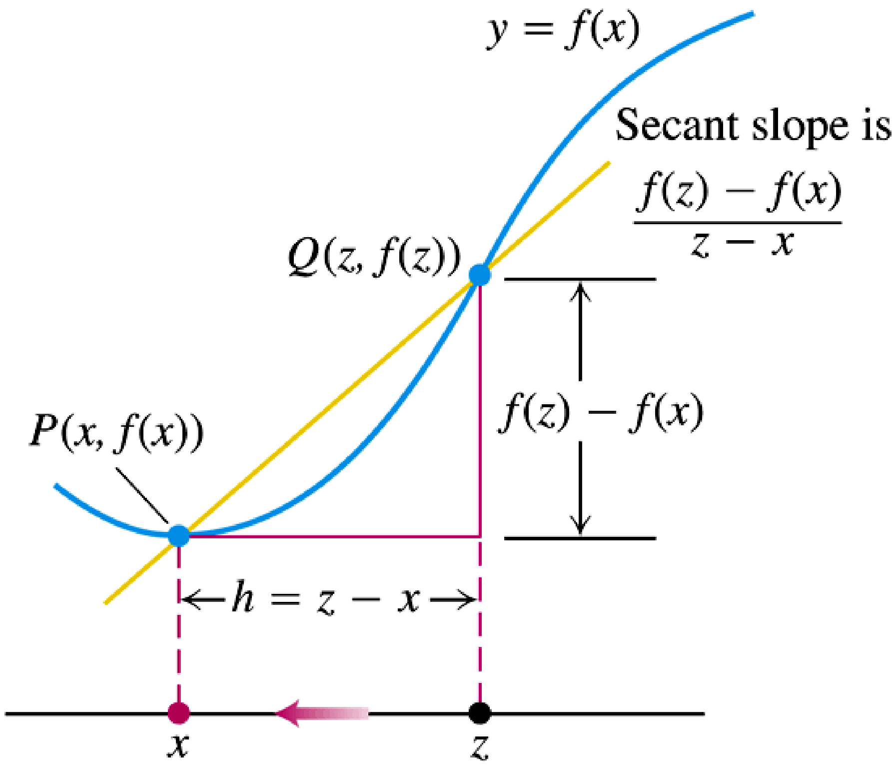
- when it existed, is called the Derivative if f at x_0.
- View derivative as a function derived from f

- If f' exists at x, f is said to be differentiable (has a derivative) at x
- If f' exists at every point in the domain of f, f is said to be differentiable.

If write $z = x + h$, then $h = z - x$

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$



Derivative of f at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

FIGURE 3.1 The way we write the difference quotient for the derivative of a function f depends on how we label the points involved.

Calculating derivatives from the definition

- Differentiation: an operation performed on a function $y = f(x)$
- d/dx operates on $f(x)$
- Write as $\frac{d}{dx} f(x)$
- f' is taken as a shorthand notation for $\frac{d}{dx} f(x)$

Example 1: Applying the definition

- Differentiate

$$f(x) = \frac{x}{x-1}$$

- Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{x+h}{x+h-1}\right) - \left(\frac{x}{x-1}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}$$

Example 2: Derivative of the square root function

- (a) Find the derivative of
- (b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$

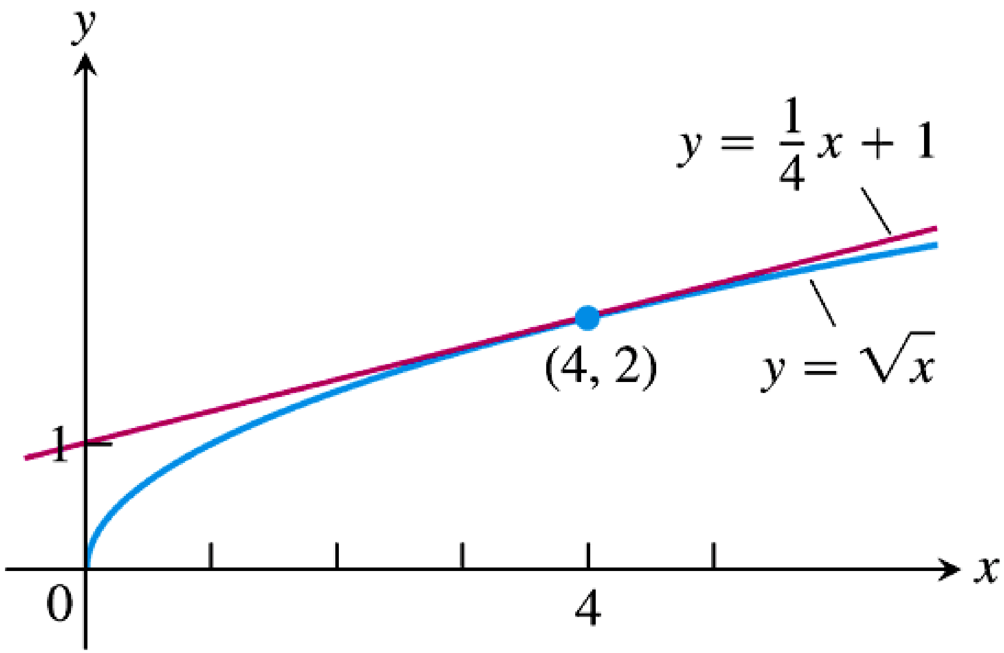


FIGURE 3.2 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

Notations

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx} f(x)\right|_{x=a}$$

Differentiable on an Interval; One sided derivatives

- A function $y = f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval.
- It is differentiable on a closed interval $[a,b]$ if it is differentiable on the interior (a,b) and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

$$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}$$

exist at the endpoints

- A function has a derivative at a point if an only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

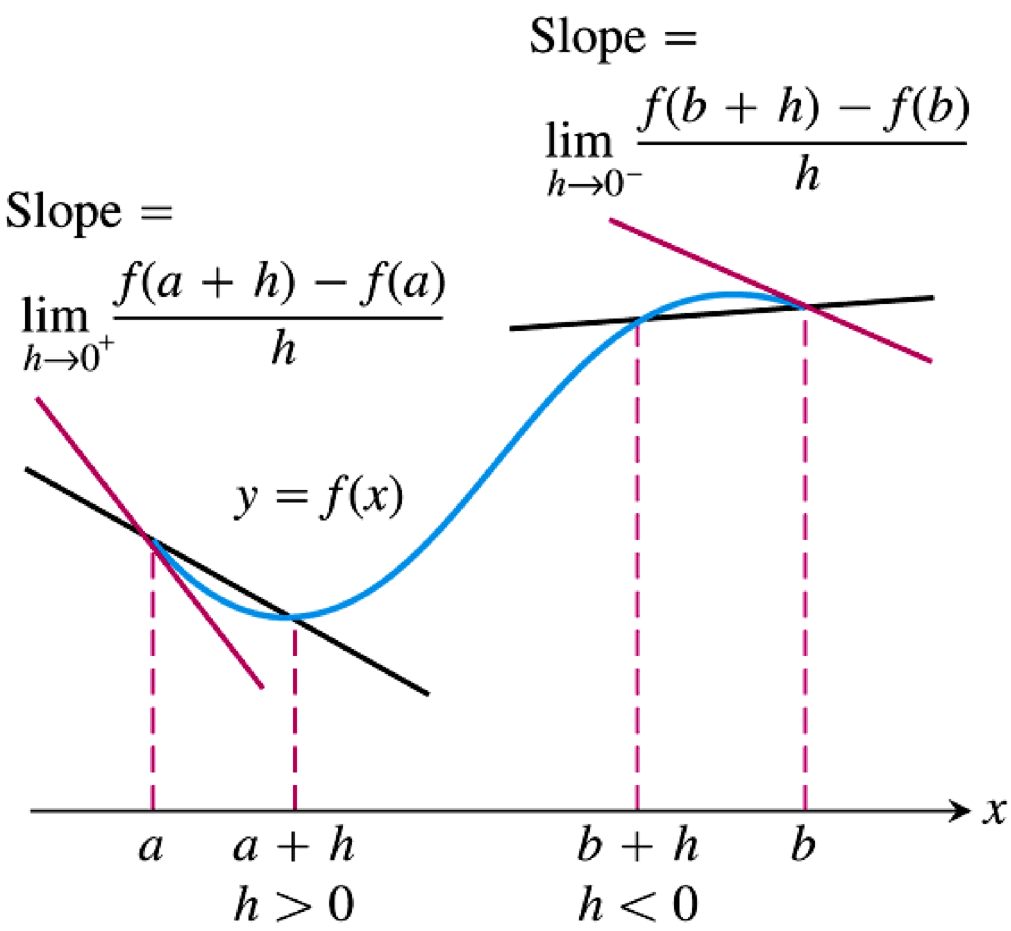


FIGURE 3.5 Derivatives at endpoints are one-sided limits.

Example 5

- $y = |x|$ is not differentiable at $x = 0$.
- Solution:
- For $x > 0$, $\frac{d|x|}{dx} = \frac{d}{dx}(x) = 1$
- For $x < 0$, $\frac{d|x|}{dx} = \frac{d}{dx}(-x) = -1$
- At $x = 0$, the right hand derivative and left hand derivative differ there. Hence $f(x)$ not differentiable at $x = 0$ but else where.

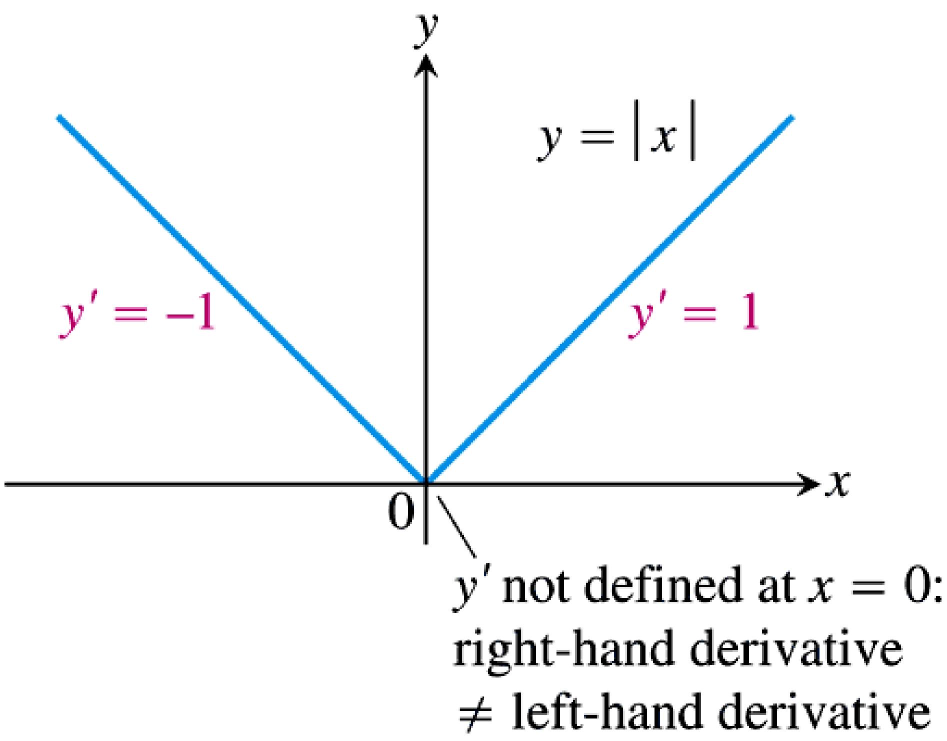


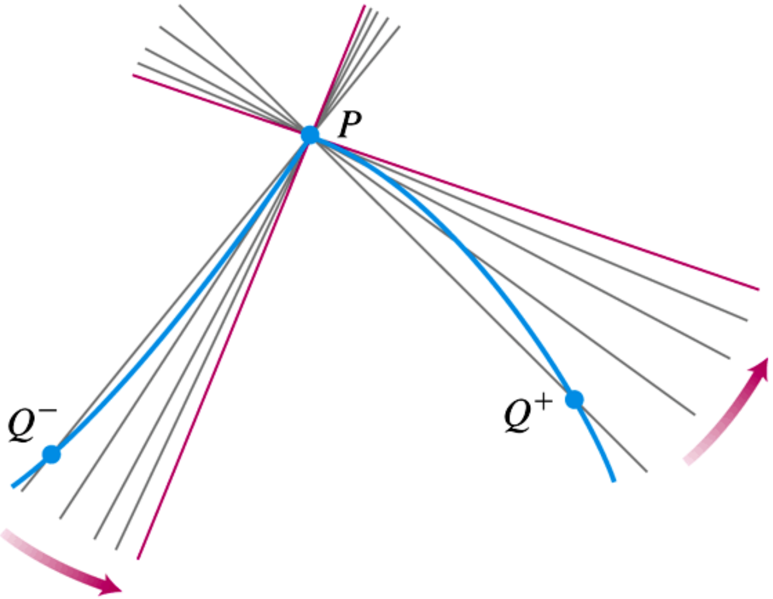
FIGURE 3.6 The function $y = |x|$ is not differentiable at the origin where the graph has a "corner."

Example 6

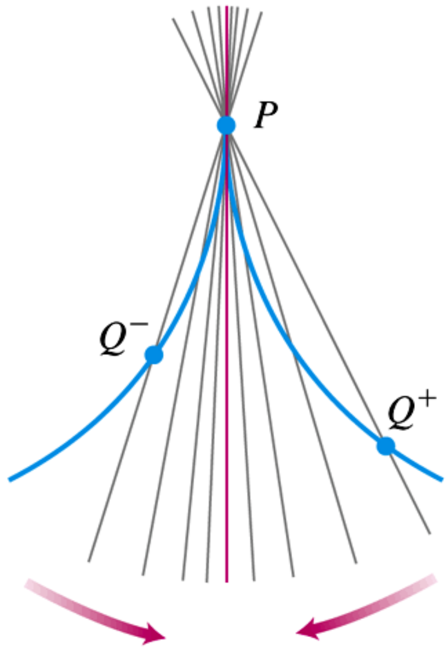
- $y = \sqrt{x}$ is not differentiable at $x = 0$
- The graph has a vertical tangent at $x = 0$

When Does a function not have a derivative at a point?

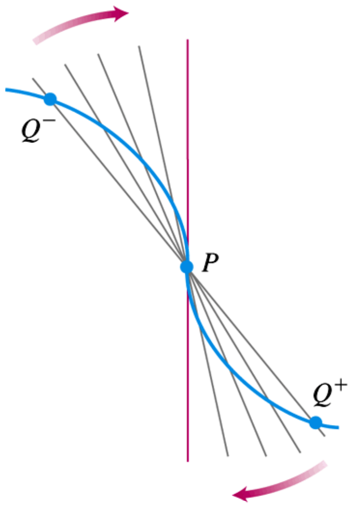
1. a *corner*, where the one-sided derivatives differ.



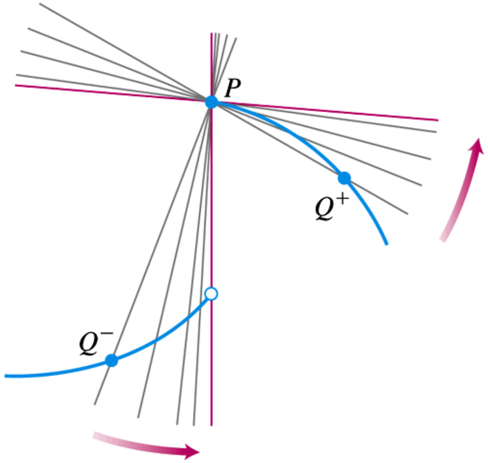
2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.

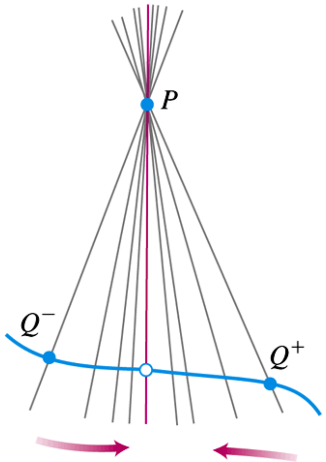


3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).



4. a *discontinuity*.





Differentiable functions are continuous

THEOREM 1 Differentiability Implies Continuity

If f has a derivative at $x = c$, then f is continuous at $x = c$.

The converse is false: continuity does not necessarily implies differentiability

Example

- $y = |x|$ is continuous everywhere, including $x = 0$, but it is not differentiable there.

The equivalent form of Theorem 1

- If f is not continuous at $x = c$, then f is not differentiable at $x = c$.
- Example: the step function is discontinuous at $x = 0$, hence not differentiable at $x = 0$.

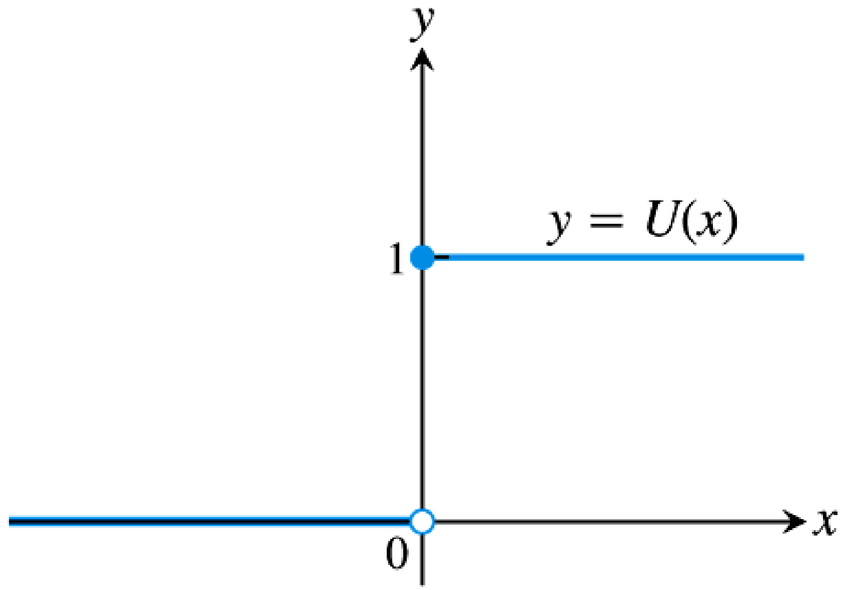


FIGURE 3.7 The unit step function does not have the Intermediate Value Property and cannot be the derivative of a function on the real line.

The intermediate value property of derivatives

THEOREM 2 Darboux's Theorem

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

- See section 4.4

3.2

Differentiation Rules

(1st lecture of week 20/08/07-25/08/07)

Powers, multiples, sums and differences

RULE 1 Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Example 1

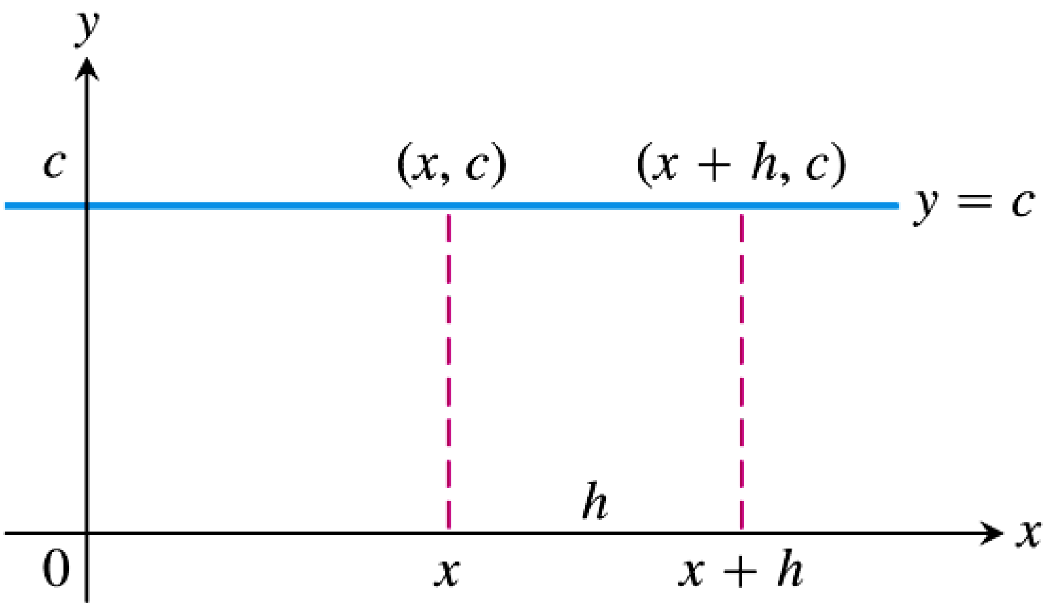


FIGURE 3.8 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

RULE 2 Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

RULE 3 Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

In particular, if $u = x^n, \frac{d}{dx}(cx^n) = cx^{n-1}$

Example 3

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x^{2-1} = 6x$$

$$\frac{d}{dx}(x^2) = 2x^{2-1} = 2x$$

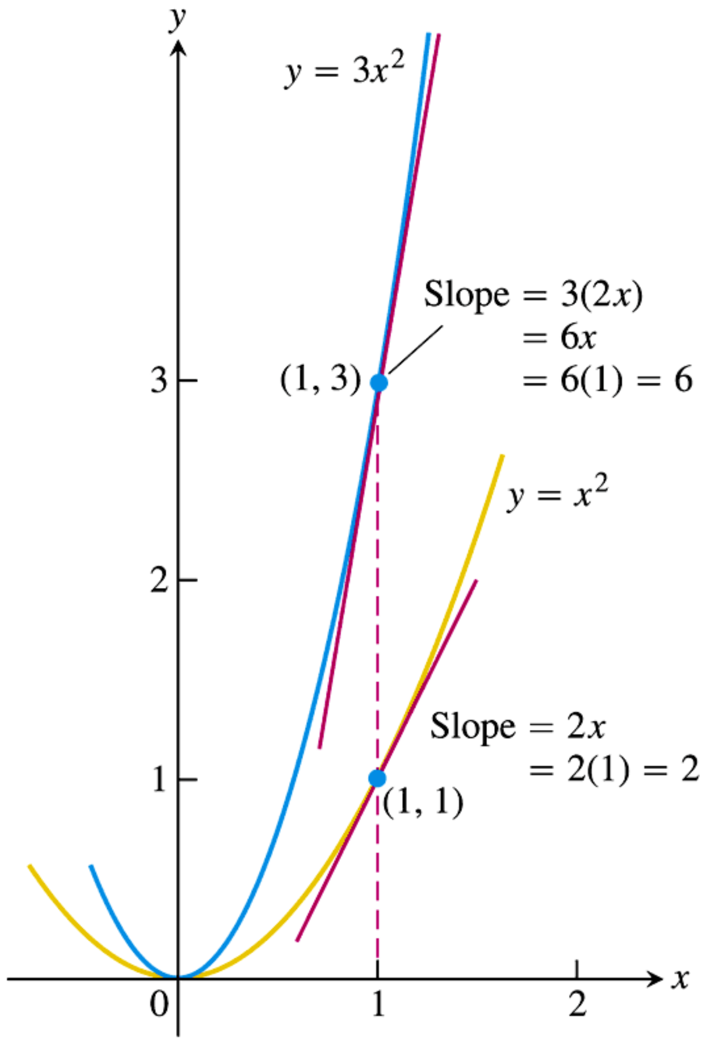


FIGURE 3.9 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinates triples the slope (Example 3).

RULE 4 Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Example 5

$$y = x^3 + \frac{4}{3}x^2 - 5x + 1$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(\frac{4}{3}x^2) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$= 3x^2 + \frac{8}{3}x - 5$$

Example 6

- Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

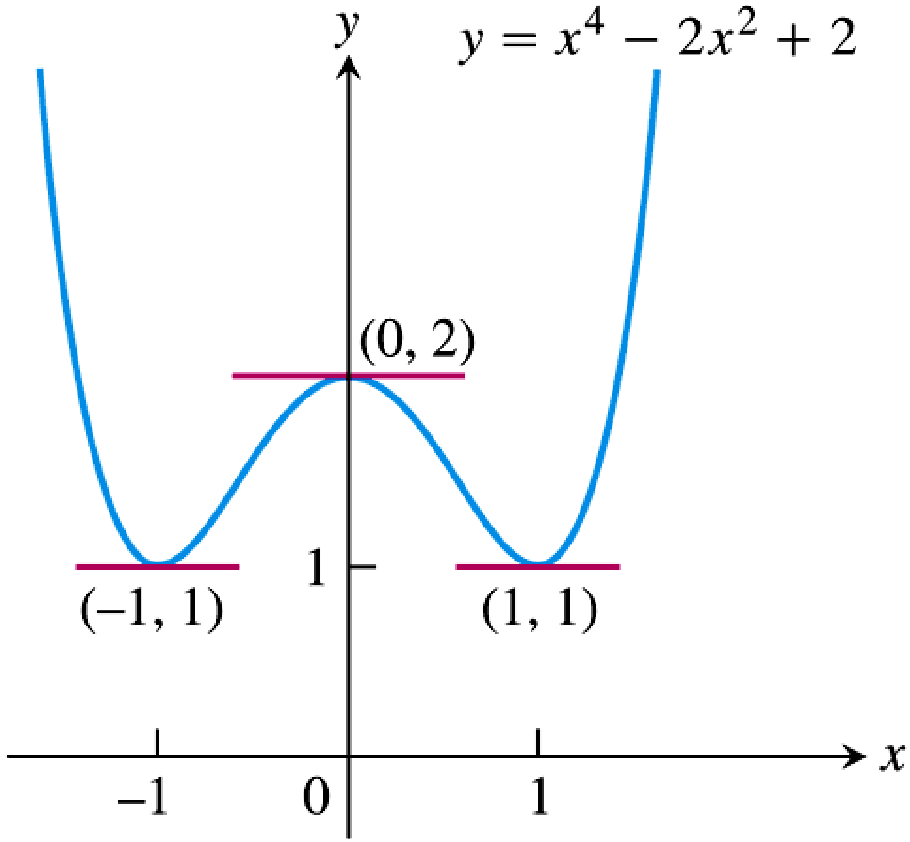


FIGURE 3.10 The curve $y = x^4 - 2x^2 + 2$ and its horizontal tangents (Example 6).

Products and quotients

- Note that $$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(x \cdot x) \neq \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1$$

RULE 5 Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example 7

- Find the derivative of $y = \frac{1}{x}\left(x^2 + \frac{1}{x}\right)$

Example 8: Derivative from numerical values

- Let $y = uv$. Find $y\,'(2)$ if $u(2) = 3$, $u'(2) = -4$, $v(2) = 1$, $v\,'(2) = 2$

Example 9

- Find the derivative of $y = \left(x^2 + 1\right)\left(x^3 + 3\right)$

RULE 6 Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Negative integer powers of *x*

- The power rule for negative integers is the same as the rule for positive integers

RULE 7 Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Example 11

$$\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{d}{dx}\left(x^{-1}\right)=\left(-1\right)x^{-1-1}=-x^{-2}$$

$$\frac{d}{dx}\left(\frac{4}{x^3}\right)=\frac{d}{dx}\left(4x^{-3}\right)=\left(-4\cdot-3\right)x^{-3-1}=12x^{-4}$$

Example 12: Tangent to a curve

- Find the tangent to the curve at the point (1,3) $y = x + \dfrac{2}{x}$

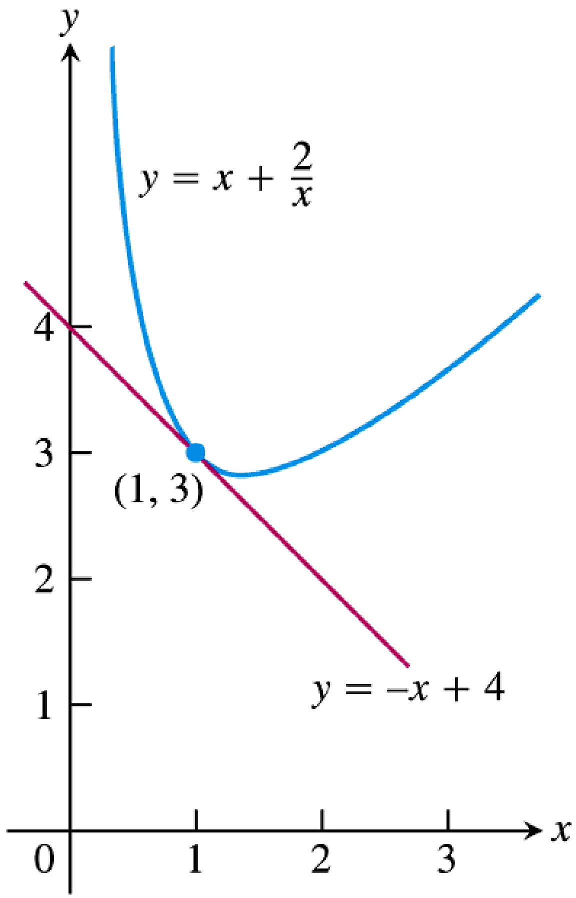


FIGURE 3.11 The tangent to the curve $y = x + (2/x)$ at $(1, 3)$ in Example 12. The curve has a third-quadrant portion not shown here. We see how to graph functions like this one in Chapter 4.

Example 13

- Find the derivative of $y=\frac{(x-1)\left(x^2-2x\right)}{x^4}$

Second- and higher-order derivative

- Second derivative

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(y')$$

$$= y'' = D^2(f)(x) = D_x^2 f(x)$$

- *n*th derivative $y^{(n)} = \frac{d}{dx} y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y$

Example 14

$$y = x^3 - 3x^2 + 2$$

$$y' = 3x^2 - 6x$$

$$y'' = 6x - 6$$

$$y''' = 6$$

$$y^{(4)} = 0$$

3.3

The Derivative as a Rate of Change

(1st lecture of week 20/08/07-25/08/07)

Instantaneous Rates of Change

DEFINITION Instantaneous Rate of Change

The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Example 1: How a circle's area changes with its diameter

- $A = \pi D^2/4$
- How fast does the area change with respect to the diameter when the diameter is 10 m?

Motion along a line

- Position $s = f(t)$
- Displacement, $\Delta s = f(t + \Delta t) - f(t)$
- Average velocity
- $v_{av} = \Delta s/\Delta t = [f(t + \Delta t) - f(t)] / \Delta t$
- The instantaneous velocity is the limit of v_{av} when $\Delta t \to 0$

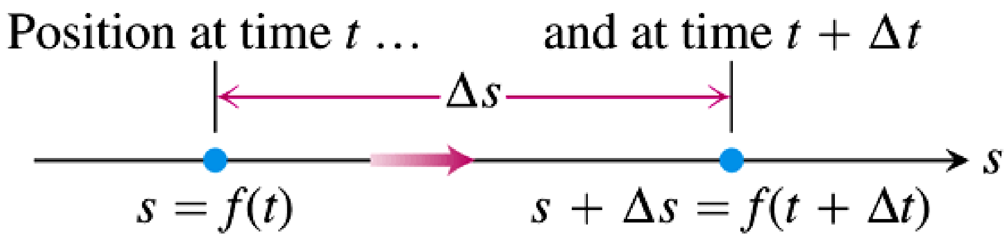


FIGURE 3.12 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$.

DEFINITION Velocity

Velocity (instantaneous velocity) is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

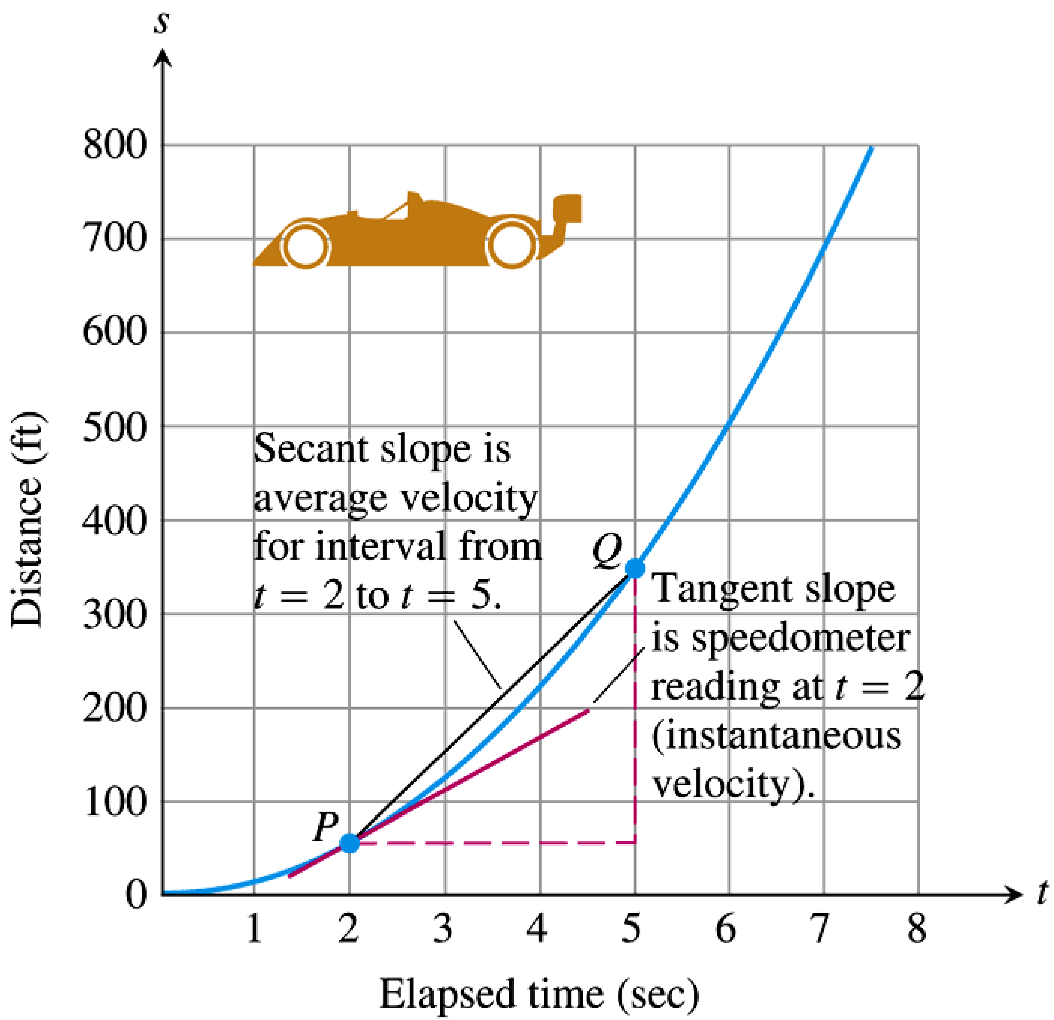


FIGURE 3.13 The time-to-distance graph for Example 2. The slope of the tangent line at P is the instantaneous velocity at $t = 2$ sec.

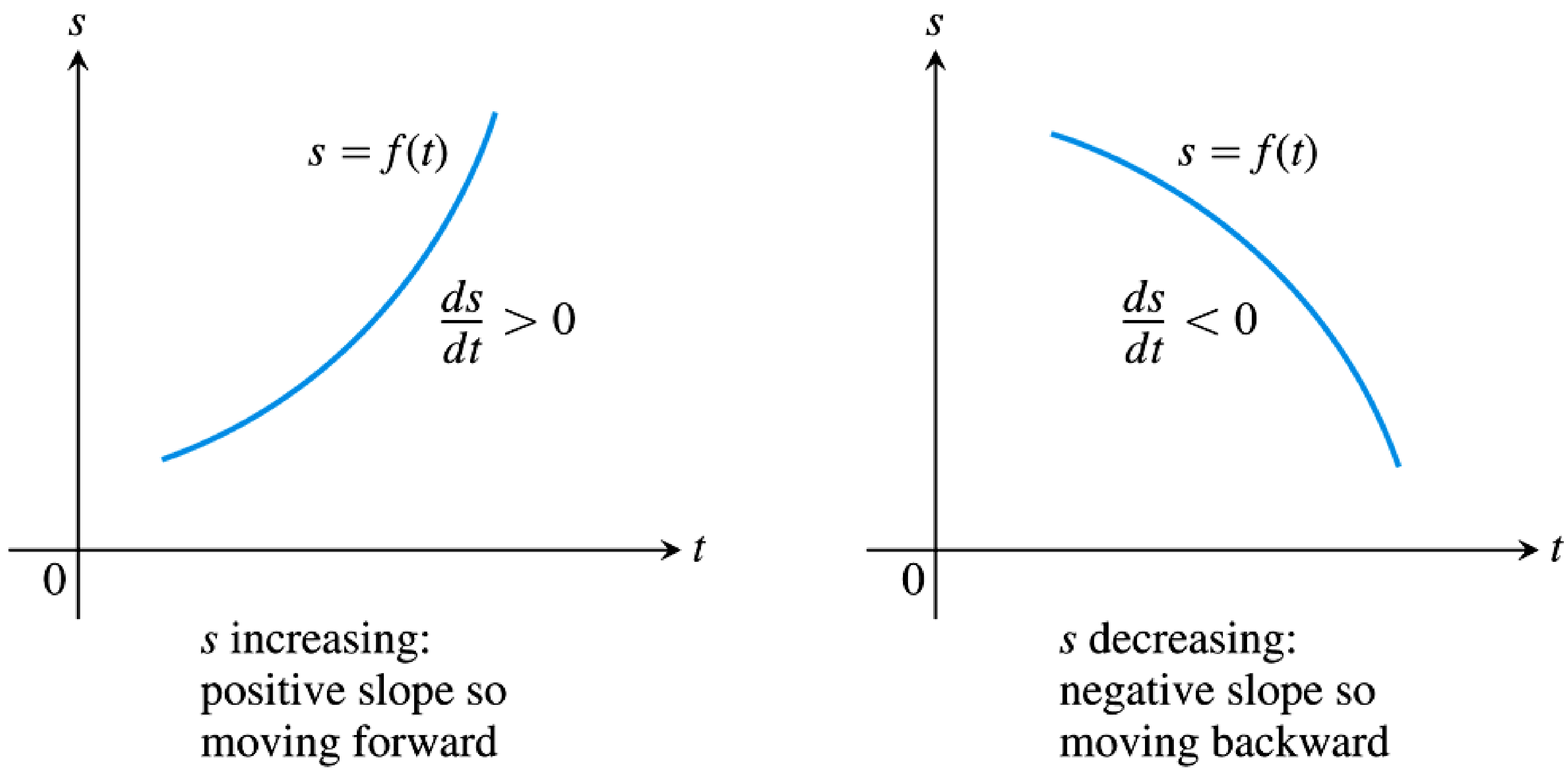


FIGURE 3.14 For motion $s = f(t)$ along a straight line, $v = ds/dt$ is positive when s increases and negative when s decreases.

DEFINITION Speed

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

Example 3

- Horizontal motion

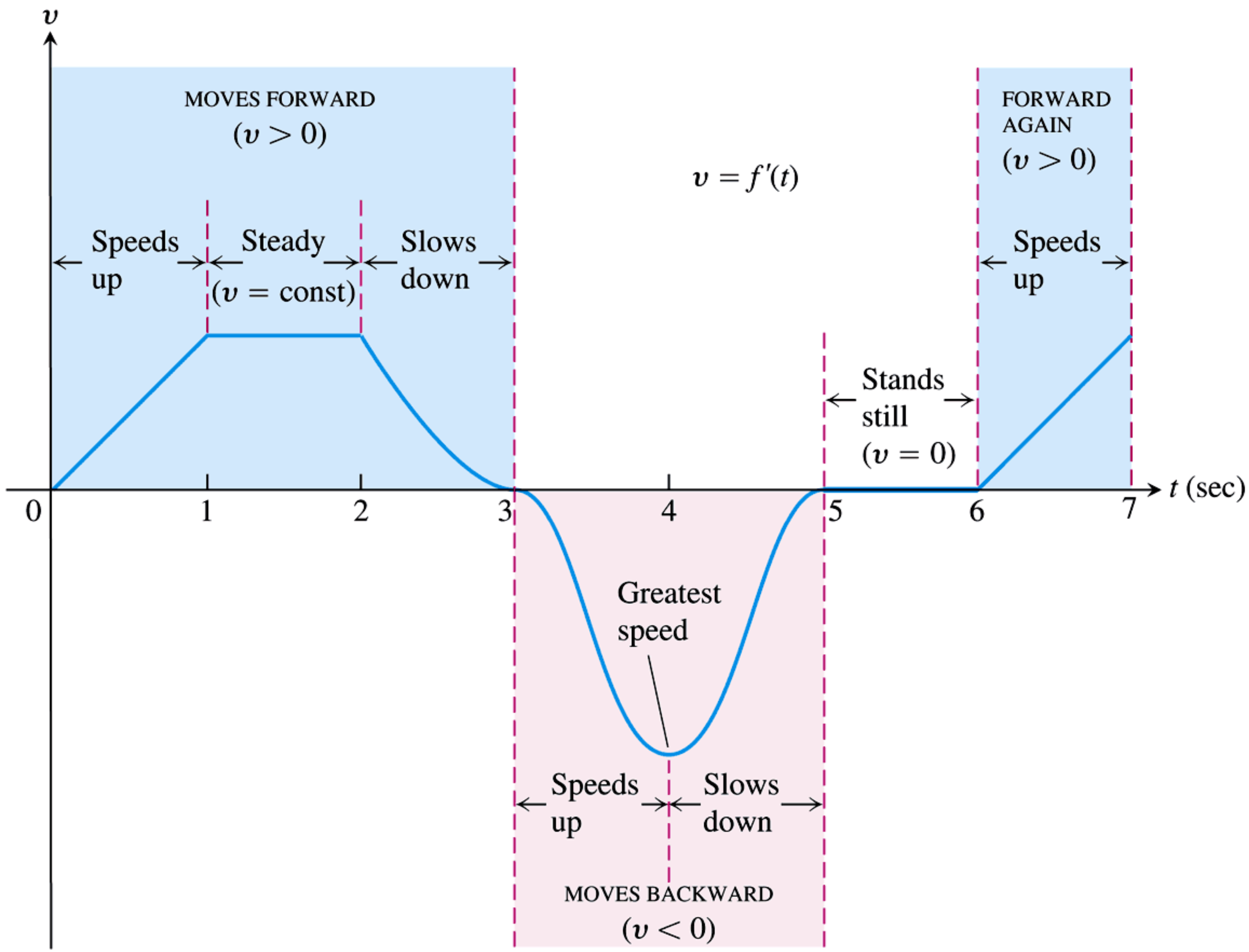


FIGURE 3.15 The velocity graph for Example 3.

DEFINITIONS Acceleration, Jerk

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Example 4

- Modeling free fall $s = \frac{1}{2}gt^2$
- Consider the free fall of a heavy ball released from rest at $t = 0$ sec.
- (a) How many meters does the ball fall in the first 2 sec?
- (b) What is the velocity, speed and acceleration then?

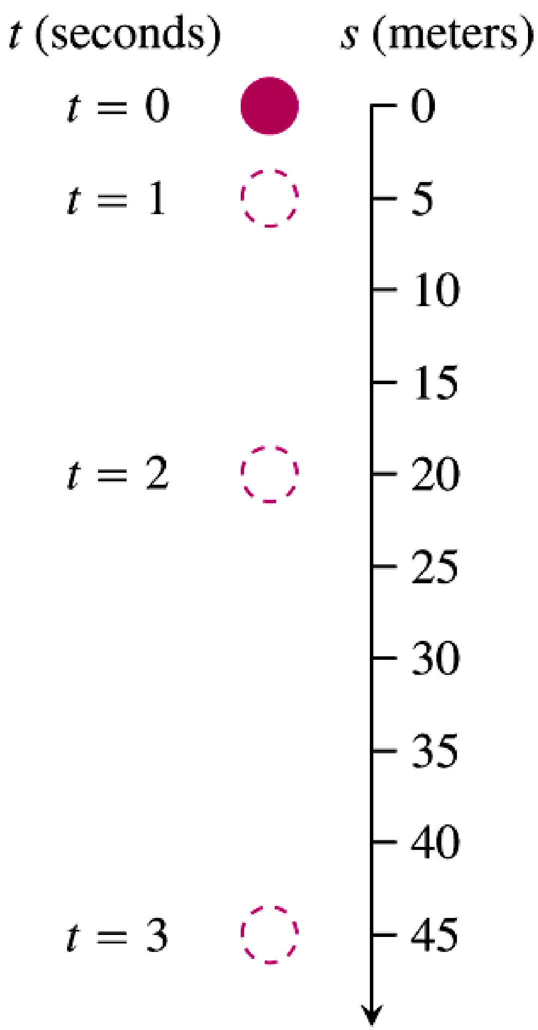


FIGURE 3.16 A ball bearing falling from rest (Example 4).

Modeling vertical motion

- A dynamite blast blows a heavy rock straight up with a launch velocity of 160 m/sec. It reaches a height of $s = 160t - 16t^2$ ft after t sec.
- (a) How high does the rock go?
- (b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?
- (c) What is the acceleration of the rock at any time t during its flight?
- (d) When does the rock hit the ground again?

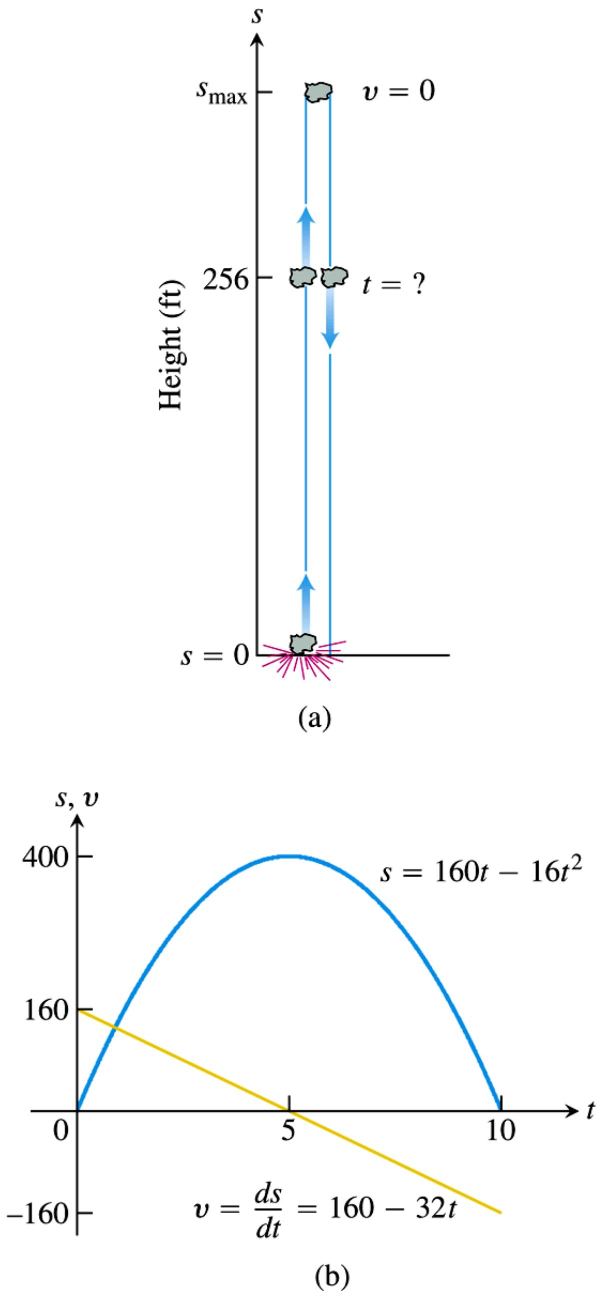


FIGURE 3.17 (a) The rock in Example 5. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

3.4

Derivatives of Trigonometric Functions

(2nd lecture of week 20/08/07-25/08/07)

Derivative of the sine function

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \cdots$$

Derivative of the cosine function

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \cdots$$

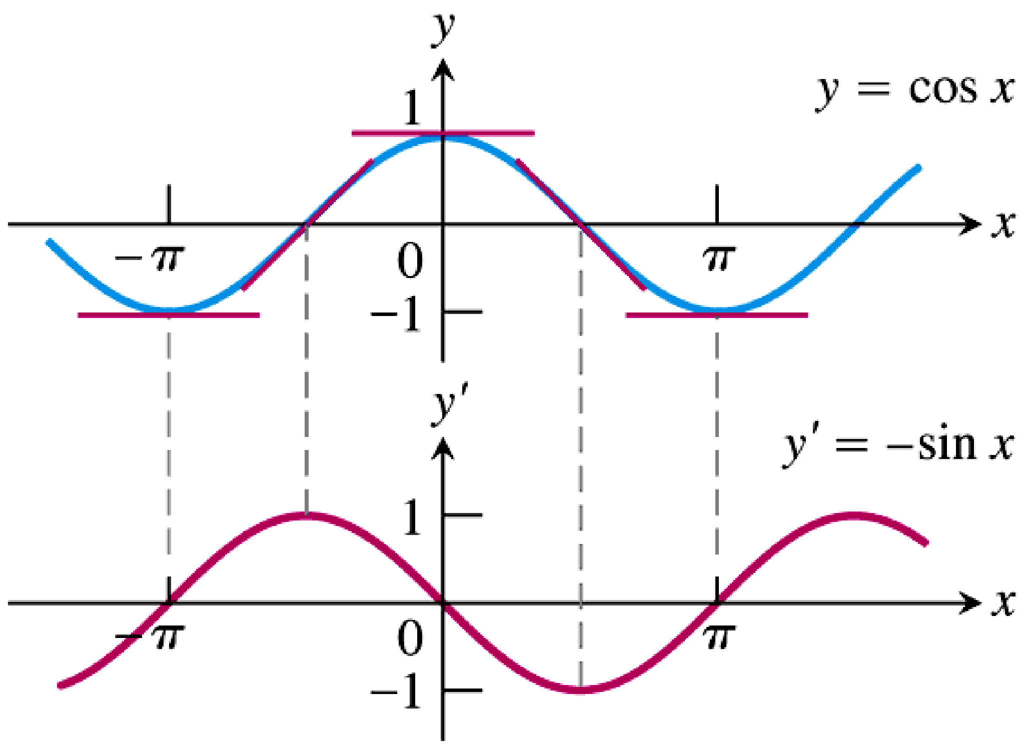


FIGURE 3.23 The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

Example 2

$$(a)\, y = 5x + \cos x$$

$$(b)\, y = \sin x \cos x$$

$$(c)\, y = \frac{\cos x}{1 - \sin x}$$

Derivative of the other basic trigonometric functions

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Example 5

- Find $d(\tan x)/dx$

Example 6

- Find y'' if $y = \sec x$

Example 7

- Finding a trigonometric limit

$$\lim_{x\to 0} \frac{\sqrt{2+\sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2+\sec 0}}{\cos(\pi - \tan 0)}$$

$$= \frac{\sqrt{2+1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

3.5

The Chain Rule and Parametric Equations

(2nd lecture of week 20/08/07-25/08/07)

Differentiating composite functions

- Example:
- $y = f(u) = \sin u$
- $u = g(x) = x^2 - 4$
- How to differentiate $F(x) = f \circ g = f[g(x)]$?
- Use chain rule

Derivative of a composite function

- Example 1 Relating derivatives
- $y = (3/2)x = (1/2)(3x)$
- $= g[u(x)]$
- $g(u) = u/2;\ u(x) = 3x$
- $dy/dx = 3/2;$
- $dg/du = ½;\ du/dx = 3;$
- $dy/dx = (dy/du)\cdot(du/dx)$ (Not an accident)

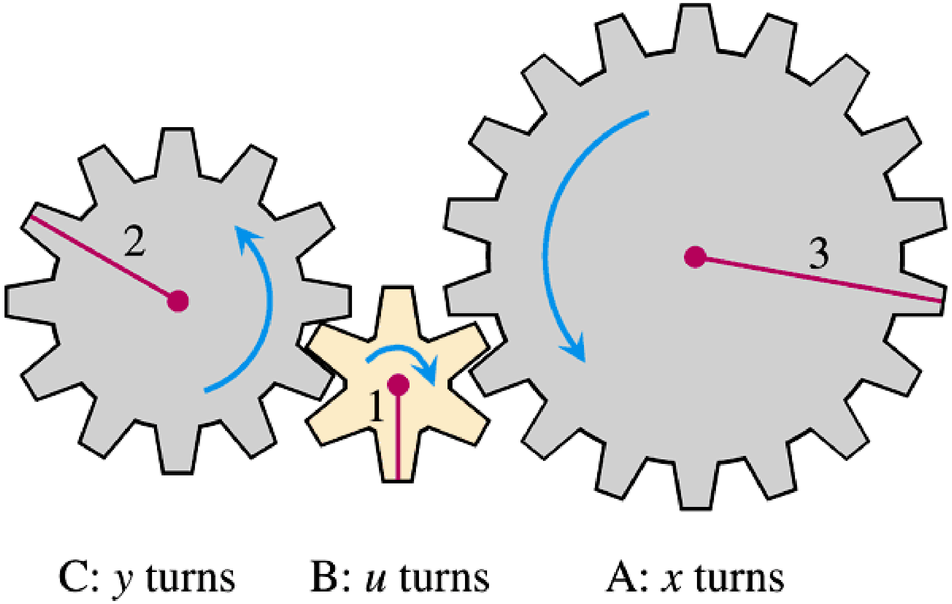


FIGURE 3.26 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ (C turns one-half turn for each B turn) and $u = 3x$ (B turns three times for A's one), so $y = 3x/2$. Thus, $dy/dx = 3/2 = (1/2)(3) = (dy/du)(du/dx)$.

Example 2

$$y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$$

$$y = u^2; u = 3x^2 + 1$$

$$\frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 6$$

$$= 2(3x^2 + 1) \cdot 6x = 36x^3 + 12x$$

c.f.

$$\frac{dy}{dx} = \frac{d}{dx}\left(9x^4 + 6x^2 + 1\right) = 36x^3 + 12x$$

THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

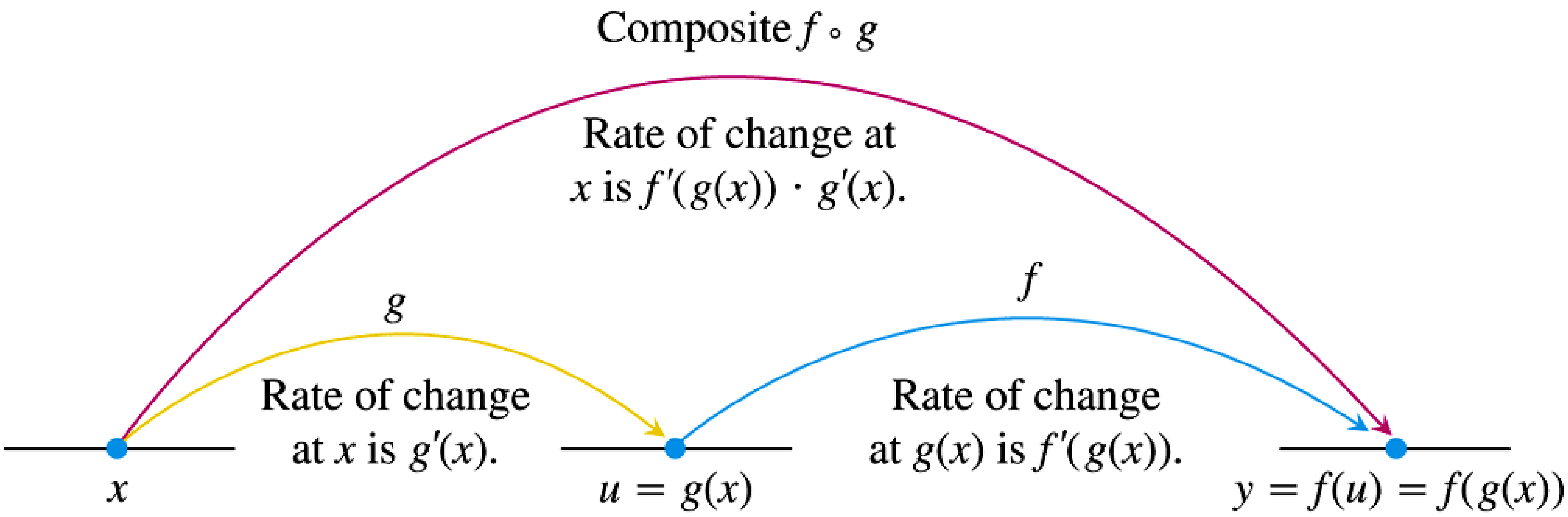


FIGURE 3.27 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x.

Example 3

- Applying the chain rule
- $x(t) = \cos(t^2 + 1)$. Find dx/dt.
- Solution:
- $x(u) = \cos(u)$; $u(t) = t^2 + 1$;
- $dx/dt = (dx/du) \cdot (du/dt) = \ldots$

Alternative form of chain rule

- If $y = f[g(x)]$, then
- $dy/dx = f'[g(x)] \cdot g'(x)$

- Think of f as 'outside function', g as 'inside-function', then
- dy/dx = differentiate the outside function and evaluate it at the inside function let alone; then multiply by the derivative of the inside function.

Example 4

- Differentiating from the outside In

$$\frac{d}{dx}\sin\underbrace{\left(x^2+x\right)}_{\substack{\text{inside}\\\text{left alone}}}=\cos\underbrace{\left(x^2+x\right)}_{\substack{\text{inside}\\\text{left alone}}}\cdot\underbrace{(2x+1)}_{\substack{\text{derivative of}\\\text{the inside}}}$$

Example 5

- A three-link ‘chain’
- Find the derivative of $g(t) = \tan(5 - \sin 2t)$

Example 6

- Applying the power chain rule

$$(a)\ \frac{d}{dx}(5x^3 - x^4)^7$$

$$(b)\ \frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$$

Example 7

- (a) Find the slope of tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$
- (b) Show that the slope of every line tangent to the curve $y = 1/(1-2x)^3$ is positive

Parametric equations

- A way of expressing both the coordinates of a point on a curve, (x,y) as a function of a third variable, t.
- The path or locus traced by a point particle on a curve is then well described by a set of two equations:
- $x = f(t)$, $y = g(t)$

DEFINITION Parametric Curve

If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

The variable t is a parameter for the curve

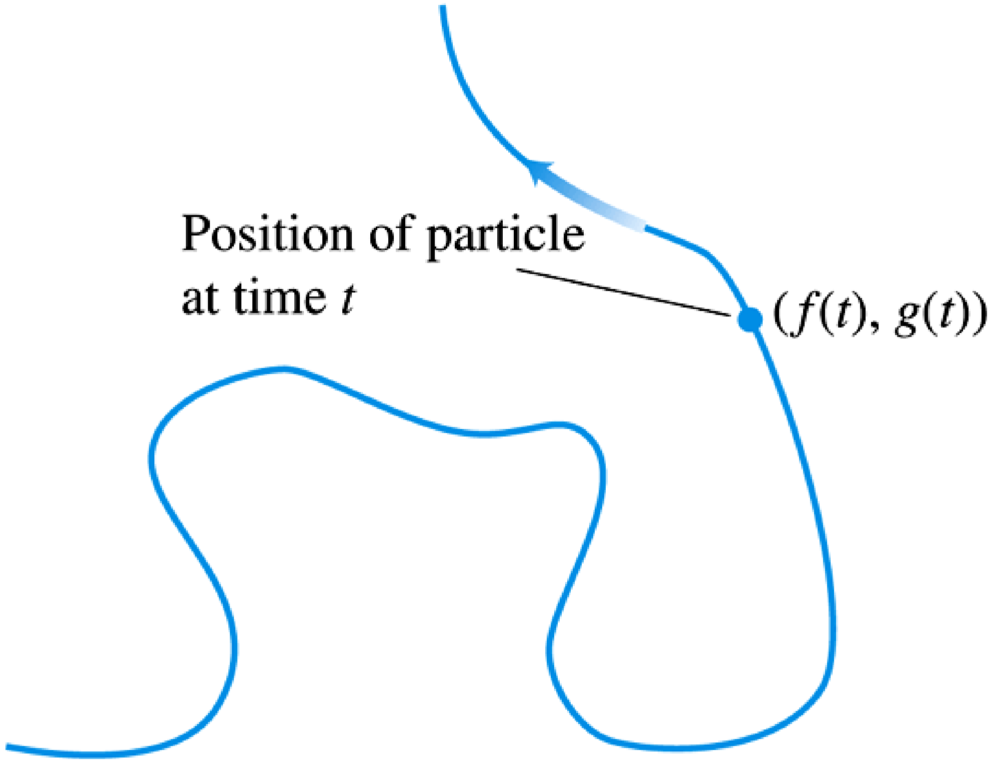


FIGURE 3.29 The path traced by a particle moving in the xy-plane is not always the graph of a function of x or a function of y.

Example 9

- Moving counterclockwise on a circle
- Graph the parametric curves
- $x = \cos t$, $y = \sin t$,

  $0 \leqslant t \leqslant 2\pi$

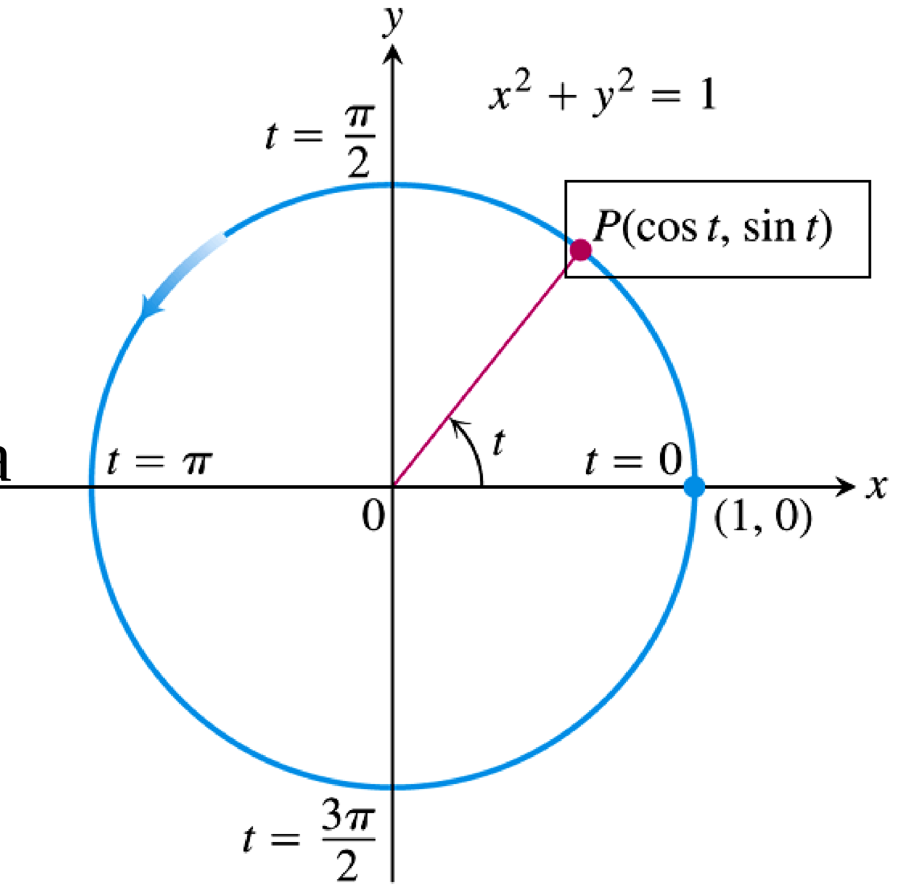


FIGURE 3.30 The equations $x = \cos t$ and $y = \sin t$ describe motion on the circle $x^2 + y^2 = 1$. The arrow shows the direction of increasing t (Example 9).

Example 10

- Moving along a parabola
- $x = \sqrt{t}, y = t, \ 0 \leqslant t$
- Determine the relation between *x* and *y* by eliminating *t*.
- $y = t = (t)^2 = x^2$
- The path traced out by *P* (the locus) is only half the parabola, $x \geqslant 0$

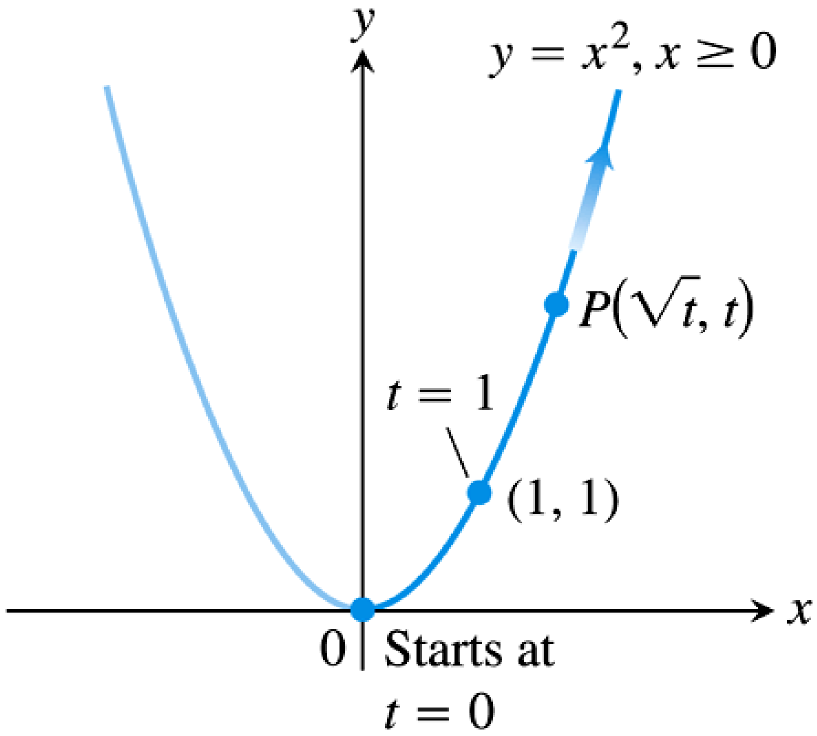


FIGURE 3.31 The equations $x = \sqrt{t}$ and $y = t$ and the interval $t \geq 0$ describe the motion of a particle that traces the right-hand half of the parabola $y = x^2$ (Example 10).

Slopes of parametrized curves

- A parametrized curved $x = f(t)$, $y = g(t)$ is differentiable at t if f and g are differentiable at t.
- At a point on a differentiable parametrised curve where y is also a differentiable function of x, i.e. $y = y(x) = y[x(\mathrm{t})]$,
- chain rule relates dx/dt, dy/dt, dy/dx via

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \qquad (2)$$

Example 12

- Differentiating with a parameter
- If $x = 2t + 3$ and $y = t^2 - 1$, find the value of dy/dx at $t = 6$.

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \tag{3}$$

(3) is just the parametric formula (2) by

$$y \to dy/dx$$

Example 14

- Finding d^2y/ dx^2 for a parametrised curve
- Find d^2y/ dx^2 as a function of t if $x = t - t^2$,
  $y = t - t^3$.

3.6

Implicit Differentiation

(3rd lecture of week 20/08/07-25/08/07)

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

Example 1: Differentiating implicitly

- Find dy/dx if $y^2 = x$

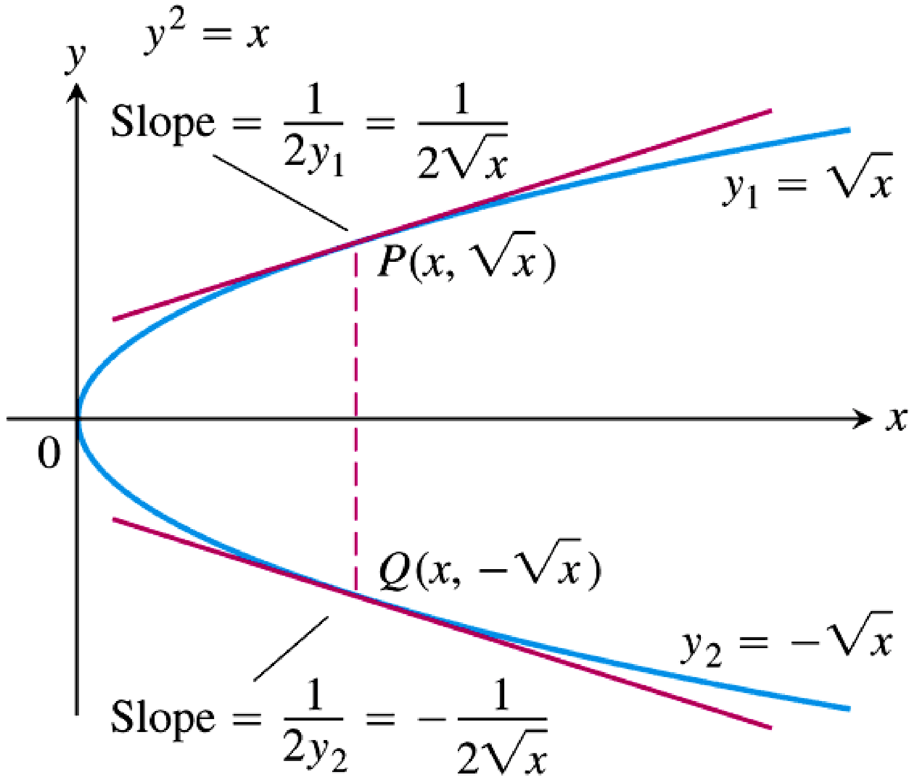


FIGURE 3.37 The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of x on the interval $x \geq 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for y.

Example 2

- Slope of a circle at a point
- Find the slope of circle $x^2 + y^2 = 25$ at (3, -4)

Example 3

- Differentiating implicitly
- Find *dy/dx* if

  $y^2 = x^2 + \sin xy$

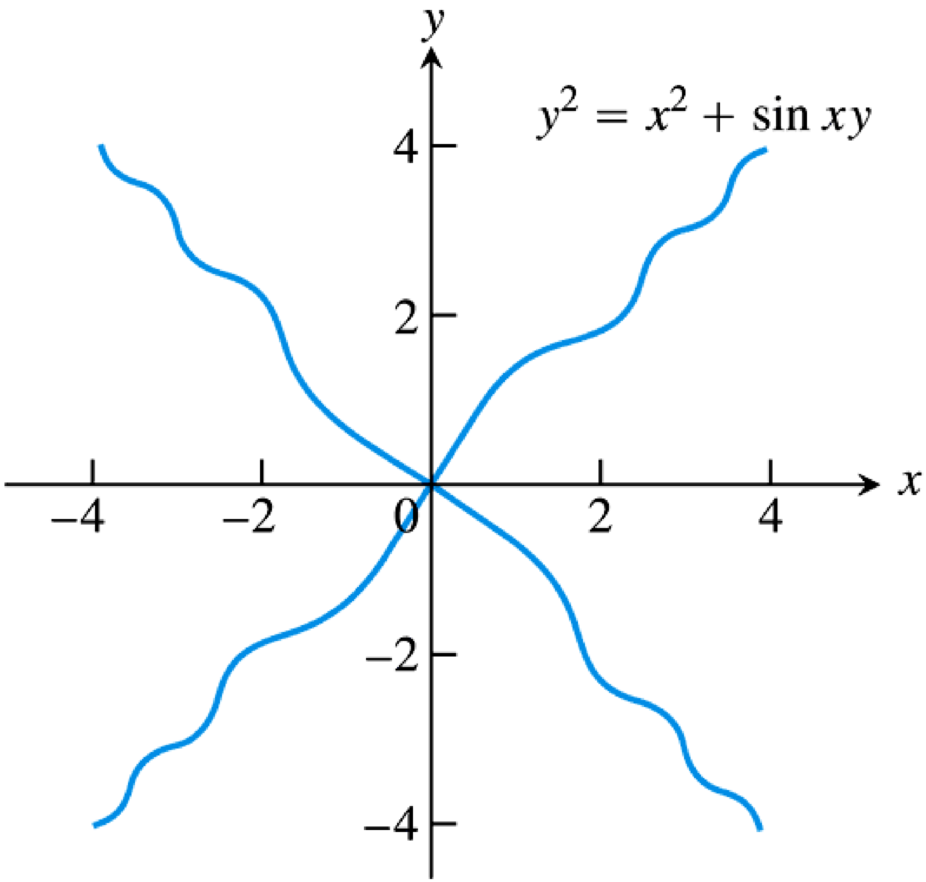


FIGURE 3.39 The graph of $y^2 = x^2 + \sin xy$ in Example 3. The example shows how to find slopes on this implicitly defined curve.

Lenses, tangents, and normal lines

If slop of tangent is m_t, the slope of normal, m_n, is given by the relation

$m_n m_t = -1$, or

$m_n = -1/m_t$

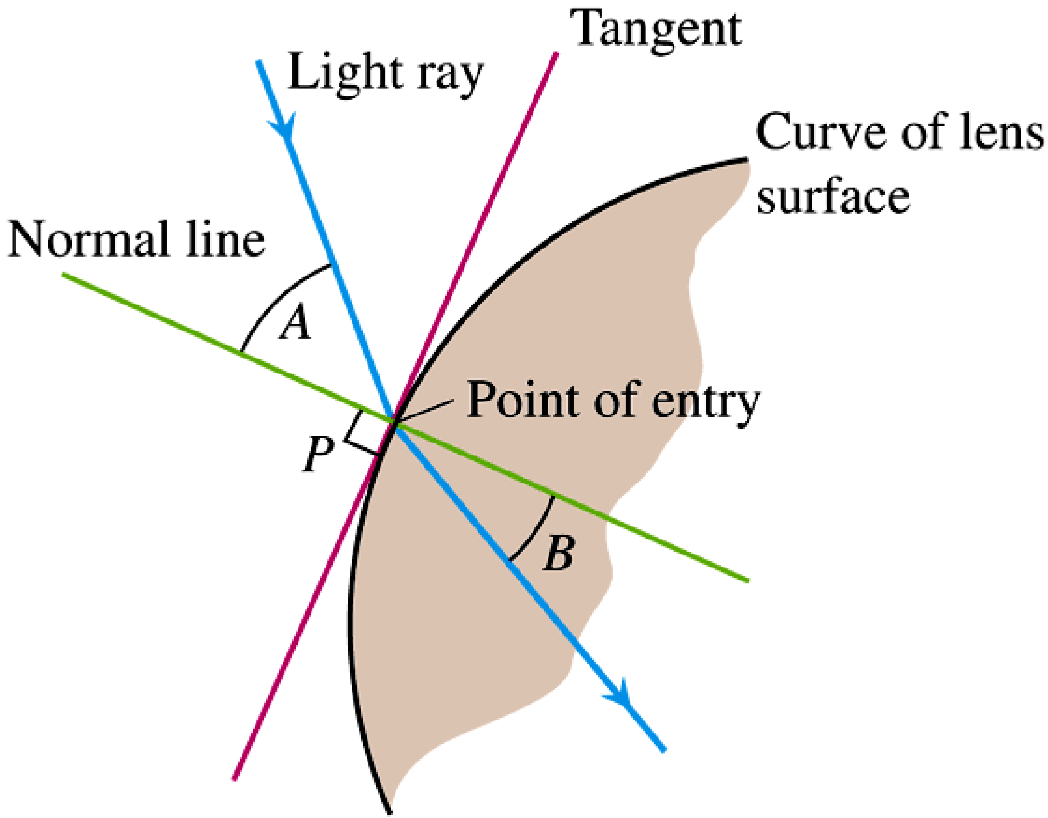


FIGURE 3.40 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

Example 4

- Tangent and normal to the folium of Descartes
- Show that the point (2,4) lies on the curve $x^2 + y^3 - 9xy = 0$. The find the tangent and normal to the curve there.

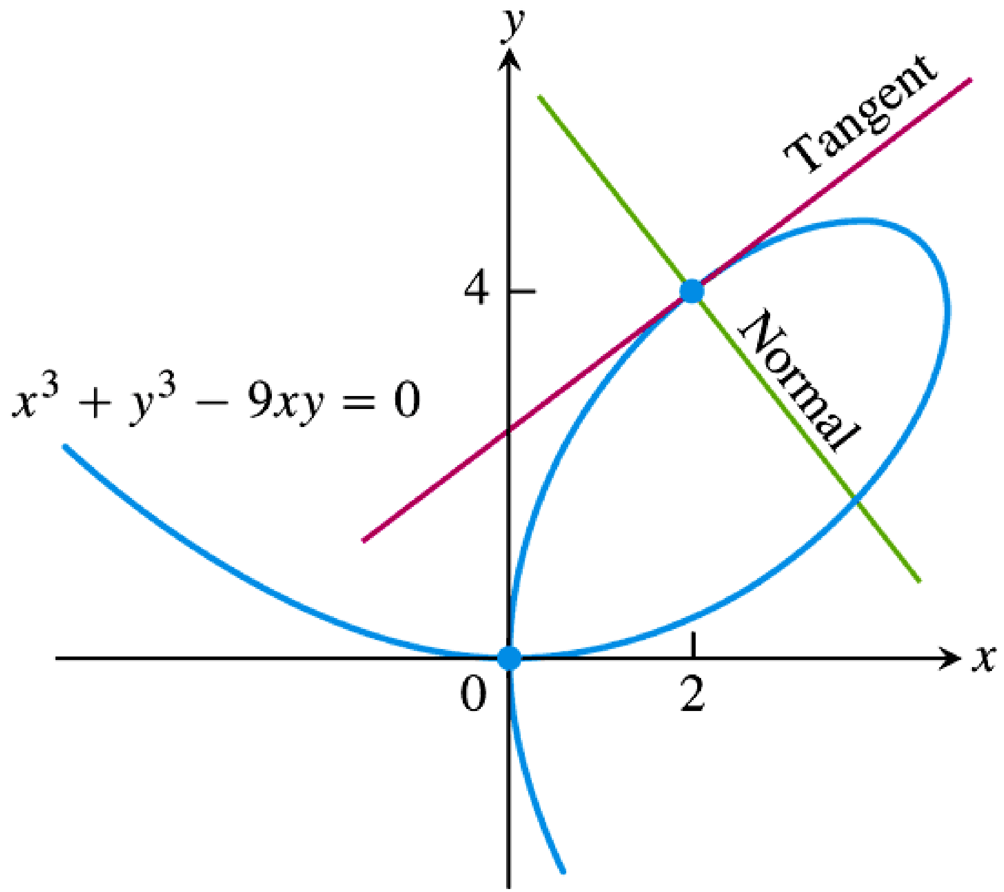


FIGURE 3.41 Example 4 shows how to find equations for the tangent and normal to the folium of Descartes at (2, 4).

Derivative of higher order

- Example 5
- Finding a second derivative implicitly
- Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Rational powers of differentiable functions

THEOREM 4 Power Rule for Rational Powers

If p/q is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$, and

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

Theorem 4 is proved based on $d/dx(x^n) = nx^{n-1}$ (where n is an integer) using implicit differentiation

- Theorem 4 provide a extension of the power chain rule to rational power:

$$\frac{d}{dx}u^{p/q} = \frac{p}{q}u^{(p/q)-1}\frac{du}{dx}$$

- $u \neq 0$ if $(p/q) < 1$, (p/q) rational number, u a differential function of x

Example 6

- Using the rational power rule
- (a) $d/dx\ (x^{1/2}) = 1/2x^{-1/2}$ for $x > 0$
- (b) $d/dx\ (x^{2/3}) = 2/3\ x^{-1/3}$ for $x \neq 0$
- (c) $d/dx\ (x^{-4/3}) = -4/3\ x^{-7/3}$ for $x \neq 0$

Proof of Theorem 4

- Let p and q be integers with $q > 0$ and

$$y = x^{p/q} \equiv y^q = x^p$$

- Explicitly differentiating both sides with respect to x…

Example 7

- Using the rational power and chain rules
- (a) Differentiate $(1-x^2)^{1/4}$
- (b) Differentiate $(\cos x)^{-1/5}$

Chapter 4

Applications of Derivatives

4.1

Extreme Values of Functions

(3rd lecture of week 20/08/07-25/08/07)

DEFINITIONS **Absolute Maximum, Absolute Minimum**

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

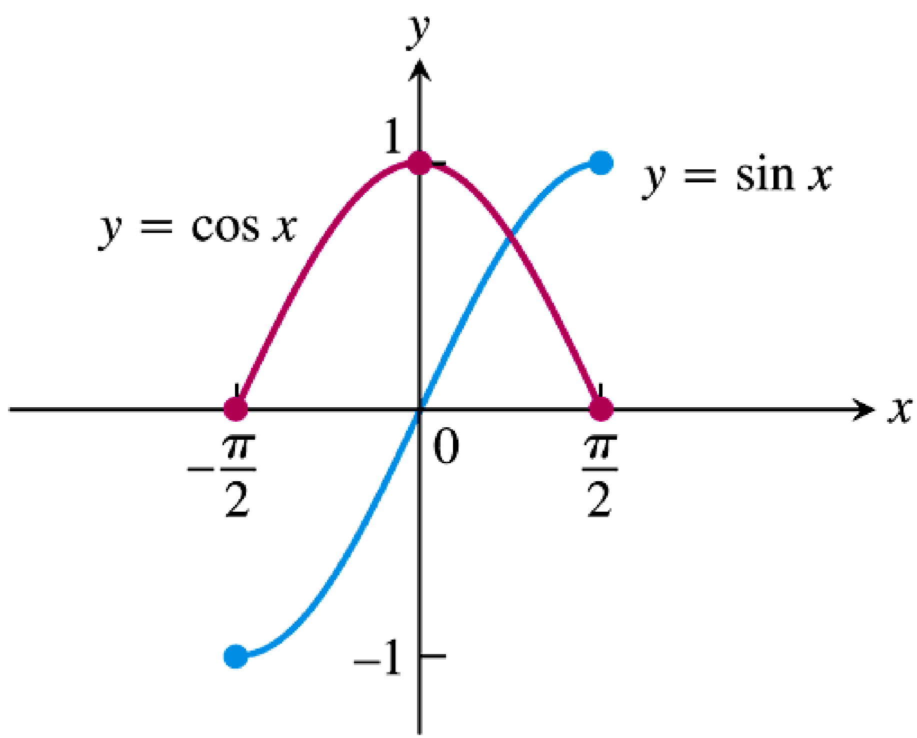


FIGURE 4.1 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

Example 1

- Exploring absolute extrema
- The absolute extrema of the following functions on their domains can be seen in Figure 4.2

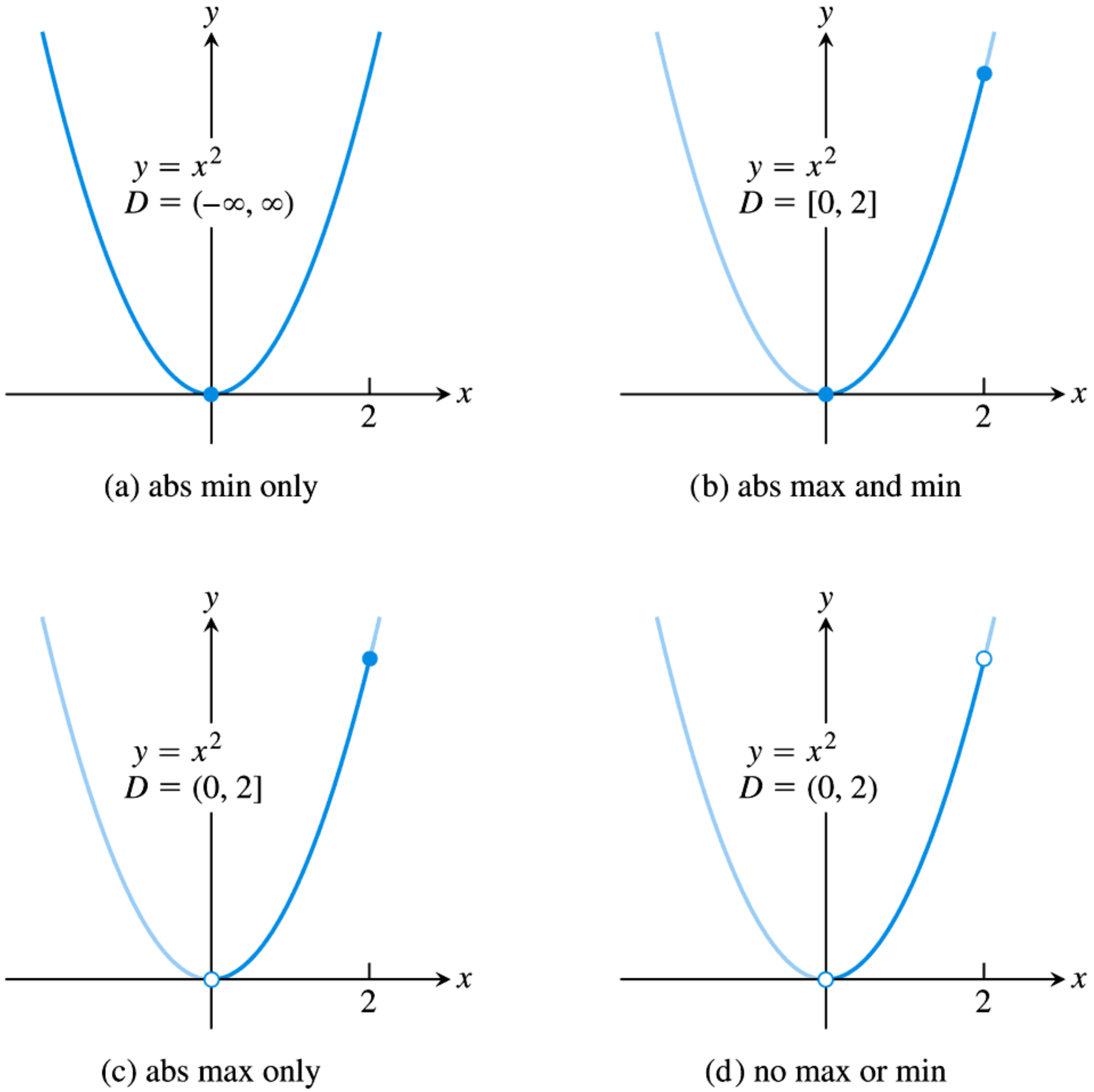


FIGURE 4.2 Graphs for Example 1.

THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$ (Figure 4.3).

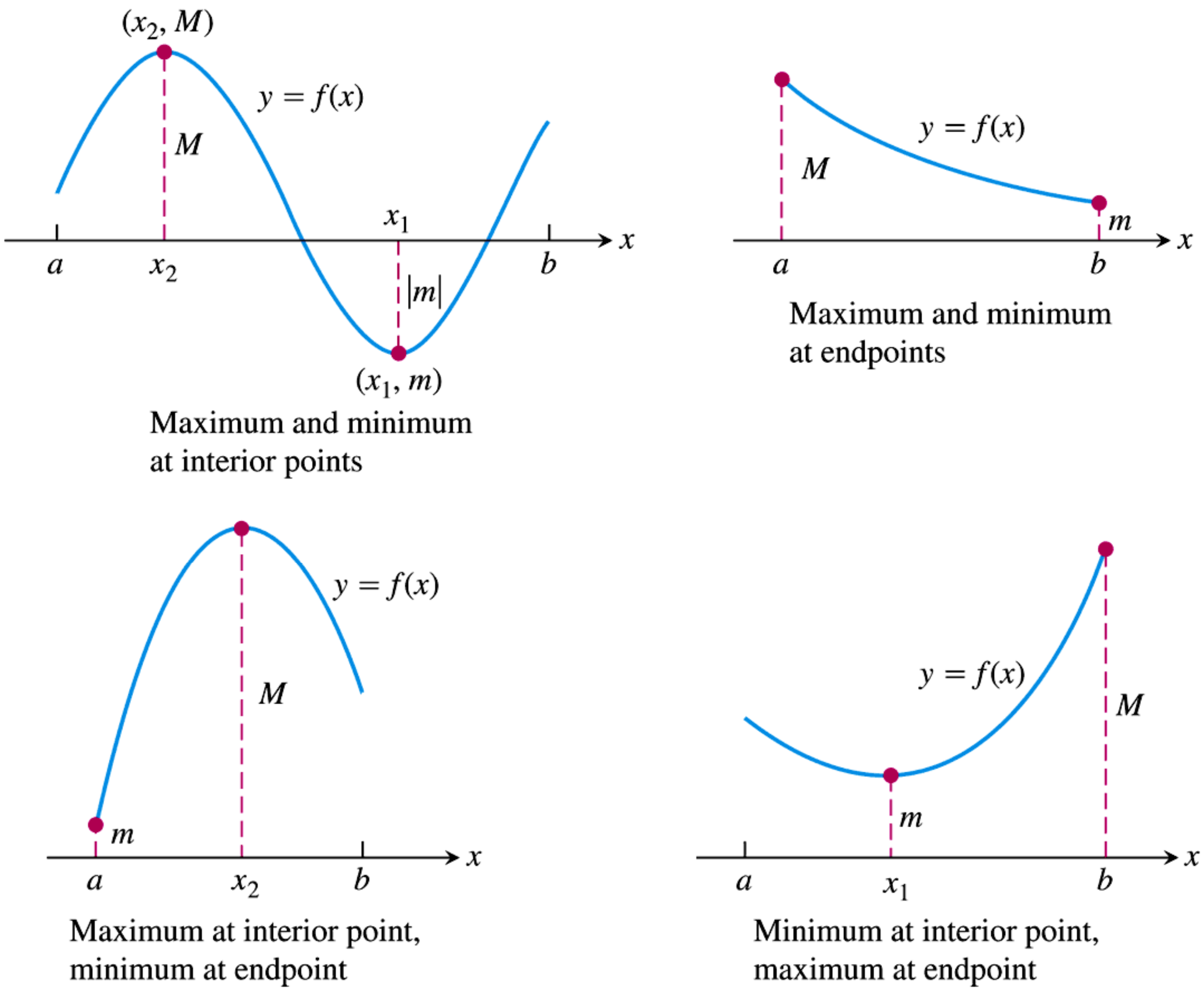


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

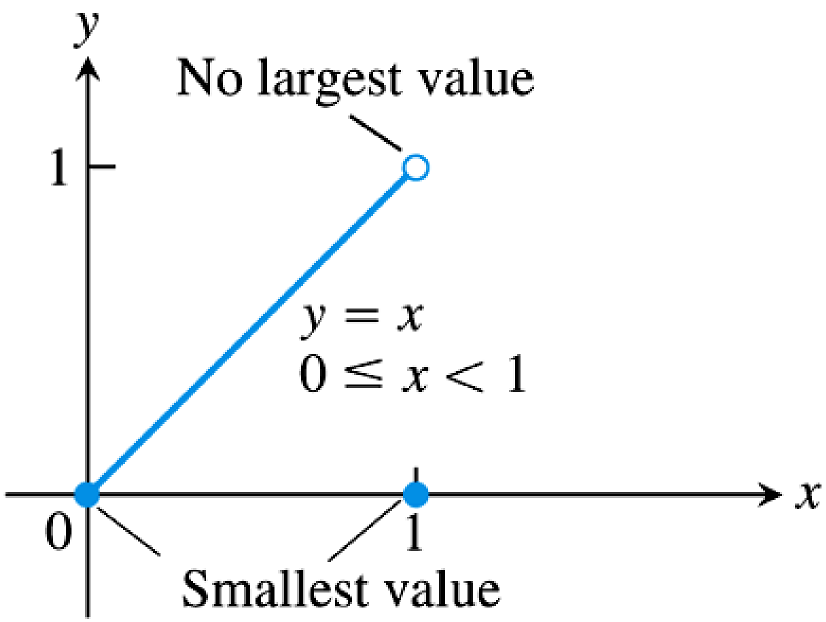


FIGURE 4.4 Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of [0, 1] except $x = 1$, yet its graph over [0, 1] does not have a highest point.

Local (relative) extreme values

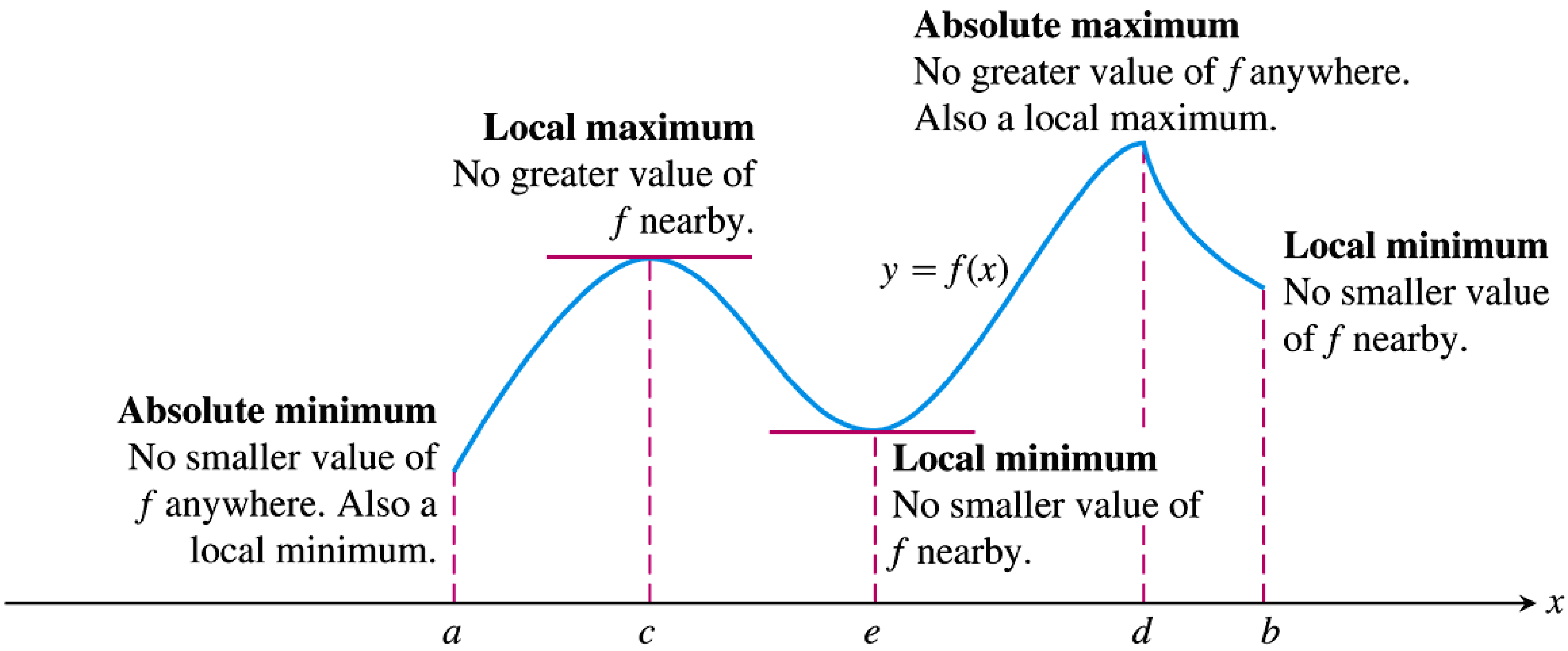


FIGURE 4.5 How to classify maxima and minima.

DEFINITIONS **Local Maximum, Local Minimum**

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

Finding Extrema

THEOREM 2 The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

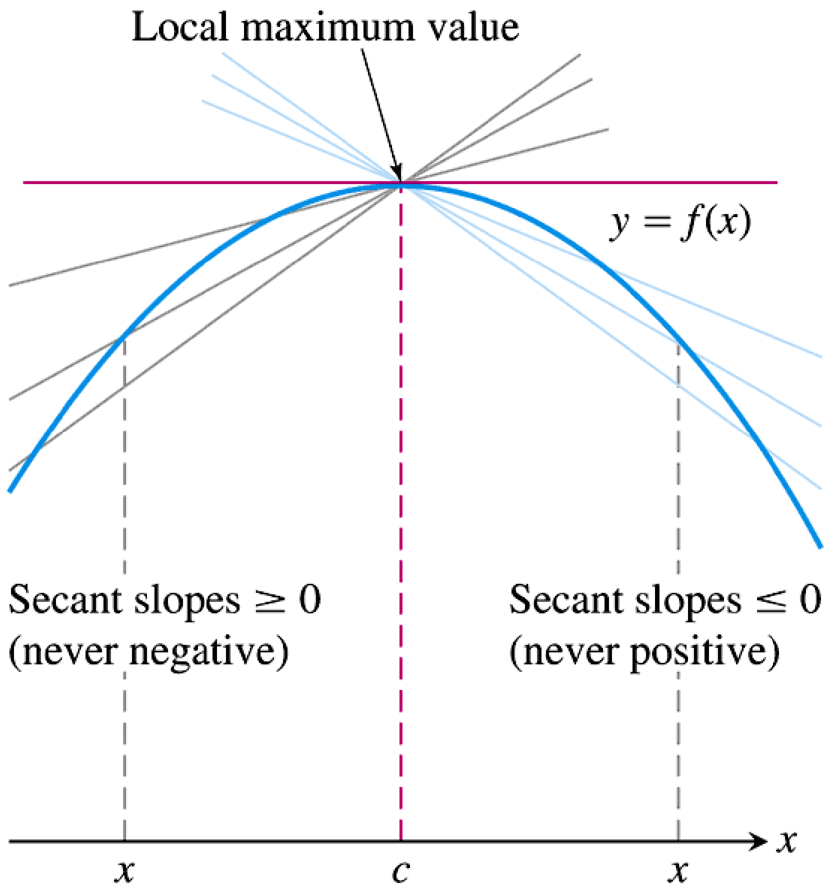


FIGURE 4.6 A curve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

DEFINITION Critical Point

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

How to find the absolute extrema of a continuous function *f* on a finite closed interval

1. Evaluate *f* at all critical point and endpoints
2. Take the largest and smallest of these values.

Example 2: Finding absolute extrema

- Find the absolute maximum and minimum of $f(x) = x^2$ on [-2,1].

Example 3:
Absolute extrema at endpoints

- Find the absolute extrema values of

  $g(t) = 8t - t^4$ on

  [-2,1].

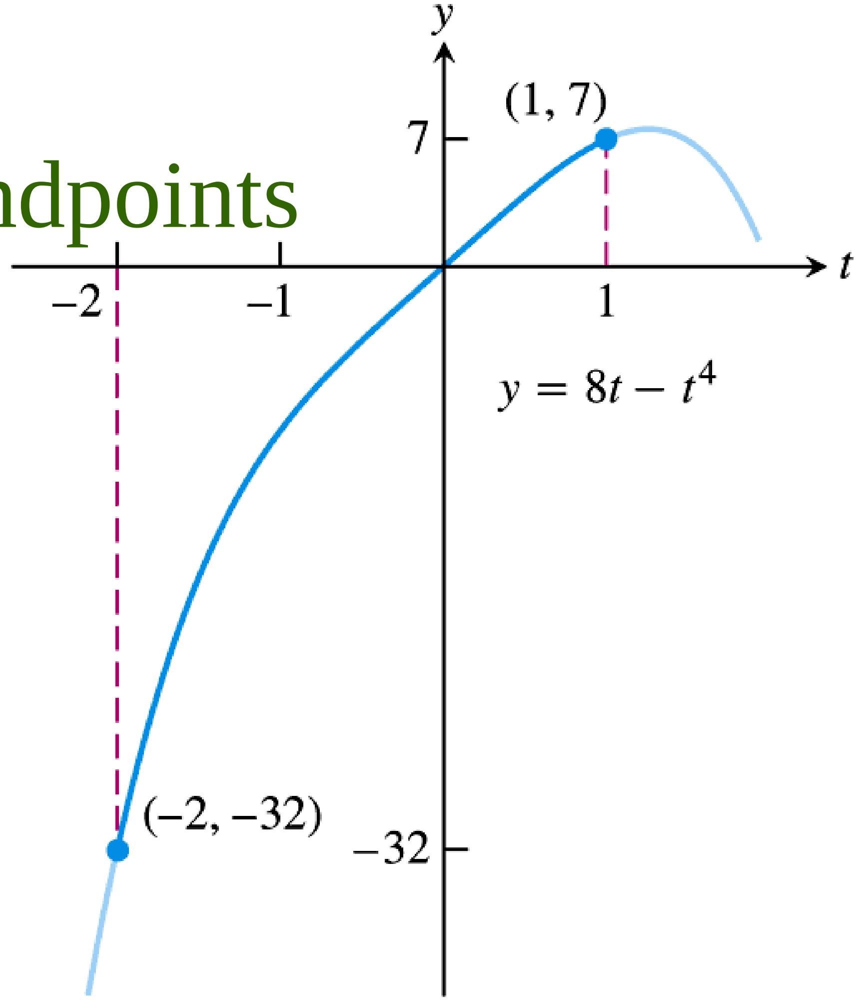


FIGURE 4.7 The extreme values of $g(t) = 8t - t^4$ on $[-2, 1]$ (Example 3).

Example 4: Finding absolute extrema on a closed interval

- Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval [-2,3].

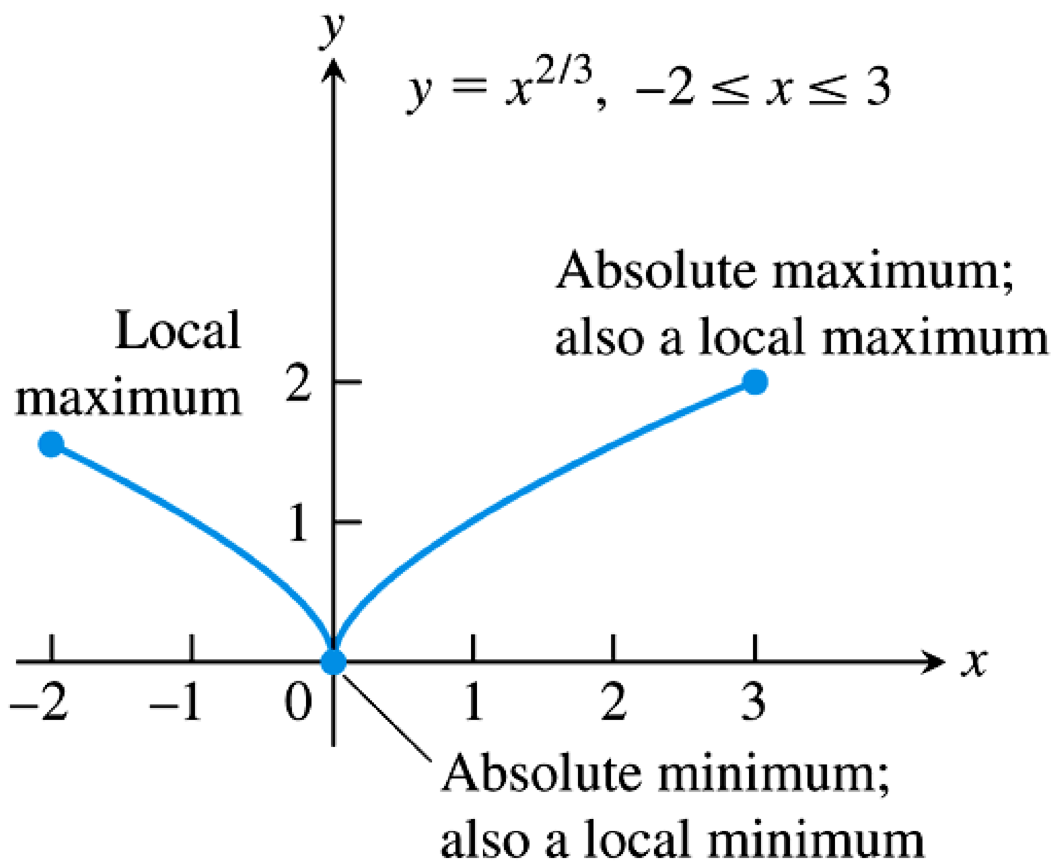
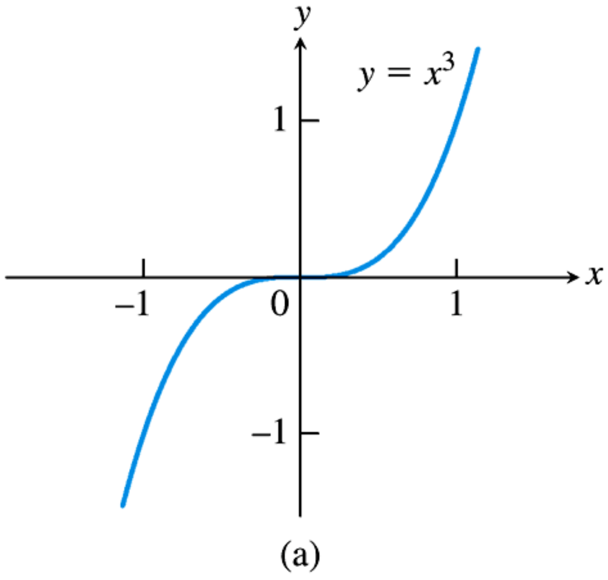
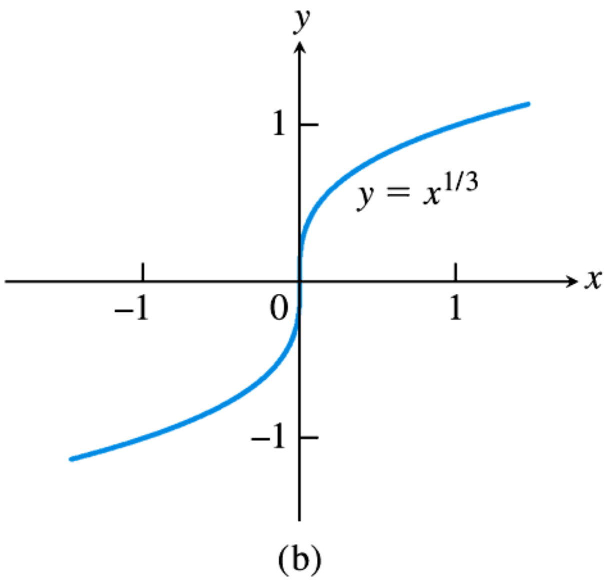


FIGURE 4.8 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).





- Not every critical point or endpoints signals the presence of an extreme value.

FIGURE 4.9 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

4.2

The Mean Value Theorem

(1st lecture of week 27/08/07-01/09/07)

THEOREM 3 Rolle's Theorem

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If

$$f(a) = f(b),$$

then there is at least one number c in (a, b) at which

$$f'(c) = 0.$$

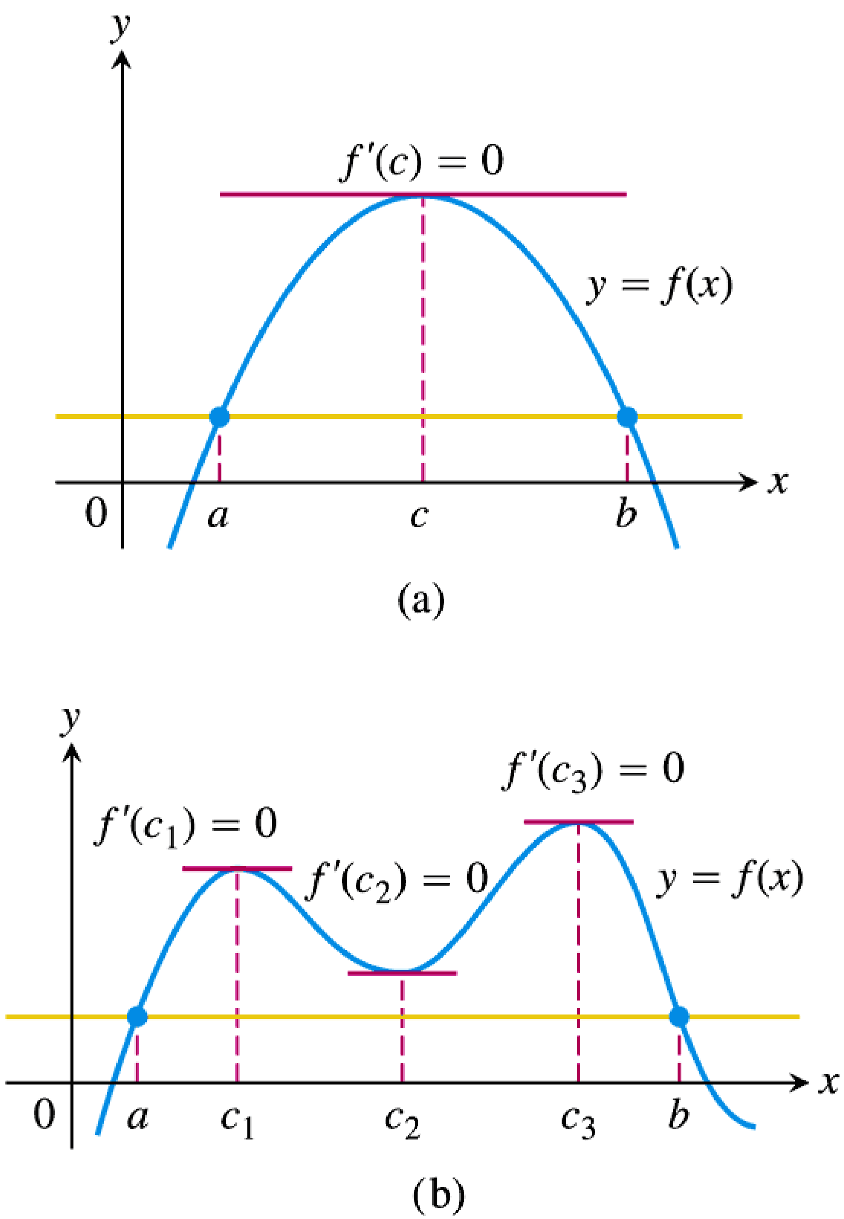
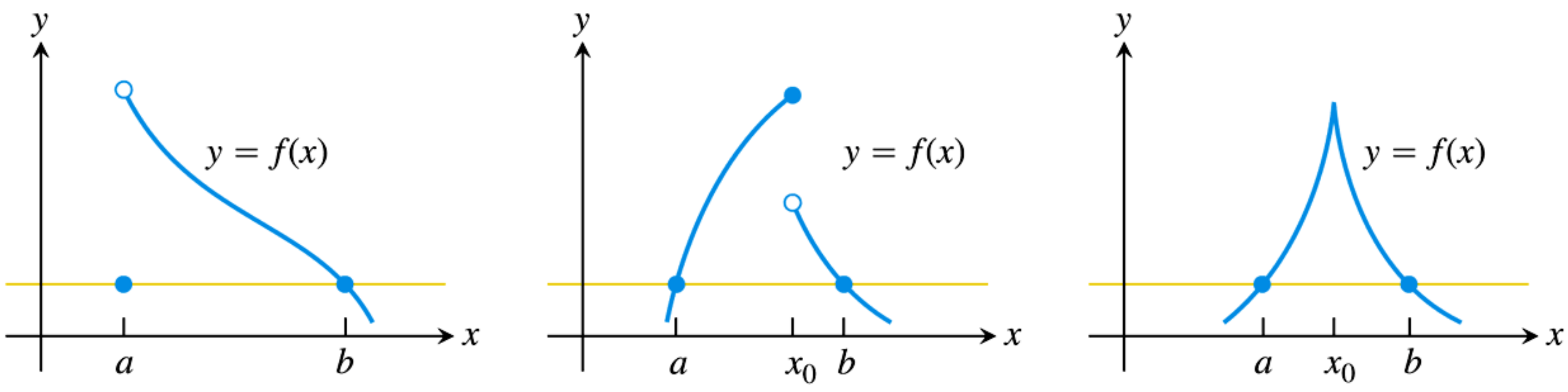


FIGURE 4.10 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

FIGURE 4.11 There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.

Example 1

- Horizontal tangents of a cubit polynomial

$$f(x) = \frac{x^3}{3} - 3x$$

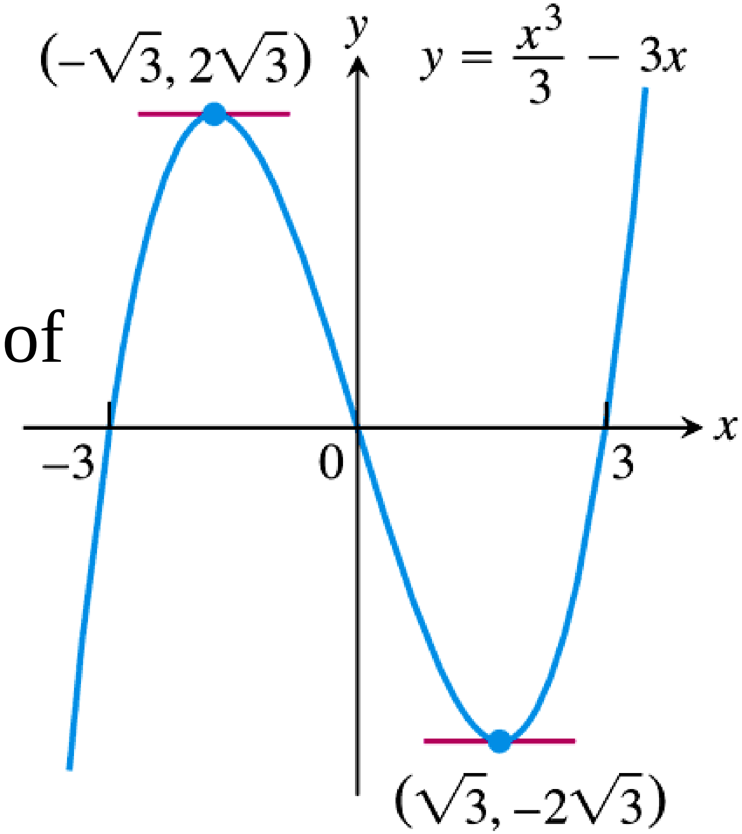


FIGURE 4.12 As predicted by Rolle's Theorem, this curve has horizontal tangents between the points where it crosses the x-axis (Example 1).

Example 2 Solution of an equation $f(x)=0$

- Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

Solution

1. Apply Intermediate value theorem to show that there exist at least one root
2. Apply Rolle's theotem to prove the uniqueness of the root.

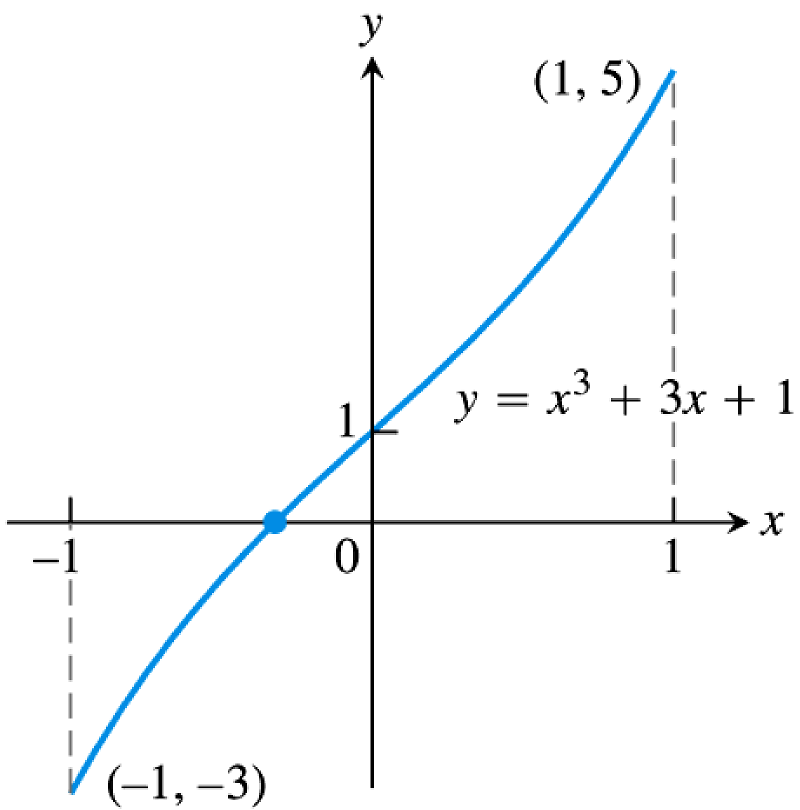


FIGURE 4.13 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the x-axis between -1 and 0 (Example 2).

The mean value theorem

THEOREM 4 The Mean Value Theorem

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \qquad (1)$$

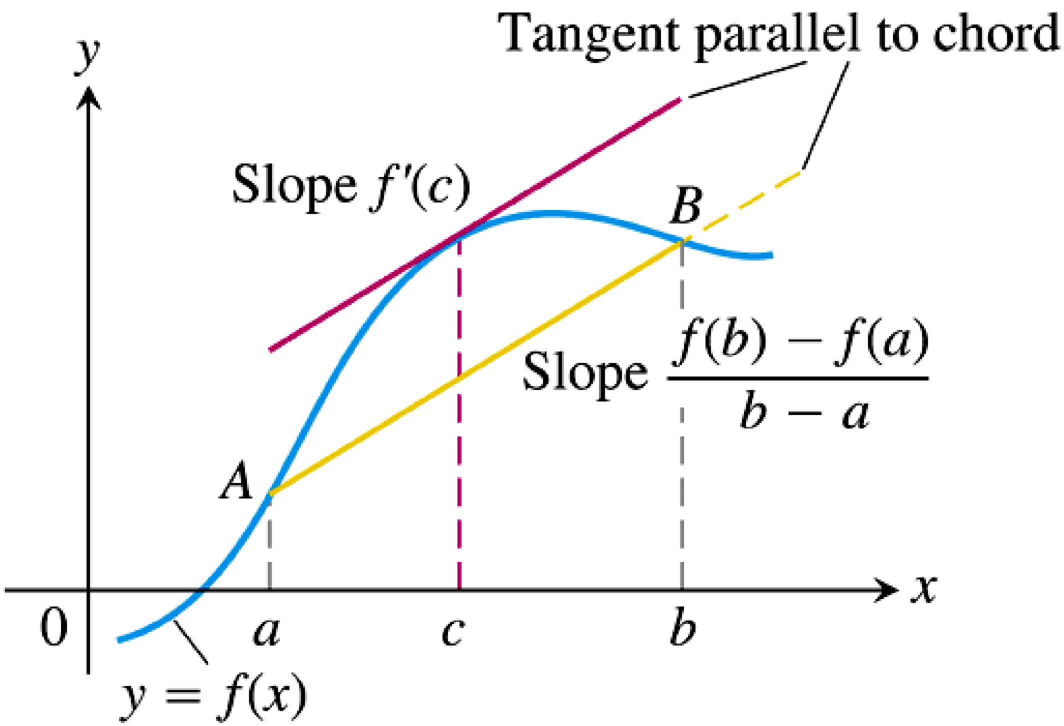


FIGURE 4.14 Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

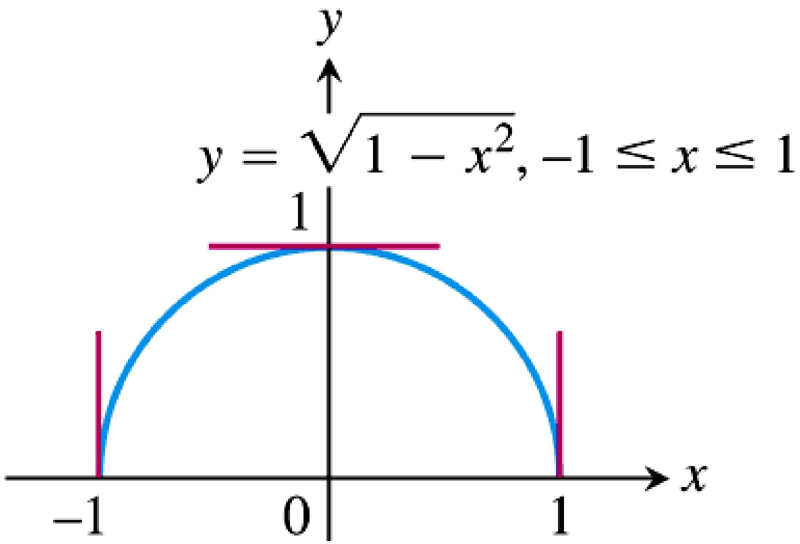


FIGURE 4.17 The function $f(x) = \sqrt{1 - x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though f is not differentiable at -1 and 1.

Example 3

- The function $f(x) = x^2$ is continuous for $0 \leqslant x \leqslant 2$ and differentiable for $0 < x < 2$.

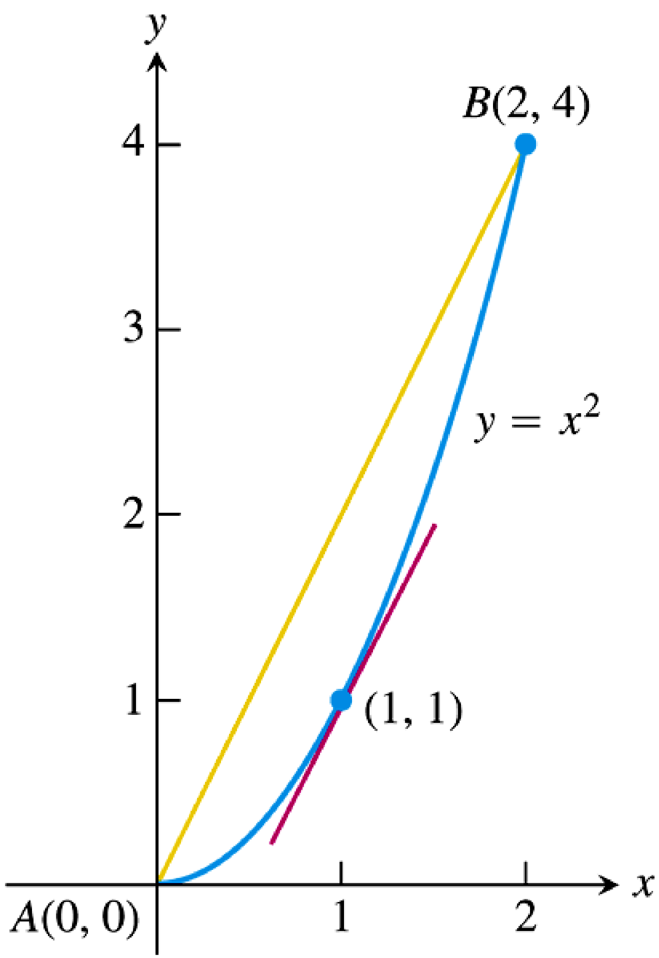


FIGURE 4.18 As we find in Example 3, $c = 1$ is where the tangent is parallel to the chord.

Mathematical consequences

COROLLARY 1 Functions with Zero Derivatives Are Constant

If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

COROLLARY 2 Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant on (a, b).

Corollary 1 can be proven using the Mean Value Theorem

- Say $x_1, x_2 \in (a,b)$ such that $x_1 < x_2$
- By the MVT on $[x_1,x_2]$ there exist some point c between x_1 and x_2 such that $f'(c) = (f(x_2) - f(x_1))/(x_2 - x_1)$
- Since $f'(c) = 0$ throughout (a,b),

  $f(x_2) - f(x_1) = 0$, hence $f(x_2) = f(x_1)$ for $x_1, x_2 \in (a,b)$.
- This is equivalent to $f(x)$ = a constant for $x \in (a,b)$.

Proof of Corollary 2

- At each point $x \in (a,b)$ the derivative of the difference between function $h = f - g$ is

  $h'(x) = f\,'(x) - g'(x) = 0$
- Thus $h(x) = C$ on (a,b) by Corollary 1. That is $f\,(x) - g(x) = C$ on (a,b), so $f\,(x) = C + g(x)$.

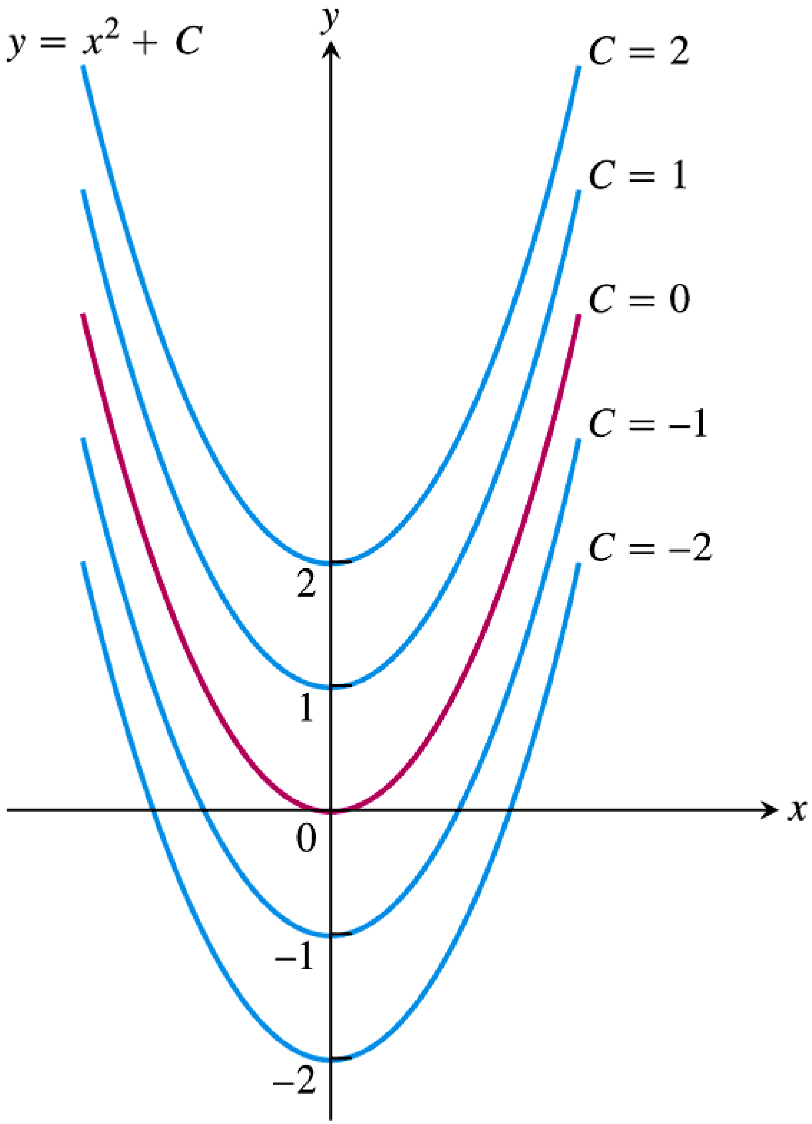


FIGURE 4.20 From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives on an interval can differ only by a vertical shift there. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for selected values of C.

Example 5

- Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point (0,2).

4.3

Monotonic Functions and
The First Derivative Test

(1st lecture of week 27/08/07-01/09/07)

Increasing functions and decreasing functions

DEFINITIONS **Increasing, Decreasing Function**

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

A function that is increasing or decreasing on I is called **monotonic** on I.

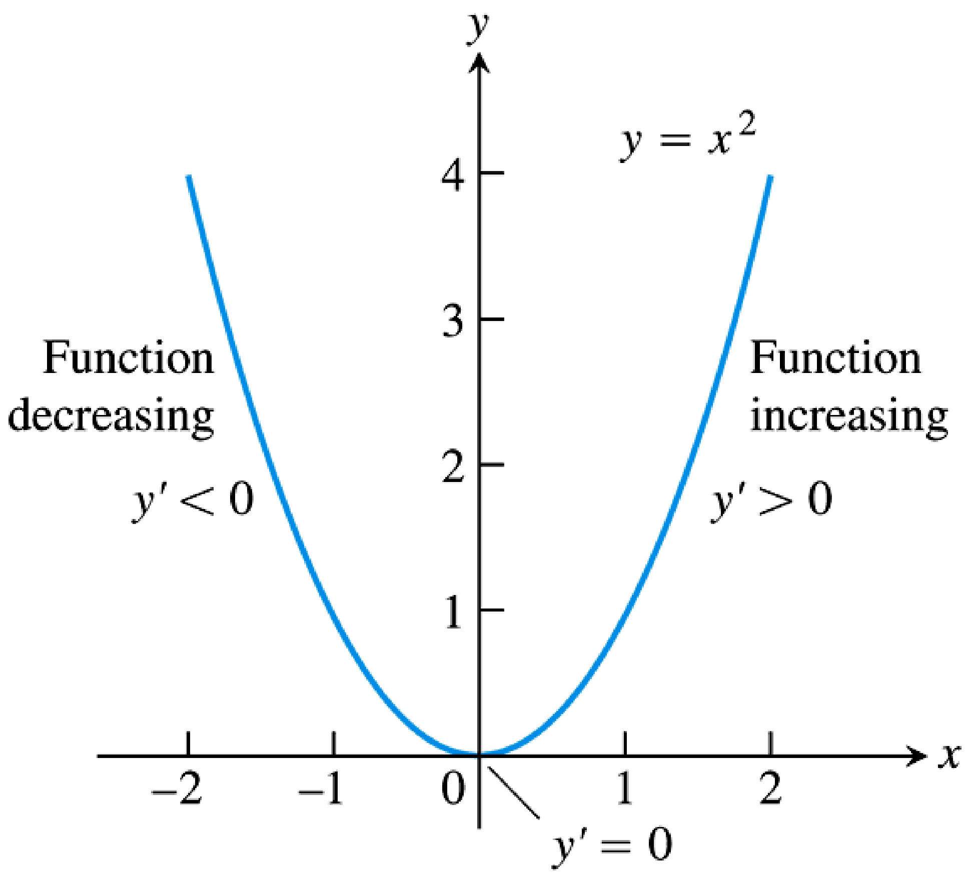


FIGURE 4.21 The function $f(x) = x^2$ is monotonic on the intervals $(-\infty, 0]$ and $[0, \infty)$, but it is not monotonic on $(-\infty, \infty)$.

COROLLARY 3 First Derivative Test for Monotonic Functions

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Mean value theorem is used to prove Corollary 3

Example 1

- Using the first derivative test for monotonic functions $f(x) = x^3 - 12x - 5$
- Find the critical point of $f(x) = x^3 - 12x - 5$

  and identify the intervals on which *f* is increasing and decreasing.

Solution

$$f'(x) = 3(x+2)(x-2)$$

$$f' + \quad \text{for } -\infty < x < -2$$

$$f' -12 \quad \text{for } -2 < x < 2$$

$$f' + \quad \text{for } 2 < x < \infty$$

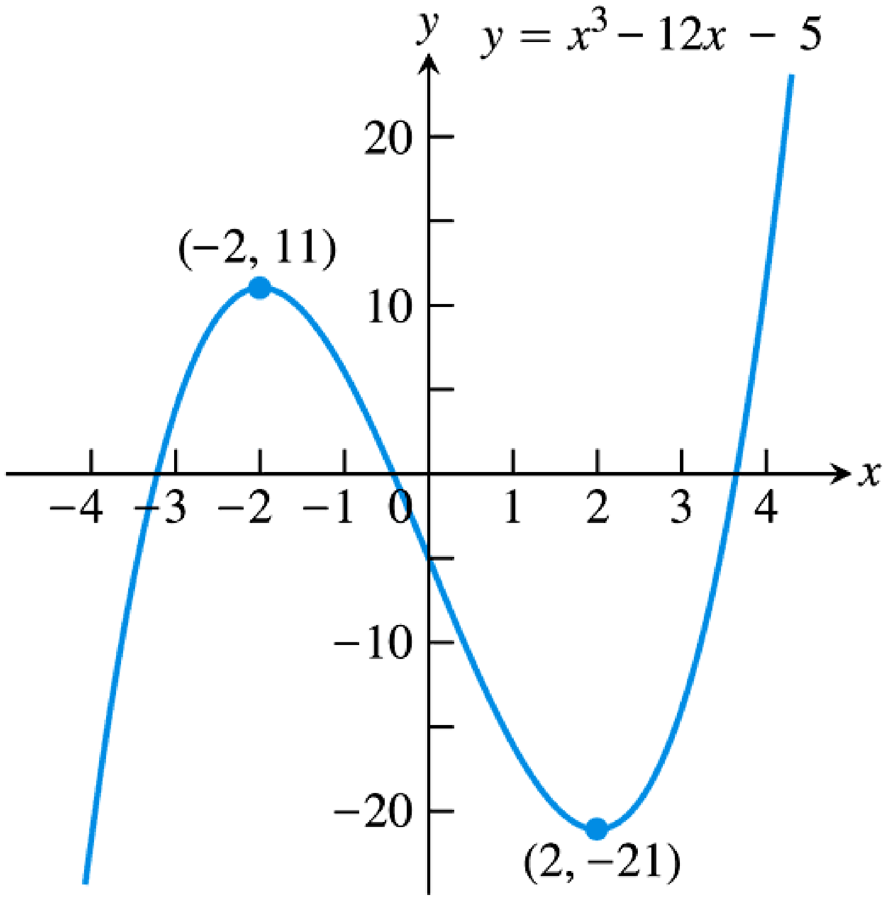


FIGURE 4.22 The function $f(x) = x^3 - 12x - 5$ is monotonic on three separate intervals (Example 1).

First derivative test for local extrema

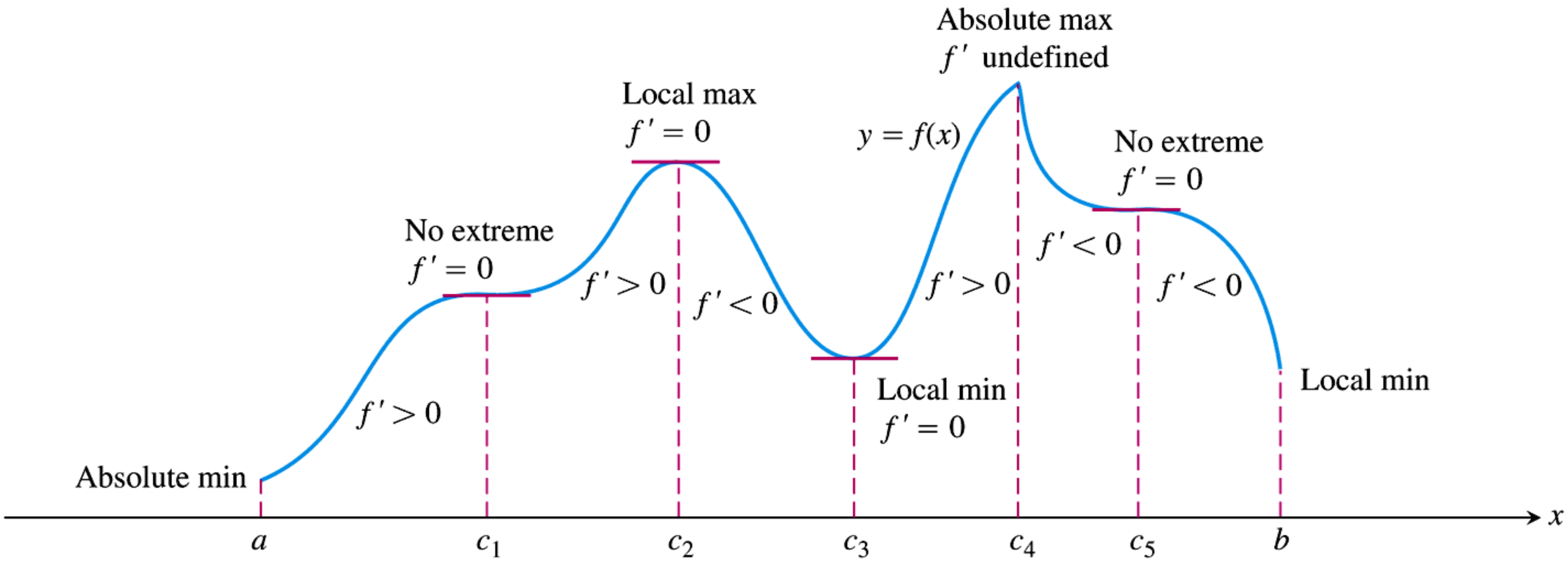


FIGURE 4.23 A function's first derivative tells how the graph rises and falls.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

Example 2: Using the first derivative test for local extrema

- Find the critical point of

$$f(x) = x^{1/3}\left(x-4\right) = x^{4/3} - 4x^{1/3}$$

- Identify the intervals on which *f* is increasing and decreasing. Find the function's local and absolute extreme values.

$$f' = \frac{4(x-1)}{3x^{2/3}};\ f' - \text{ve} \quad \text{for } x < 0;$$

$$f' - \text{ve} \quad \text{for } 0 < x < 1;\ f' + \text{ve} \quad \text{for } x > 1$$

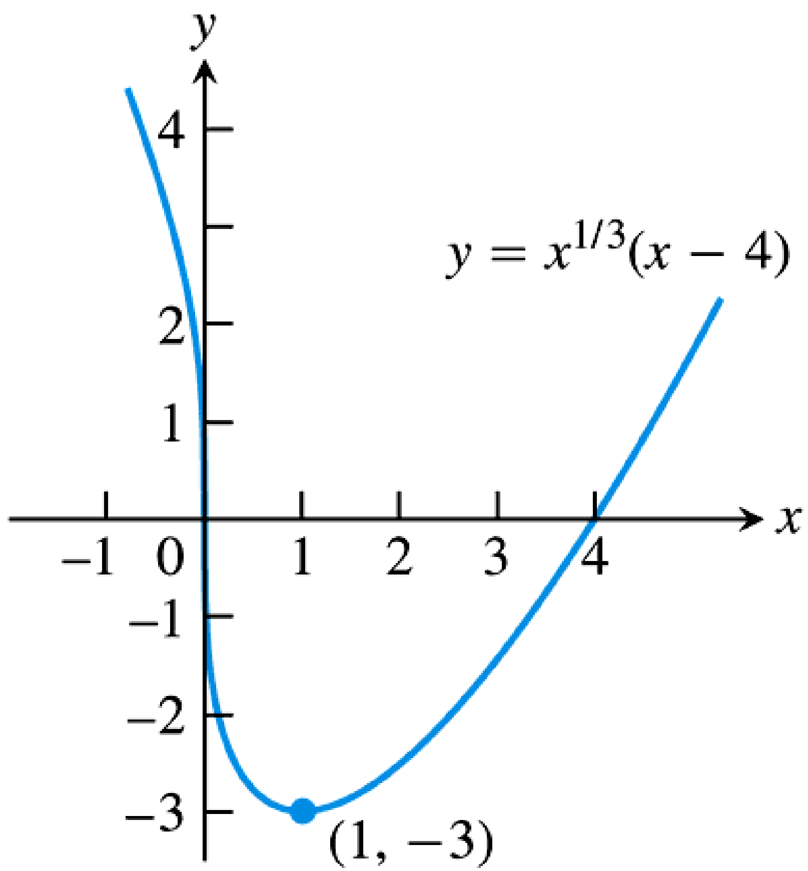


FIGURE 4.24 The function $f(x) = x^{1/3}(x - 4)$ decreases when $x < 1$ and increases when $x > 1$ (Example 2).

4.4

Concavity and Curve Sketching

(2nd lecture of week 27/08/07-01/09/07)

Concavity

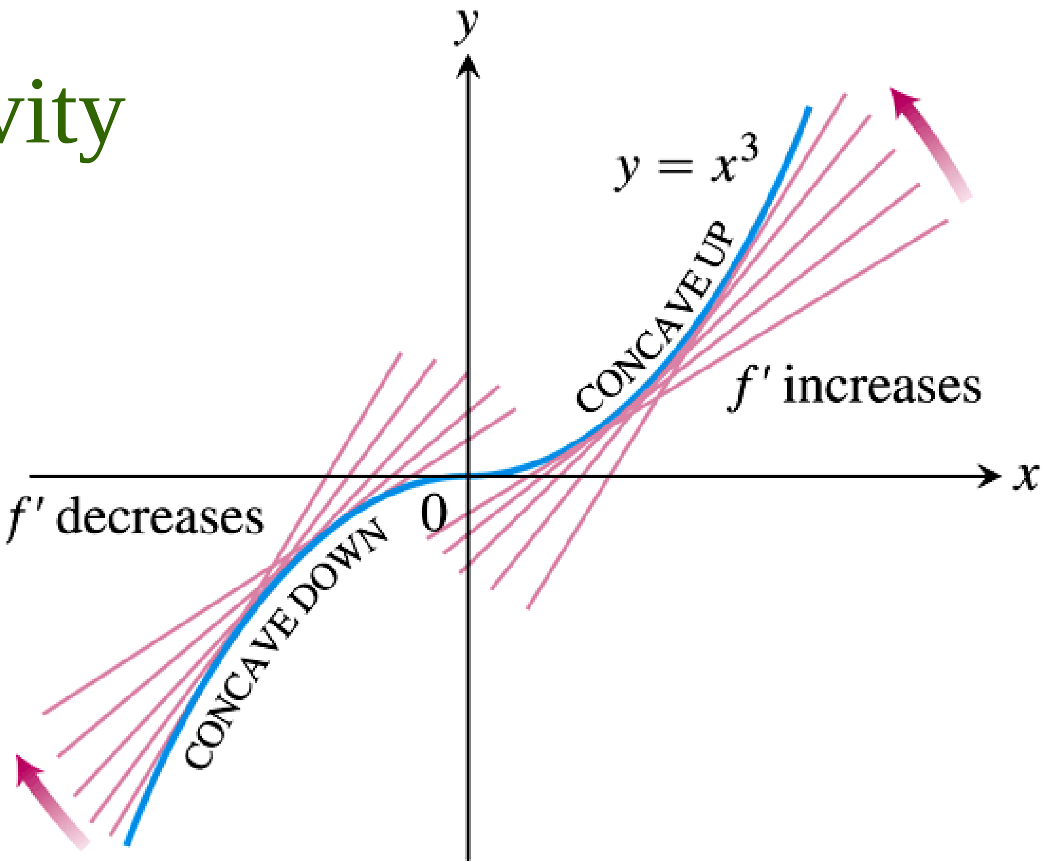


FIGURE 4.25 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

go back

DEFINITION **Concave Up, Concave Down**

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I

(b) **concave down** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Example 1(a): Applying the concavity test

- Check the concavity of the curve $y = x^3$
- Solution: $y'' = 6x$
- $y'' < 0$ for $x < 0$; $y'' > 0$ for $x > 0$;

Link to Figure 4.25

Example 1(b): Applying the concavity test

- Check the concavity of the curve $y = x^2$
- Solution: $y'' = 2 > 0$

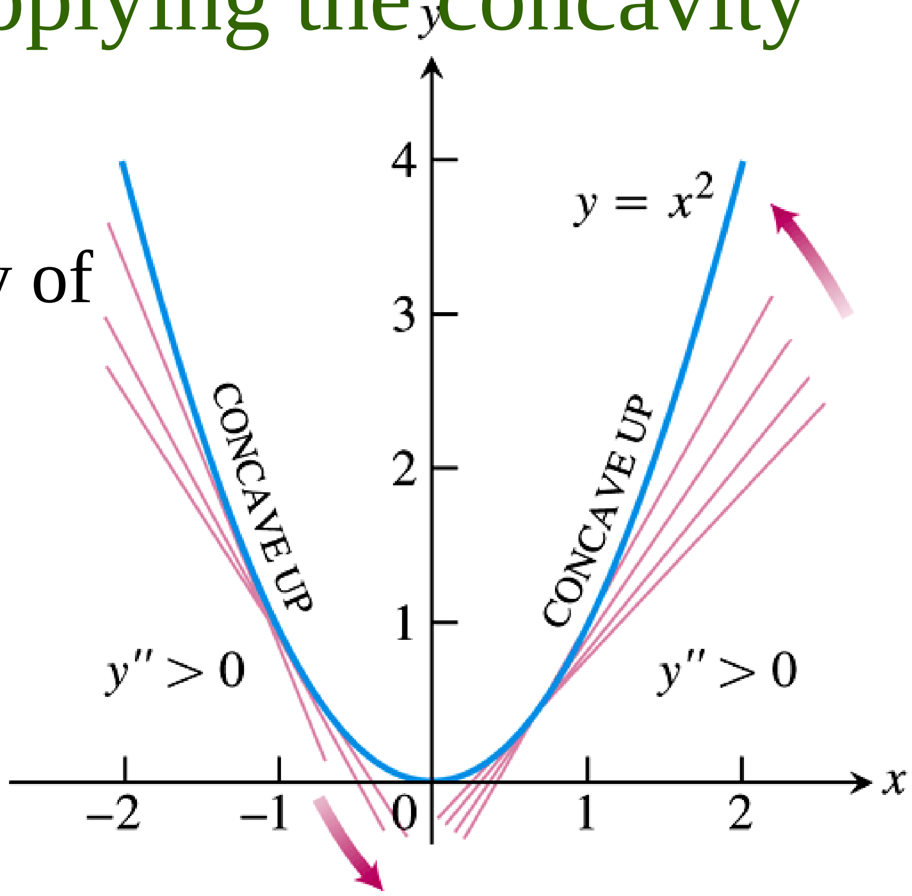


FIGURE 4.26 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

Example 2

- Determining concavity
- Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

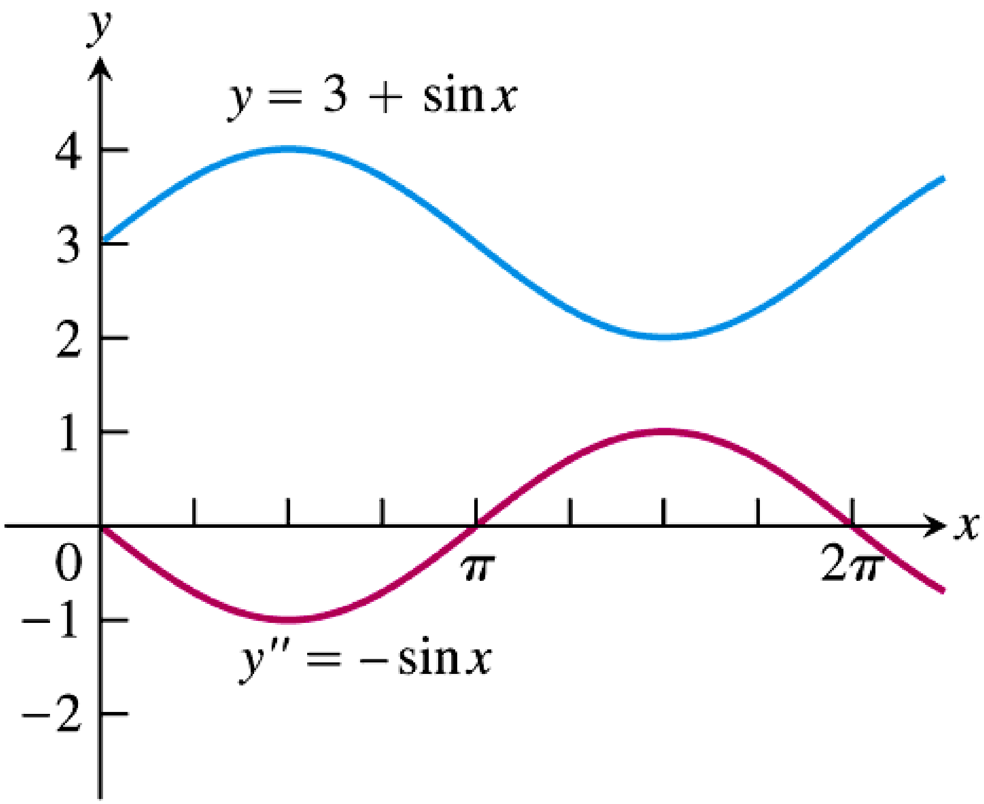


FIGURE 4.27 Using the graph of y'' to determine the concavity of y (Example 2).

Point of inflection

DEFINITION Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

Example 3

- An inflection point may not exist where An inflection point may not exist where $y'' = 0$
- The curve $y = x^4$ has no inflection point at $x=0$. Even though $y'' = 12x^2$ is zero there, it does not change sign.

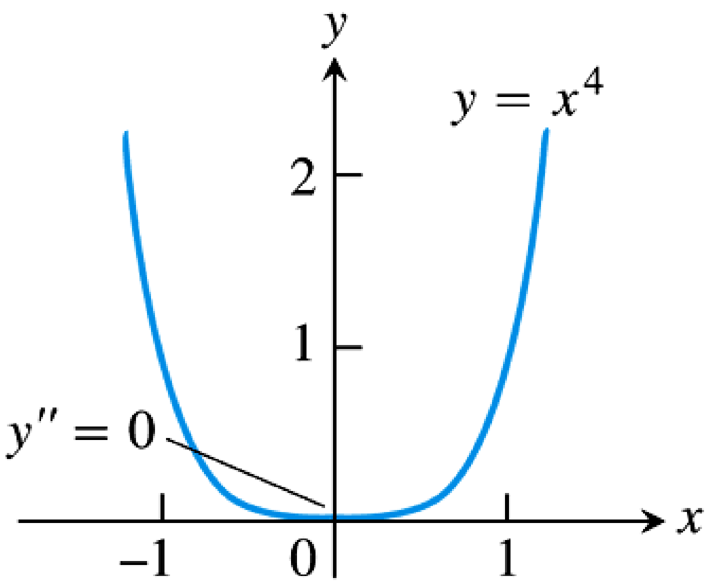


FIGURE 4.28 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 3).

Example 4

- An inflection point may not occur where $y'' = 0$ does not exist
- The curve $y = x^{1/3}$ has a point of inflection at $x=0$ but y'' does not exist there.
- $y'' = (2/9)x^{-5/3}$

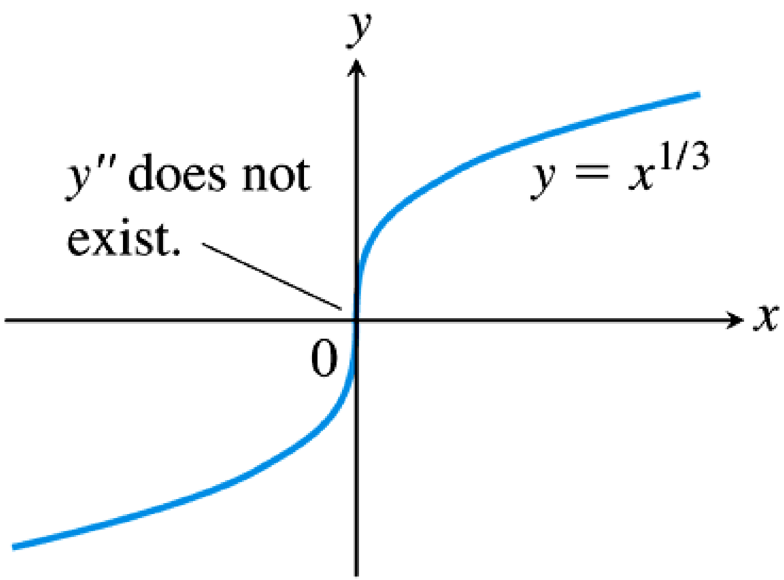


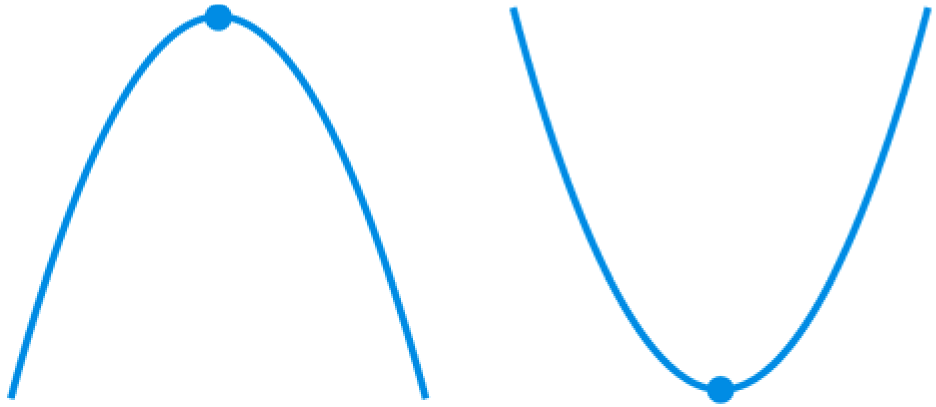
FIGURE 4.29 A point where y'' fails to exist can be a point of inflection (Example 4).

Second derivative test for local extrema

THEOREM 5 Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



$f' = 0, f'' < 0$ $\Rightarrow$ local max

$f' = 0, f'' > 0$ $\Rightarrow$ local min

Example 6: Using f' and f'' to graph f

- Sketch a graph of the function
  $f(x) = x^4 - 4x^3 + 10$

using the following steps.

(a) Identify where the extrema of *f* occur

(b) Find the intervals on which *f* is increasing and the intervals on which *f* is decreasing

(c) Find where the graph of *f* is concave up and where it is concave down.

(d) Sketch the general shape of the graph for *f*.

(e) Plot the specific points. Then sketch the graph.

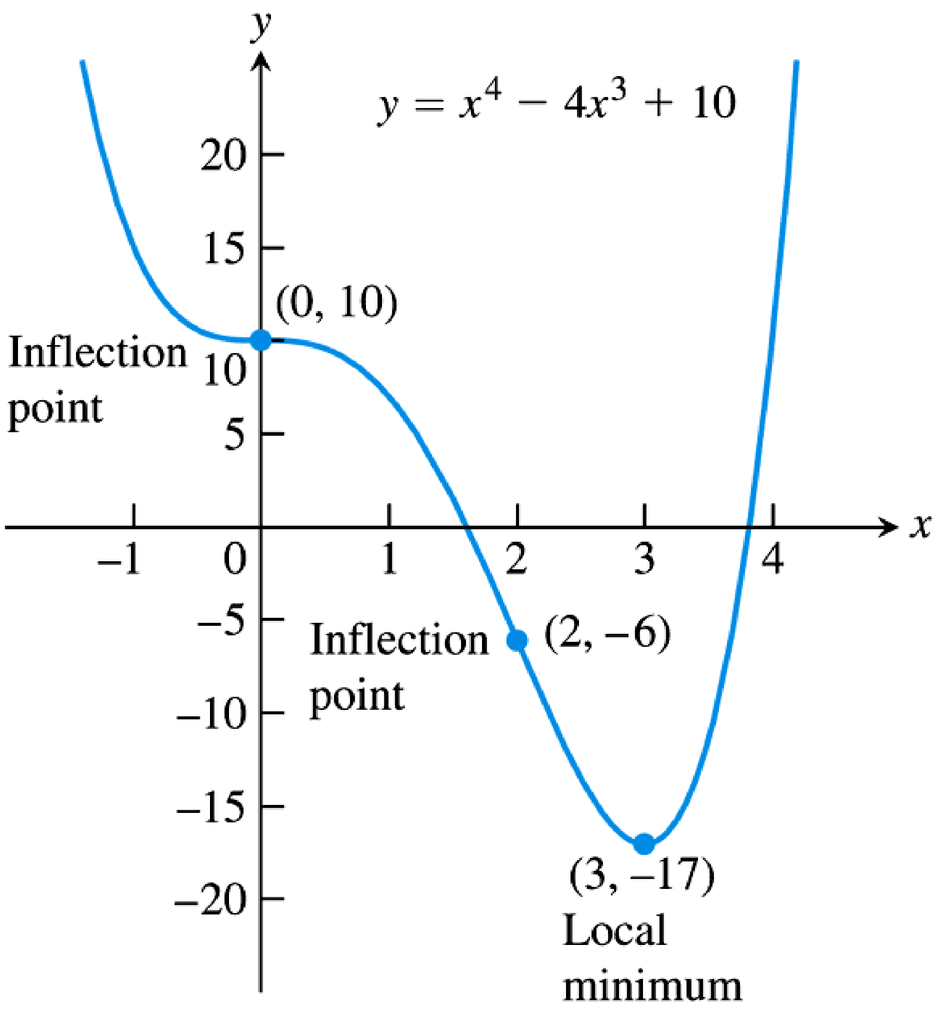


FIGURE 4.30 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 6).

Example

- Using the graphing strategy
- Sketch the graph of
- $f(x) = (x + 1)^2 / (x + 1)$.

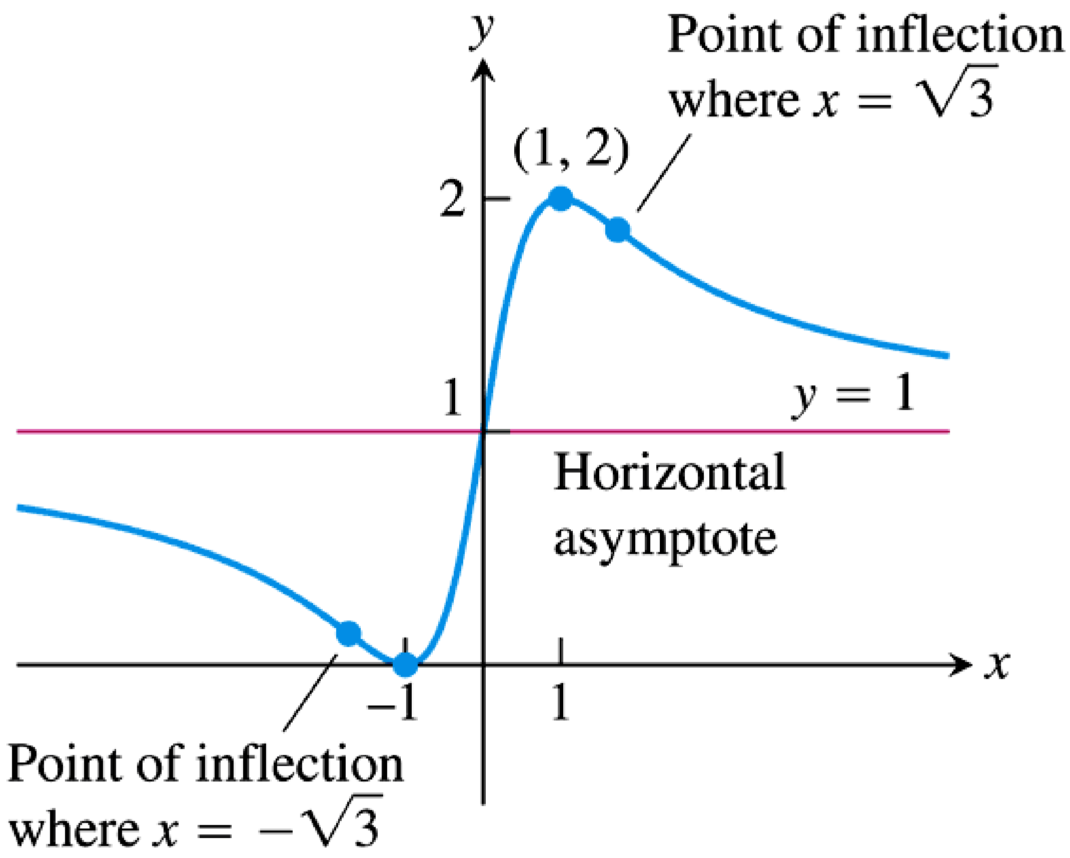
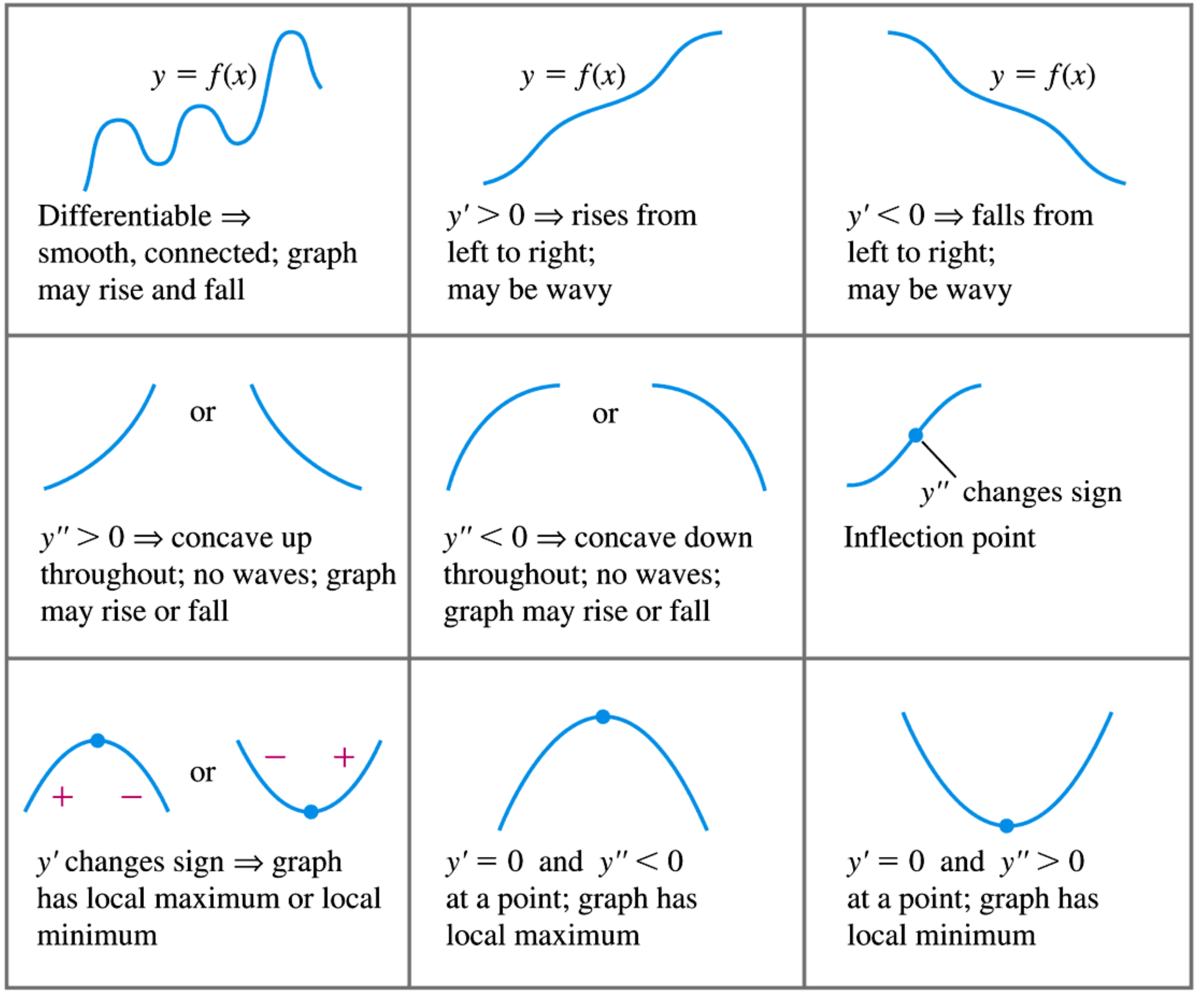


FIGURE 4.31 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$ (Example 7).

Learning about functions from derivatives



4.5

Applied Optimization Problems

(2^{nd} lecture of week 27/08/07-01/09/07)

Example 1

- An open-top box is to be cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

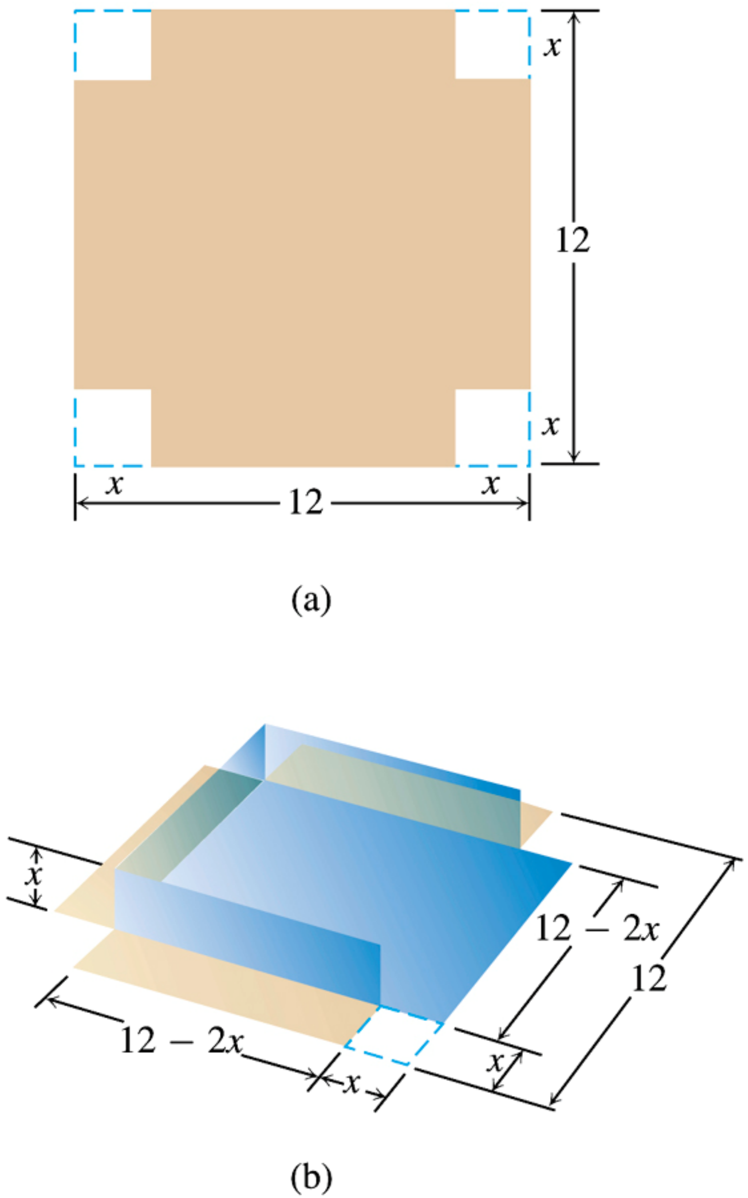


FIGURE 4.32 An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume (Example 1)?

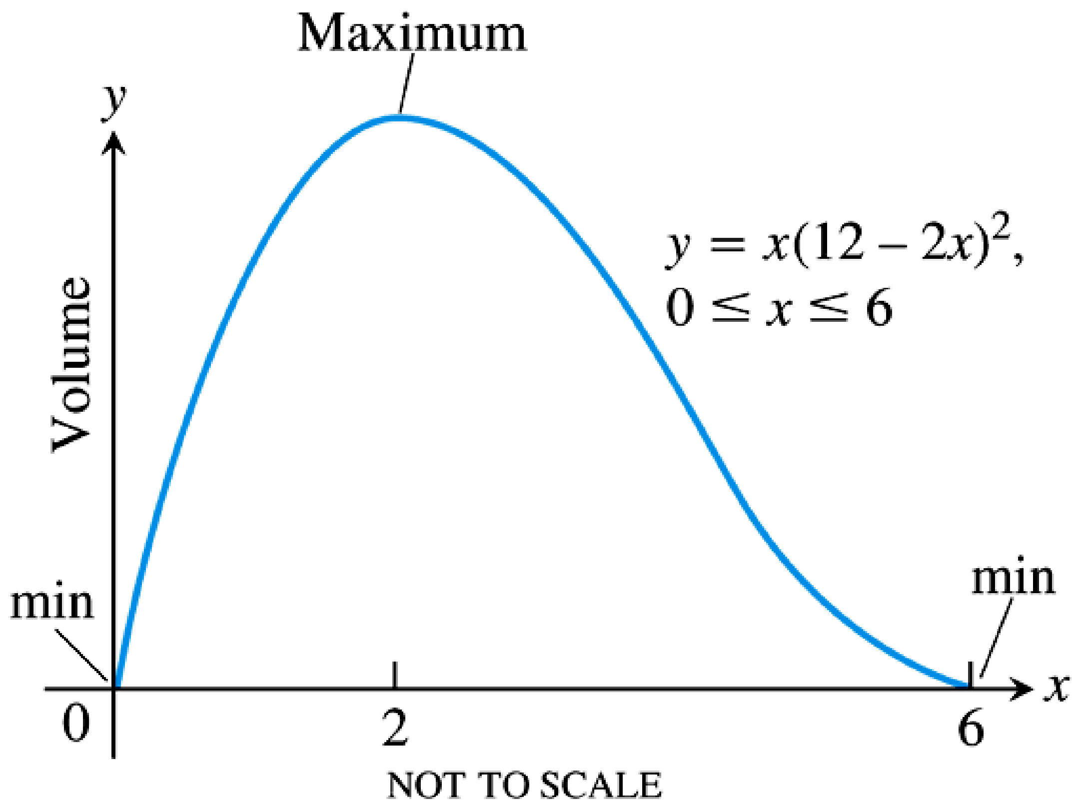


FIGURE 4.33 The volume of the box in Figure 4.32 graphed as a function of x.

Example 2

- Designing an efficient cylindrical can
- Design a 1-liter can shaped like a right circular cylinder. What dimensions will use the least material?

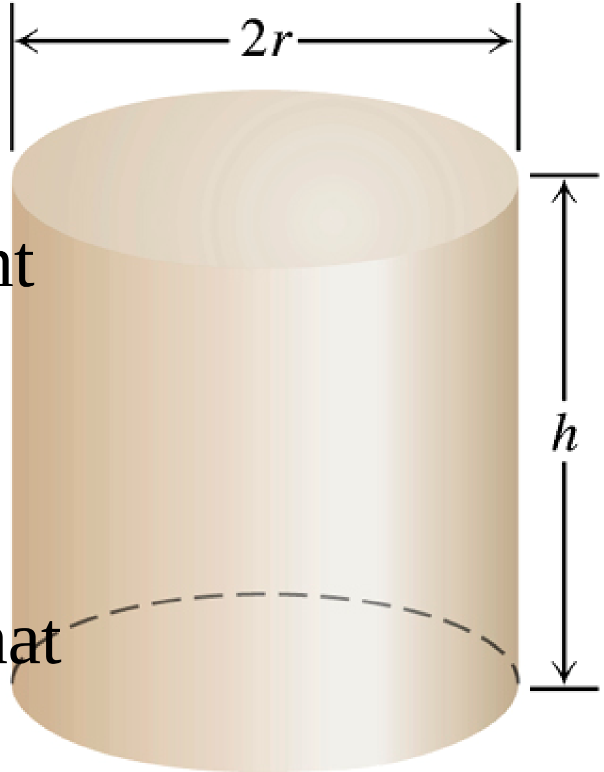


FIGURE 4.34 This 1-L can uses the least material when $h = 2r$ (Example 2).

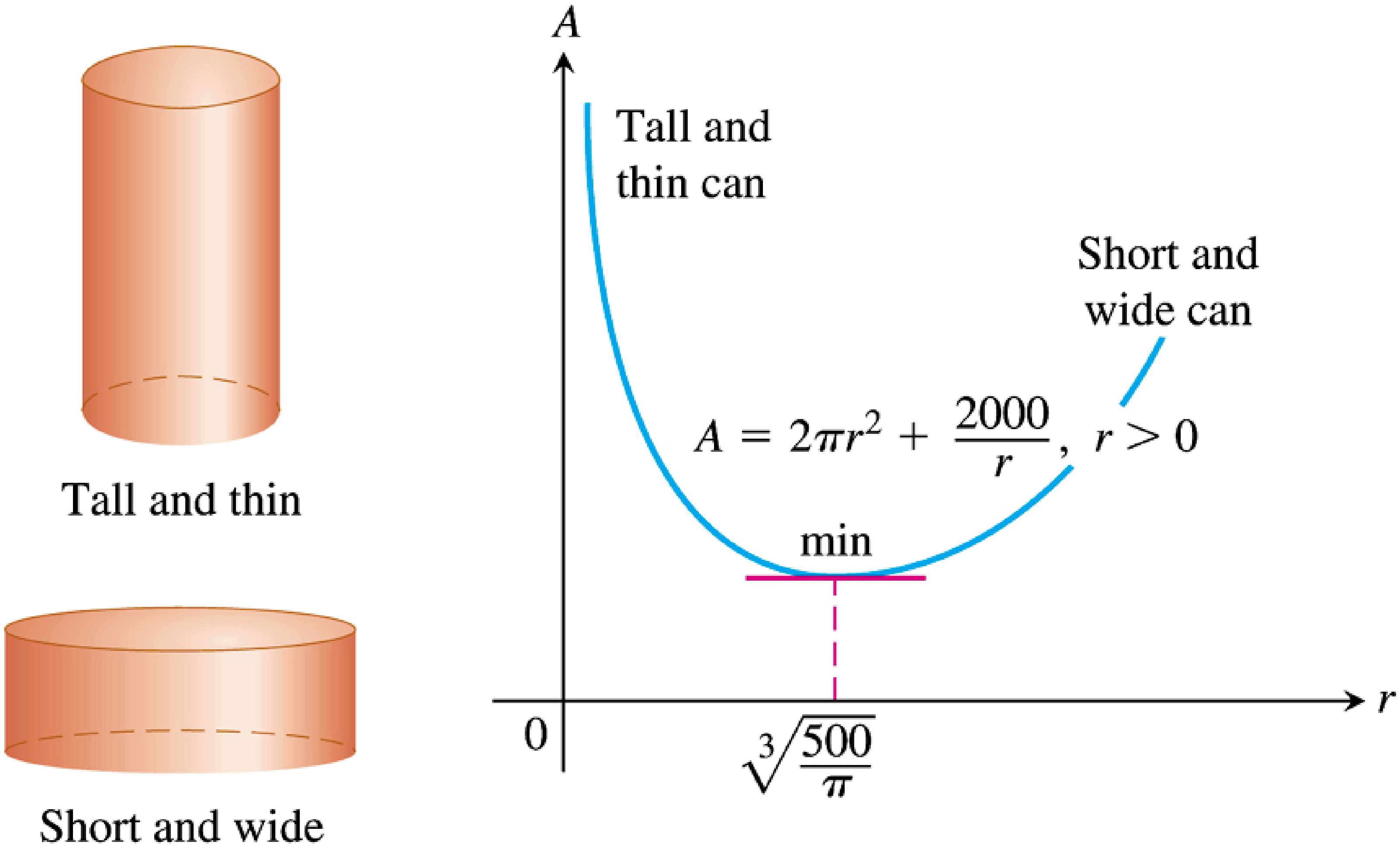


FIGURE 4.35 The graph of $A = 2\pi r^2 + 2000/r$ is concave up.

Example 3

- Inscribing rectangles
- A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

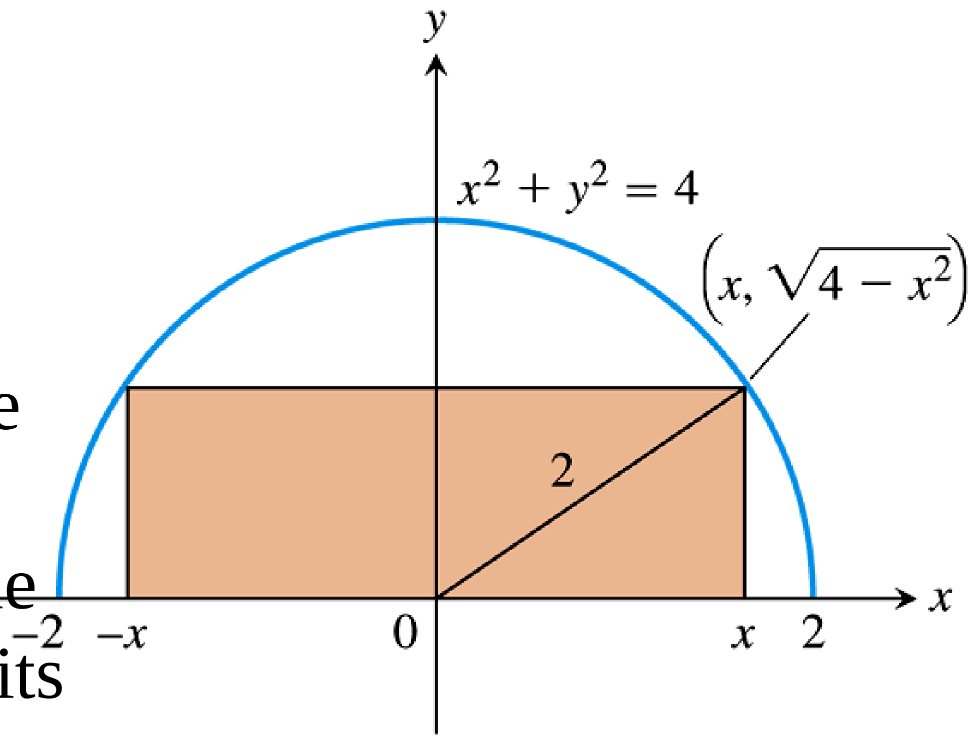


FIGURE 4.36 The rectangle inscribed in the semicircle in Example 3.

4.6

Indeterminate Forms and L'Hôpital's Rule

(3rd lecture of week 27/08/07-01/09/07)

Indeterminate forms 0/0

THEOREM 6 L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Example 1

- Using L$'$ Hopital's Rule
- (a) $\lim_{x\to 0}\frac{3x-\sin x}{x}=\frac{3-\cos x}{1}\Big|_{x=0}=2$
- (b) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{\frac{1}{2\sqrt{1+x}}}{1}\Bigg|_{x=0}=\frac{1}{2}$

THEOREM 7 L'Hôpital's Rule (Stronger Form)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

Example 2(a)

- Applying the stronger form of L' Hopital's rule
- (a) $\lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2}=\lim_{x\to 0}\frac{(1/2)(1+x)^{-1/2}-1/2}{2x}$

$$=\lim_{x\to 0}\frac{-(1/4)(1+x)^{-3/2}}{2}=\frac{-1}{8}$$

Example 2(b)

- Applying the stronger form of L′ Hopital's rule
- (b) $\lim_{x \to 0} \frac{x - \sin x}{x^3}$

THEOREM 8 Cauchy's Mean Value Theorem

Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Example 3

- Incorrectly applying the stronger form of L′ Hopital's

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2}$$

Example 4

- Using l′ Hopital's rule with one-sided limits

$$(a)\ \lim_{x\to 0^+}\frac{\sin x}{x^2}=\lim_{x\to 0^+}\frac{\cos x}{2x}=\ldots$$

$$(b)\ \lim_{x\to 0^-}\frac{\sin x}{x^2}=\lim_{x\to 0^-}\frac{\cos x}{2x}=\ldots$$

Indeterminate forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$

- Example 5(a)
- Working with the indeterminate form ∞/∞

$$(a)\ \lim_{x\to\pi/2} \frac{\sec x}{1+\tan x}$$

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{1+\tan x} = \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

$$\lim_{x\to(\pi/2)^+} \frac{\sec x}{1+\tan x}. = \ldots$$

Example 5(b)

$$(b)\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5x}=\ldots$$

Example 6

- Working with the indeterminate form $\infty \cdot 0$

$$\lim_{x \to \infty} \left(x \sin \frac{1}{x} \right)$$

Example 7

- Working with the indeterminate form ∞ - ∞

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right)=\ldots$$

4.8

Antiderivatives

(3rd lecture of week 27/08/07-01/09/07)

Finding antiderivatives

DEFINITION Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example 1

- Finding antiderivatives
- Find an antiderivative for each of the following functions
- (a) $f(x) = 2x$
- (b) $f(x) = \cos x$
- (c) $h(x) = 2x + \cos x$

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Example 2 Finding a particular antiderivative

- Find an antiderivative of $f(x) = \sin x$ that satisfies $F(0) = 3$

TABLE 4.2 Antiderivative formulas

	Function	General antiderivative
1.	x^n	$\frac{x^{n+1}}{n+1} + C, \quad n \neq -1, n$ rational
2.	$\sin kx$	$-\frac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\frac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

Example 3 Finding antiderivatives using table 4.2

- Find the general antiderivative of each of the following functions.
- (a) $f(x) = x^5$
- (b) $g(x) = 1/x^{1/2}$
- (c) $h(x) = \sin 2x$
- (d) $i(x) = \cos(x/2)$

Example 4 Using the linearity rules for antiderivatives

- Find the general antiderivative of
- $f(x) = 3/x^{1/2} + \sin 2x$

DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Example of indefinite integral notation

$$\int 2x \, dx = x^2 + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int (2x + \cos x) \, dx = x^2 + \sin x + C$$

Example 7 Indefinite integration done term-by term and rewriting the constant of integration

- Evaluate

$$\int \left(x^2 - 2x + 5\right) dx = \int x^2 dx - \int 2x dx + \int 5 dx = \ldots$$

Chapter 5

Integration

5.1

Estimating with Finite Sums

(1st lecture of week 03/09/07-08/09/07)

Riemann Sums

Approximating area bounded by the graph between [a,b]

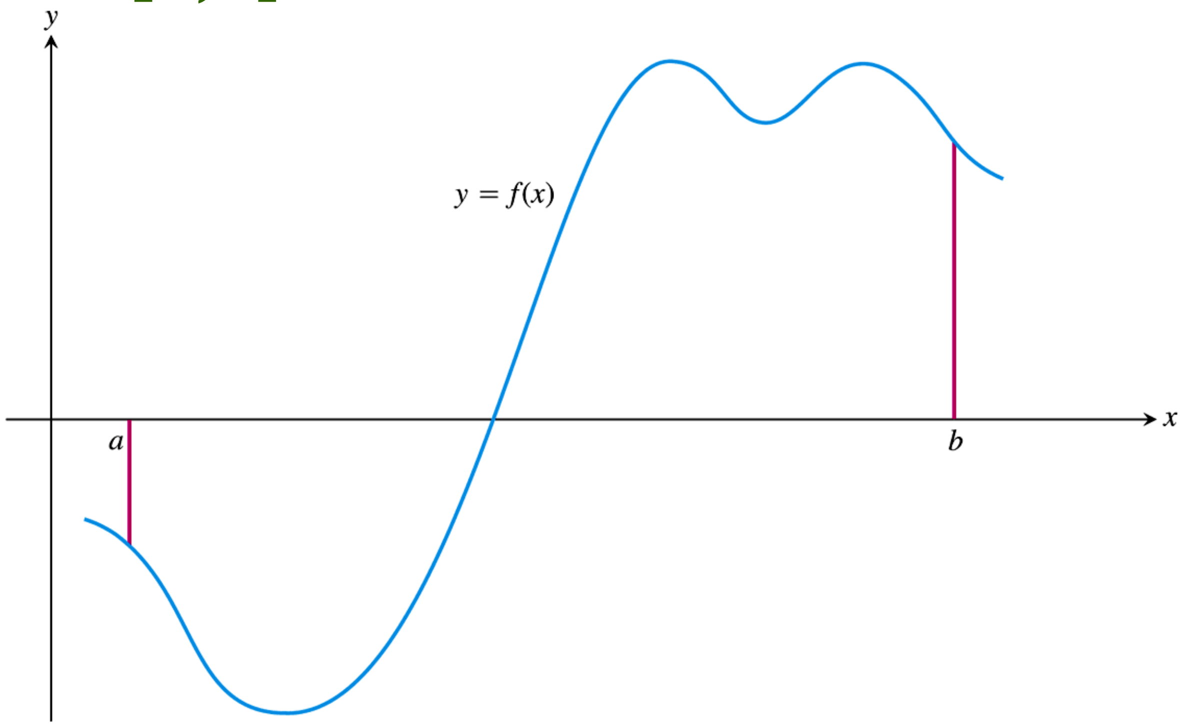


FIGURE 5.8 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

Area is approximately given by

$$f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \ldots + f(c_n)\Delta x_n$$

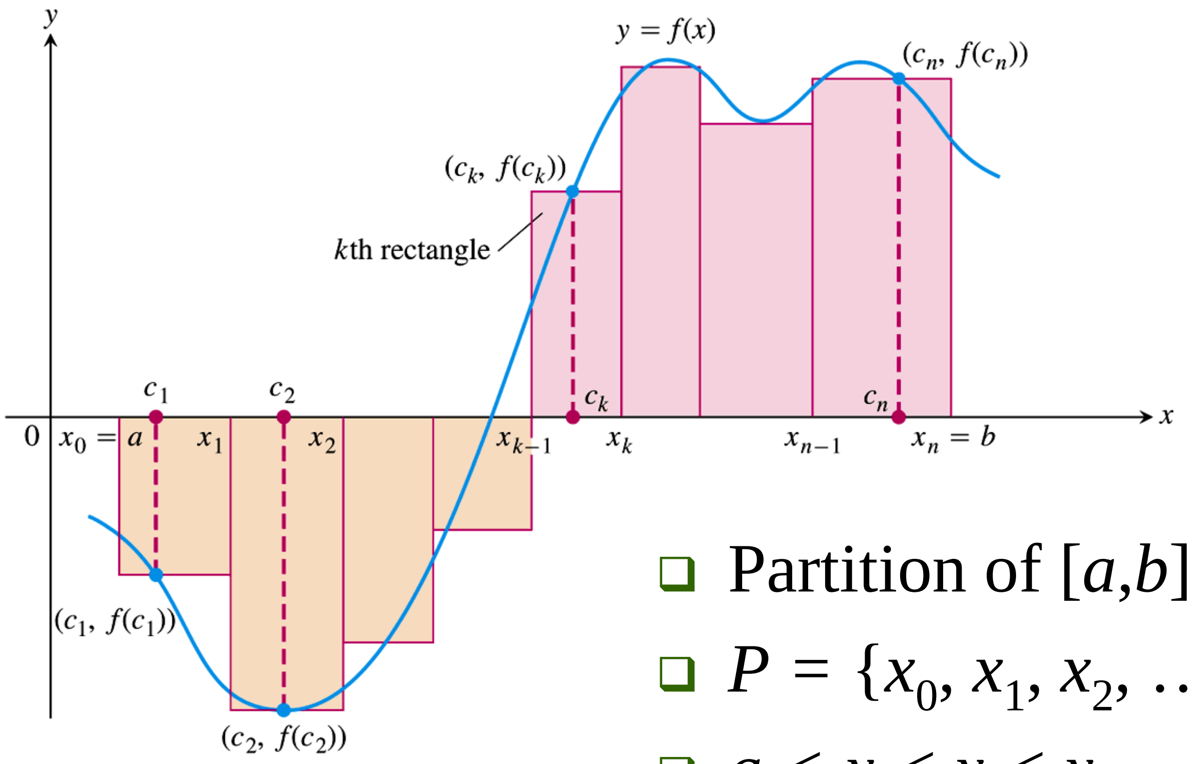
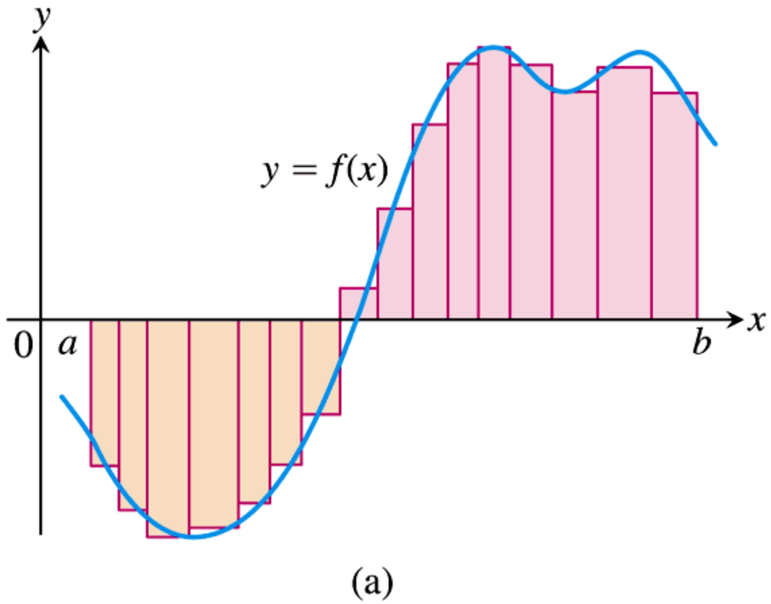


FIGURE 5.9 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis.

- Partition of $[a,b]$ is the set of
- $P = \{x_0, x_1, x_2, \ldots x_{n-1}, x_n\}$
- $a < x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = b$
- $c_n \in [x_{n-1}, x_n]$
- $||P||$ = norm of P = the largest of all subinterval width



Riemann sum for f on $[a,b]$

$$R_n = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \ldots + f(c_n)\Delta x_n$$

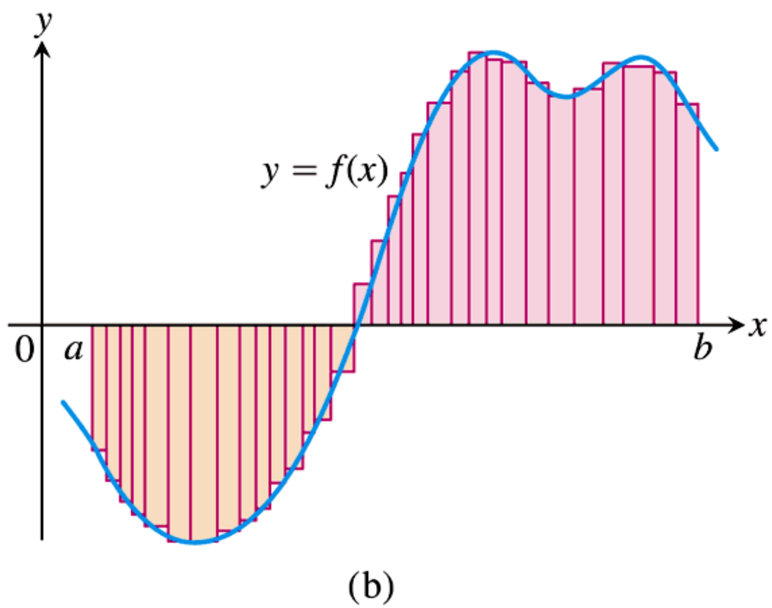
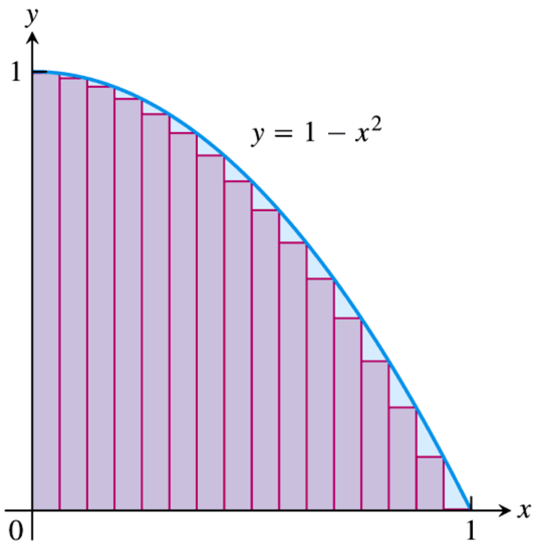
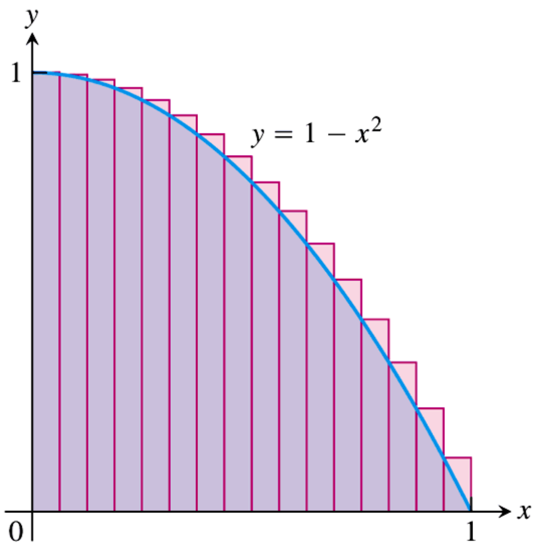


FIGURE 5.10 The curve of Figure 5.9 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x-axis with increasing accuracy.



(a)



(b)

- Let the true value of the area is *R*
- Two approximations to *R*:
- $c_n = x_n$ corresponds to case (a). This under estimates the true value of the area *R* if *n* is finite.
- $c_n = x_n$ corresponds to case (b). This over estimates the true value of the area *S* if *n* is finite.

go back

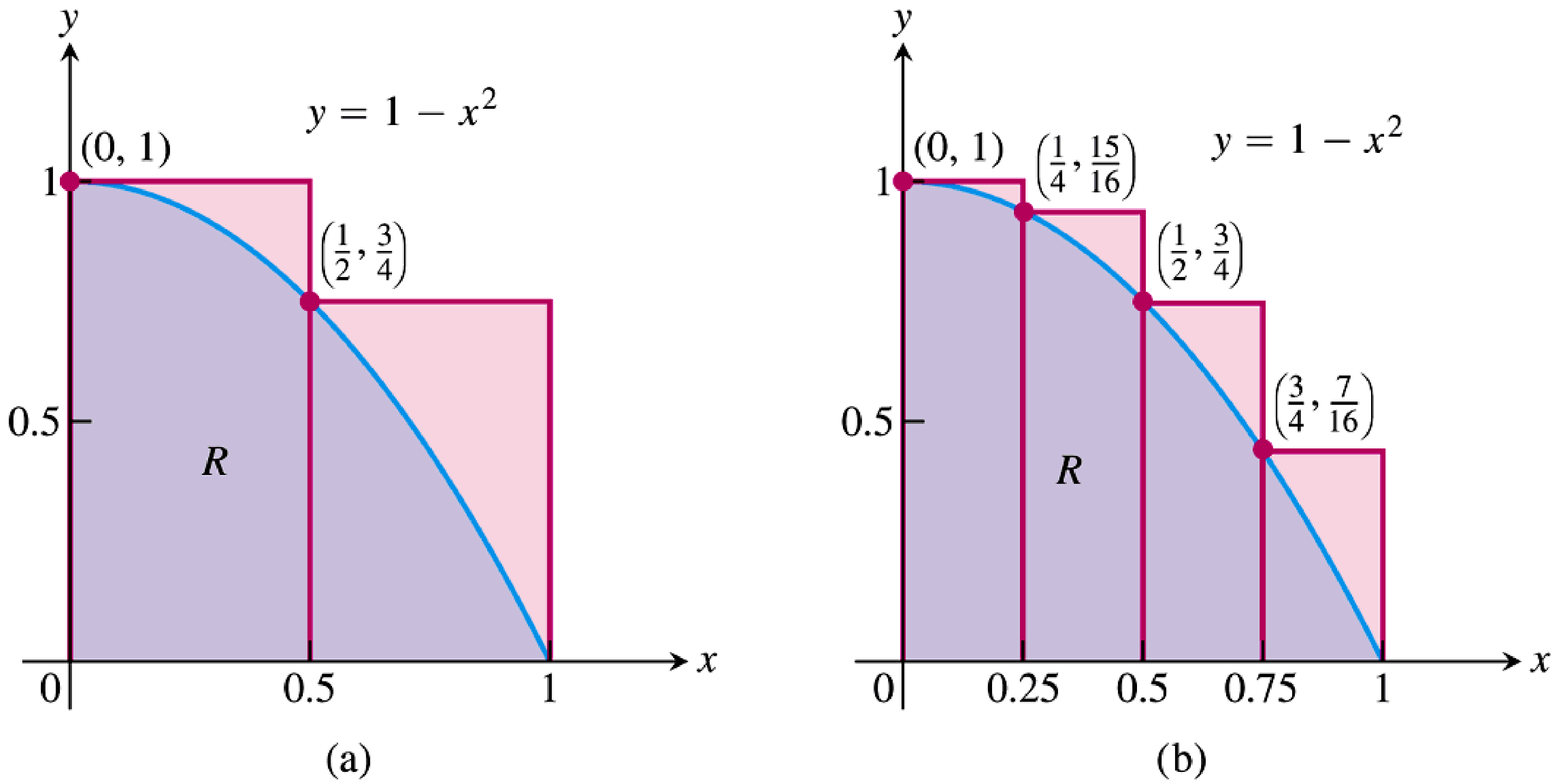


FIGURE 5.2 (a) We get an upper estimate of the area of R by using two rectangles containing R. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area.

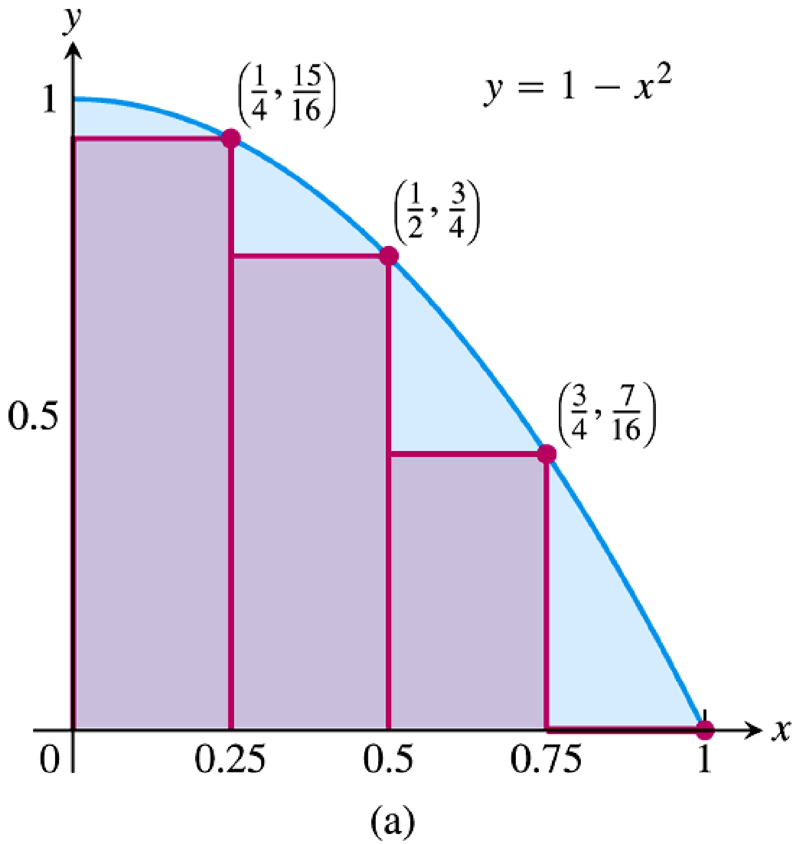


FIGURE 5.3 (a) Rectangles contained in R give an estimate for the area that undershoots the true value.

Limits of finite sums

- Example 5 The limit of finite approximation to an area
- Find the limiting value of lower sum approximation to the area of the region *R* below the graphs $f(x) = 1 - x^2$ on the interval [0,1] based on Figure 5.4(a)

Solution

- $\Delta x_k = (1 - 0)/n = 1/n \equiv \Delta x;\ k = 1,2,\ldots n$
- Partition on the x-axis: $[0,1/n]$, $[1/n, 2/n]$,…, $[(n-1)/n,1]$.
- $c_k = x_k = k\Delta x = k/n$
- The sum of the stripes is

$R_n = \Delta x_1\, f(c_1) + \Delta x_2\, f(c_2) + \Delta x_3\, f(c_3) + \ldots + \Delta x_n\, f(c_n)$

$= \Delta x\, f(1/n) + \Delta x\, f(2/n) + \Delta x\, f(3/n) + \ldots + \Delta x_n\, f(1)$

$= \sum_{k=1}^{n} \Delta x\, f(k\Delta x) = \Delta x \sum_{k=1}^{n} f(k/n)$

$= (1/n) \sum_{k=1}^{n} [1 - (k/n)^2]$

$= \sum_{k=1}^{n} 1/n - k^2/n^3 = 1 - (\sum_{k=1}^{n} k^2)/n^3$

$= 1 - [(n)(n+1)(2n+1)/6]/n^3 = 1 - [2n^3 + 3n^2 + n]/(6n^3)$

$\sum_{k=1}^{n} k^2 = (n)(n+1)(2n+1)/6$

- Taking the limit of $n \rightarrow \infty$

$$\lim_{n\to\infty} R_n = R = \left(1 - \frac{2n^3 + 3n^2 + n}{6n^3}\right) = 1 - 2/6 = 2/3$$

- The same limit is also obtained if $c_n = x_{n-1}$ is chosen instead.
- For all choice of $c_n \in [x_{n-1}, x_n]$ and partition of P, the same limit for S is obtained when $n \rightarrow \infty$

5.3

The Definite Integral

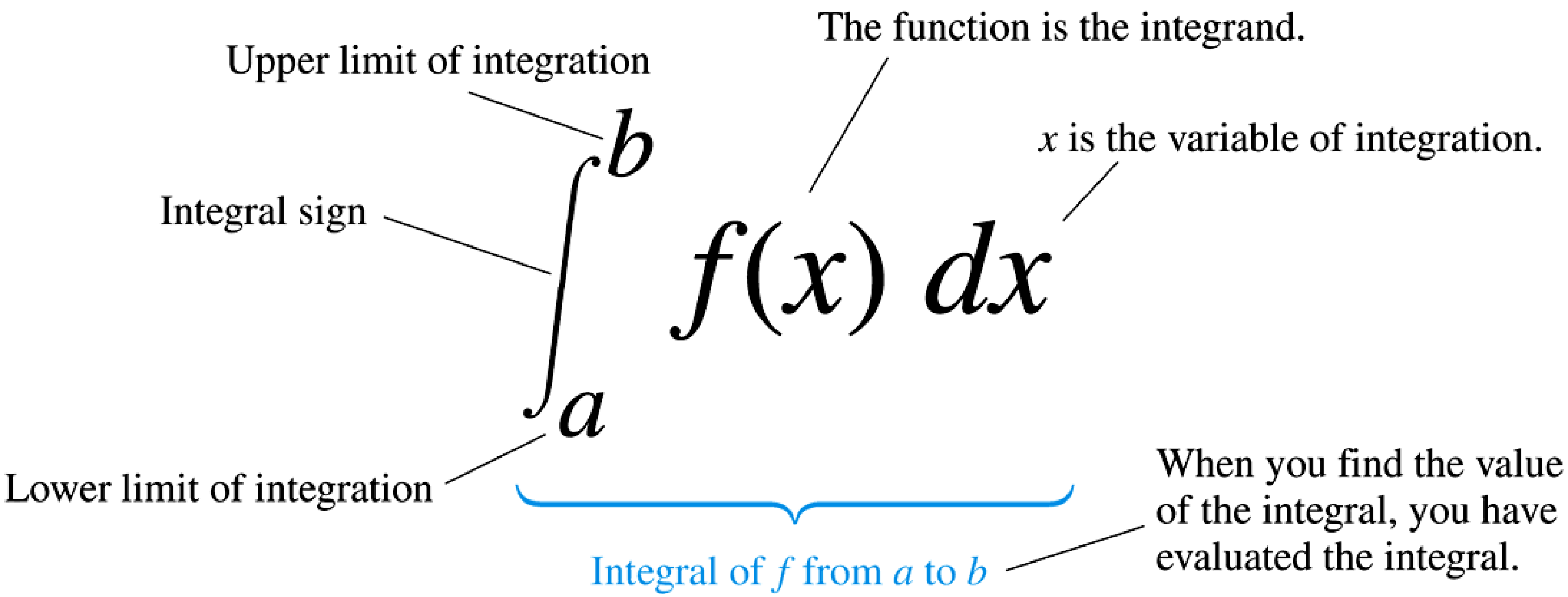
(2nd lecture of week 03/09/07-08/09/07)

DEFINITION The Definite Integral as a Limit of Riemann Sums

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number I is the **definite integral of f over $[a, b]$** and that I is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - I \right| < \epsilon.$$



"The integral from *a* to *b* of *f* of *x* with respect to *x*"

- The limit of the Riemann sums of f on $[a,b]$ converge to the finite integral I

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = I = \int_a^b f(x)dx$$

- We say f is integrable over $[a,b]$
- Can also write the definite integral as

$$I = \int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du$$

$$= \int_a^b f(\text{what ever}) \quad d(\text{what ever})$$

- The variable of integration is what we call a dummy variable

THEOREM 1 The Existence of Definite Integrals

A continuous function is integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

Question: is a non continuous function integrable?

Integral and nonintegrable functions

- Example 1
- A nonintegrable function on [0,1]

$$f(x)=\begin{cases}1, & \text{if } x \text{ is rational}\\ 0, & \text{if } x \text{ is irrational}\end{cases}$$

- Not integrable

Properties of definite integrals

THEOREM 2

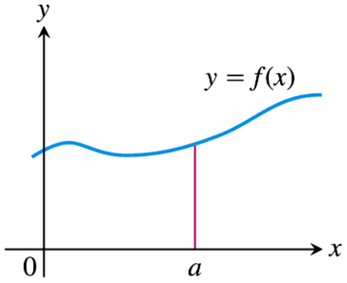
When f and g are integrable, the definite integral satisfies Rules 1 to 7 in Table 5.3.

TABLE 5.3 Rules satisfied by definite integrals

1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	Also a Definition
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any Number k
		$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5.	*Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6.	*Max-Min Inequality:*	If f has maximum value max f and minimum value min f on $[a, b]$, then	

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$$

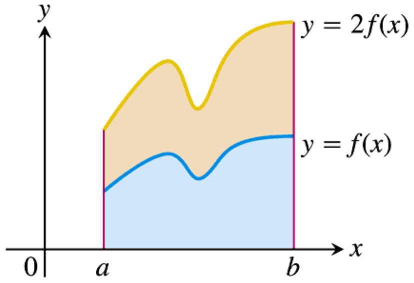
7.	*Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
		$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special Case)



(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0.$$

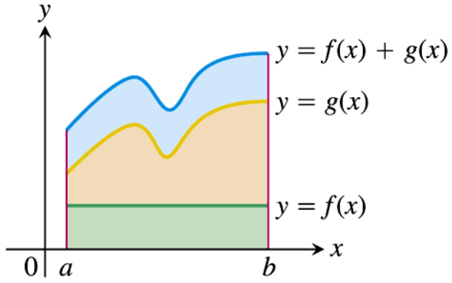
(The area over a point is 0.)



(b) *Constant Multiple:*

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

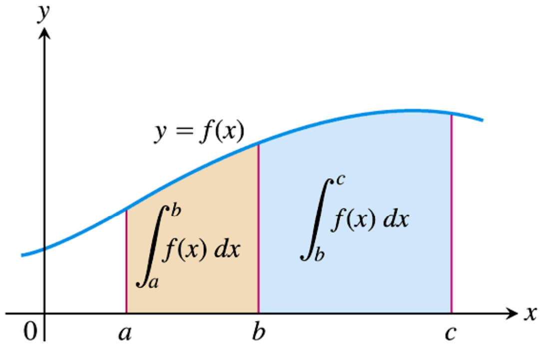
(Shown for $k = 2$.)



(c) *Sum:*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

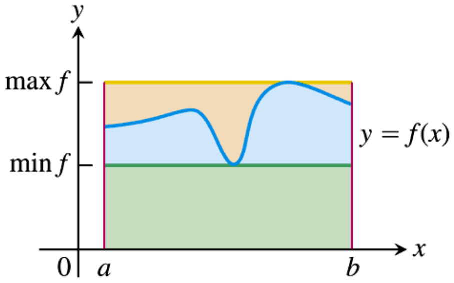
(Areas add)



(d) *Additivity for definite integrals:*

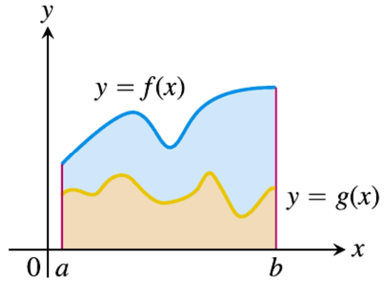
$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

FIGURE 5.11



(e) *Max-Min Inequality*:

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$$



(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

Example 3 Finding bounds for an integral

- Show that the value of $\int_0^1 \sqrt{1+\cos x}\,dx$ is less than 3/2

- **Solution**
- Use rule 6 Max-Min Inequality

Area under the graphs of a nonnegative function

DEFINITION Area Under a Curve as a Definite Integral

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\, dx.$$

Example 4 Area under the line $y = x$

- Compute $\int_0^b x\,dx$ (the Riemann sum)

  and find the area A under $y = x$ over the interval $[0,b]$, $b>0$

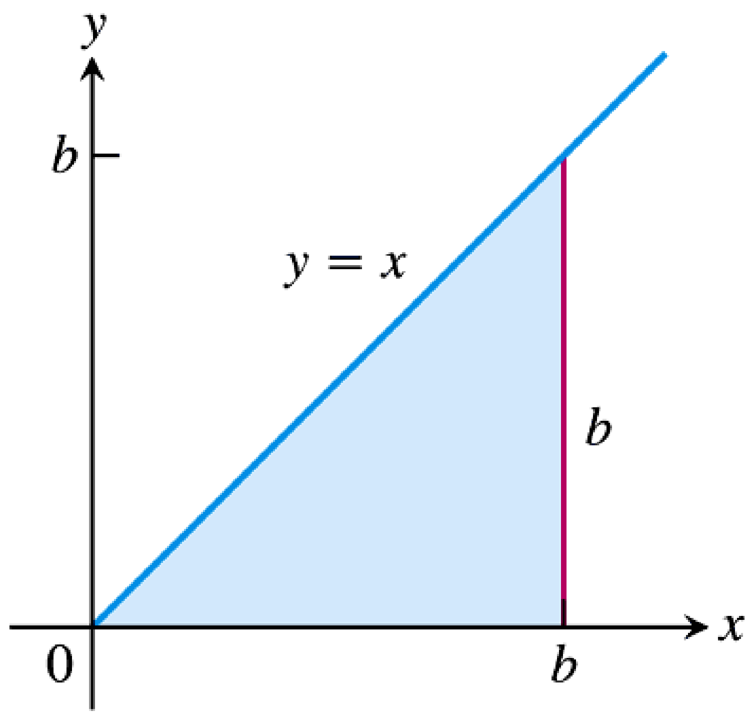


FIGURE 5.12 The region in Example 4 is a triangle.

Solution

By geometrical consideration:

A=(1/2)×high×width= (1/2)×b×b= $b^2/2$

Riemann sum

$$\lim_{n\to\infty}\sum_{k=1}^{n}\Delta x f(c_k) = \lim_{n\to\infty}\Delta x\sum_{k=1}^{n} f(x_k)$$

$$= \lim_{n\to\infty}\Delta x\sum_{k=1}^{n} x_k = \lim_{n\to\infty}\Delta x\sum_{k=1}^{n} k\Delta x$$

$$= \lim_{n\to\infty}\Delta x^2\sum_{k=1}^{n} k = \lim_{n\to\infty}\left(\frac{b}{n}\right)^2\sum_{k=1}^{n} k$$

$$= \lim_{n\to\infty}\left(\frac{b}{n}\right)^2\frac{n(n+1)}{2} = \lim_{n\to\infty}\left(\frac{b}{n}\right)^2\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty}\frac{b^2}{2}\left(1+\frac{1}{n}\right) = \frac{b^2}{2}$$

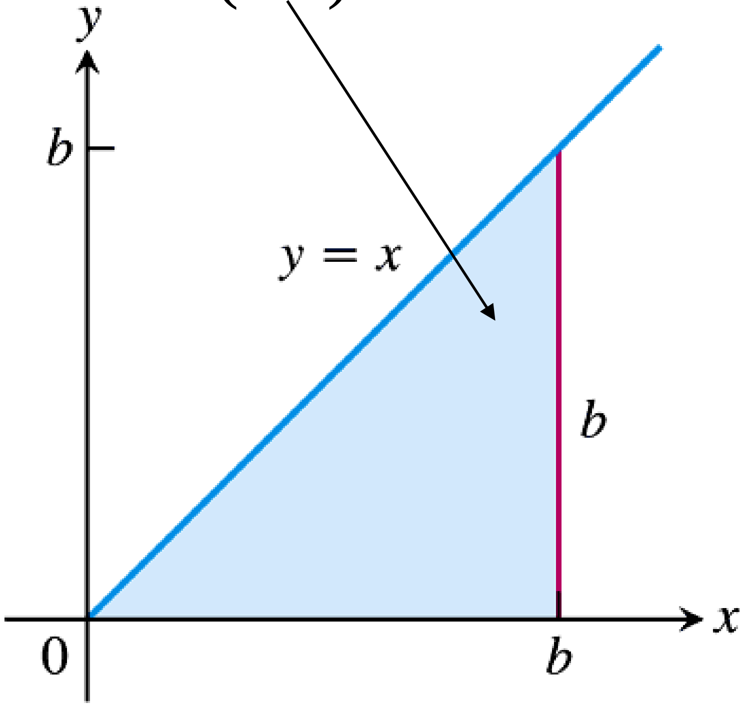


FIGURE 5.12 The region in Example 4 is a triangle.

$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}, \qquad a < b \tag{1}$$

Using geometry, the area is the area of a trapezium

$A = (1/2)(b-a)(b+a)$

$\quad = b^2/2 - a^2/2$

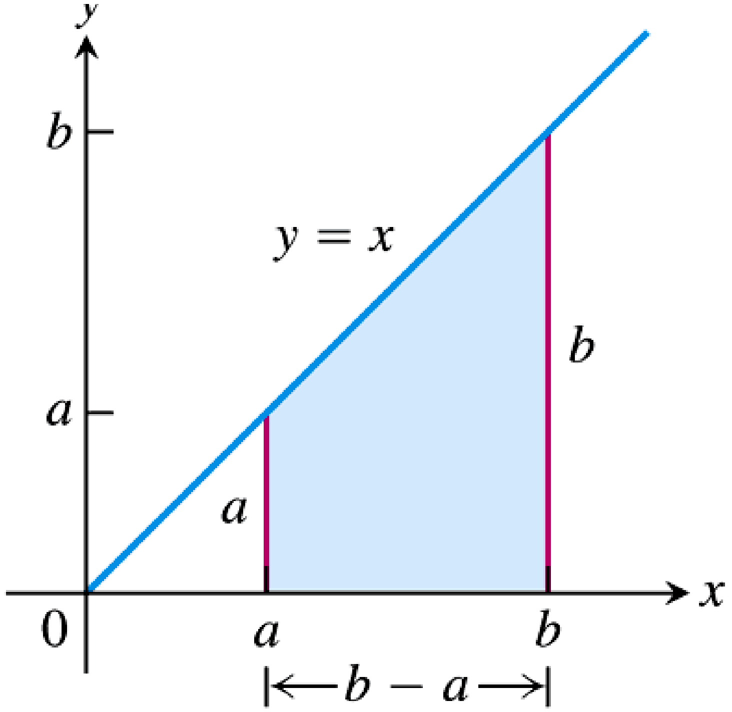


FIGURE 5.13 The area of this trapezoidal region is $A = (b^2 - a^2)/2$.

$$\int_a^b c\,dx = c(b - a), \qquad c \text{ any constant} \tag{2}$$

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}, \qquad a < b \tag{3}$$

Average value of a continuous function revisited

- Average value of nonnegative continuous function *f* over an interval [*a*,*b*] is

$$\frac{f(c_1)+f(c_2)+\cdots f(c_n)}{n}=\frac{1}{n}\sum_{k=1}^{n} f(c_k)$$

$$=\frac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k)=\frac{1}{b-a}\sum_{k=1}^{n}\Delta x f(c_k)$$

- In the limit of $n \rightarrow \infty$, the average =

$$\frac{1}{b-a}\int_a^b f(x)dx$$

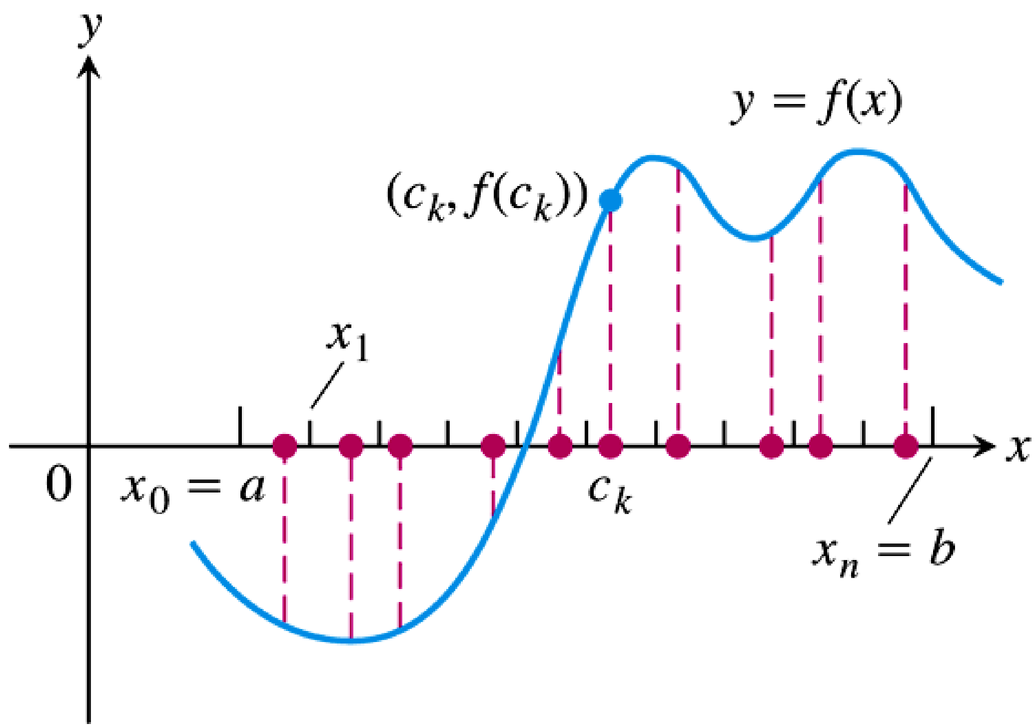


FIGURE 5.14 A sample of values of a function on an interval $[a, b]$.

DEFINITION The Average or Mean Value of a Function

If f is integrable on $[a, b]$, then its **average value on** $[a, b]$, also called its **mean value**, is

$$\text{av}(f) = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

Example 5 Finding average value

- Find the average value of $f(x) = \sqrt{4 - x^2}$ over [-2,2]

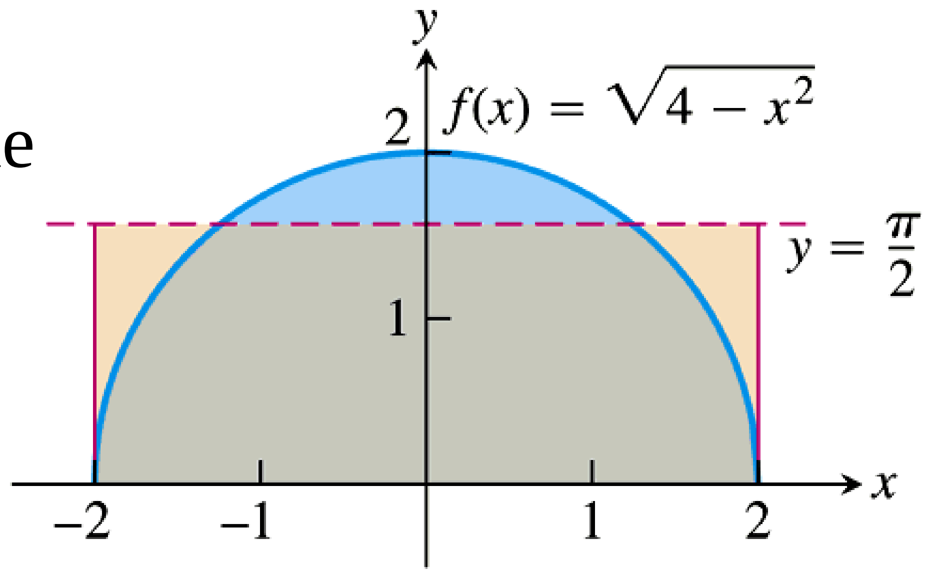


FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 5).

5.4

The Fundamental Theorem of Calculus

(2nd lecture of week 03/09/07-08/09/07)

Mean value theorem for definite integrals

THEOREM 3 The Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

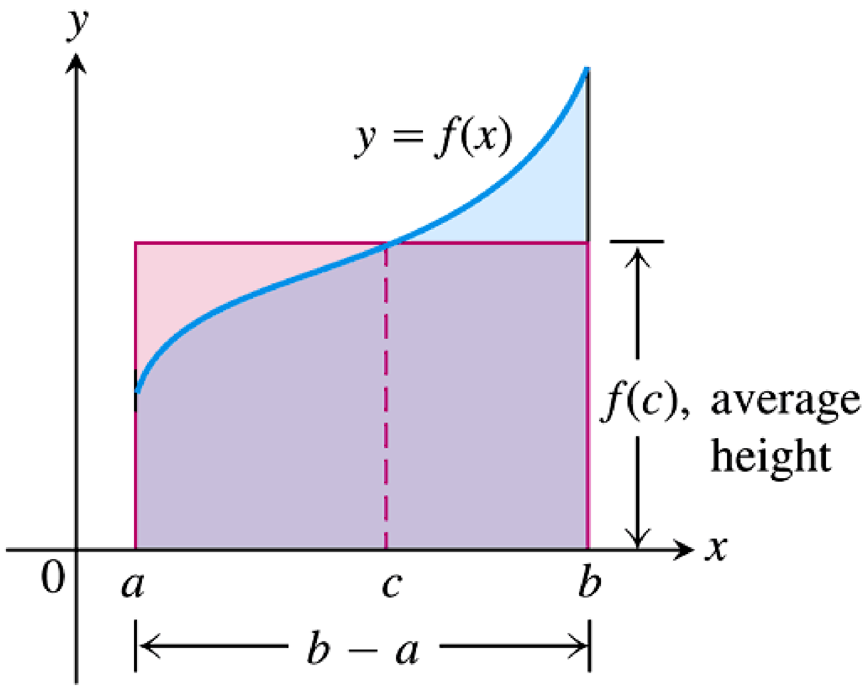


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

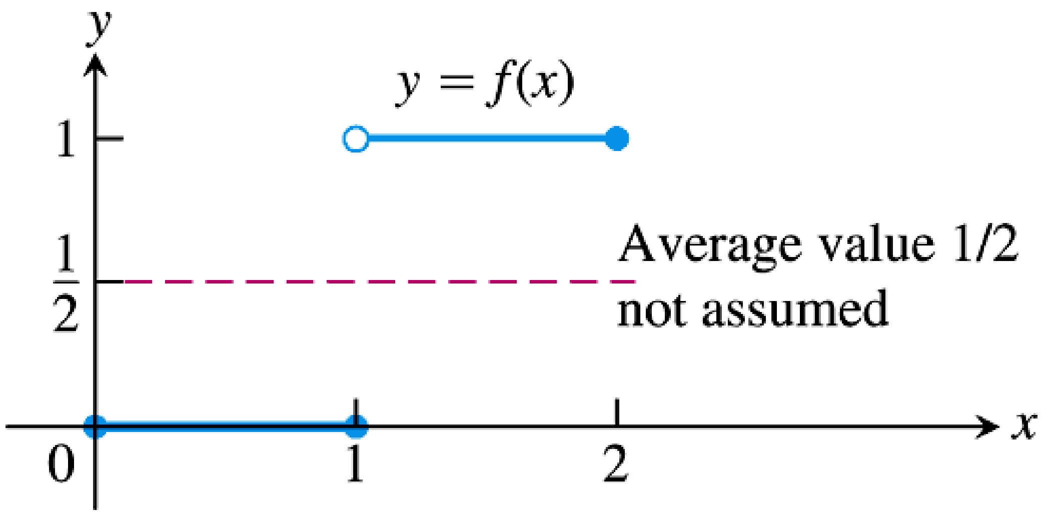


FIGURE 5.17 A discontinuous function need not assume its average value.

Example 1 Applying the mean value theorem for integrals

- Find the average value of $f(x)=4-x$ on [0,3] and where f actually takes on this value as some point in the given domain.

- **Solution**
- Average = 5/2
- Happens at $x=3/2$

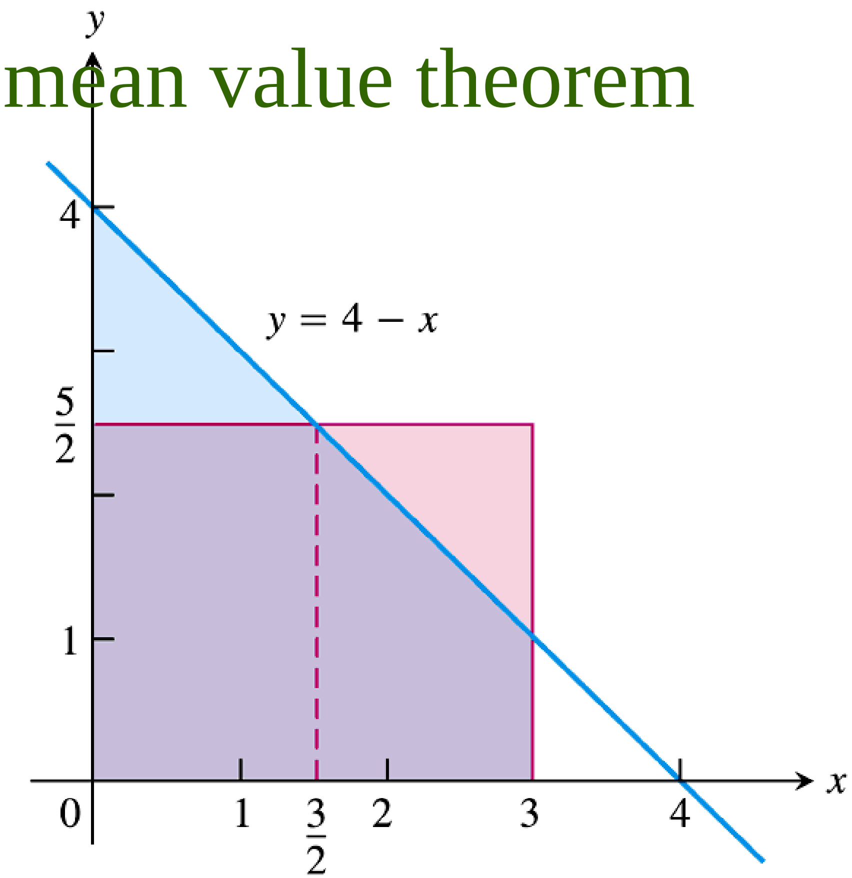


FIGURE 5.18 The area of the rectangle with base [0, 3] and height 5/2 (the average value of the function $f(x) = 4 - x$) is equal to the area between the graph of f and the x-axis from 0 to 3 (Example 1).

Fundamental theorem Part 1

- Define a function $F(x)$: $F(x) = \int_a^x f(t)\,dt$
- $x, a \in I$, an interval over which $f(t) > 0$ is integrable.
- The function $F(x)$ is the area under the graph of $f(t)$ over $[a,x]$, $x > a \geqslant 0$

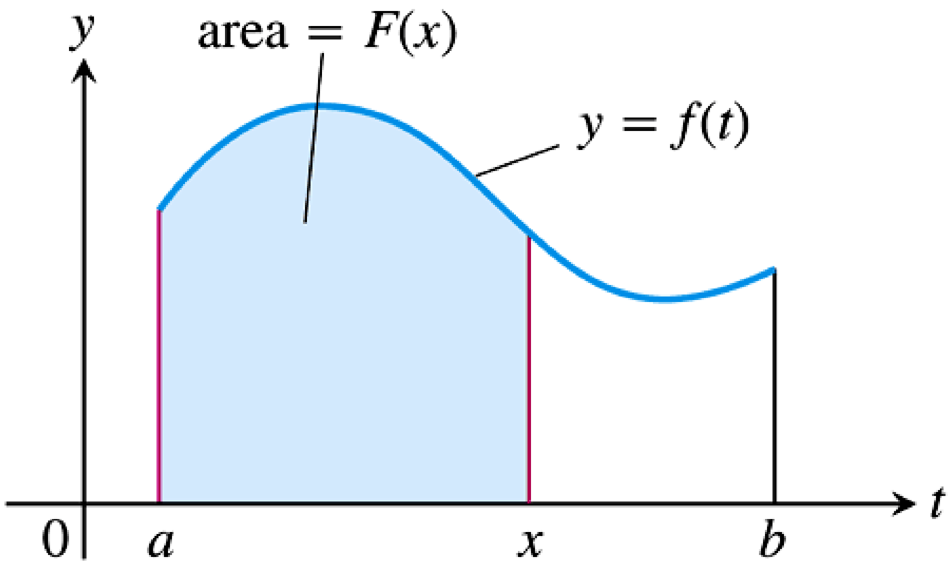


FIGURE 5.19 The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.

Fundamental theorem Part 1 (cont.)

$$F(x+h) - F(x) \approx hf(x)$$

$$\frac{F(x+h) - F(x)}{h} \approx f(x)$$

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = F'(x) = f(x)$$

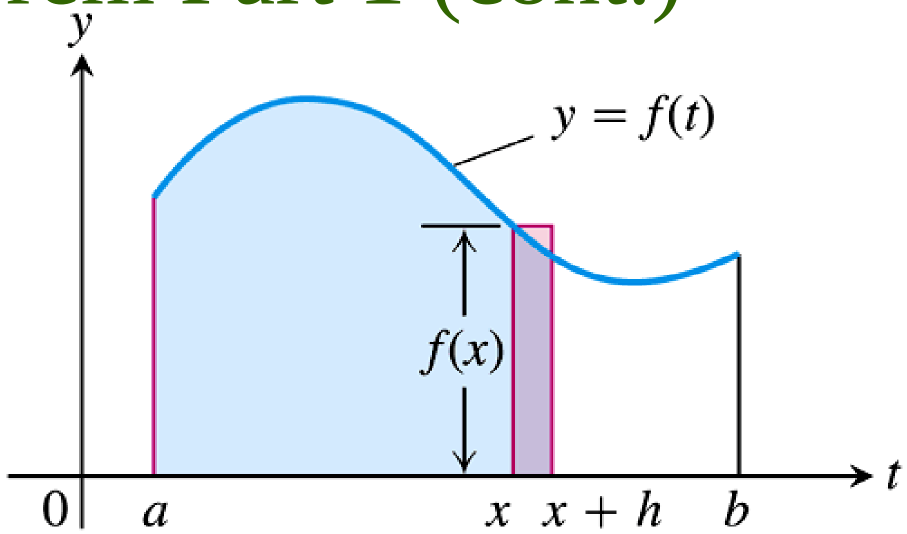


FIGURE 5.20 In Equation (1), $F(x)$ is the area to the left of x. Also, $F(x + h)$ is the area to the left of $x + h$. The difference quotient $[F(x + h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

The above result holds true even if *f* is not positive definite over [*a*,*b*]

THEOREM 4 The Fundamental Theorem of Calculus Part 1

If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

Example 3 Applying the fundamental theorem

- Use the fundamental theorem to find

$$(a)\ \frac{d}{dx}\int_a^x \cos t\,dt \qquad (b)\ \frac{d}{dx}\int_a^x \frac{1}{1+t^2}\,dt$$

$$(c)\ \frac{dy}{dx}\ \text{if } y=\int_x^5 3t\sin t\,dt \qquad (d)\ \frac{dy}{dx}\ \text{if } y=\int_1^{x^2} \cos t\,dt$$

Example 4 Constructing a function with a given derivative and value

- Find a function $y = f(x)$ on the domain $(-\pi/2, \pi/2)$ with derivative $dy/dx = \tan x$

  that satisfy $f(3)=5$.

Solution $k(x) = \int_a^x \tan t\,dt$

- Set the constant $a = 3$, and then add to $k(3) = 0$ a value of 5, that would make $k(3) + 5 = 5$
- Hence the function that will do the job is

$$f(x) = k(x) + 5 = \int_3^x \tan t\,dt + 5$$

Fundamental theorem, part 2 (The evaluation theorem)

THEOREM 4 (Continued) **The Fundamental Theorem of Calculus Part 2**

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

To calculate the definite integral of *f* over [*a*,*b*], do the following

- 1. Find an antiderivative *F* of *f*, and
- 2. Calculate the number

$$\int_a^b f(x)dx = F(b) - F(a)$$

To summarise

$$\frac{d}{dx}\int_a^x f(t)dt = \frac{dF(x)}{dx} = f(x)$$

$$\int_a^x \left(\frac{dF(t)}{dt}\right)dt = \int_a^x f(t)dt = F(x) - F(a)$$

Example 5 Evaluating integrals

$$(a)\int_{0}^{\pi} \cos x\,dx$$

$$(b)\int_{-\pi/4}^{0} \sec x \tan x\,dx$$

$$(c)\int_{1}^{x^2} \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)dx$$

Example 7 Canceling areas

- Compute
- (a) the definite integral of $f(x)$ over $[0,2\pi]$
- (b) the area between the graph of $f(x)$ and the x-axis over $[0,2\pi]$

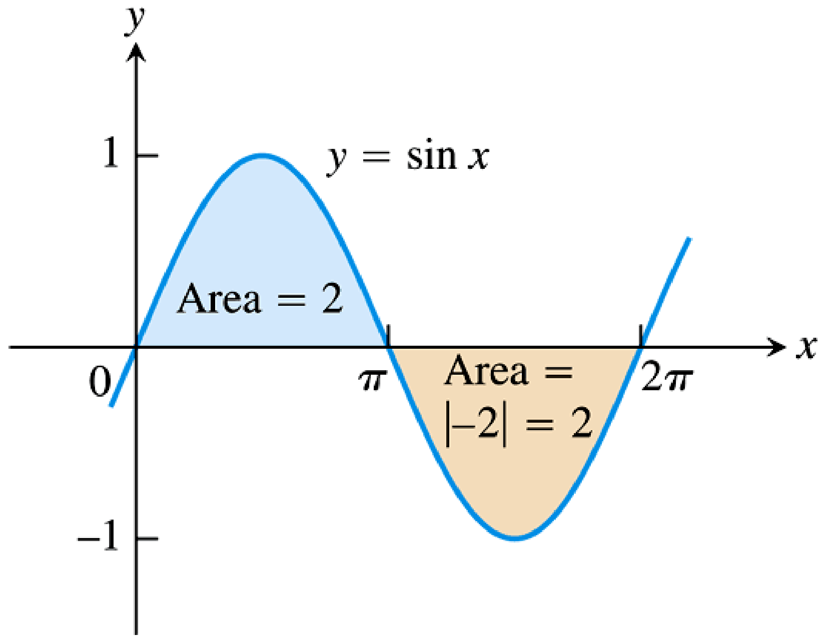


FIGURE 5.22 The total area between $y = \sin x$ and the x-axis for $0 \le x \le 2\pi$ is the sum of the absolute values of two integrals (Example 7).

Example 8 Finding area using antiderivative

- Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leqslant x \leqslant 2$.

- **Solution**
- First find the zeros of f.
- $f(x) = x(x+1)(x-2)$

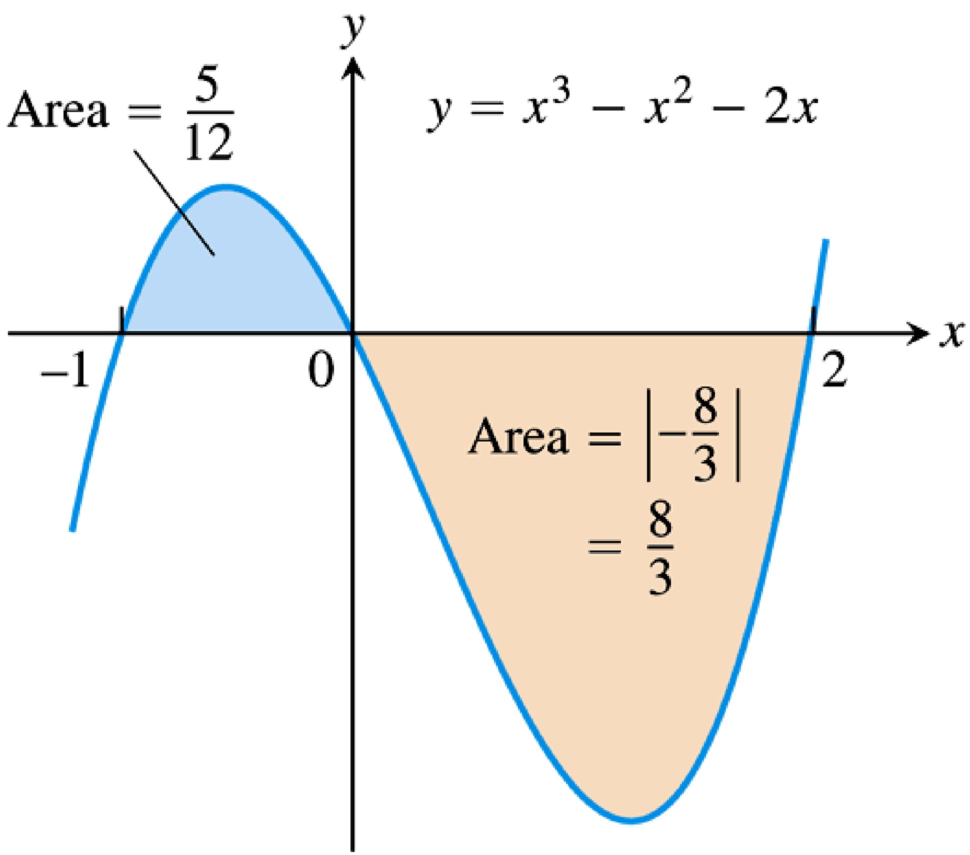


FIGURE 5.23 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 8).

5.5

Indefinite Integrals and the Substitution Rule

(3rd lecture of week 03/09/07-08/09/07)

Note

- The indefinite integral of *f* with respect to *x*

$$\int f(x)dx$$

is a function plus an arbitrary constant

- A definite integral $\int_a^b f(x)dx$ is a number.

The power rule in integral form

- From $\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx} \rightarrow \int \left(u^n \frac{du}{dx}\right) dx = \left(\frac{u^{n+1}}{n+1}\right)$

$\int \left(u^n \frac{du}{\cancel{dx}}\right) \cancel{dx} = \int u^n du$ differential of $u(x), du$ is $du = \frac{du}{dx} dx$

- We obtain the following rule

If u is any differentiable function, then

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1, n \text{ rational}). \qquad (1)$$

Example 1 Using the power rule

$$\int \sqrt{1+y^2} \cdot 2y \; dy = \int \sqrt{u} \cdot \frac{du}{dy} \cdot \; dy$$

$$= \int \sqrt{u} \; du = \ldots$$

Example 2 Adjusting the integrand by a constant

$$\int \sqrt{4t-1}\, dt = \int \frac{1}{4}\sqrt{4t-1} \cdot\ 4dt$$

$$= \frac{1}{4}\int \sqrt{u} \cdot \left(\frac{du}{dt}\right) dt = \frac{1}{4}\int \sqrt{u}\ du = \ldots$$

Substitution: Running the chain rule backwards

THEOREM 5 The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

let $u = g(x); \int f[g(x)] \cdot g'(x)dx = \int f(u) \cdot \frac{du}{dx}dx = \int f(u)du$

Used to find the integration with the integrand in the form of the product of $f[g(x)] \cdot g'(x)dx$

$$\int \underbrace{f[g(x)]}_{f(u)} \cdot \underbrace{g'(x)dx}_{du} = \int f(u)du$$

Example 3 Using substitution

$$\int \cos \underbrace{(7x+5)}_{u} \underbrace{dx}_{\frac{1}{7}du} = \int \cos u \cdot \frac{du}{7} = \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7x+5) + C$$

Example 4 Using substitution

$$\int x^2 \sin x^3 \; dx = \int \sin \underbrace{x^3}_{u} \; \underbrace{x^2 dx}_{\frac{1}{3}du} =$$

Example 5 Using Identities and substitution

$$\int \frac{1}{\cos^2 2x}\, dx = \int \sec^2 2x\, dx = \int \sec^2 \underbrace{2x}_{u}\ \underbrace{dx}_{\frac{1}{2}du} =$$

$$\frac{1}{2}\int \underbrace{\sec^2 u}_{\frac{d}{du}\tan u}\, du = \frac{1}{2}\int d(\tan u) = \frac{1}{2}\tan u + C = \frac{1}{2}\tan 2x + C$$

Example 6 Using different substitutions

$$\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz = \int \underbrace{\left(z^2+1\right)^{-1/3}}_{u^{-1/3}} \underbrace{2z\,dz}_{du} = \int u^{-1/3}\,du = \ldots$$

The integrals of $\sin^2 x$ and $\cos^2 x$

- Example 7

$$\int \sin^2 x\, dx = \frac{1}{2}\int 1 - \cos 2x\, dx$$

$$= \frac{x}{2} - \frac{1}{2}\int \cos \underbrace{2x}_{u} \underbrace{dx}_{\frac{1}{2}du}$$

$$= \frac{x}{2} - \frac{1}{4}\int \cos u\, du = \ldots$$

The integrals of $\sin^2 x$ and $\cos^2 x$

- Example 7(b)

$$\int \cos^2 x \, dx = \frac{1}{2}\int \cos 2x + 1 \, dx = \ldots$$

Example 8 Area beneath the curve $y=x^2$

- For Figure 5.24, find
- (a) the definite integral of $g(x)$ over $[0,2\pi]$.
- (b) the area between the graph and the x-axis over $[0,2\pi]$.

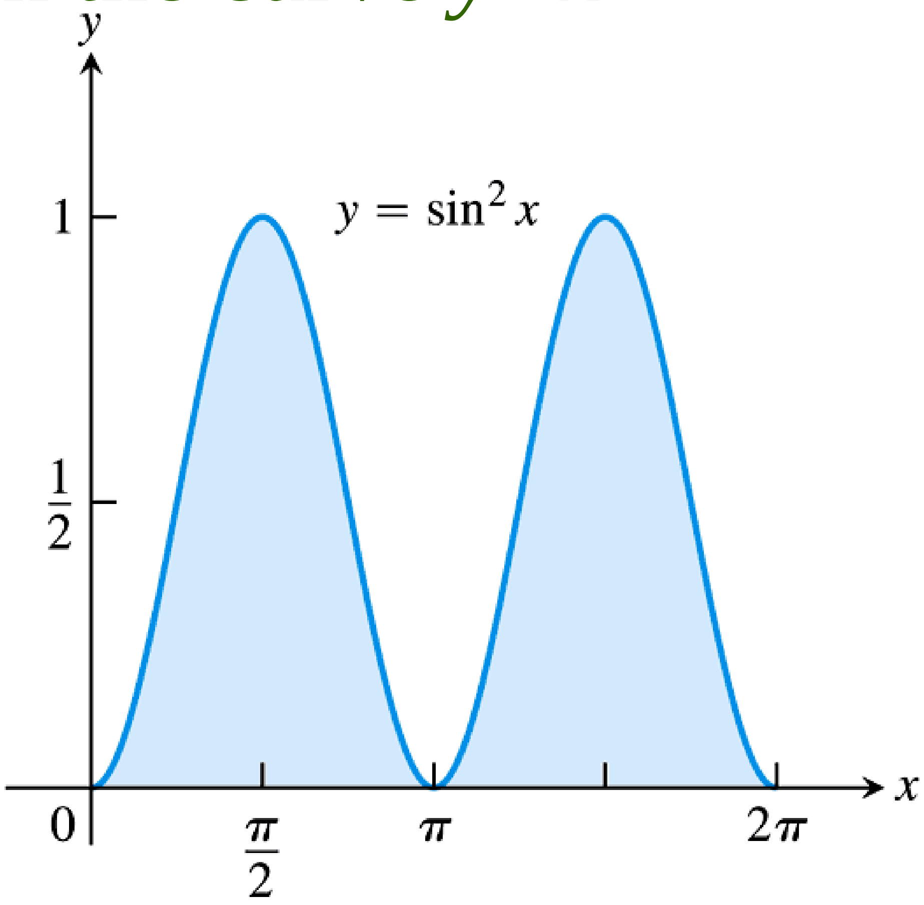


FIGURE 5.24 The area beneath the curve $y = \sin^2 x$ over $[0, 2\pi]$ equals π square units (Example 8).

5.6

Substitution and
Area Between Curves

(3rd lecture of week 03/09/07-08/09/07)

Substitution formula

THEOREM 6 Substitution in Definite Integrals

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

$$\text{let } u = g(x); \int_{x=a}^{x=b} f[g(x)] \cdot g'(x)dx = \int_{x=a}^{x=b} f[u] \cdot \frac{du}{dx} dx = \int_{u=g(a)}^{u=g(b)} f(u)du$$

Example 1 Substitution

- Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$

$$\int_{x=-1}^{x=1} \underbrace{\sqrt{x^3+1}}_{u^{1/2}} \cdot \underbrace{3x^2dx}_{du} = \int_{u(x=1)}^{u(x=1)} u^{1/2} \cdot du = \ldots$$

Example 2 Using the substitution formula

$$\int_{x=\pi/4}^{x=\pi/2} \cot x \csc^2 x dx = ?$$

$$\int \cot x \csc^2 x dx = \int \underbrace{\cot x}_{u} \cdot \underbrace{\csc^2 x dx}_{-du} = -\int u du = -\frac{u^2}{2} + c$$

$$= -\frac{\cot^2 x}{2} + c$$

$$\int_{\pi/4}^{\pi/2} \cot x \csc^2 x dx = -\frac{\cot^2 x}{2}\Bigg|_{\pi/4}^{\pi/2} = \frac{\cot^2 x}{2}\Bigg|_{\pi/2}^{\pi/4} = \frac{1}{2}\left[\underbrace{\cot^2(\pi/4)}_{1} - \underbrace{\cot^2(\pi/2)}_{0}\right] = \frac{1}{2}$$

Definite integrals of symmetric functions

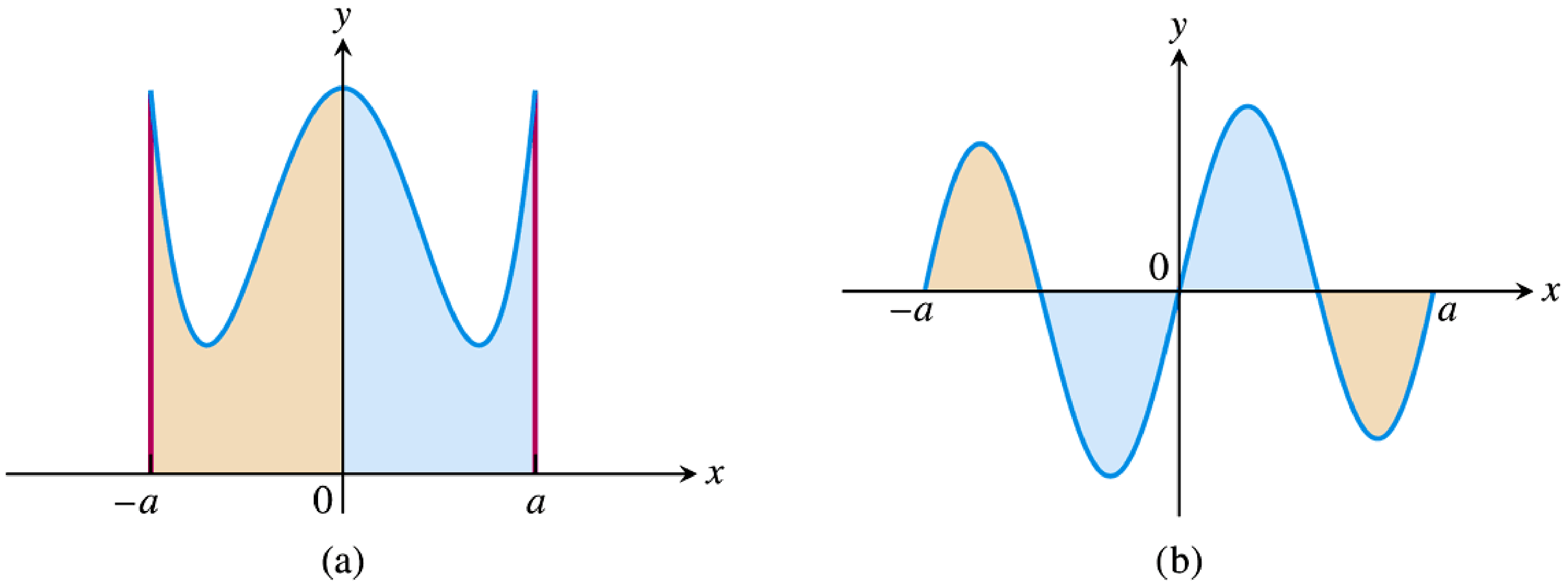


FIGURE 5.26 (a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ (b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

Theorem 7

Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Example 3 Integral of an even function

Evaluate $\int_{-2}^{2}\left(x^4-4x^2+6\right)dx$

Solution:

$f(x)=x^4-4x^2+6;$

$f(-x)=\left(-x\right)^4-4\left(-x\right)^2+6=x^4-4x^2+6=f(x)$

even function

How about integration of the same function from *x*=-1 to *x*=2

Area between curves

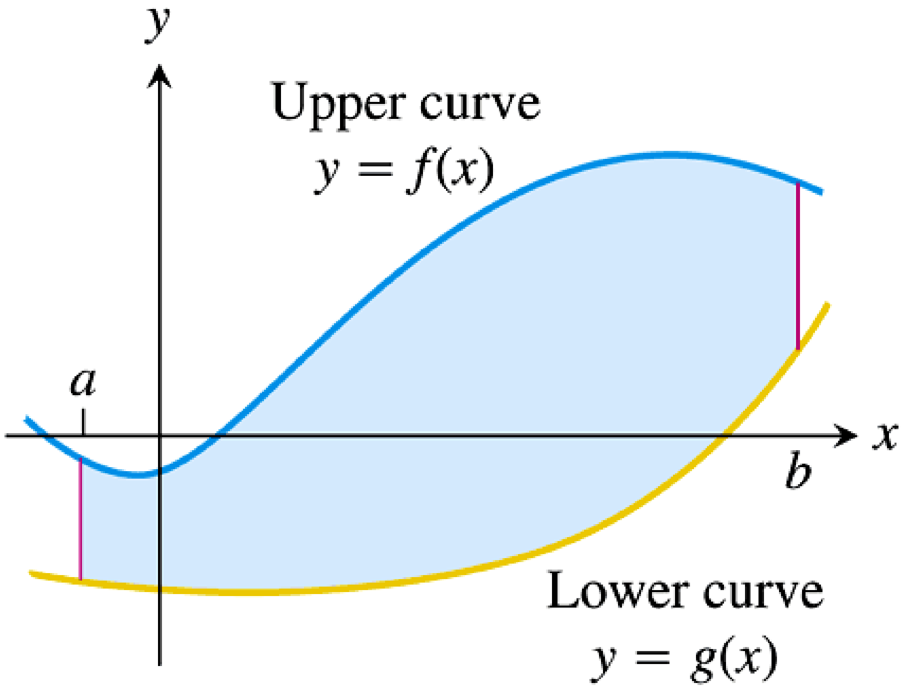


FIGURE 5.27 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

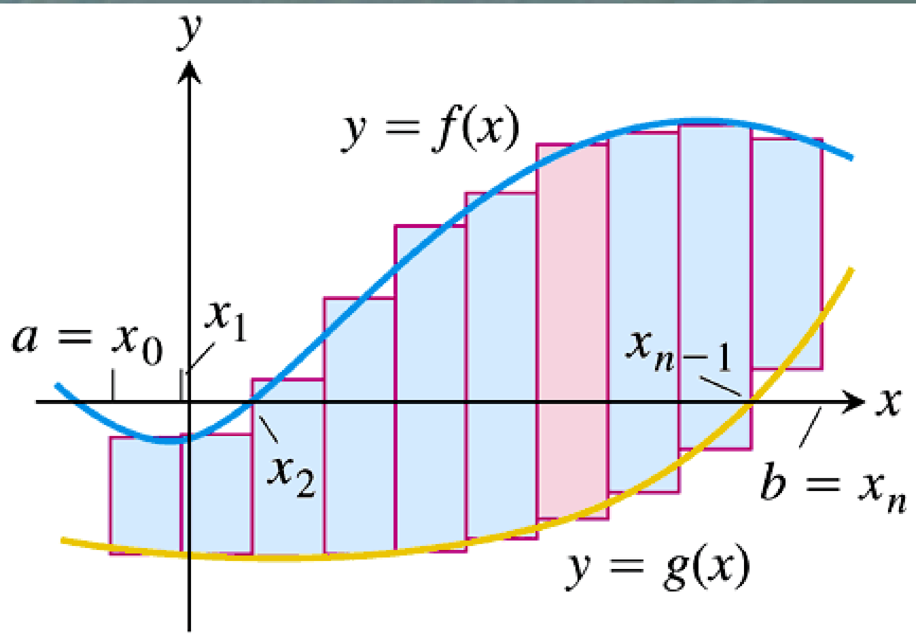


FIGURE 5.28 We approximate the region with rectangles perpendicular to the x-axis.

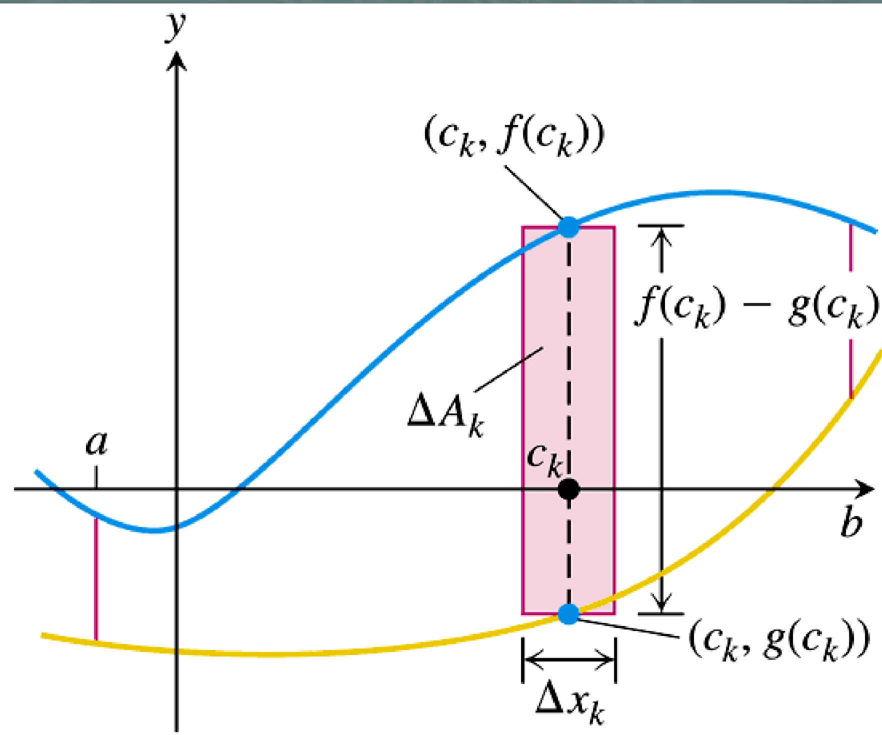


FIGURE 5.29 The area ΔA_k of the kth rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k.

$$A \approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} \Delta x_k \left[\left(f(c_k) - g(c_k) \right) \right]$$

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \Delta x_k \left[\left(f(c_k) - g(c_k) \right) \right] = \int_a^b \left[f(x) - g(x) \right] dx$$

DEFINITION **Area Between Curves**

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Example 4 Area between intersecting curves

- Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

$$\Delta A = \left(f(x) - g(x)\right) \cdot \Delta x$$

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} \Delta A_k = \int_0^A dA;$$

$$A = \int_{a=-1}^{b=2} [\underbrace{f(x)}_{2-x^2} - \underbrace{g(x)}_{x}]dx$$

$$= \int_{-1}^{2} \left(2 - x^2 - x\right) dx = \ldots$$

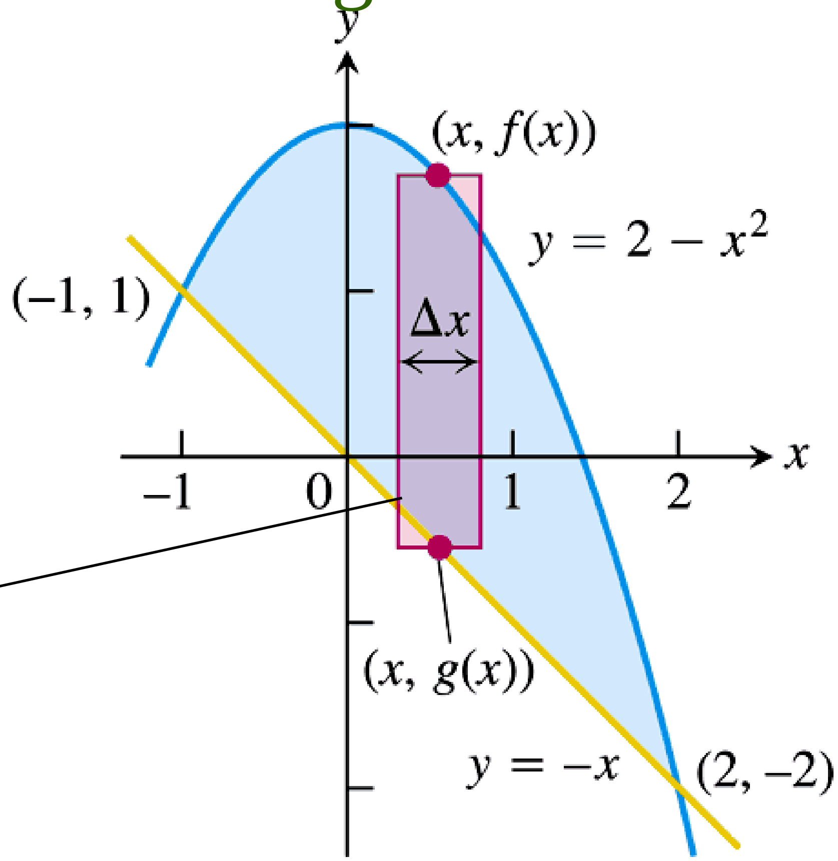


FIGURE 5.30 The region in Example 4 with a typical approximating rectangle.

Example 5 Changing the integral to match a boundary change

- Find the area of the shaded region

$$Area = A + B$$

$$A = \int_0^2 \sqrt{x}dx;$$

$$B = \int_2^4 \sqrt{x} - (x-2)dx$$

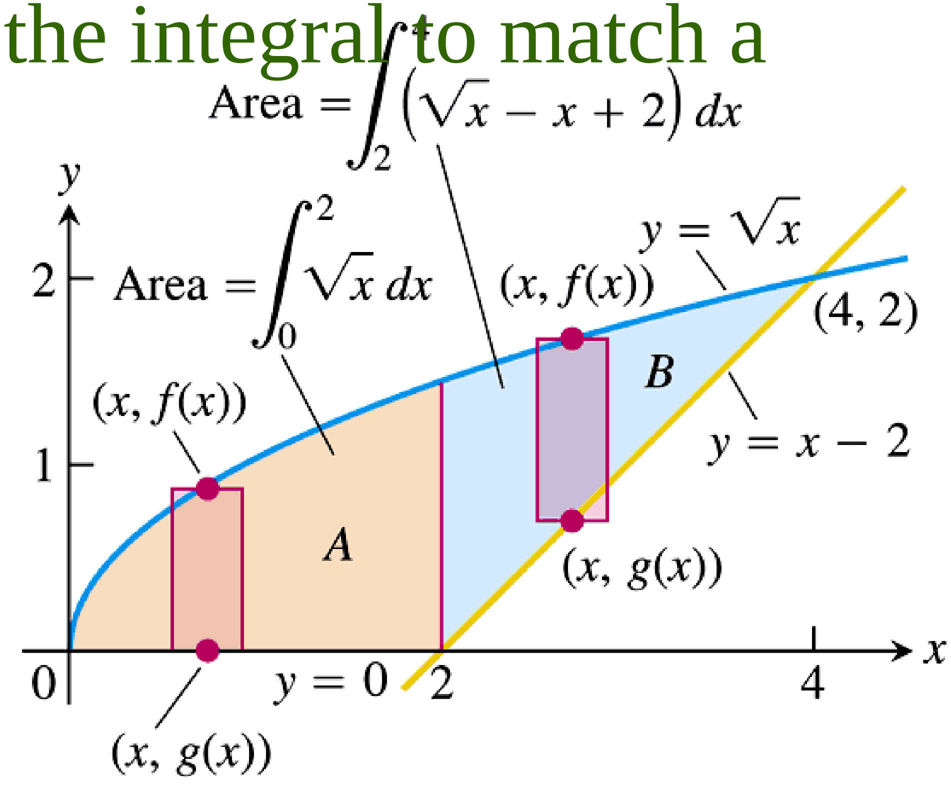


FIGURE 5.31 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 5.

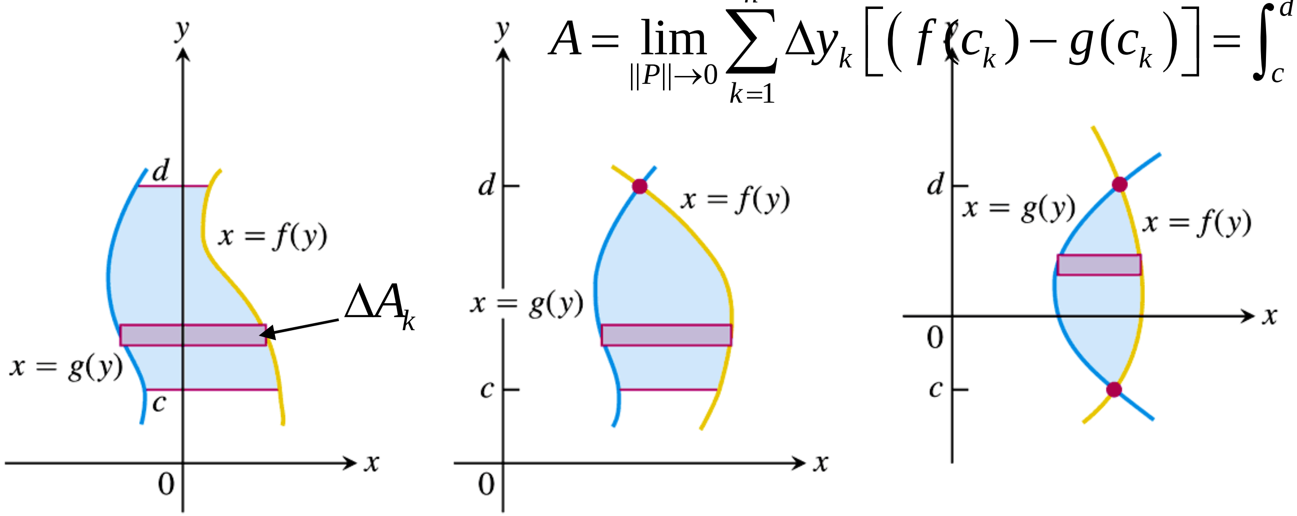
Integration with Respect to *y*

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these

$$A \approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} \Delta y_k \left[\left(f(c_k) - g(c_k) \right) \right]$$

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \Delta y_k \left[\left(f(c_k) - g(c_k) \right) \right] = \int_c^d [f(y) - g(y)]\,dy$$



use the formula

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

Example 6 Find the area of the region in Example 5 by integrating with respect to *y*

$$\Delta A = (f(y) - g(y)) \cdot \Delta y$$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta A_k = \int_{y=0}^{y=4} [f(y) - g(y)]dy$$

$$= \int_{-1}^{2} (y+2) - (y^2)\, dy = \ldots$$

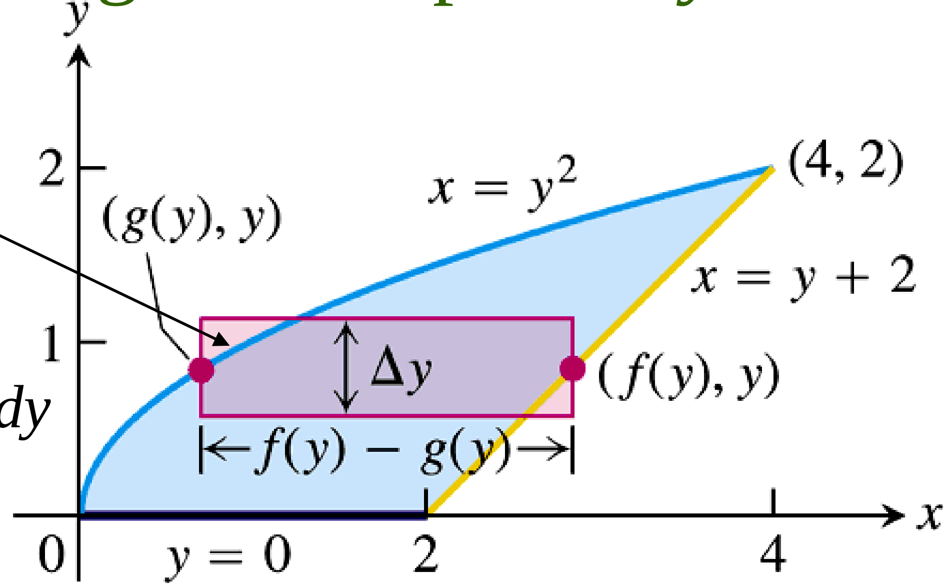


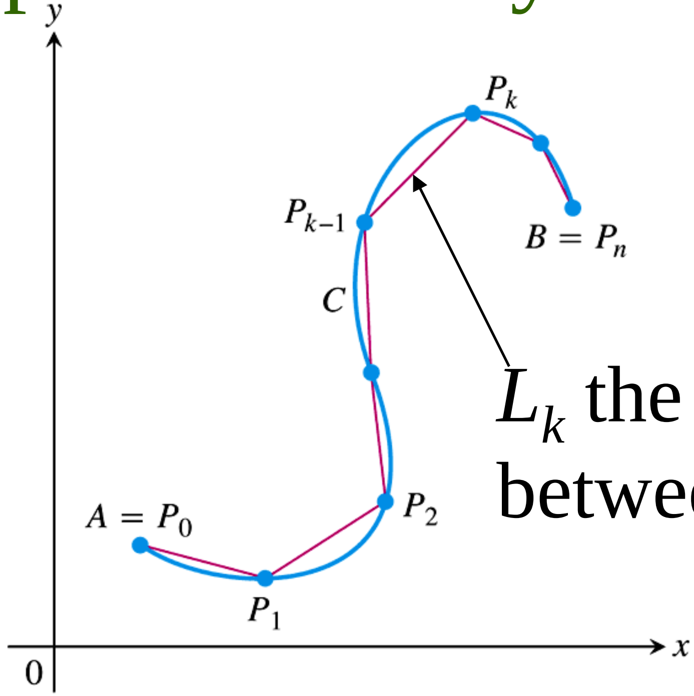
FIGURE 5.32 It takes two integrations to find the area of this region if we integrate with respect to *x*. It takes only one if we integrate with respect to *y* (Example 6).

6.3

Lengths of Plane Curves

(1st lecture of week 10/09/07-15/09/07)

Length of a parametrically defined curve



L_k the line segment between P_k and $P_{k\text{-}1}$

$$L = \lim_{\|P\| \to 0} \sum_{k}^{n} L_k$$

FIGURE 6.24 The curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1, and so on, ending at $B = P_n$.

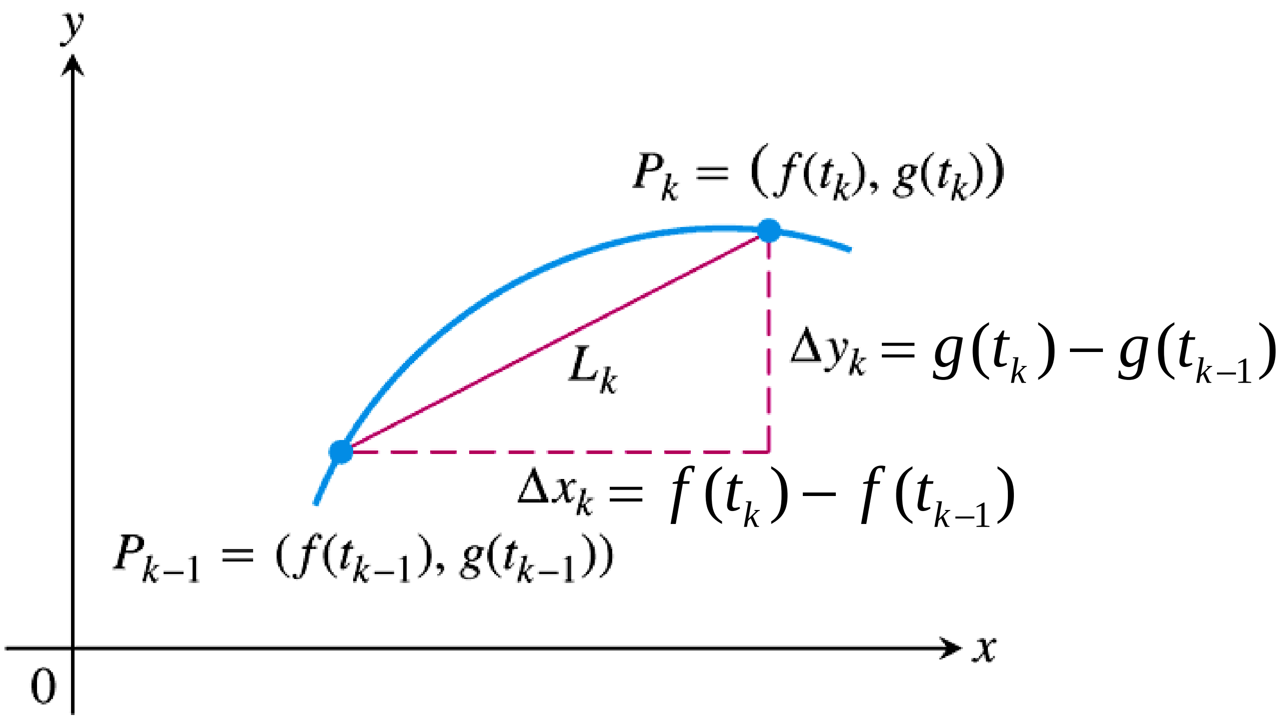


FIGURE 6.25 The arc $P_{k-1}P_k$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

$$\Delta y_k = g(t_k) - g(t_{k-1}) = g'(t_k^*)\cdot\left(t_k - t_{k-1}\right) = g'(t_k^*)\cdot\Delta t;$$

$$\Delta x_k = f(t_k) - f(t_{k-1}) = f'(t_k^{**})\cdot\left(t_k - t_{k-1}\right) = f'(t_k^{**})\cdot\Delta t$$

due to mean value theorem

$$L_k = \sqrt{\left(\Delta y_k\right)^2 + \left(\Delta x_k\right)^2} = \Delta t\sqrt{\left(g'(t_k^*)\right)^2 + \left(f'(t_k^{**})\right)^2}$$

$$L = \lim_{n\to\infty}\sum_k^n L_k = \lim_{\|P\|\to 0}\sum_k^n L_k$$

$$= \lim_{\|P\|\to 0}\sum_k^n \Delta t\sqrt{\left(g'(t_k^*)\right)^2 + \left(f'(t_k^{**})\right)^2}$$

$$= \int_a^b \sqrt{\left(g'(t)\right)^2 + \left(f'(t)\right)^2}\,dt = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\,dt$$

DEFINITION **Length of a Parametric Curve**

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of *C*** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Example 1 The circumference of a circle

- Find the length of the circle of radius *r* defined parametrically by
- $x = r\cos t$ and $y = r\sin t$, $0 \leqslant t \leqslant 2\pi$

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt \equiv \int_0^{2\pi} \sqrt{(r\cos t)^2 + (r\sin t)^2}\, dt$$

$$= r\int_0^{2\pi} dt = 2\pi r$$

Length of a curve $y = f(x)$

Assign the parameter $x = t$, the length of the curve $y = f(x)$ is then given by

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$$y = y[x(t)] \Rightarrow \frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} = \frac{dy}{dx}$$

$$L = \int_a^b dt \sqrt{\left(\frac{dy}{dx}\cdot\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \int_a^b dx \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}$$

$$= \int_a^b dx \sqrt{[f'(x)]^2 + 1}$$

Formula for the Length of $y = f(x)$, $a \le x \le b$

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \qquad (2)$$

Example 3 Applying the arc length formula for a graph

- Find the length of the curve

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1, \quad 0 \le x \le 1$$

Dealing with discontinuity in *dy*/*dx*

- At a point on a curve where *dy*/*dx* fails to exist and we may be able to find the curve's length by expressing *x* as a function of *y* and applying the following

Formula for the Length of $x = g(y)$, $c \leq y \leq d$

If g is continuously differentiable on $[c, d]$, the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \qquad (3)$$

Example 4 Length of a graph which has a discontinuity in *dy*/*dx*

- Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.
- **Solution**
- $dy/dx = (1/3)\,(2/x)^{1/3}$ is not defined at $x=0$.
- $dx/dy = 3y^{1/2}$ is continuous on [0,1].

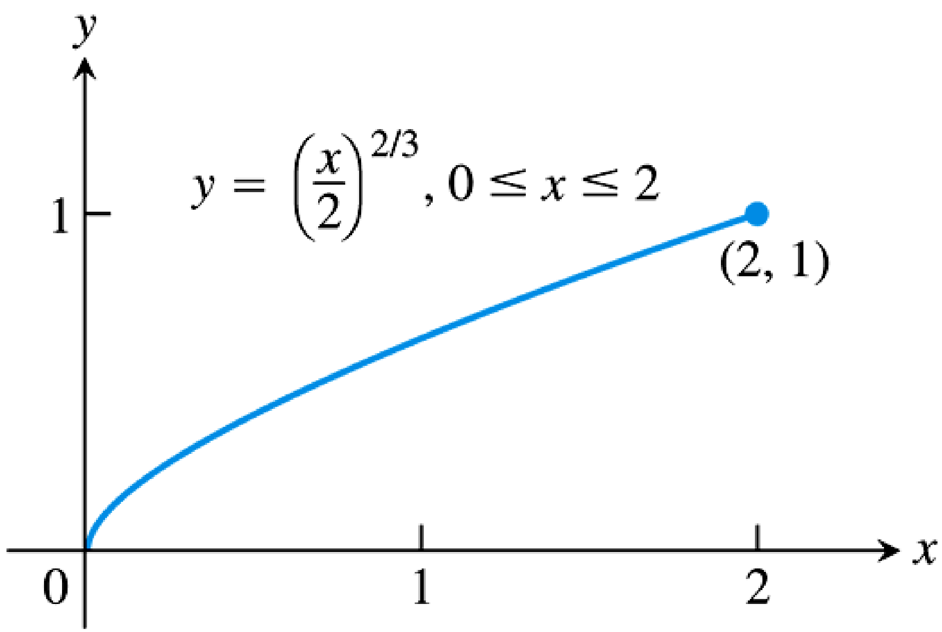


FIGURE 6.27 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Example 4).

Chapter 7

Transcendental Functions

7.1

Inverse Functions and Their Derivatives

(1st lecture of week 10/09/07-15/09/07)

DEFINITION One-to-One Function

A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Example 1 Domains of one-to-one functions

- (a) $f(x) = x^{1/2}$ is one-to-one on any domain of nonnegative numbers
- (b) $g(x) = \sin x$ is NOT one-to-one on $[0,\pi]$ but one-to-one on $[0,\pi/2]$.

The Horizontal Line Test for One-to-One Functions

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

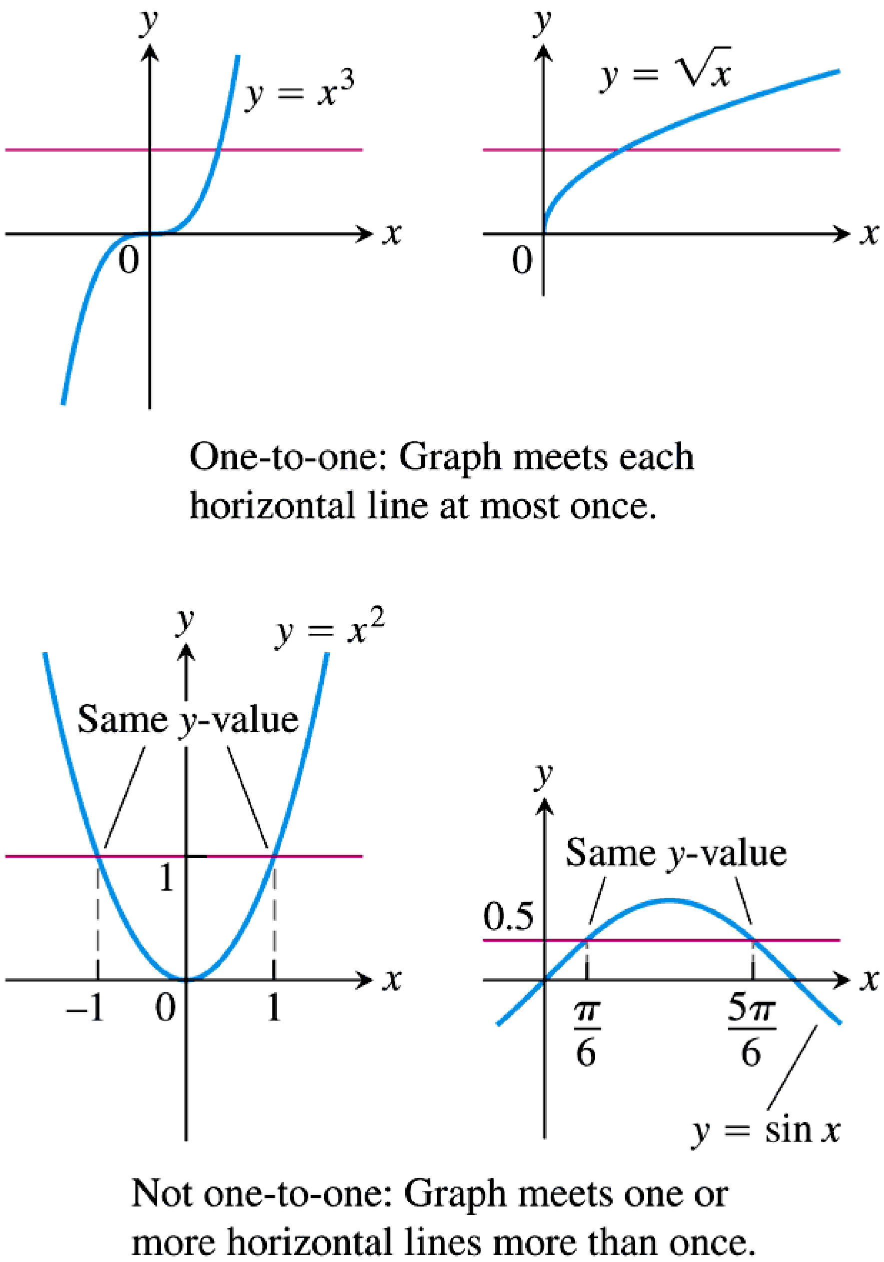


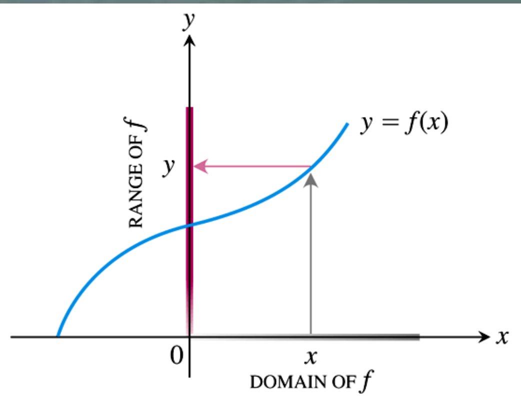
FIGURE 7.1 Using the horizontal line test, we see that $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$, but $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

DEFINITION **Inverse Function**

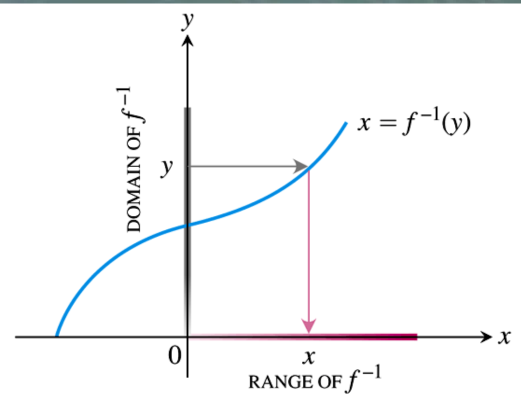
Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a.$$

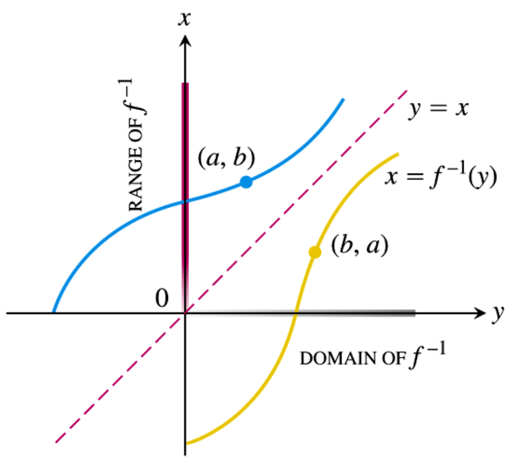
The domain of f^{-1} is R and the range of f^{-1} is D.



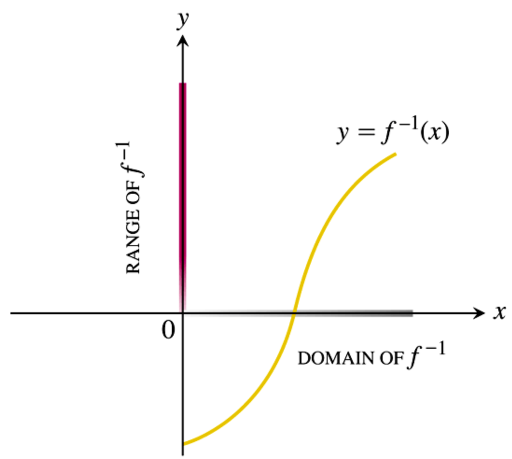
(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.



(b) The graph of f is already the graph of f^{-1}, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.



(c) To draw the graph of f^{-1} in the more usual way, we reflect the system in the line $y = x$.



(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

FIGURE 7.2 Determining the graph of $y = f^{-1}(x)$ from the graph of $y = f(x)$.

Finding inverses

- 1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.

2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where $f^{-1}(x)$ is expressed in the conventional format with x as the independent variable and y as the dependent variables.

Example 2 Finding an inverse function

- Find the inverse of $y = x/2 + 1$, expressed as a function of x.

- **Solution**
- 1. solve for x in terms of y: $x = 2(y - 1)$
- 2. interchange x and y: $y = 2(x - 1)$
- The inverse function $f^{-1}(x) = 2(x - 1)$
- Check: $f^{-1}[f(x)] = 2[f(x) - 1] = 2[(x/2 + 1) - 1] = x = f[f^{-1}(x)]$

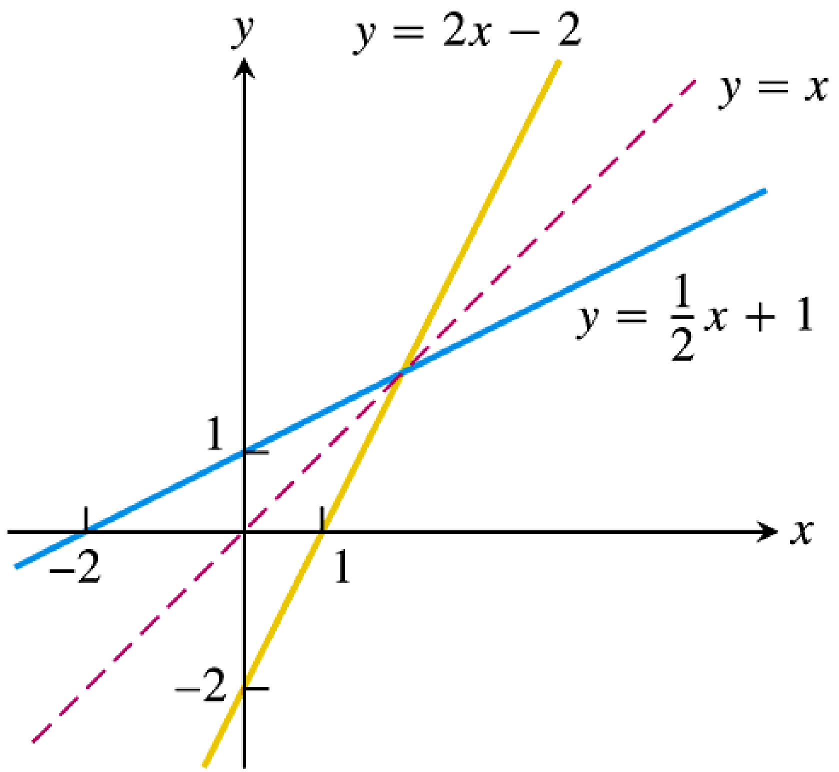


FIGURE 7.3 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$. The slopes are reciprocals of each other (Example 2).

Example 3 Finding an inverse function

- Find the inverse of $y = x^2$, $x \geqslant 0$, expressed as a function of x.
- **Solution**
- 1. solve for x in terms of y: $x = \sqrt{y}$
- 2. interchange x and y: $y = \sqrt{x}$
- The inverse function $f^{-1}(x) = \sqrt{x}$

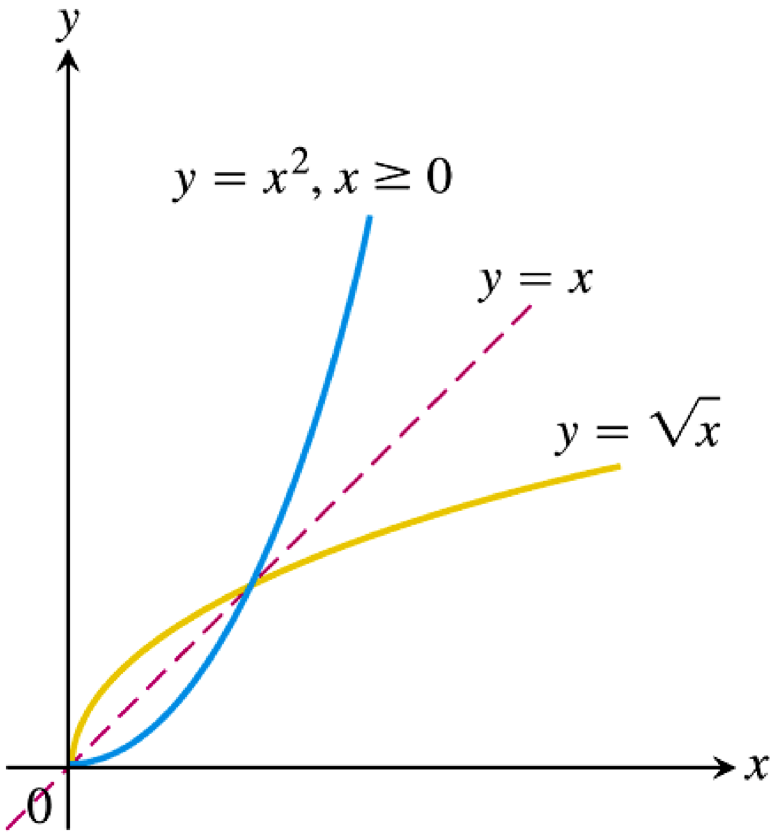


FIGURE 7.4 The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 3).

Derivatives of inverses of differentiable functions

- From example 2 (a linear function)
- $f(x) = x/2 + 1$; $f^{-1}(x) = 2(x + 1)$;
- $df(x)/dx = 1/2$; $df^{-1}(x)/dx = 2$,
- i.e. $df(x)/dx = 1/df^{-1}(x)/dx$
- Such a result is obvious because their graphs are obtained by reflecting on the $y = x$ line.
- In general, the reciprocal relationship between the slopes of f and f^{-1} holds for other functions.

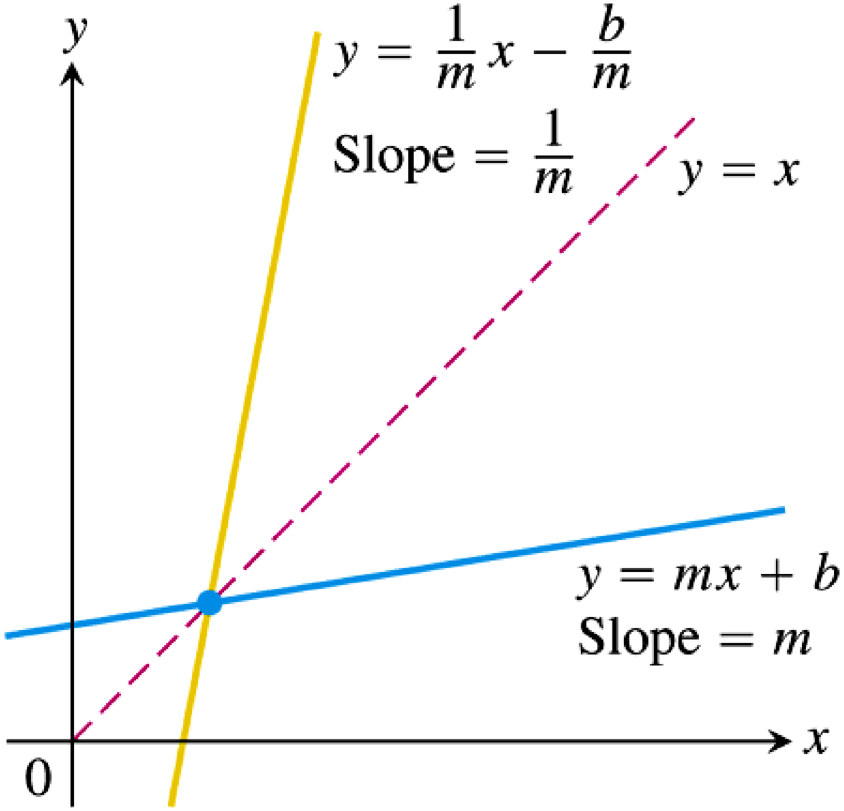
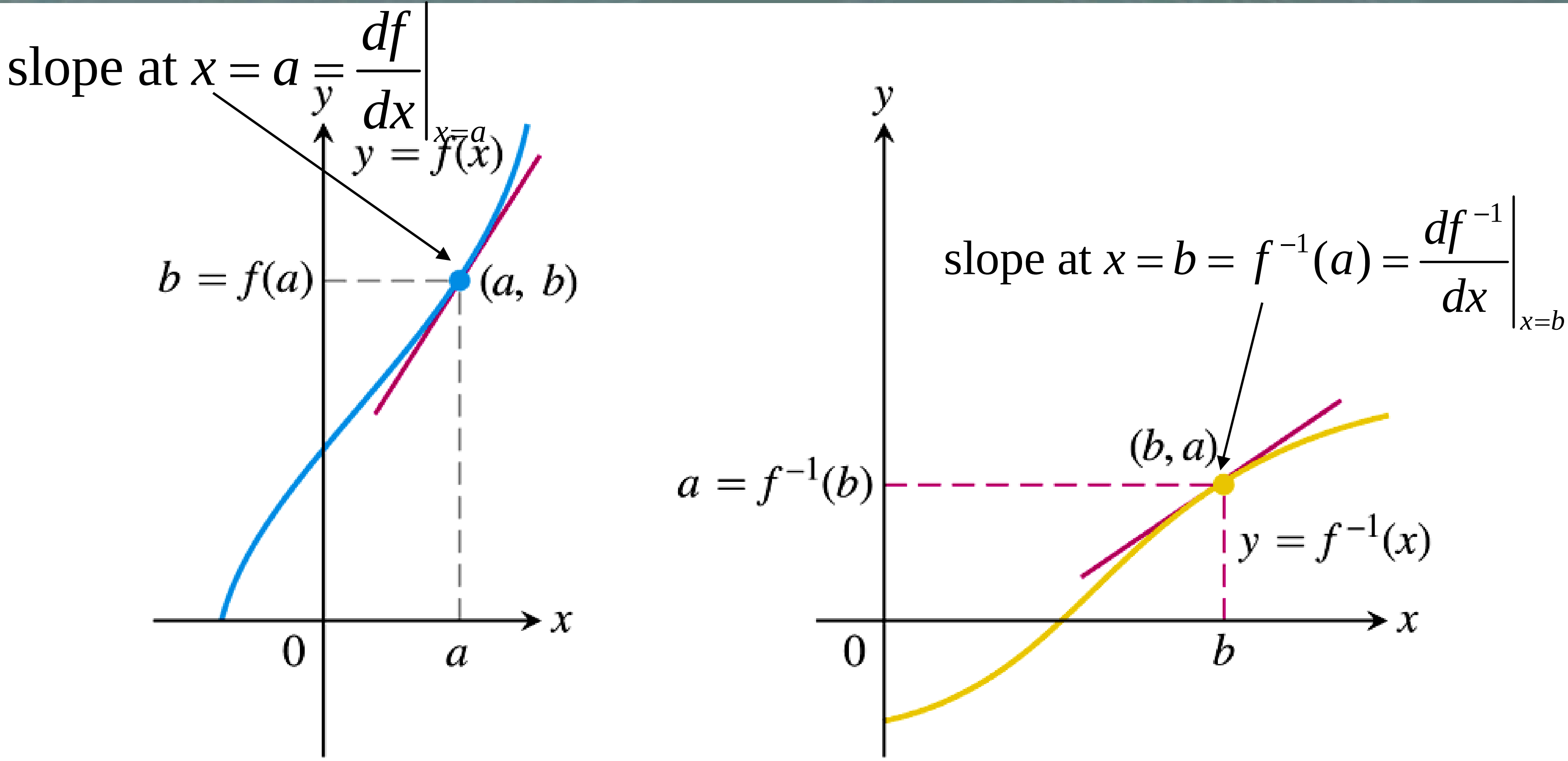


FIGURE 7.5 The slopes of nonvertical lines reflected through the line $y = x$ are reciprocals of each other.



The slopes are reciprocal: $(f^{-1})'(b) = \frac{1}{f'(a)}$ or $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$

FIGURE 7.6 The graphs of inverse functions have reciprocal slopes at corresponding points.

THEOREM 1 The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}} \qquad (1)$$

Example 4 Applying theorem 1

- The function $f(x) = x^2$, $x \geqslant 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$, and $(f^{-1})'(x) = 1/(2\sqrt{x})$.
- Theorem 1 predicts that the derivative of $f^{-1}(x)$ is

$$(f^{-1})'(x) = 1/f'[f^{-1}(x)] = 1/f'[\sqrt{x}]$$
$$= 1/(2\sqrt{x})$$

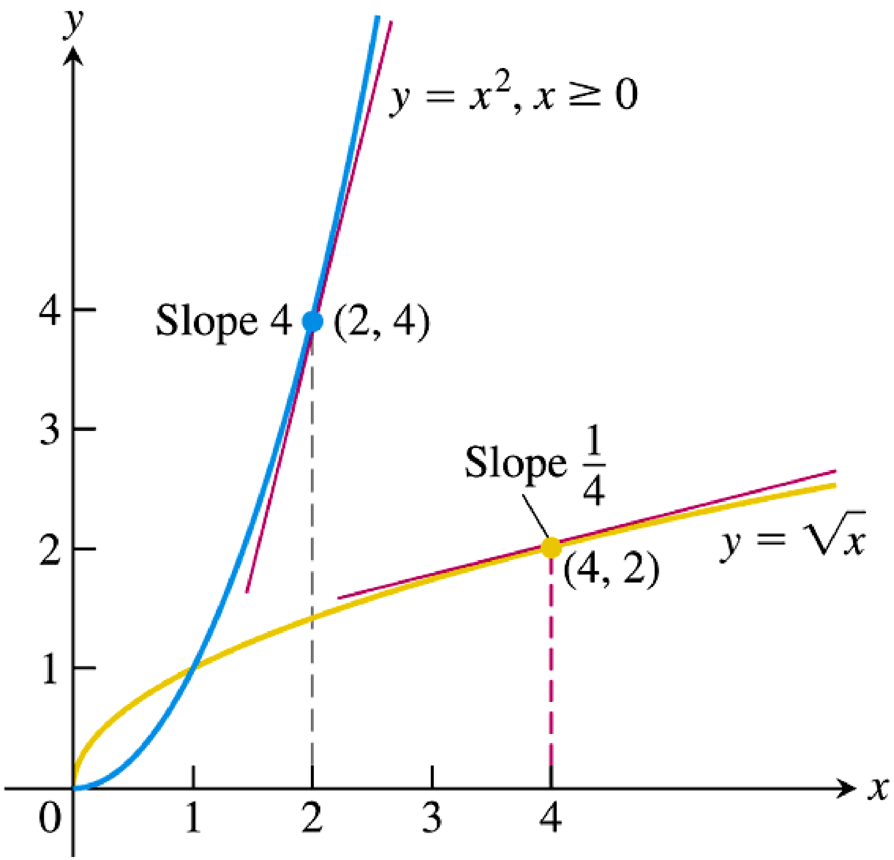


FIGURE 7.7 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 4).

Example 5 Finding a value of the inverse derivative

- Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without a formula for f^{-1}.
- The point for f is (2,6); The corresponding point for f^{-1} is (6,2).
- **Solution**
- $df/dx = 3x^2$
- $df^{-1}/dx|_{x=6} = 1/(df/dx|_{x=2}) = 1/(df/dx|_{x=2})$

  $= 1/3x^2|_{x=2} = 1/12$

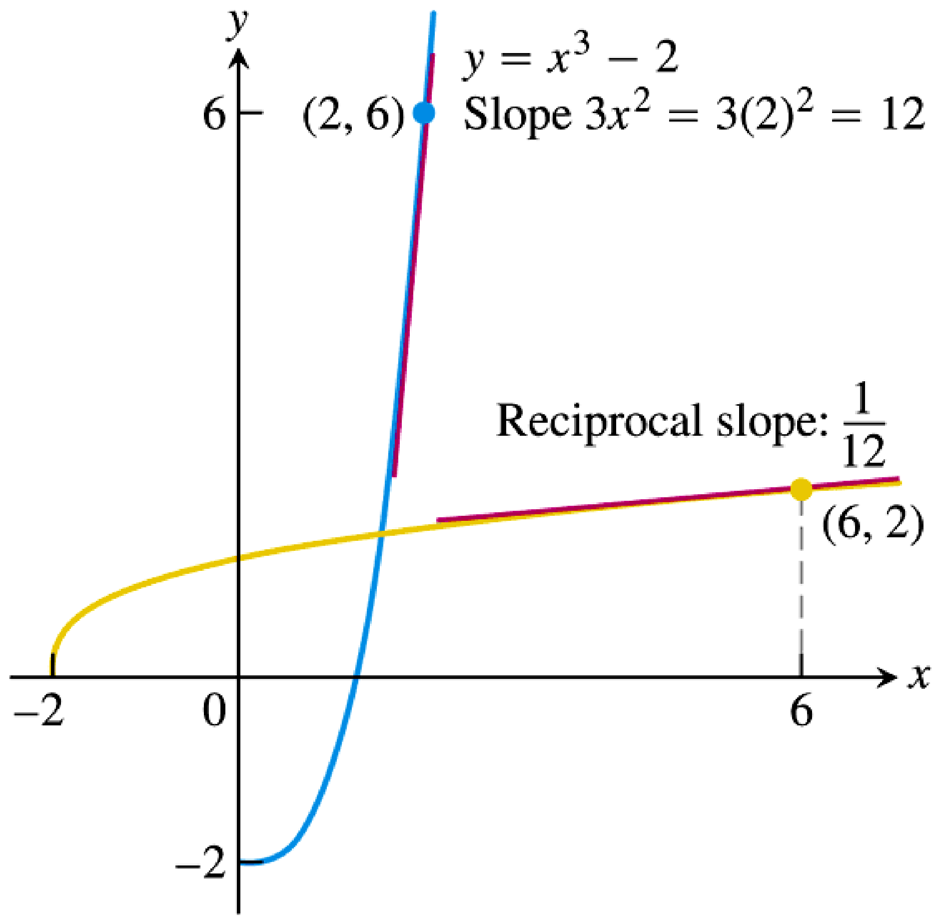


FIGURE 7.8 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 5).

7.2

Natural Logarithms

(2nd lecture of week 10/09/07-15/09/07)

Definition of natural logarithmic fuction

DEFINITION **The Natural Logarithm Function**

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$

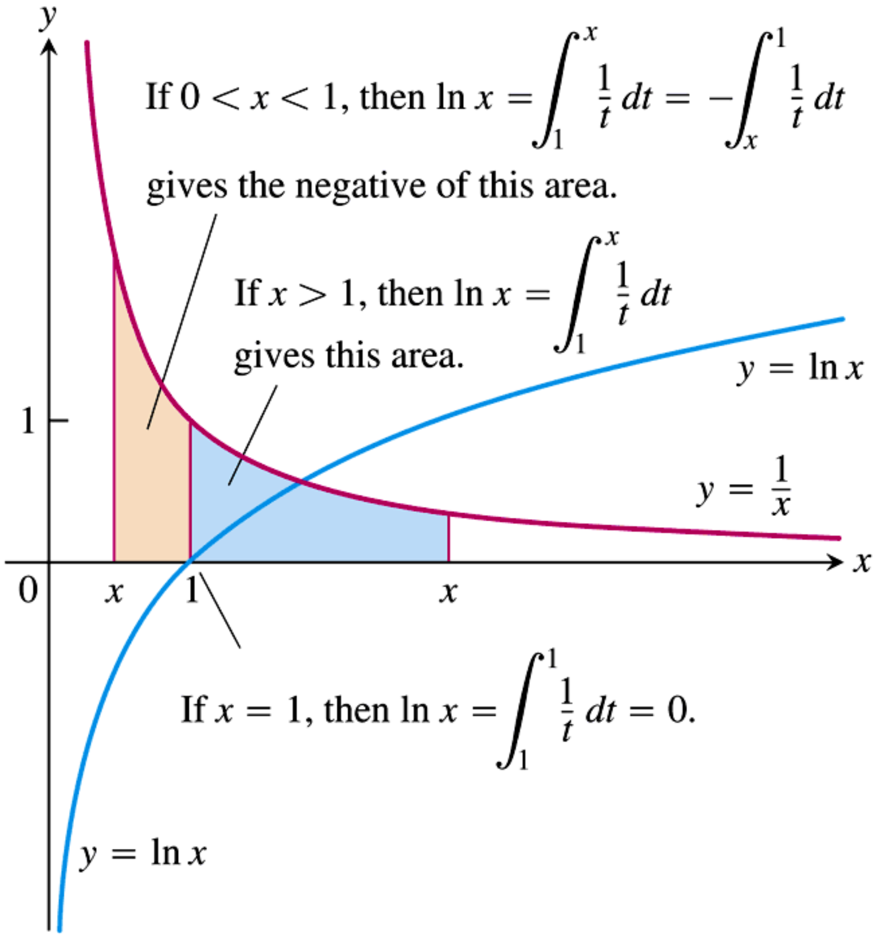


FIGURE 7.9 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left.

$$\ln x = \int_1^x \frac{1}{x}\,dx$$

TABLE 7.1 Typical 2-place values of ln x

x	ln x
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

e lies between 2 and 3

$\ln x = 1$

DEFINITION The Number e

The number e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1$$

By definition, the antiderivative of ln *x* is just 1/*x*

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0 \qquad (1)$$

Let $u = u(x)$. By chain rule,

$d/dx\,[\ln u(x)] = d/du(\ln u)\cdot du(x)/dx$

$=(1/u)\cdot du(x)/dx$

Example 1 Derivatives of natural logarithms

$$(a)\ \frac{d}{dx}\ln 2x =$$

$$(b)\quad u = x^2 + 3; \frac{d}{dx}\ln u = \frac{du}{dx}\frac{1}{u} =$$

Properties of logarithms

THEOREM 2 Properties of Logarithms

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1.	*Product Rule*:	$\ln ax = \ln a + \ln x$	
2.	*Quotient Rule*:	$\ln \frac{a}{x} = \ln a - \ln x$	
3.	*Reciprocal Rule*:	$\ln \frac{1}{x} = -\ln x$	Rule 2 with $a = 1$
4.	*Power Rule*:	$\ln x^r = r \ln x$	r rational

Example 2 Interpreting the properties of logarithms

$$(a)\ln 6 = \ln\left(2\cdot 3\right) = \ln 2 + \ln 3;$$

$$(b)\ln 4 - \ln 5 = \ln\left(4/5\right) = \ln 0.8$$

$$(c)\ln(1/8) = \ln 1 - \ln 2^3 = -3\ln 2$$

Example 3 Applying the properties to function formulas

$$(a)\ln 4+\ln\sin x=\ln\left(4\sin x\right);$$

$$(b)\ln\frac{x+1}{2x-3}=\ln\left(x+1\right)-\ln(2x-3)$$

$$(c)\ln(\sec x)=\ln\frac{1}{\cos x}=-\ln\cos x$$

$$(d)\ln\sqrt[3]{x+1}=\ln(x+1)^{1/3}=(1/3)\ln(x+1)$$

Proof of ln *ax* = ln *a* + ln *x*

- ln *ax* and ln *x* have the same derivative:

$$\frac{d}{dx}\ln ax = \frac{d(ax)}{dx}\frac{1}{ax} = a\frac{1}{ax} = \frac{1}{x} = \frac{d}{dx}\ln x$$

- Hence, by the corollary 2 of the mean value theorem, they differs by a constant *C*

$$\ln ax = \ln x + C$$

- We will prove that *C* = ln *a* by applying the definition ln *x* at *x* = 1.

Estimate the value of ln 2

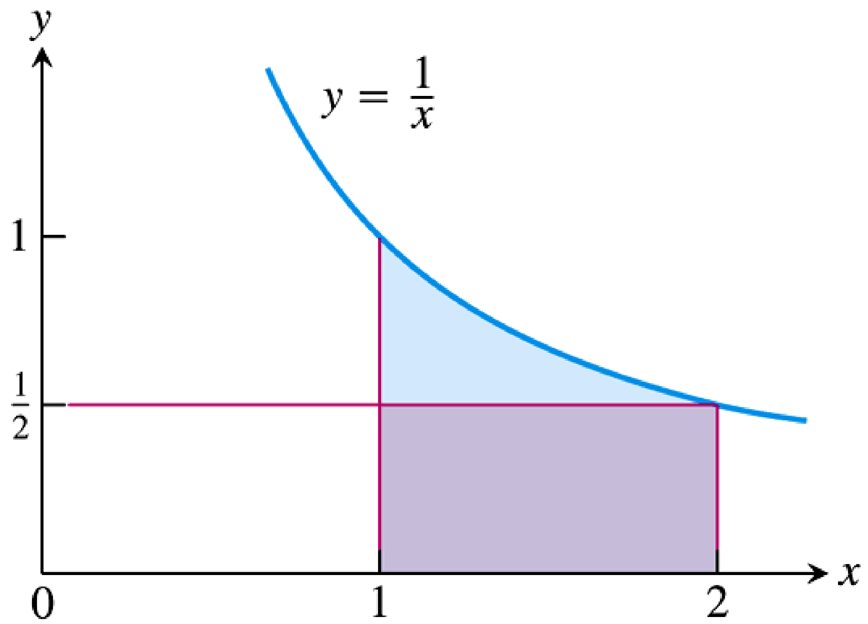


FIGURE 7.10 The rectangle of height $y = 1/2$ fits beneath the graph of $y = 1/x$ for the interval $1 \leq x \leq 2$.

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$

$$\frac{1}{2}\cdot(2-1) < \int_1^2 \frac{1}{x}\,dx < 2\cdot 2 = 1\cdot(2-1)$$

$$\frac{1}{2} < \ln 2 < 2$$

The integral $\int (1/u)\, du$

From $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

Let $u > 0$

Taking the integration on both sides gives

$$\int \frac{d}{dx}\ln u dx = \int \frac{1}{u}\frac{du}{dx}dx$$

$$\int d\ln u = \int \frac{du}{u} \rightarrow \ln u + C' = \int \frac{du}{u};$$

For $u < 0$:

$-u > 0,$

$$\int \frac{d}{dx}\ln(-u) dx = \int \frac{1}{(-u)}\frac{d(-u)}{dx}dx$$

$$\int d\ln(-u) = \int \frac{du}{u} \rightarrow \ln(-u) + C'' = \int \frac{du}{u}$$

Combining both cases of $u > 0, u < 0$,

$$\int \frac{du}{u} = \ln|u| + C$$

recall: $\int u^n du = \frac{u^{n+1}}{n+1} + C, n$ rational, $\neq 1$

If u is a differentiable function that is never zero,

$$\int \frac{1}{u}\,du = \ln|u| + C. \qquad (5)$$

From $\int u^{-1} du = \ln|u| + C.$

let $u = f(x)$.

$$\int u^{-1} du = \int \frac{du}{u} = \int \frac{df(x)}{f(x)} = \int \frac{\frac{df(x)}{dx}dx}{f(x)}$$

$$\Rightarrow \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

Example 4 Applying equation (5)

$$\text{(a)} \int \frac{2x\,dx}{x^2-5} = \int \frac{d(x^2-5)}{x^2-5} = \ln|x^2-5| + C$$

$$\text{(b)} \int_{-\pi/2}^{\pi/2} \frac{4\cos x}{3+2\sin x}\,dx = \ldots$$

The integrals of tan *x* and cot *x*

$$\int \tan u \, du = -\ln |\cos u| + C = \ln |\sec u| + C$$

$$\int \cot u \, du = \ln |\sin u| + C = -\ln |\csc x| + C$$

Example 5

$$\int \tan 2x\,dx = \int \frac{\sin 2x}{\cos 2x}dx = \int \frac{-\frac{1}{2}\frac{d}{dx}\cos 2x}{\cos 2x}dx$$

$$= -\frac{1}{2}\int \frac{d\cos 2x}{\cos 2x} = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C$$

$$= -\frac{1}{2}\ln|\cos 2x| + C$$

$$= \frac{1}{2}\ln|\sec 2x| + C$$

Example 6 Using logarithmic differentiation

- Find *dy/dx* if $y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, x > 1$

$$\ln y = \ln(x^2+1) + (1/2)\ln(x+3) - \ln(x-1)$$

$$\frac{d}{dy}\ln y = \frac{d}{dy}\ln(x^2+1) + \frac{1}{2}\frac{d}{dy}\ln(x+3) - \frac{d}{dy}\ln(x-1)$$

$$= \ldots$$

7.3

The Exponential Function

(2nd lecture of week 10/09/07-15/09/07)

The inverse of ln *x* and the number *e*

- ln *x* is one-to-one, hence it has an inverse. We name the inverse of ln *x*, $\ln^{-1} x$ as exp (*x*)
- ln *x* is an increasing function since

  $dy/dx = 1/x > 0$
- Domain of ln *x* = $(0,\infty)$
- Range of ln *x* = $(-\infty,\infty)$

  $$\lim_{x\to\infty} \ln^{-1} x = \infty, \lim_{x\to-\infty} \ln^{-1} x = 0$$
- The graph of the inverse of ln *x*

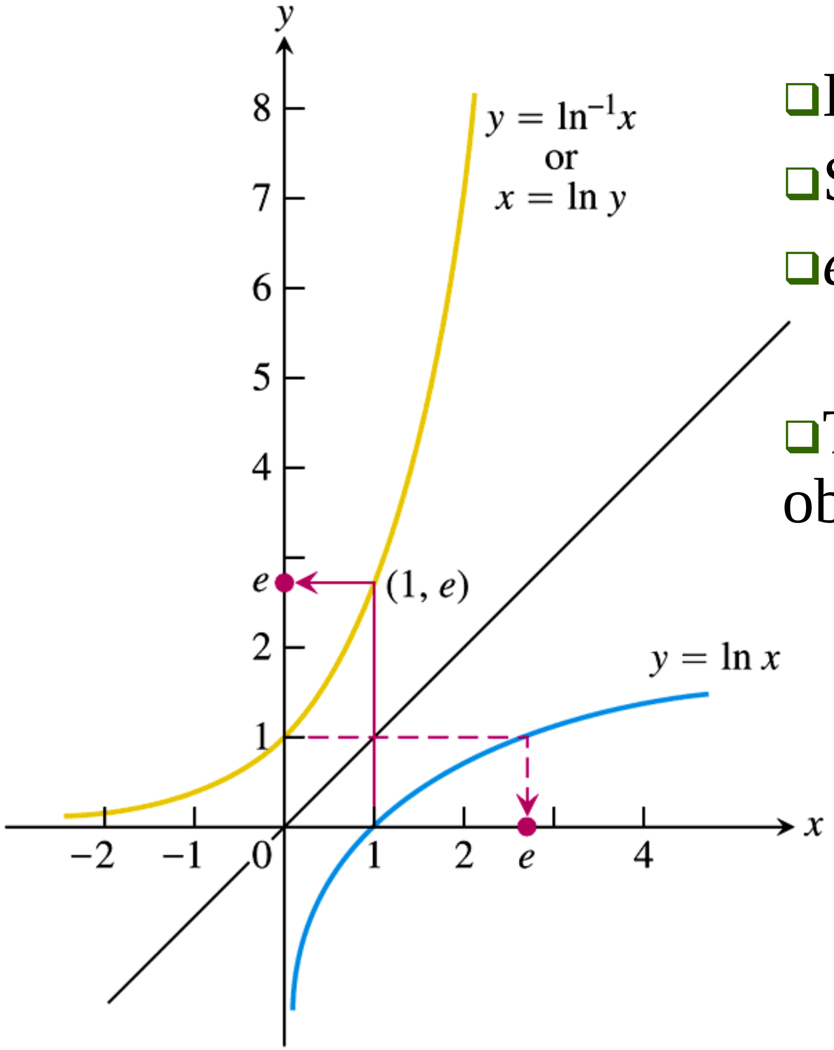


FIGURE 7.11 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

- Definition of e as $\ln e = 1$.
- So, $e = \ln^{-1}(1) = \exp(1)$
- $e = 2.718281828459045\ldots$ (an irrational number)
- The approximate value for e is obtained numerically (later).

The function $y = e^x$

- Raising the number e to a rational power r:
- $e^2 = e \cdot e$, $e^{-2} = 1/ e^{-2}$, $e^{1/2} = \sqrt{e}$ etc.
- Taking the logarithm of e^r, we get
- $\ln e^r = \ln (e \cdot e \cdot e \cdot e \ldots)$

  $= \ln e + \ln e + \ln e + \ldots + \ln e = r \ln e = r$
- From $\ln e^r = r$, we take the inverse to obtain
- $\ln^{-1} (\ln e^r) = \ln^{-1} (r)$
- $e^r = \ln^{-1} (r) = \exp r$, for r rational.
- How do we define e^x where x is irrational?
- This can be defined by assigning e^x as $\exp x$ since $\ln^{-1} (x)$ is defined (as the inverse function of $\ln x$ is defined for all real x).

DEFINITION The Natural Exponential Function

For every real number x, $e^x = \ln^{-1} x = \exp x$.

For the first time we have a precise meaning for an irrational exponent. (previously a^x is defined for only rational x)

Note: please do make a distinction between e^x and $\exp x$. They have different definitions.

e^x is the number e raised to the power of real number x.

$\exp x$ is defined as the inverse of the logarithmic function, $\exp x = \ln^{-1} x$

Typical values of e^x

x	e^x (rounded)
−1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0) \qquad (2)$$

$$\ln(e^x) = x \qquad (\text{all } x) \qquad (3)$$

- (2) follows from the definition of
- From $e^x = \exp x$, let $x \rightarrow \ln x$
- $e^{\ln x} = \ln[\exp x] = x$ (by definition). (2)
- From $e^x = \exp x$, take logarithm both sides, $\rightarrow \ln e^x = \ln[\exp x] = x$ (by definition)

Example 1 Using inverse equations

$$(a) \ln e^2 = 2$$

$$(b) \ln e^{-1} = -1$$

$$(c) \ln \sqrt{e} = \ln e^{1/2} = 1/2$$

$$(d) \ln e^{\sin x} = \sin x$$

$$(f) e^{\ln 2} = 2$$

$$(g) e^{\ln\left(x^2+1\right)} = x^2 + 1$$

$$(h) e^{3\ln 2} = e^{\ln 2^3} = 2^3 = 8$$

$$(i) e^{3\ln 2} = e^{3\cdot\ln 2} = \left(e^{\ln 2}\right)^3 = 2^3 = 8$$

Example 2 Solving for an exponent

- Find k if $e^{2k}=10$.

The general exponential function a^x

- Since $a = e^{\ln a}$ for any positive number a
- $a^x = (e^{\ln a})^x = e^{x \ln a}$

DEFINITION General Exponential Functions

For any numbers $a > 0$ and x, the exponential function with base a is

$$a^x = e^{x \ln a}.$$

Example 3 Evaluating exponential functions

$$(a)2^{\sqrt{3}} = \left(e^{\ln 2}\right)^{\sqrt{3}} = e^{\sqrt{3}\ln 2} \approx e^{1.20} \approx 3.32$$

$$(b)2^{\pi} = \left(e^{\ln 2}\right)^{\pi} = e^{\pi \ln 2} \approx e^{2.18} \approx 8.8$$

Laws of exponents

THEOREM 3 Laws of Exponents for e^x

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$

2. $e^{-x} = \dfrac{1}{e^x}$

3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Laws of exponents

THEOREM 3 Laws of Exponents for e^x

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2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Theorem 3 also valid for a^x

Proof of law 1

$$y_1 = e^{x_1}, y_2 = e^{x_2}$$

$$\Rightarrow x_1 = \ln y_1, x_2 = \ln y_2$$

$$\Rightarrow x_1 + x_2 = \ln y_1 + \ln y_2 = \ln y_1 y_2$$

$$\Rightarrow \exp(x_1 + x_2) = \exp(\ln y_1 y_2)$$

$$e^{x_1 + x_2} = y_1 y_2 = e^{x_1} e^{x2}$$

Example 4 Applying the exponent laws

$$(a) e^{x+\ln 2} =$$

$$(b) e^{-\ln x} =$$

$$(c) \frac{e^{2x}}{e} =$$

$$(d) \left(e^{3}\right)^{x} =$$

The derivative and integral of e^x

$$f(x) = \ln x, y = e^x = \ln^{-1} x = f^{-1}(x)$$

$$\frac{dy}{dx} = \frac{d}{dx} e^x = \frac{d}{dx} f^{-1}(x) = \frac{1}{\left.\frac{df(x)}{dx}\right|_{x=f^{-1}(x)}}$$

$$= \frac{1}{(1/x)\big|_{x=f^{-1}(x)}} = \frac{1}{(1/x)\big|_{x=y}} = y = e^x$$

$$\frac{d}{dx} e^x = e^x \qquad (5)$$

Example 5 Differentiating an exponential

$$\frac{d}{dx}\left(5e^{x}\right)=$$

By the virtue of the chain rule, we obtain

If u is any differentiable function of x, then

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}. \tag{6}$$

$$f(u) = e^u; u = u(x);$$

$$\frac{d}{dx}\left(e^{u(x)}\right) = \frac{d}{dx} f(u) = \frac{df(u)}{du}\frac{du(x)}{dx} = e^u \frac{du}{dx}$$

$$\int e^u \, du = e^u + C.$$

This is the integral equivalent of (6)

Example 7 Integrating exponentials

$$(a)\int_0^{\ln 2} e^{3x}dx =$$

$$(b)\int_0^{\pi/2} e^{\sin x}\cos x\ dx = \int_0^{\pi/2} \underbrace{e^{\sin x}}_{e^{f(x)}}\underbrace{\cos x dx}_{df(x)}$$

$$= \int_{f(0)}^{f(\pi/2)} e^{f(x)}df(x) = \int_{f(0)}^{f(\pi/2)} de^{f(x)}$$

$$= e^{f(x)}\Big|_{f(0)}^{f(\pi/2)} = e^{f(\pi/2)} - e^{f(0)} = e^{\sin(\pi/2)} - e^{\sin(0)} = e - 1$$

The number e expressed as a limit

THEOREM 4 The Number e as a Limit

The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

Proof

- If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$. But by definition of derivative,
- $$f'(y) = \lim_{h\to 0} \frac{f(y+h) - f(y)}{h}$$
  $$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x\to 0} \frac{f(1+x) - f(x)}{x}$$
  $$= \lim_{x\to 0} \frac{\ln(1+x) - \ln(1)}{x} = \lim_{x\to 0} \frac{\ln(1+x)}{x}$$
  $$= \lim_{x\to 0} \left[\ln(1+x)\right]^{\frac{1}{x}} = \ln\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right] = 1 \quad (\text{since } f'(1) = 1)$$
  $$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}(1+\frac{1}{y})^{y} = e$$

Define x^n for any real $x > 0$ as $x^n = e^{n \ln x}$.

Here n need not be rational but can be any real number as long as x is positive.

Then we can take the logarithm of x^n :

$$\ln x^n = \ln\left(e^{n \ln x}\right) = n \ln x.$$

Once x^n is defined, we can take its differentiation :

$$\frac{d}{dx} x^n = \frac{d}{dx}\left(e^{\overbrace{n \ln x}^{u(x)}} \right) = \frac{du}{dx}\frac{de^u}{du} = \frac{n}{x} e^{n \ln x} = \frac{n}{x} x^n = nx^{n-1}$$

$$\Rightarrow \frac{d}{dx} x^n = nx^{n-1}$$

- By virtue of chain rule,

$$u = u(x);$$

$$\frac{d}{dx}u^n = \frac{du(x)}{dx}\frac{du^n}{du} = \frac{du(x)}{dx}nu^{n-1}$$

Power Rule (General Form)

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

Example 9 using the power rule with irrational powers

$$(a)\frac{d}{dx}x^{\sqrt{2}} \equiv \frac{du^{n}}{dx} = \frac{du}{dx}nu^{n-1}$$

$$\frac{du}{dx}nu^{n-1} \equiv \frac{dx}{dx}\sqrt{2}x^{\sqrt{2}-1} = \sqrt{2}x^{\sqrt{2}-1}$$

$$(b)\frac{d}{dx}(2+\sin 3x)^{\pi} \equiv \frac{du^{n}}{dx} = \frac{du}{dx}nu^{n-1}$$

$$\frac{du}{dx}nu^{n-1} \equiv \frac{d(2+\sin 3x)}{dx}\pi u^{\pi-1} = 3\pi(2+\sin 3x)^{\pi-1}\cos 3x$$

7.4

a^x and $\log_a x$

(3rd lecture of week 10/09/07-15/09/07)

The derivative of a^x

$$a^x = e^{x \ln a}$$

$$\frac{d}{dx} a^x = \frac{d}{dx}\left(e^{\overbrace{x \ln a}^{u}} \right) = \frac{d}{dx}(x \ln a) \frac{d}{du}(e^u)$$

$$= e^u \ln a = e^{x \ln a} \ln a = a^x \ln a$$

By virtue of the chain rule,

$$\frac{d}{dx} a^{u(x)} = \frac{du}{dx}\frac{d}{du}(a^u) = a^u \ln a \frac{du}{dx}$$

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u \ln a \, \frac{du}{dx}. \quad (1)$$

C

Example 1 Differentiating general exponential functions

$$(a)\frac{d}{dx}3^x=\frac{d}{dx}\left(e^{\overbrace{x\ln 3}^{u}}\right)=\frac{d}{dx}(x\ln 3)\frac{d}{du}\left(e^u\right)$$

$$=\ln 3\cdot e^{x\ln 3}=3^x\ln 3$$

$$(b)\frac{d}{dx}3^{-x}=-\frac{d}{d(-x)}3^{\overbrace{(-x)}^{u}}=-\frac{d}{du}3^u=-\frac{d}{du}3^u$$

$$=-3^u\ln 3=-3^{(-x)}\ln 3=-\ln 3/3^x$$

$$(c)\frac{d}{dx}3^{\overbrace{\sin x}^{u}}=\frac{du}{dx}\frac{d}{du}3^u=\frac{d(\sin x)}{dx}3^u\ln 3=3^{\sin x}\ln 3\cdot\cos x$$

Other power functions

- Example 2 Differentiating a general power function
- Find dy/dx if $y = x^x$, $x > 0$.
- **Solution:** Write x^x as a power of e
- $x^x = e^{x\ln x}$

$$\frac{d}{dx}\left(e^{\overbrace{x\ln x}^{u}}\right) = \frac{du}{dx}\frac{d}{du}\left(e^{u}\right) = \frac{d}{dx}(x\ln x)\cdot\left(e^{u}\right) = \ldots$$

Integral of a^u

From $\frac{d}{dx}a^{u(x)} = a^u \ln a \frac{du}{dx}$, devide by $\ln a$:

$$\Rightarrow \frac{1}{\ln a}\frac{d}{dx}a^{u(x)} = a^u \frac{du}{dx}$$

$\Rightarrow \frac{d}{dx}a^{u(x)} = a^u \ln a \frac{du}{dx}$, integrate both sides wrp to dx:

$$\Rightarrow \int\left(\frac{d}{dx}a^u\right)dx = \int\left(a^u \ln a \frac{du}{dx}\right)dx:$$

$$\Rightarrow \int da^u = \ln a \int a^u du + C$$

$$\Rightarrow \int a^u du = \frac{1}{\ln a}\int da^u = \frac{a^u}{\ln a}$$

$$\int a^u\, du = \frac{a^u}{\ln a} + C. \qquad (2)$$

Example 3 Integrating general exponential functions

$$(a)\int 2^x\,dx = \frac{2^x}{\ln 2} + C$$

$$(b)\int 2^{\overbrace{\sin x}^{u}}\overbrace{\cos dx}^{du} = \int 2^u\,du = \ldots$$

Logarithm with base *a*

DEFINITION $\log_a x$

For any positive number $a \neq 1$,

$$\log_a x \text{ is the inverse function of } a^x.$$

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0) \qquad (3)$$

$$\log_a (a^x) = x \qquad (\text{all } x) \qquad (4)$$

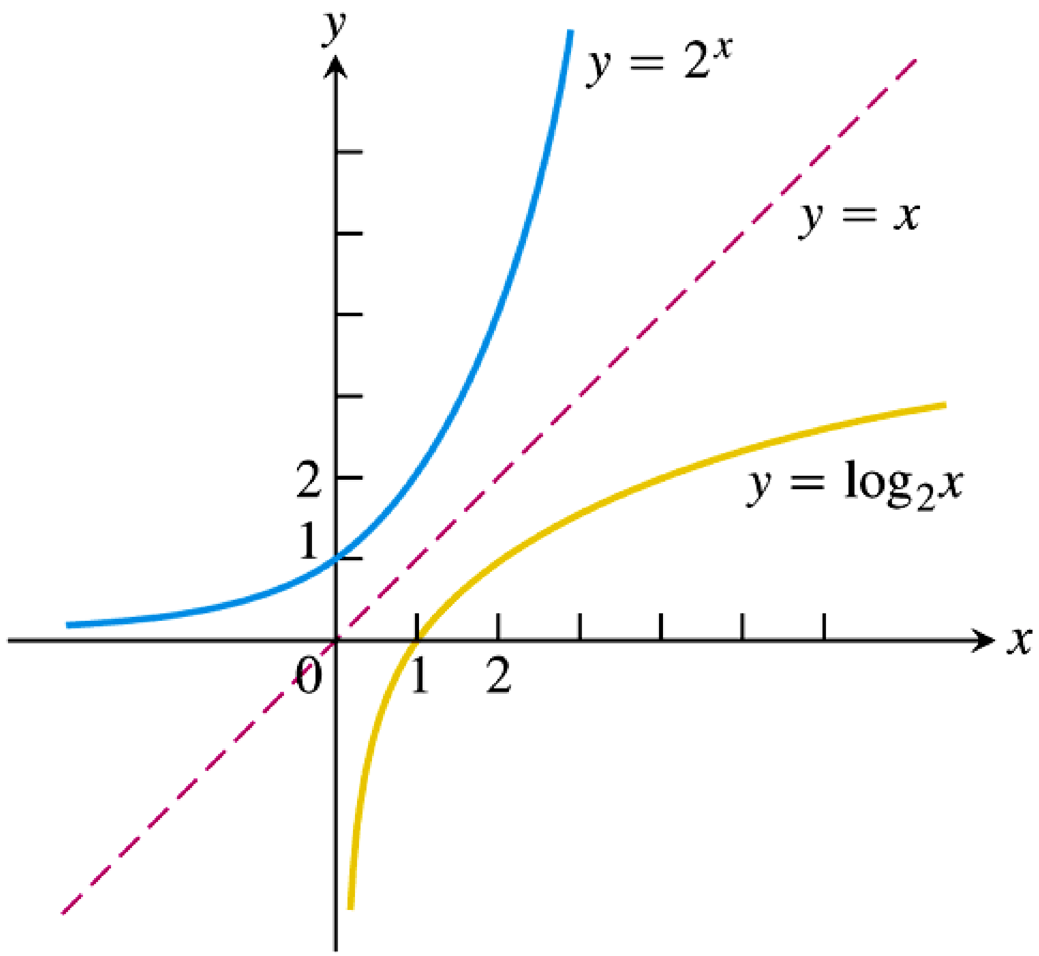


FIGURE 7.13 The graph of 2^x and its inverse, $\log_2 x$.

Example 4 Applying the inverse equations

$$(a)\log_2 2^5 = 5$$

$$(b)2^{\log_2 3} = 3$$

$$(c)\log_{10} 10^{(-7)} = -7$$

$$(d)10^{\log_{10} 4} = 4$$

Evaluation of $\log_a x$

$a^{\log_a x} = x$ Taking ln on both sides,

$\ln(a^{\log_a x}) = \ln x$

$\text{LHS}, \ln(a^{\log_a x}) = \log_a x \ln a$

equating LHS to RHS yields

$\log_a x \ln a = \ln x$

$$\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a} \qquad (5)$$

- Example: $\log_{10} 2 = \ln 2 / \ln 10$

TABLE 7.2 Rules for base *a* logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:*
   $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule:*
   $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Reciprocal Rule:*
   $\log_a \frac{1}{y} = -\log_a y$
4. *Power Rule:*
   $\log_a x^y = y \log_a x$

- Proof of rule 1:

$$\ln xy = \ln x + \ln y$$

divide both sides by $\ln a$

$$\frac{\ln(xy)}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a}$$

$$\log_a(xy) = \log_a x + \log_a y$$

Derivatives and integrals involving $\log_a x$

$$\frac{d}{dx}\left(\log_a u\right)=\frac{du}{dx}\frac{d\left(\log_a u\right)}{du}=\frac{du}{dx}\frac{d\left(\log_a u\right)}{du}$$

$$\frac{d}{du}\left(\log_a u\right)=\frac{d}{du}\left(\frac{\ln u}{\ln a}\right)=\frac{1}{\ln a}\frac{d}{du}\left(\ln u\right)=\frac{1}{\ln a}\frac{1}{u}$$

$$\frac{d}{dx}\left(\log_a u\right)=\frac{du}{dx}\cdot\left(\frac{1}{\ln a}\frac{1}{u}\right)=\frac{1}{\ln a}\left(\frac{1}{u}\right)\cdot\frac{du}{dx}$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}$$

Example 5

$$(a)\frac{d}{dx}\left(\log_{10}\overbrace{(3x+1)}^{u}\right)=\frac{du}{dx}\frac{d\left(\log_{10}u\right)}{du}$$

$$=\frac{d}{dx}(3x+1)\frac{1}{\ln 10}\frac{d\left(\ln u\right)}{du}=\frac{3}{\ln 10}\frac{1}{(3x+1)}$$

$$(b)\int\frac{\log_2 x}{x}dx=\frac{1}{\ln 2}\int\underbrace{\ln x}_{u}\underbrace{\frac{dx}{x}}_{d(\ln x)\equiv du}=\frac{1}{\ln 2}\int u\,du=\ldots$$

7.5

Exponential Growth and Decay

(3rd lecture of week 10/09/07-15/09/07)

The law of exponential change

- For a quantity y increases or decreases at a rate proportional to it size at a give time t follows the law of exponential change, as per $\frac{dy}{dt} \propto y(t) \Rightarrow \frac{dy}{dt} = ky(t).$

  k is the proportional constant.

  Very often we have to specify the value of y at some specified time, for example the initial condition $y(t=0) = y_0$

Rearrange the equation $\frac{dy}{dt} = ky$:

$$\frac{1}{y}\frac{dy}{dt} = k \rightarrow \int \frac{1}{y}\frac{dy}{dt}dt = \int kdt$$

$$\rightarrow \int \frac{1}{y}dy = k\int dt = kt \rightarrow \ln|y| = kt + C$$

$$\rightarrow \ln|y| = kt + C \rightarrow y = \pm Ce^{kt} = Ae^{kt}, A = \pm C.$$

Put in the initial value of y at $t = 0$ is y_0:

$$\rightarrow y(0) = y_0 = Ae^{k\cdot 0} = A \rightarrow y = y_0e^{kt}$$

The Law of Exponential Change

$$y = y_0 e^{kt} \qquad (2)$$

Growth: $k > 0$ Decay: $k < 0$

The number k is the **rate constant** of the equation.

Example 1 Reducing the cases of infectious disease

- Suppose that in the course of any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1000? Assume the law of exponential change applies.

Example 3 Half-life of a radioactive element

- The effective radioactive lifetime of polonium-210 is very short (in days). The number of radioactive atoms remaining after t days in a sample that starts with y_0 radioactive atoms is $y = y_0 \exp(-5\times10^{-3}t)$. Find the element's half life.

Solution

- Radioactive elements decay according to the exponential law of change. The half life of a given radioactive element can be expressed in term of the rate constant *k* that is specific to a given radioactive species. Here $k=-5\times10^{-3}$.
- At the half-life, $t = t_{1/2}$,

  $y(t_{1/2}) = y_0/2 = y_0 \exp(-5\times10^{-3}\, t_{1/2})$

  $\exp(-5\times10^{-3}\, t_{1/2}) = 1/2$

$\rightarrow \ln(1/2) = -5\times10^{-3}\, t_{1/2}$

$\rightarrow t_{1/2} = -\ln(1/2)/5\times10^{-3} = \ln(2)/5\times10^{-3} = \ldots$

7.7

Inverse Trigonometric Functions

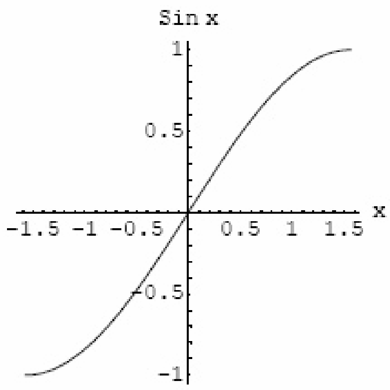
(1st lecture of week 17/09/07-22/09/07)

Defining the inverses

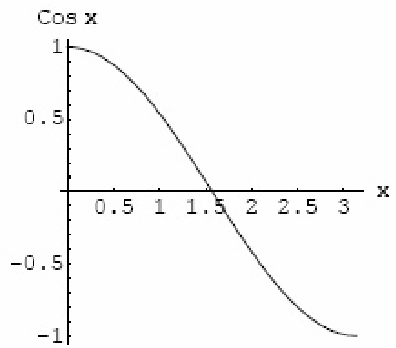
- Trigo functions are periodic, hence not one-to-one in the their domains.
- If we restrict the trigonometric functions to intervals on which they are one-to-one, then we can define their inverses.

Domain restriction that makes the trigonometric functions one-to-one

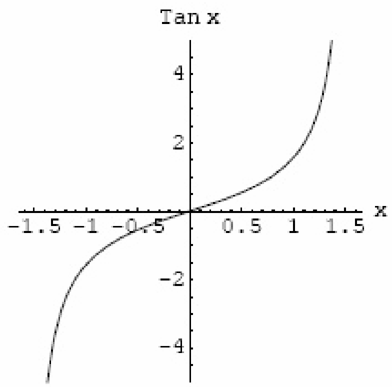
Function : $\sin x$ Domain : $[-\pi/2, \pi/2]$ Range : $[-1, 1]$



Function : $\cos x$ Domain : $[0, \pi]$ Range : $[-1, 1]$

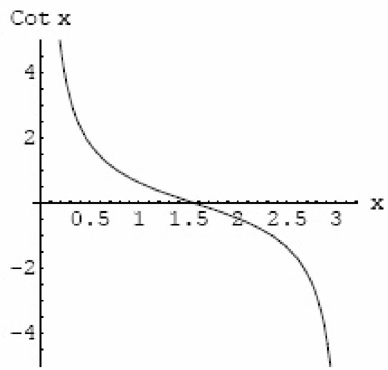


Function : $\tan x$ Domain : $[-\pi/2, \pi/2]$ Range : $(-\infty, \infty)$

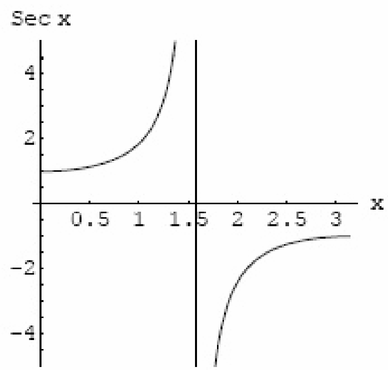


Domain restriction that makes the trigonometric functions one-to one

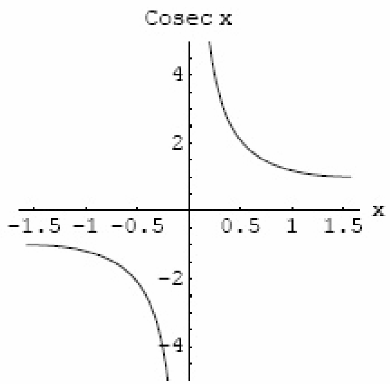
Function : cot x   Domain : [0, π]   Range : (-∞, ∞)



Function : sec x   Domain : [0, π / 2) ⋃ (π / 2, π]   Range : (-∞, -1], [1, ∞)



Function : cosec x   Domain : [-π / 2, 0) ⋃ (0, π / 2]   Range : (-∞, -1] ⋃ [1, ∞)



Inverses for the restricted trigo functions

$$y = \sin^{-1} x = \arcsin x$$

$$y = \cos^{-1} x = \arccos x$$

$$y = \tan^{-1} x = \arctan x$$

$$y = \cot^{-1} x = \operatorname{arc}\cot x$$

$$y = \sec^{-1} x = \operatorname{arc}\sec x$$

$$y = \csc^{-1} x = \operatorname{arc}\csc x$$

- The graphs of the inverse trigonometric functions can be obtained by reflecting the graphs of the restricted trigo functions through the line $y = x$.

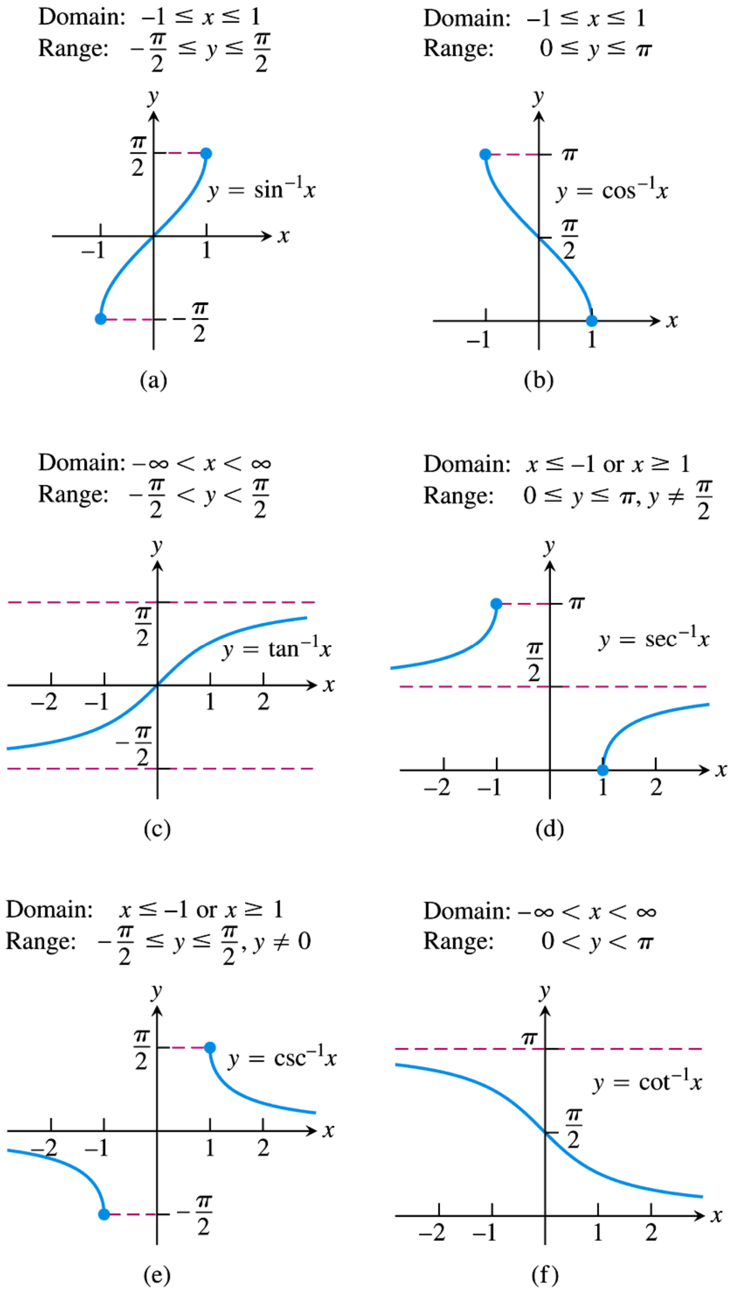


FIGURE 7.17 Graphs of the six basic inverse trigonometric functions.

DEFINITION   Arcsine and Arccosine Functions

$\boldsymbol{y = \sin^{-1} x}$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$\boldsymbol{y = \cos^{-1} x}$ is the number in $[0, \pi]$ for which $\cos y = x$.

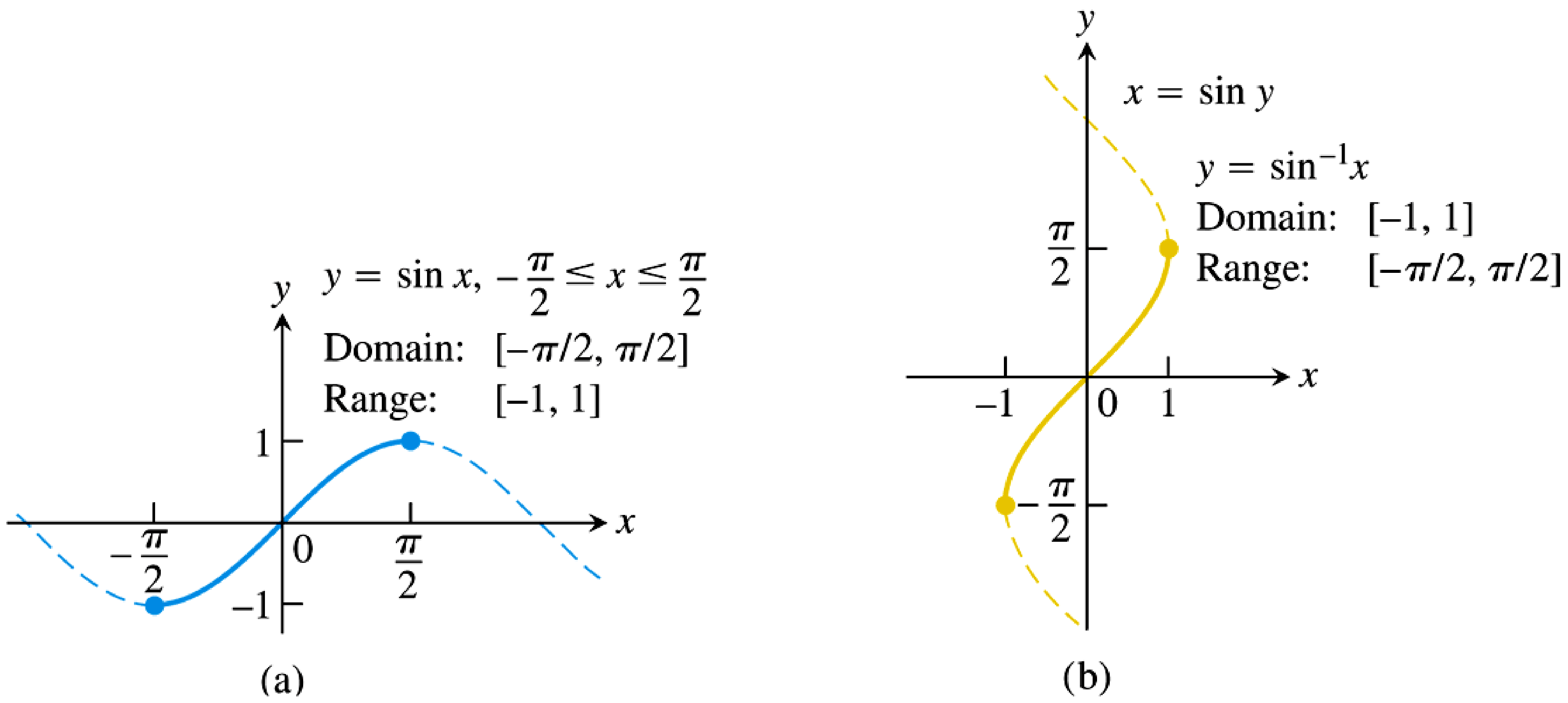


FIGURE 7.18 The graphs of (a) $y = \sin x, -\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

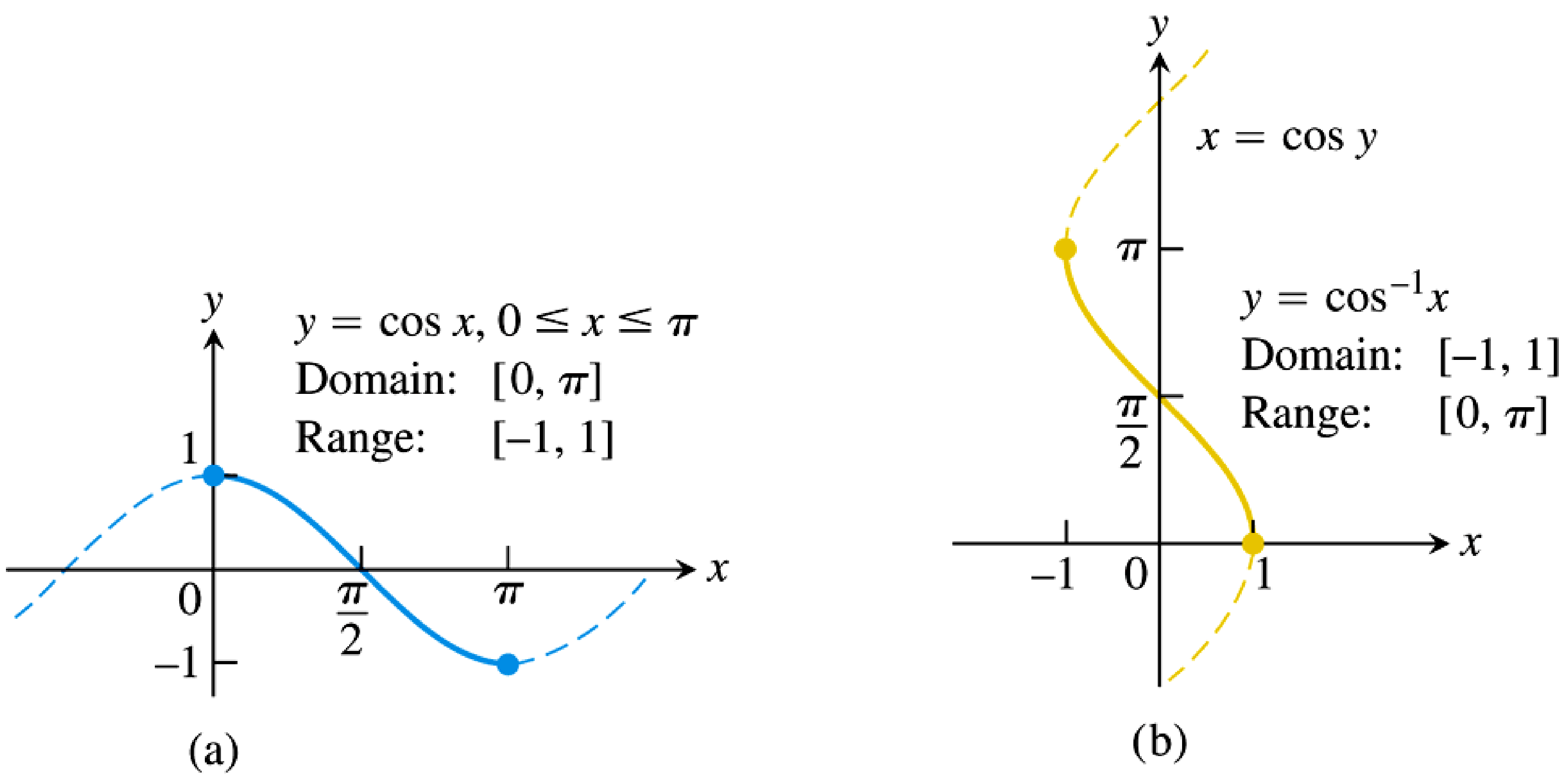


FIGURE 7.19 The graphs of (a) $y = \cos x, 0 \le x \le \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

Some specific values of $\sin^{-1} x$ and $\cos^{-1} x$

x	$\sin^{-1} x$
$\sqrt{3}/2$	$\pi/3$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/6$
$-1/2$	$-\pi/6$
$-\sqrt{2}/2$	$-\pi/4$
$-\sqrt{3}/2$	$-\pi/3$

x	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/3$
$-1/2$	$2\pi/3$
$-\sqrt{2}/2$	$3\pi/4$
$-\sqrt{3}/2$	$5\pi/6$

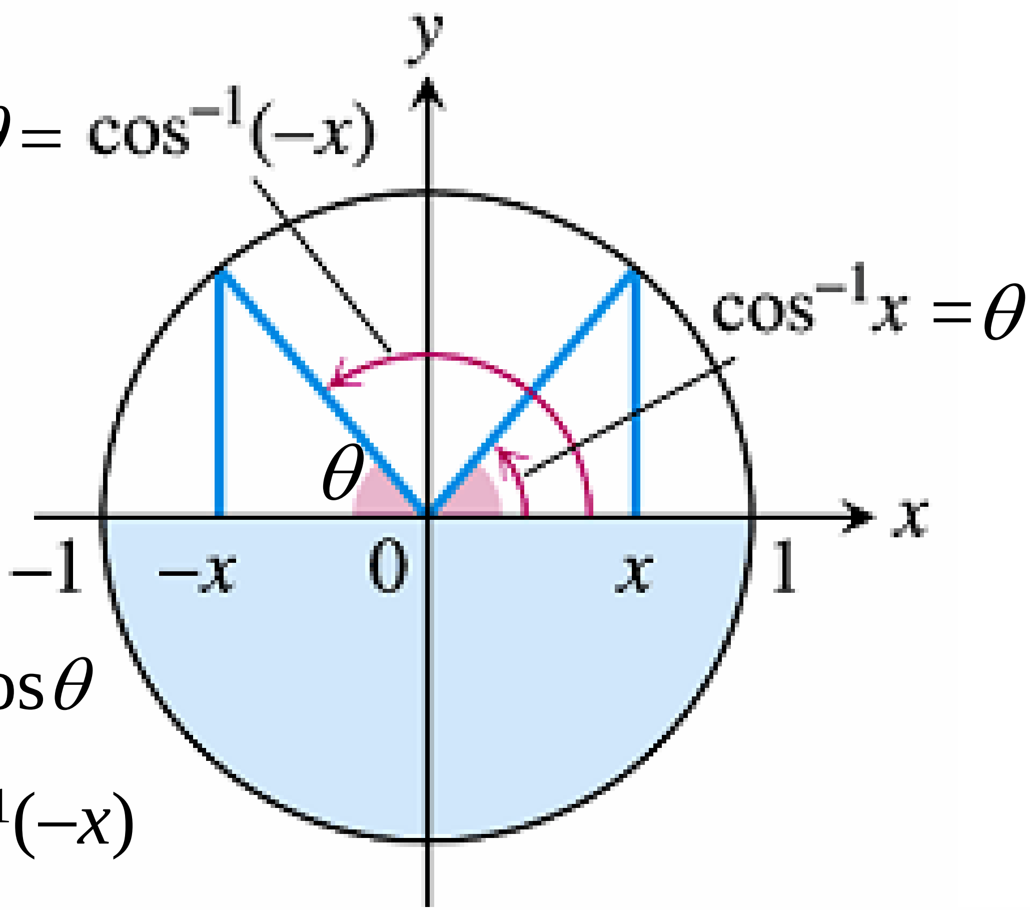


FIGURE 7.20 $\cos^{-1}x$ and $\cos^{-1}(-x)$ are supplementary angles (so their sum is π).

$\theta = \cos^{-1}x;$

$\cos\phi = \cos(\pi - \theta) = -\cos\theta$

$\phi = \cos^{-1}(-\cos\theta) = \cos^{-1}(-x)$

Add up θ and ϕ:

$\theta + \phi = \cos^{-1}x + \cos^{-1}(-x)$

$$\pi = \cos^{-1}x + \cos^{-1}(-x)$$

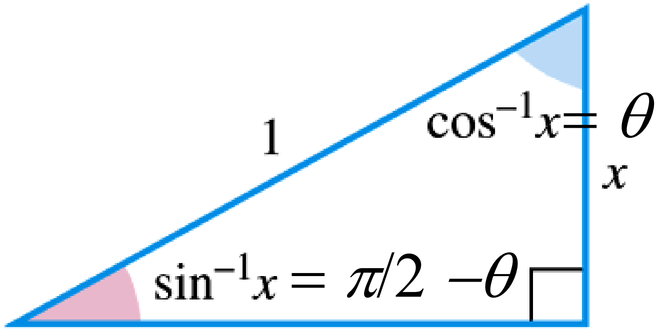


FIGURE 7.21 $\sin^{-1} x$ and $\cos^{-1} x$ are complementary angles (so their sum is $\pi/2$).

$$\cos^{-1} x = \theta; \sin^{-1} x = \left(\frac{\pi}{2} - \theta\right);$$

$$\cos^{-1} x + \sin^{-1} x + = \theta + \left(\frac{\pi}{2} - \theta\right) = \frac{\pi}{2}$$

DEFINITION **Arctangent and Arccotangent Functions**

$\mathbf{y} = \mathbf{tan^{-1}}\, \boldsymbol{x}$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$\mathbf{y} = \mathbf{cot^{-1}}\, \boldsymbol{x}$ is the number in $(0, \pi)$ for which $\cot y = x$.

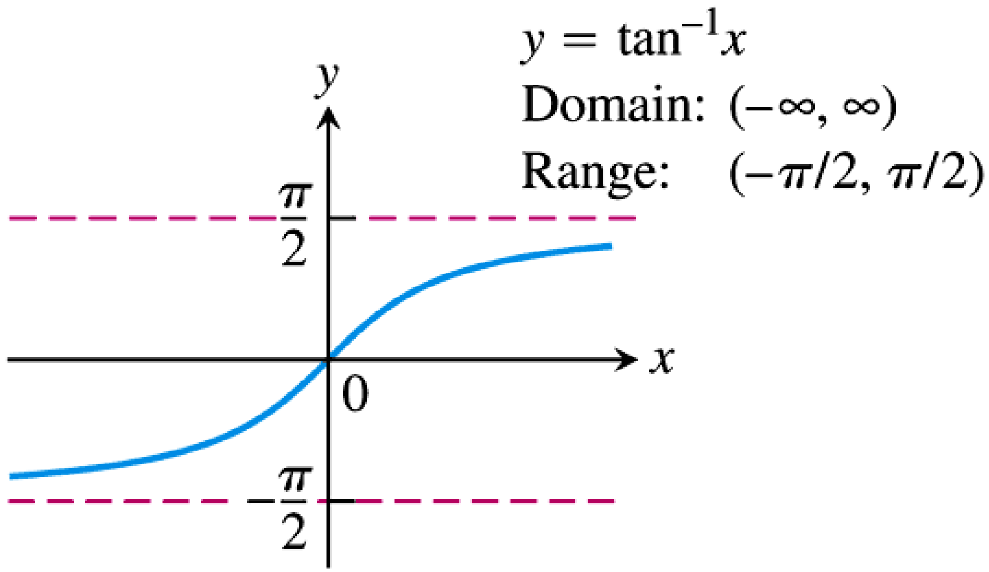


FIGURE 7.22 The graph of $y = \tan^{-1} x$.

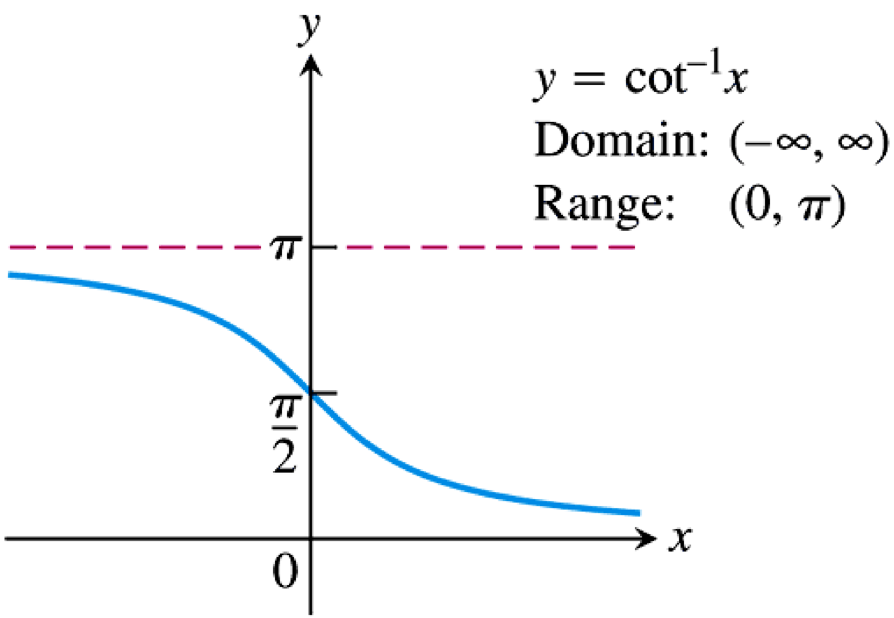


FIGURE 7.23 The graph of $y = \cot^{-1} x$.

$y = \sec^{-1} x$

Domain: $|x| \geq 1$

Range: $[0, \pi/2) \cup (\pi/2, \pi]$

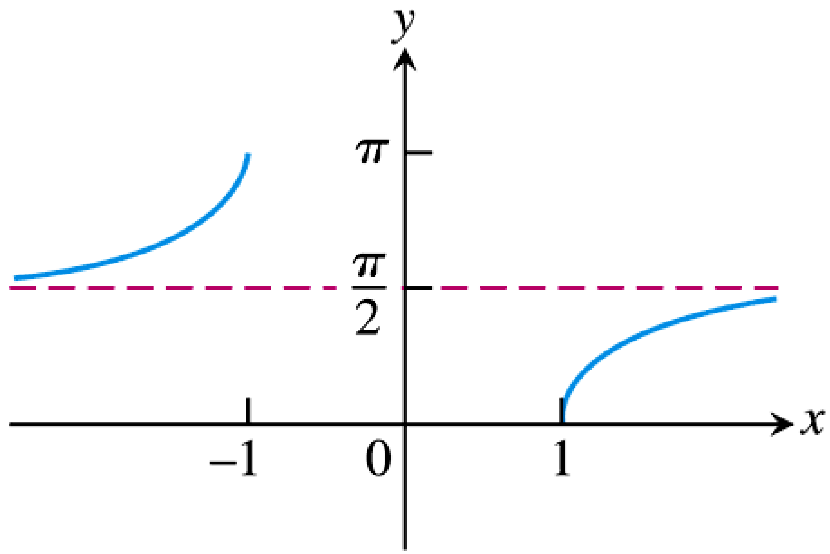


FIGURE 7.24 The graph of $y = \sec^{-1} x$.

Domain: $|x| \geq 1$

Range: $0 \leq y \leq \pi,\ y \neq \frac{\pi}{2}$

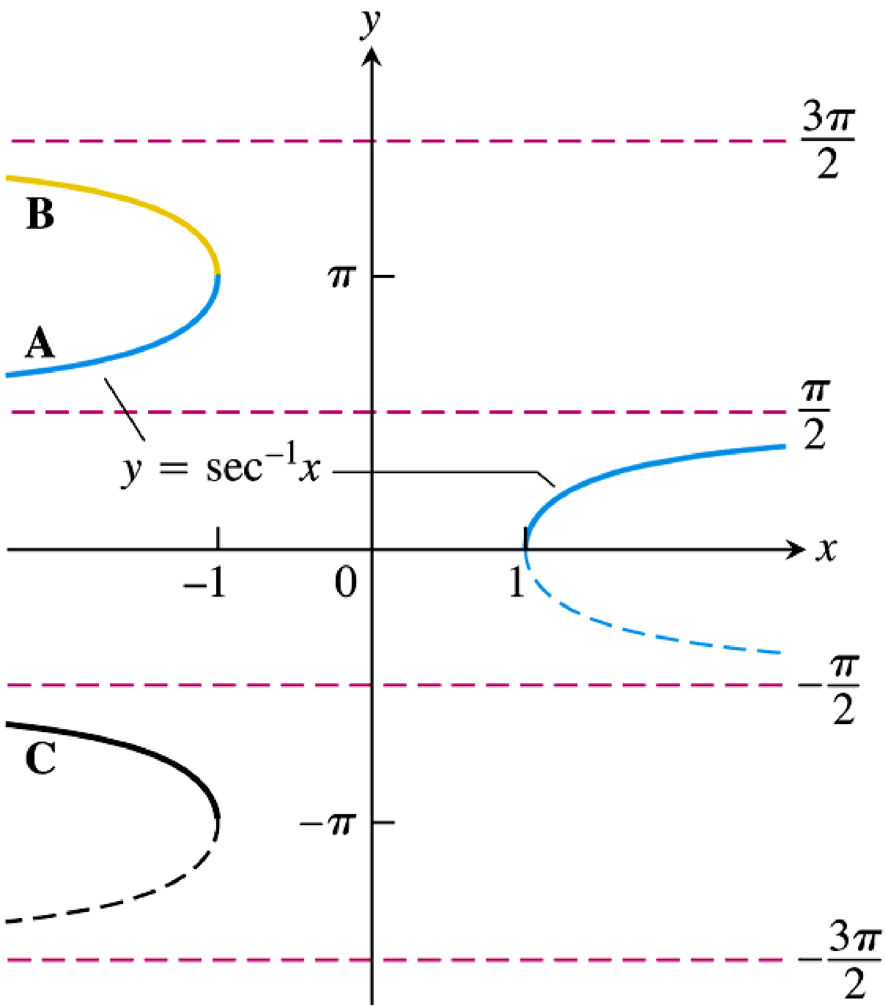


FIGURE 7.26 There are several logical choices for the left-hand branch of $y = \sec^{-1} x$. With choice **A**, $\sec^{-1} x = \cos^{-1}(1/x)$, a useful identity employed by many calculators.

$y = \csc^{-1} x$
Domain: $|x| \geq 1$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$

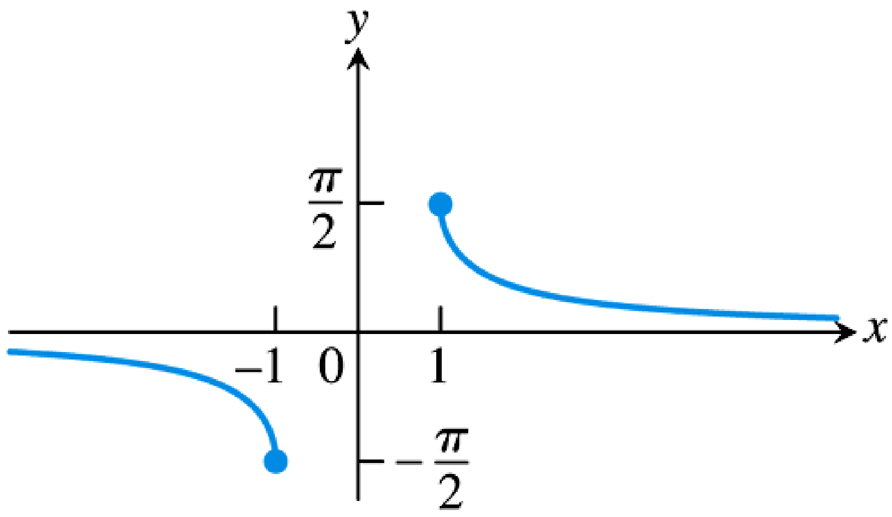


FIGURE 7.25 The graph of $y = \csc^{-1} x$.

Some specific values of $\tan^{-1} x$

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

Example 4

- Find cos α, tan α, sec α, csc α if $\alpha = \sin^{-1}(2/3)$.

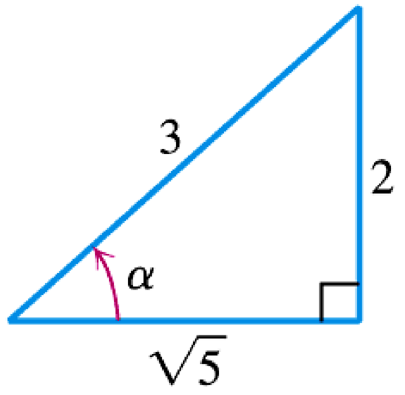


FIGURE 7.27 If $\alpha = \sin^{-1}(2/3)$, then the values of the other basic trigonometric functions of α can be read from this triangle (Example 4).

- $\sin \alpha = 2/3$
- ...

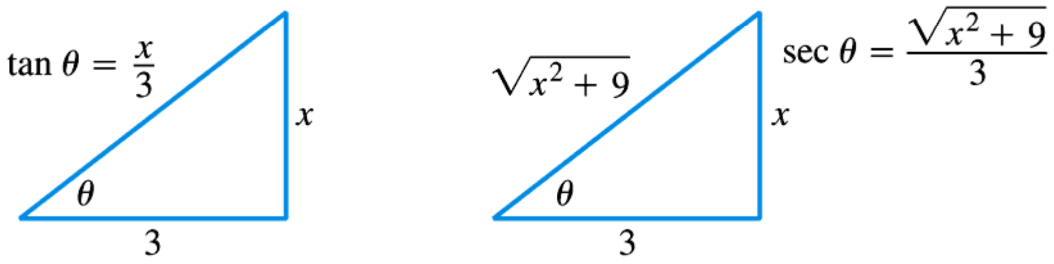
EXAMPLE 5 Find $\sec\left(\tan^{-1}\frac{x}{3}\right)$.

Solution We let $\theta = \tan^{-1}(x/3)$ (to give the angle a name) and picture θ in a right triangle with

$$\tan\theta = \text{opposite}/\text{adjacent} = x/3.$$

The length of the triangle's hypotenuse is

$$\sqrt{x^2 + 3^2} = \sqrt{x^2 + 9}.$$



Thus,

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \sec\theta$$

$$= \frac{\sqrt{x^2+9}}{3}. \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

The derivative of $y = \sin^{-1} x$

$$f(x) = \sin^{-1} x \Rightarrow f^{-1}(x) = \sin x;$$

$$\frac{df(x)}{dx} = \frac{1}{\left.\frac{df^{-1}(x)}{dx}\right|_{x=f(x)}} = \frac{1}{\cos x|_{x=f(x)}} = \frac{1}{\cos f(x)}$$

$$\text{Let } y = f(x) = \sin^{-1} x \rightarrow x = \sin y \Rightarrow \cos y = \sqrt{1-x^2}$$

$$\frac{1}{\cos(f(x))} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

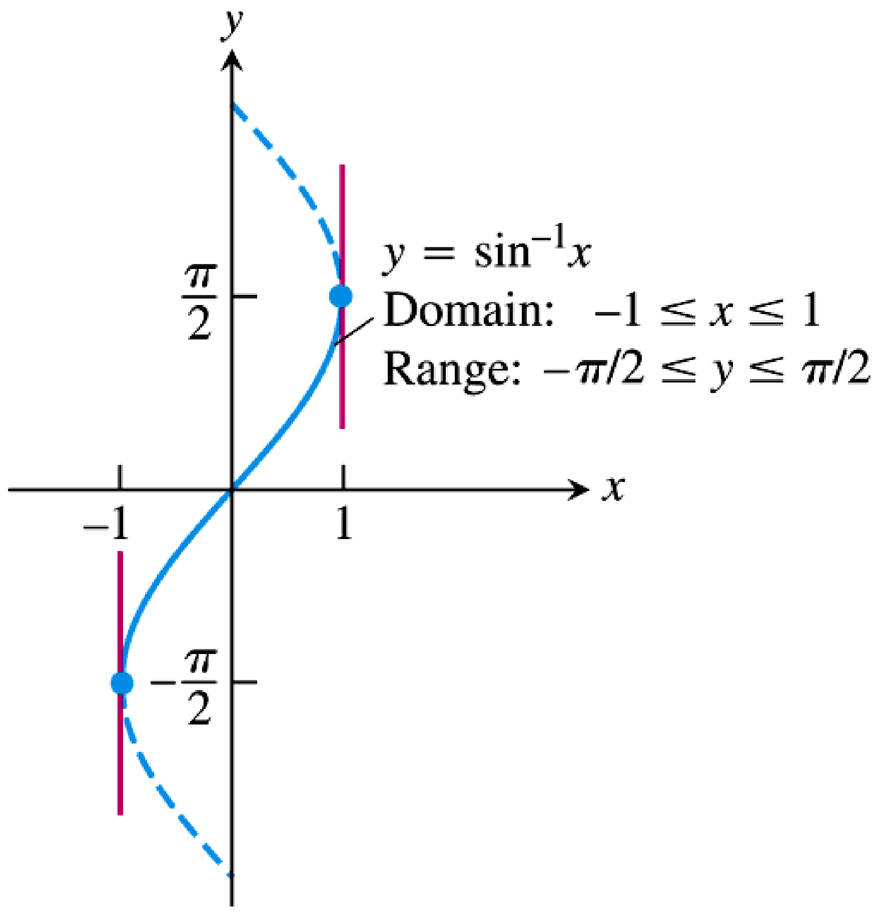


FIGURE 7.29 The graph of $y = \sin^{-1} x$.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

Note that the graph is not differentiable at the end points of $x=\pm1$ because the tangents at these points are vertical.

The derivative of $y = \sin^{-1} u$

If $u = u(x)$ is an diffrentiable function of x,

$$\frac{d}{dx}\sin^{-1} u = ?$$

Use chain rule: Let $y = \sin^{-1} u$

$$\frac{d}{dx}\sin^{-1} u = \frac{du}{dx}\frac{d}{du}\left(\sin^{-1} u\right) = \frac{du}{dx}\frac{1}{\sqrt{1-u^2}}$$

Note that $|u| < 1$ for the formula to apply

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1.$$

Example 7 Applying the derivative formula

$$\frac{d}{dx}\sin^{-1} x^2 = \ldots$$

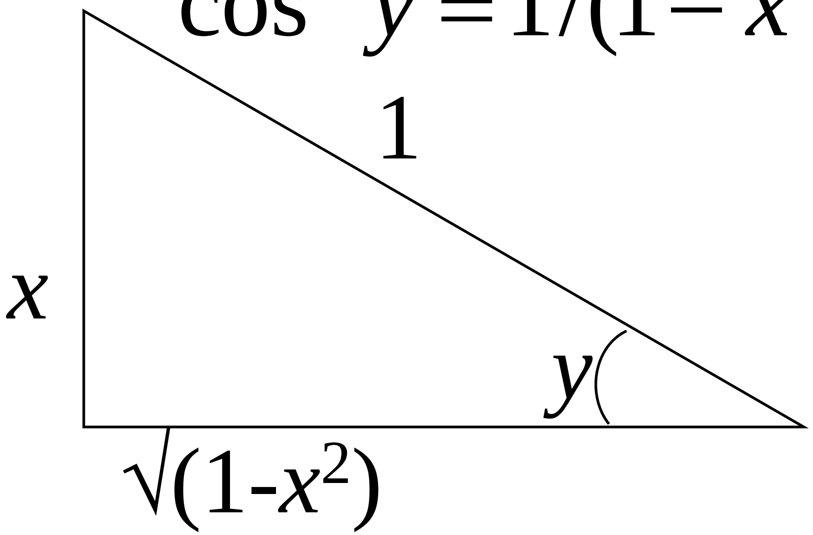
The derivative of $y = \tan^{-1} u$

$$y = \tan^{-1} x \Rightarrow x = \tan y$$

$$1 = \frac{d}{dx}(\tan y) = \frac{dy}{dx}\sec^2 y$$

$$\frac{dy}{dx} = \cos^2 y = 1/(1-x^2)$$

$$\cos^2 y = 1/(1-x^2)$$



By virtue of chain rule, we obtain

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}.$$

Example 8

$$x(t) = \tan^{-1}\sqrt{t}.$$

$$\left.\frac{dx}{dt}\right|_{t=16} = ?$$

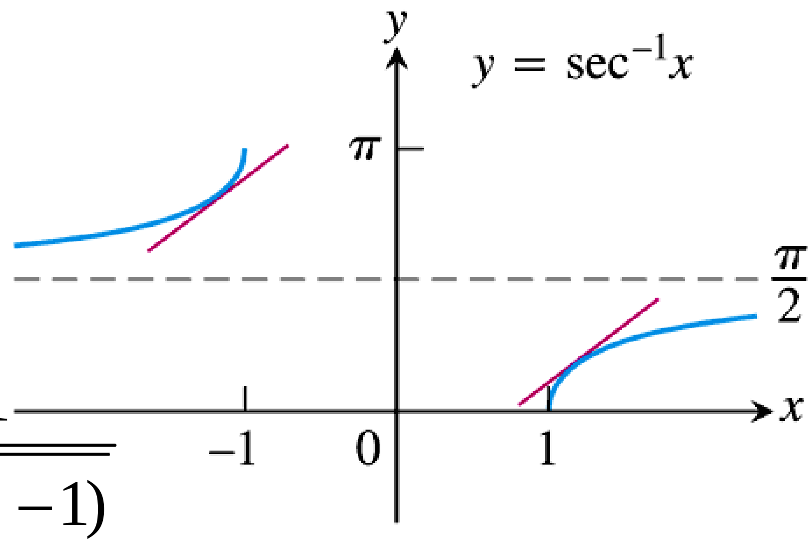
The derivative of $y = \sec^{-1} x$

$$y = \sec^{-1} x \Rightarrow x = \sec y$$

$$1 = \frac{d}{dx}(\sec y) = \frac{dy}{dx}\sec y \tan y$$

$$\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$$

$$\frac{d}{dx}\sec^{-1} x = \cos y \cot y = \pm\frac{1}{x}\frac{1}{\sqrt{(x^2-1)}}$$



$$\because \frac{dy}{dx} > 0 \text{ (from Figure 7.30)},$$

$$\frac{dy}{dx} = \frac{1}{|x|}\frac{1}{\sqrt{(x^2-1)}}$$

FIGURE 7.30 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

The derivative of $y = \sec^{-1} u$

By virtue of chain rule, we obtain

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad |u| > 1.$$

Example 5 Using the formula

$$\frac{d}{dx}\sec^{-1}\left(5x^4\right)=\ldots$$

Derivatives of the other three

- The derivative of $\cos^{-1}x$, $\cot^{-1}x$, $\csc^{-1}x$ can be easily obtained thanks to the following identities:

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{du/dx}{\sqrt{1 - u^2}}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{du/dx}{\sqrt{1 - u^2}}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{du/dx}{1 + u^2}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{du/dx}{1 + u^2}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{du/dx}{|u|\sqrt{u^2 - 1}}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{u^2 - 1}}, \quad |u| > 1$$

Example 10 A tangent line to the arccotangent curve

- Find an equation for the tangent to the graph of $y = \cot^{-1} x$ at $x = -1$.

Integration formula

- By integrating both sides of the derivative formulas in Table 7.3, we obtain three useful integration formulas in Table 7.4.

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1. $$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$ (Valid for $u^2 < a^2$)

2. $$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$ (Valid for all u)

3. $$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$ (Valid for $|u| > a > 0$)

Example 11 Using the integral formulas

$$(a)\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} =$$

$$(b)\int_{0}^{1} \frac{dx}{1+x^2} =$$

$$(c)\int_{2/\sqrt{3}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} =$$

Example 13 Completing the square

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{-(x^2 - 4x)}} = \int \frac{dx}{\sqrt{-[(x-2)^2 - 4]}}$$

$$= \int \frac{dx}{\sqrt{4 - (x-2)^2}} = \int \frac{du}{\sqrt{2^2 - u^2}} = \ldots$$

Example 15 Using substitution

$$\int \frac{dx}{\sqrt{e^{2x}-6}} = \int \frac{dx}{\sqrt{\left(e^x\right)^2 - \left(\sqrt{6}\right)^2}} =$$

$$\int \frac{1}{\sqrt{\left(e^x\right)^2 - \left(\sqrt{6}\right)^2}} \frac{de^x}{e^x} = \int \frac{1}{\sqrt{u^2 - \left(\sqrt{6}\right)^2}} \frac{du}{u} = \ldots$$

7.8

Hyperbolic Functions

(1st lecture of week 17/09/07-22/09/07)

Even and odd parts of the exponential function

- $f(x) = ½ [f(x) + f(-x)] + ½ [f(x) - f(-x)]$
- $½ [f(x) + f(-x)]$ is the even part
- $½ [f(x) - f(-x)]$ is the odd part

- $f(x) = e^x = ½ (e^x + e^{-x}) + ½ (e^x – e^{-x})$
- The odd part $½ (e^x - e^{-x}) \equiv \cosh x$ (hyperbolic cosine of x)
- The odd part $½ (e^x + e^{-x}) \equiv \sinh x$ (hyperbolic sine of x)

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Proof of $\sinh 2x = 2\cosh x \sinh x$

$$\sinh 2x = \frac{1}{2}(e^{2x} - e^{-2x}) = \frac{1}{2}\frac{(e^{4x} - 1)}{e^{2x}}$$

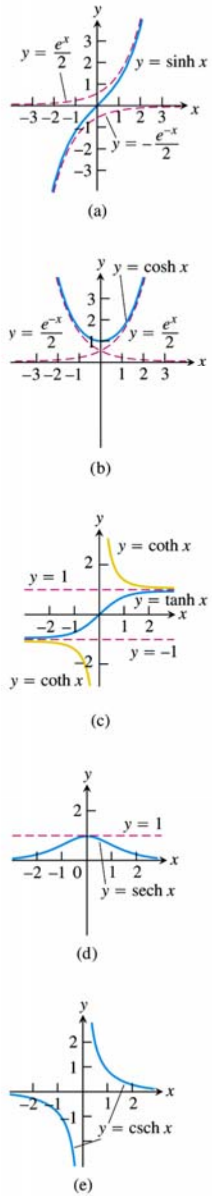
$$= \frac{1}{2}\frac{(e^{2x} - 1)}{e^{x}}\frac{(e^{2x} + 1)}{e^{x}} = \frac{2}{2}\cdot\frac{1}{2}(e^{x} - e^{-x})(e^{x} + e^{-x})$$

$$= 2\cdot\frac{1}{2}(e^{x} - e^{-x})\cdot\frac{1}{2}(e^{x} + e^{-x}) = 2\sinh x \cosh x$$

TABLE 7.5 The six basic hyperbolic functions

Hyperbolic sine of x:	$\sinh x = \dfrac{e^x - e^{-x}}{2}$
Hyperbolic cosine of x:	$\cosh x = \dfrac{e^x + e^{-x}}{2}$
Hyperbolic tangent:	$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
Hyperbolic cotangent:	$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
Hyperbolic secant:	$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$
Hyperbolic cosecant:	$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

FIGURE 7.31



Derivatives and integrals

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.8 Integral formulas for hyperbolic functions

$$\int \sinh u \, du = \cosh u + C$$

$$\int \cosh u \, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

$$\frac{d}{dx}\sinh u = \frac{du}{dx}\frac{d}{dx}\sinh x$$

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{1}{2}(e^{x} - e^{-x}) = \frac{1}{2}(e^{x} + e^{-x}) = \cosh x$$

$$\therefore \frac{d}{dx}\sinh u = \frac{du}{dx}\cosh x$$

Example 1 Finding derivatives and integrals

$$(a)\frac{d}{dx}\tanh\overbrace{\sqrt{1+t^2}}^{u}=\frac{du}{dx}\frac{d}{du}\tanh u$$

$$(b)\int\coth\overbrace{5x}^{u}dx=\frac{1}{5}\int\coth u\,du=\frac{1}{5}\int\frac{\cosh u\;du}{\sinh u}$$

$$=\frac{1}{5}\int\frac{\overbrace{d(\sinh u)}^{dv}}{\underbrace{\sinh u}_{v}}=\frac{1}{5}\int\frac{dv}{v}=\frac{1}{5}\ln|v|+C=\frac{1}{5}\ln|\sinh 5x|+C$$

$$(c)\int\sinh^2 x\;dx=\frac{1}{2}\int(\cosh 2x-1)\;dx=\ldots$$

$$(d)\int 4e^x\sinh x\;dx=4\int\frac{\overbrace{e^x}^{u}-e^{-x}}{2}\;de^x=2\int u-u^{-1}\;du$$

$$=2\left(\frac{u^2}{2}-\ln|u|\right)+C=(e^x)^2-\ln e^{2x}+C=e^{2x}-2x+C$$

Inverse hyperbolic functions

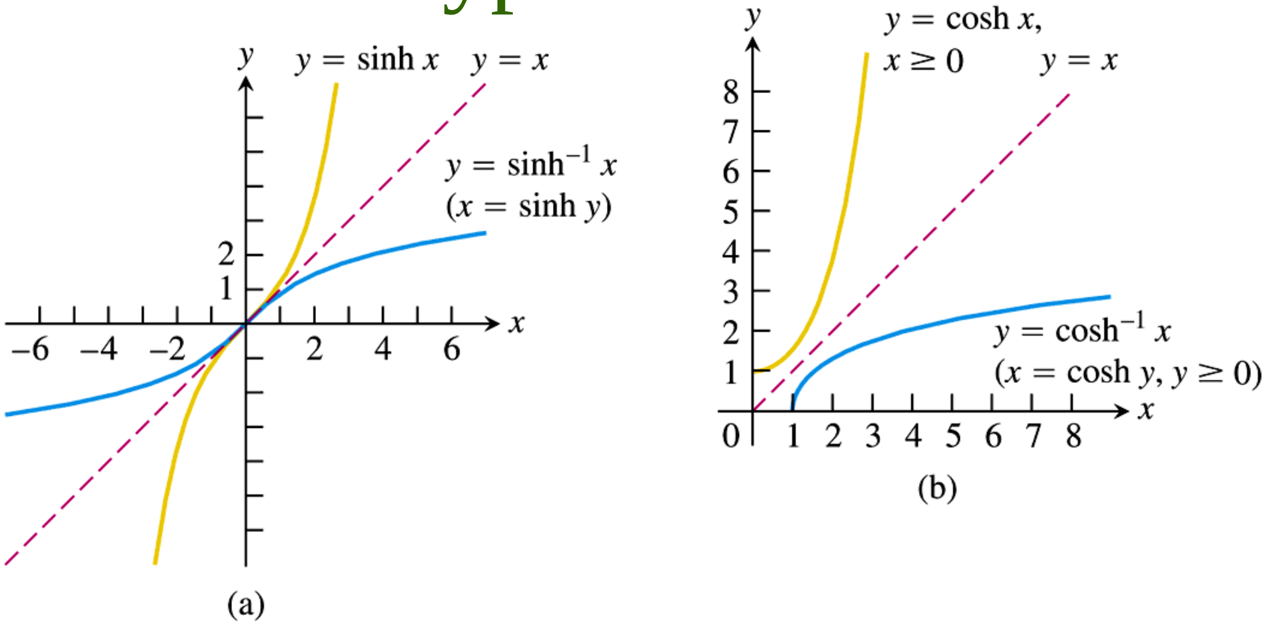


FIGURE 7.32 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

The inverse is useful in integration.

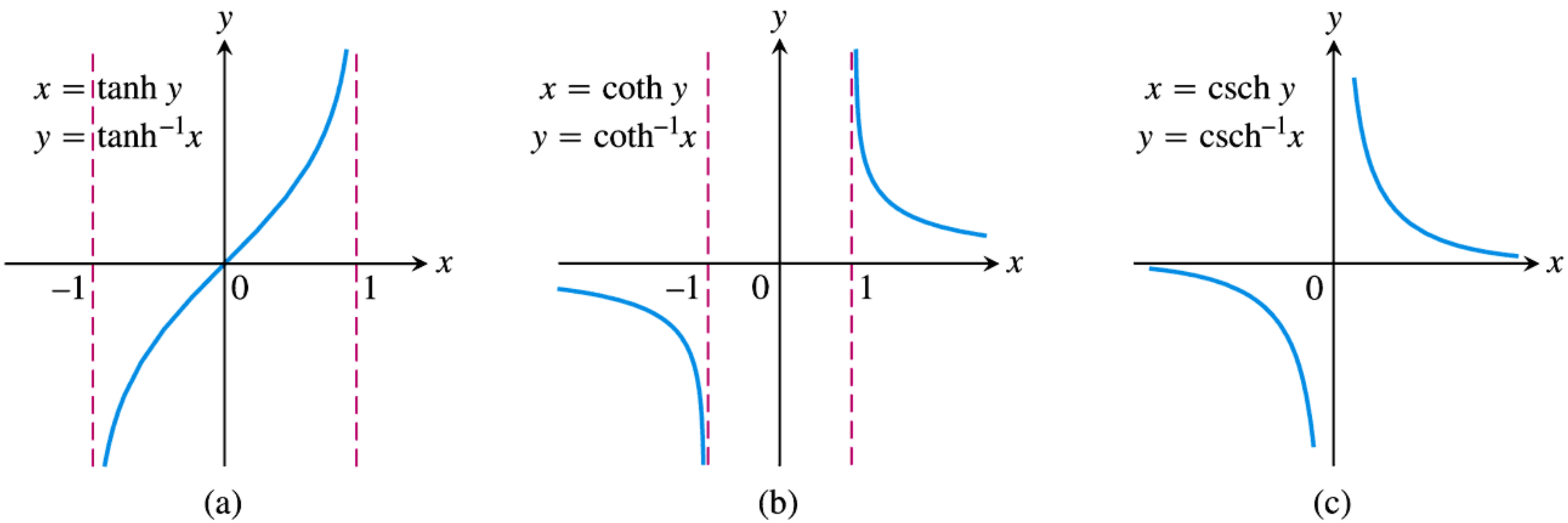


FIGURE 7.33 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

Useful Identities

TABLE 7.9 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

Proof

$$\operatorname{sech}^{-1}x = \cosh^{-1}\frac{1}{x}.$$

Take sech of $\cosh^{-1}\dfrac{1}{x}$.

$$\operatorname{sech}\left(\cosh^{-1}\frac{1}{x}\right) = \frac{1}{\cosh\left(\cosh^{-1}\frac{1}{x}\right)} = \frac{1}{\frac{1}{x}} = x$$

$$\operatorname{sech}\left(\cosh^{-1}\frac{1}{x}\right) = x$$

Take sech^{-1} on both sides:

$$\operatorname{sech}^{-1}\left(\operatorname{sech}\left(\cosh^{-1}\frac{1}{x}\right)\right) = \operatorname{sech}^{-1}x \Rightarrow \left(\cosh^{-1}\frac{1}{x}\right) = \operatorname{sech}^{-1}x$$

TABLE 7.10 Derivatives of inverse hyperbolic functions

$\dfrac{d(\sinh^{-1} u)}{dx} = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$	
$\dfrac{d(\cosh^{-1} u)}{dx} = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx},$	$u > 1$
$\dfrac{d(\tanh^{-1} u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx},$	$\lvert u\rvert < 1$
$\dfrac{d(\coth^{-1} u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx},$	$\lvert u\rvert > 1$
$\dfrac{d(\operatorname{sech}^{-1} u)}{dx} = \dfrac{-du/dx}{u\sqrt{1-u^2}},$	$0 < u < 1$
$\dfrac{d(\operatorname{csch}^{-1} u)}{dx} = \dfrac{-du/dx}{\lvert u\rvert\sqrt{1+u^2}},$	$u \neq 0$

Integrating these formulas will allows us to obtain a list of useful integration formula involving hyperbolic functions

e.g.

$$\frac{1}{\sqrt{1+x^2}} = \frac{d}{dx}\sinh^{-1}x$$

$$\rightarrow \int \frac{1}{\sqrt{1+x^2}}dx = \int \frac{d}{dx}\sinh^{-1}x\ dx$$

$$\int \frac{1}{\sqrt{1+x^2}}dx = \sinh^{-1}x + C$$

Proof

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}.$$

let $y = \sinh^{-1}x$

$$x = \sinh y \to \frac{d}{dx}x = \frac{d}{dx}\sinh y = \frac{dy}{dx}\cosh y$$

$$\to \frac{dy}{dx} = \operatorname{sech} y = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

$\Rightarrow$ By virtue of chain rule,

$$\frac{d}{dx}\sinh^{-1}u = \frac{du}{dx}\frac{1}{\sqrt{1+u^2}}$$

Example 2 Derivative of the inverse hyperbolic cosine

- Show that

$$\frac{d}{dx}\cosh^{-1}u = \frac{1}{\sqrt{1+u^2}}.$$

Let $y = \cosh^{-1} x$...

Example 3 Using table 7.11

$$\int_0^1 \frac{2dx}{\sqrt{3+4x^2}}$$

Let $y = 2x$

$$\int_0^1 \frac{2dx}{\sqrt{3+4x^2}} = \int_0^2 \frac{dy}{\sqrt{3+y^2}}$$

Scale it again to normalise the constant 3 to 1

$$\text{Let } z = \frac{y}{\sqrt{3}} \rightarrow \int_0^2 \frac{dy}{\sqrt{3+y^2}} = \int_0^{2/\sqrt{3}} \frac{\sqrt{3}dz}{\sqrt{3+3z^2}} = \int_0^{2/\sqrt{3}} \frac{dz}{\sqrt{1+z^2}}$$

$$= \sinh^{-1} z\Big|_0^{2/\sqrt{3}} = \sinh^{-1}(2/\sqrt{3}) - \sinh^{-1}(0) = \sinh^{-1}(2/\sqrt{3}) - 0$$

$$= \sinh^{-1}(2/\sqrt{3})$$

$$\sinh^{-1}(2/\sqrt{3}) = ?$$

$$\text{Let } \ q = \sinh^{-1}(2/\sqrt{3})$$

$$\sinh q = 2/\sqrt{3} \rightarrow \frac{1}{2}\left(e^{q} - e^{-q}\right) = \frac{2}{\sqrt{3}}$$

$$e^{2q} - \frac{4}{\sqrt{3}}e^{q} - 1 = 0$$

$$e^{q} = \frac{\frac{4}{\sqrt{3}} + \sqrt{\left(-\frac{4}{\sqrt{3}}\right)^{2} - 4(-1)}}{2} = \frac{\frac{4}{\sqrt{3}} + \sqrt{\frac{296}{9}}}{2} = 2.682$$

$$\sinh^{-1}(2/\sqrt{3}) = q = \ln 2.682 = 0.9866$$

TABLE 7.11 Integrals leading to inverse hyperbolic functions

1. $$\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

2. $$\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$$

3. $$\int \frac{du}{a^2 - u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & \text{if } u^2 > a^2 \end{cases}$$

4. $$\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$$

5. $$\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$$

Chapter 8

Techniques of Integration

8.1

Basic Integration Formulas

(2nd lecture of week 17/09/07-22/09/07)

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C \qquad \text{(any number } k\text{)}$
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \dfrac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{u}{a}\right| + C$
21. $\int \dfrac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (a > 0)$
22. $\int \dfrac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (u > a > 0)$

Example 1 Making a simplifying substitution

$$\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \int \frac{d(\overbrace{x^2-9x}^{u})}{\sqrt{x^2-9x+1}}$$

$$= \int \frac{du}{\sqrt{u+1}} = \int \frac{d(u+1)}{\sqrt{u+1}} = \int \frac{dv}{\sqrt{v}} = 2v^{1/2}+C$$

$$= 2(u+1)^{1/2}+C = 2\left(x^2-9x+1\right)^{1/2}+C$$

Example 2 Completing the square

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x-4)^2}} =$$

$$\int \frac{d(x-4)}{\sqrt{16 - (x-4)^2}} = \int \frac{du}{\sqrt{4^2 - u^2}}$$

$$= \sin^{-1}\frac{u}{4} + C = \sin^{-1}\left(\frac{x-4}{4}\right) + C$$

Example 3 Expanding a power and using a trigonometric identity

$$\int (\sec x + \tan x)^2 \, dx$$

$$= \int (\sec^2 x + \tan^2 x + 2\sec x \tan x) dx.$$

$$\text{Racall:} \tan^2 x = \sec^2 x - 1; \frac{d}{dx} \tan x = \sec^2 x; \frac{d}{dx} \sec x = \tan x \sec x;$$

$$= \int (2\sec^2 x - 1 + 2\sec x \tan x) dx$$

$$= 2\tan x + -x + 2\sec x + C$$

Example 4 Eliminating a square root

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx =$$

$$\cos 4x = \cos 2(2x) = 2\cos^2(2x) - 1$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\,dx = \sqrt{2}\int_0^{\pi/4} |\cos 2x|\,x$$

$$= \sqrt{2}\int_0^{\pi/4} \cos 2x\,dx = \ldots$$

Example 5 Reducing an improper fraction

$$\int \frac{3x^2 - 7x}{3x+2}\,dx$$

$$= \int x - 3 + \frac{6}{3x+2}\,dx$$

$$= \int x - 3 + \frac{2}{x+2/3}\,dx$$

$$= \frac{1}{2}x^2 - 3x + 2\ln|x + \frac{2}{3}| + C$$

Example 6 Separating a fraction

$$\int \frac{3x+2}{\sqrt{1-x^2}}dx$$

$$= 3\int \frac{x}{\sqrt{1-x^2}}dx + \int \frac{2}{\sqrt{1-x^2}}dx$$

$$= 3\int \frac{\frac{1}{2}d(x^2)}{\sqrt{1-x^2}} + 2\int \frac{1}{\sqrt{1-x^2}}dx$$

$$= \frac{3}{2}\int \frac{du}{\sqrt{1-u}} + 2\sin^{-1}x + C$$

$$\int \frac{du}{(1-u)^{1/2}} = -2(1-u)^{1/2} + C'$$

$$= \frac{3}{2}[-2(1-u)^{1/2}] + 2\sin^{-1}x + C''$$

$$= -3\sqrt{(1-x^2)} + 2\sin^{-1}x + C''$$

Example 7 Integral of $y = \sec x$

$$\int \sec x dx = ?$$

$$d \sec x = \sec x \tan x dx$$

$$d \tan x = \sec^2 x dx = \sec x \sec x dx$$

$$d(\sec x + \tan x) = \sec x(\sec x + \tan x)dx$$

$$\sec x dx = \frac{d(\sec x + \tan x)}{\sec x + \tan x}$$

$$\int \sec x dx = \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} = \ln|\sec x + \tan x| + C$$

TABLE 8.2 The secant and cosecant integrals

1. $$\int \sec u \, du = \ln |\sec u + \tan u| + C$$

2. $$\int \csc u \, du = -\ln |\csc u + \cot u| + C$$

Procedures for Matching Integrals to Basic Formulas

Procedure	Example
Making a simplifying substitution	$\dfrac{2x - 9}{\sqrt{x^2 - 9x + 1}}\,dx = \dfrac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x - x^2} = \sqrt{16 - (x - 4)^2}$
Using a trigonometric identity	$(\sec x + \tan x)^2 = \sec^2 x + 2 \sec x \tan x + \tan^2 x$ $= \sec^2 x + 2 \sec x \tan x + (\sec^2 x - 1)$ $= 2 \sec^2 x + 2 \sec x \tan x - 1$
Eliminating a square root	$\sqrt{1 + \cos 4x} = \sqrt{2 \cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$
Reducing an improper fraction	$\dfrac{3x^2 - 7x}{3x + 2} = x - 3 + \dfrac{6}{3x + 2}$
Separating a fraction	$\dfrac{3x + 2}{\sqrt{1 - x^2}} = \dfrac{3x}{\sqrt{1 - x^2}} + \dfrac{2}{\sqrt{1 - x^2}}$
Multiplying by a form of 1	$\sec x = \sec x \cdot \dfrac{\sec x + \tan x}{\sec x + \tan x}$ $= \dfrac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$

8.2

Integration by Parts

(2nd lecture of week 17/09/07-22/09/07)

Product rule in integral form

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$$

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int g(x)\frac{d}{dx}[f(x)]dx + \int f(x)\frac{d}{dx}[g(x)]dx$$

$$f(x)g(x) = \int g(x)f'(x)dx + \int f(x)g'(x)dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Integration by parts formula

Alternative form of the integration by parts formula

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$$

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int g(x)\frac{d}{dx}[f(x)]dx + \int f(x)\frac{d}{dx}[g(x)]dx$$

$$f(x)g(x) = \int g(x)df(x) + \int f(x)dg(x)$$

Let $u = f(x); v = g(x)$.The above formular is recast into the form

$$uv = \int vdu + \int udv$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

Example 4 Repeated use of integration by parts

$$\int x^2 e^x dx = ?$$

Example 5 Solving for the unknown integral

$$\int e^x \cos x dx = ?$$

Evaluating by parts for definite integrals

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \qquad (3)$$

Example 6 Finding area

- Find the area of the region in Figure 8.1

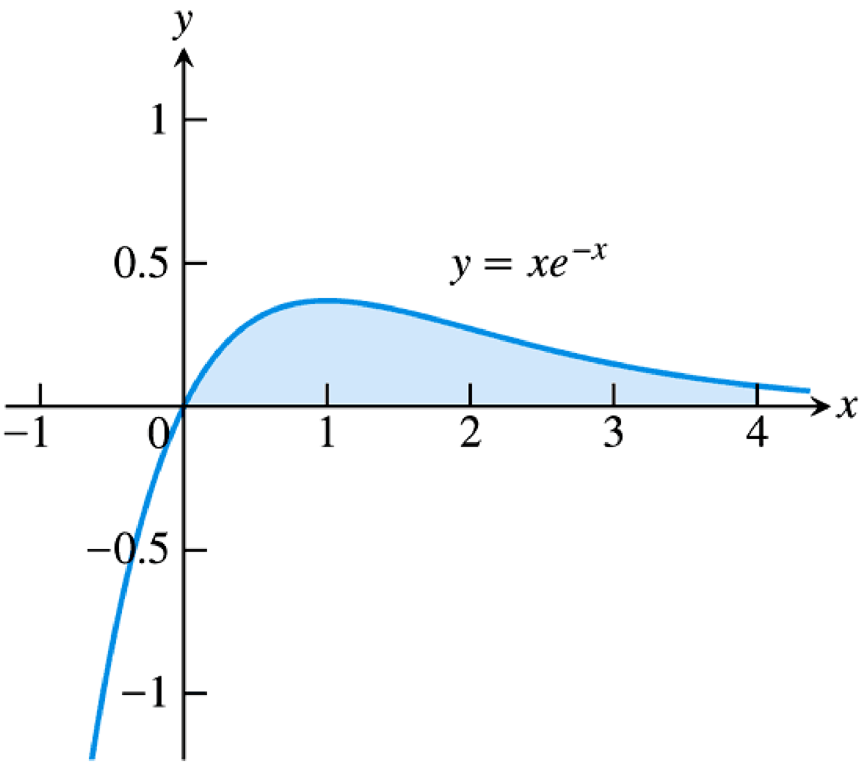


FIGURE 8.1 The region in Example 6.

Solution

$$\int_0^4 xe^{-x}dx = \ldots$$

Example 9 Using a reduction formula

- Evaluate $\int \cos^3 x\,dx$

8.3

Integration of Rational Functions by Partial Fractions

(3rd lecture of week 17/09/07-22/09/07)

General description of the method

- A rational function $f(x)/g(x)$ can be written as a sum of partial fractions. To do so:
- (a) The degree of $f(x)$ must be less than the degree of $g(x)$. That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term.
- We must know the factors of $g(x)$. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors.

Reducibility of a polynomial

- A polynomial is said to be **reducible** if it is the product of two polynomials of lower degree.
- A polynomial is **irreducible** if it is not the product of two polynomials of lower degree.

- THEOREM (Ayers, Schaum's series, pg. 305)
- Consider a polynomial $g(x)$ of order $n \geqslant 2$ (with leading coefficient 1). Two possibilities.

1. $g(x) = (x-r)h(x)$, where $h_1(x)$ is a polynomial of degree n-1, or
2. $g(x) = (x^2+px+q)\ h_2(x)$, where $h_2(x)$ is a polynomial of degree n-2, with the irreducible quadratic factor (x^2+px+q).

Example

$$g(x) = x^3 - 4x = \underbrace{(x-2)}_{\text{linear factor}} \cdot \underbrace{x(x+2)}_{\text{poly. of degree 2}}$$

$$g(x) = x^3 + 4x = \underbrace{(x^2+4)}_{\text{irreducible quadratic factor}} \cdot \underbrace{x}_{\text{poly. of degree 1}}$$

$$g(x) = x^4 - 9 = \underbrace{(x^2+3)}_{\text{irreducible quadratic factor}} \cdot \underbrace{(x+\sqrt{3})(x-\sqrt{3})}_{\text{poly. or degree 2}}$$

$$g(x) = x^3 - 3x^2 - x + 3 = \underbrace{(x+1)}_{\text{linear factor}} \underbrace{(x-2)^2}_{\text{poly. or degree 2}}$$

Quadratic polynomial

- A quadratic polynomial (polynomial or order $n = 2$) is either reducible or not reducible.
- Consider: $g(x)= x^2+px+q$.
- If $(p^2-4q) \geqslant 0$, $g(x)$ is reducible, i.e. $g(x) = (x+r_1)(x+r_2)$.
- If $(p^2-4q) < 0$, $g(x)$ is irreducible.

- In general, a polynomial of degree *n* can always be expressed as the product of linear factors and irreducible quadratic factors:

$$P_n(x) = (x - r_1)^{n_1}(x - r_2)^{n_2} \ldots (x - r_l)^{n_l} \times$$
$$(x^2 + p_1 x + q_1)^{m_1}(x^2 + p_2 x + q_2)^{m_2} \ldots (x^2 + p_k x + q_k)^{m_k}$$

$$n = (n_1 + n_2 + \ldots + n_l) + 2(m_1 + m_2 + \ldots + m_l)$$

Integration of rational functions by partial fractions

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1 x + C_1}{x^2 + px + q} + \frac{B_2 x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_n x + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example 1 Distinct linear factors

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \ldots$$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)} = \ldots$$

Example 2 A repeated linear factor

$$\int \frac{6x+7}{(x+2)^2}\,dx = \ldots$$

$$\frac{6x+7}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

Example 3 Integrating an improper fraction

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx = \ldots$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{5x - 3}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)} = \ldots$$

Example 4 Integrating with an irreducible quadratic factor in the denominator

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \ldots$$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2} = \ldots$$

Example 5 A repeated irreducible quadratic factor

$$\int \frac{1}{x(x^2+1)^2}\,dx = ?$$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2} = \ldots$$

Other ways to determine the coefficients

- Example 8 Using differentiation
- Find *A*, *B* and *C* in the equation

$$\frac{x-1}{(x+1)^3}=\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

$$\frac{A(x+1)^2+B(x+1)+C}{(x+1)^3}=\frac{x-1}{(x+1)^3}$$

$$\Rightarrow A(x+1)^2+B(x+1)+C=x-1$$

$$x=-1\rightarrow C=-2$$

$$\Rightarrow A(x+1)^2+B(x+1)=x+1$$

$$\Rightarrow A(x+1)+B=1$$

$$\frac{d}{dx}[A(x+1)+B]=\frac{d}{dx}(1)=0$$

$$A=0$$

$$B=1$$

Example 9 Assigning numerical values to *x*

- Find *A*, *B* and *C* in

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}$$

$$=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$$

$$A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)\equiv f(x)$$

$$=x^2+1$$

$$f(1)=2A+=1^2+1=2\Rightarrow A=1$$

$$f(2)=-B=2^2+1=5;\Rightarrow B=-5$$

$$f(3)=2C=3^2+1=10;\Rightarrow C=5$$

8.4

Trigonometric Integrals

(3rd lecture of week 17/09/07-22/09/07)

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the work into three cases.

Case 1 If m is odd, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \qquad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Example 1 *m* is odd

$$\int \sin^3 x \cos^2 x \; dx = ?$$

$$\int \sin^3 x \cos^2 x \; dx = -\int \sin^2 x \cos^2 x \; d\left(\cos x\right)$$

$$= \int (\cos^2 x - 1) \cos^2 x \; d\left(\cos x\right)$$

$$= \int (u^2 - 1) u^2 du = \ldots$$

Example 2 *m* is even and *n* is odd

$$\int \cos^5 x \; dx = ?$$

$$\int \cos^3 x \cos^2 x \; dx = \int \cos^2 x \cos^2 x \; d \sin x$$

$$= \int (1\text{-}\sin^2 x)(1\text{-}\sin^2 x) \; d \sin x$$

$$= \int (1\text{-}u^2)(1\text{-}u^2) \; du = \ldots$$

Example 3 *m* and *n* are both even

$$\int \cos^2 x \sin^4 x \quad dx = ?$$

$$\int \cos^2 x \sin^4 x \ dx =$$

$$\int \left(\frac{1\text{-}\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 \ dx$$

$$= \frac{1}{4}\int (1\text{-}\cos 2x)(1+\cos 2x)^2 \ dx$$

$$= \frac{1}{4}\int \left(1+\cos 2x-\cos^2 2x-\cos \ \ 2x\right) \ dx = \ldots$$

Example 4 Eliminating square roots

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx = ?$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx$$

$$= \int_0^{\pi/4} \sqrt{2\cos^2 2x}\,dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\,dx = \ldots$$

Example 6 Integrals of powers of tan x and sec x

$$\int \sec^3 x dx = ?$$

Use integration by parts.

$$\int \sec^3 x dx = \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x dx}_{dv};$$

$$dv = \sec^2 x dx \rightarrow v = \int \sec^2 x dx = \tan x$$

$$u = \sec \rightarrow du = \sec x \tan x dx$$

$$\int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x dx}_{dv}$$

$$= \sec x \tan x - \int \tan x \cdot \underbrace{\sec x \tan x dx}_{du}$$

$$= \sec x \tan x - \int \tan x^2 \sec x dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx$$

$$\int \sec^3 x dx = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \ldots$$

$$\int \sec x dx = \int \sec x \frac{(\tan x + \sec x)}{\tan x + \sec x} dx$$

$$= \int \frac{(\sec x \tan x + \sec^2 x)}{\tan x + \sec x} dx$$

$$= \int \frac{d(\sec x + \tan x)}{\tan x + \sec x}$$

$$= \ln|\sec x + \tan x| + C$$

Example 7 Products of sines and cosines

$$\int \cos 5x \sin 3x dx = ?$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x];$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x];$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

$$\int \cos 5x \sin 3x dx$$

$$= \frac{1}{2}\int [\sin(-2x) + \sin 8x] dx$$

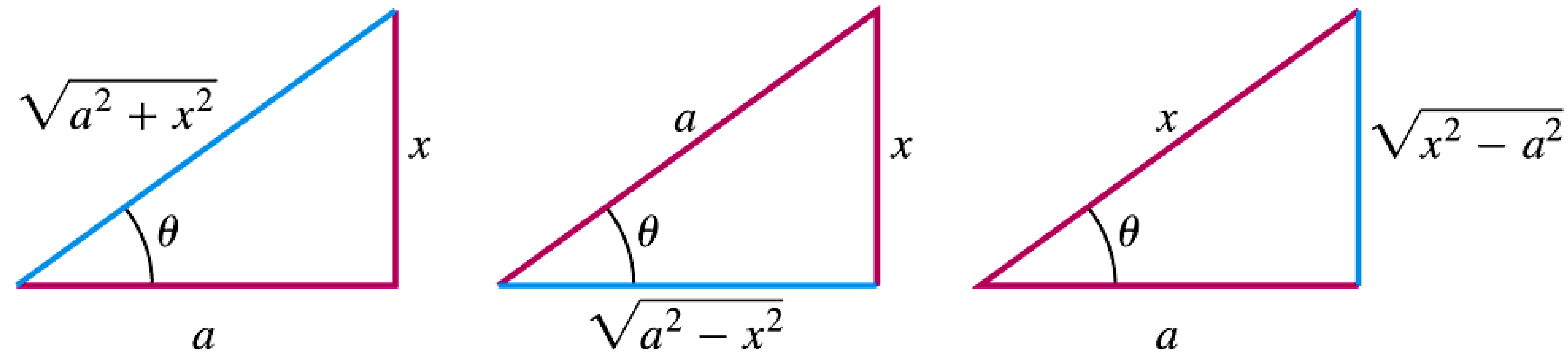
$$= \ldots$$

8.5

Trigonometric Substitutions

(1st lecture of week 24/09/07-29/09/07)

Three basic substitutions



$x = a \tan \theta$ $\qquad$ $x = a \sin \theta$ $\qquad$ $x = a \sec \theta$

$\sqrt{a^2 + x^2} = a|\sec \theta|$ $\qquad$ $\sqrt{a^2 - x^2} = a|\cos \theta|$ $\qquad$ $\sqrt{x^2 - a^2} = a|\tan \theta|$

FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

Useful for integrals involving $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$

Example 1 Using the substitution $x = a\tan\theta$

$$\int \frac{dx}{\sqrt{4+x^2}} = ?$$

$$x = 2\tan y \rightarrow dx = 2\sec^2 y dy = 2(\tan^2 y + 1)dy$$

$$\int \frac{dx}{\sqrt{4+4\tan^2 y}} = \int \frac{2(\tan^2 y+1)}{\sqrt{4+4\tan^2 y}} dy$$

$$= \int \frac{(\tan^2 y+1)}{\sqrt{1+\tan^2 y}} dy = \int \sqrt{\sec^2 y} dy = \int |\sec y| \, dy$$

$$= \ln|\sec y + \tan y| + C$$

Example 2 Using the substitution $x = a\sin\theta$

$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = ?$$

$$x = 3\sin y \to dx = 3\cos y \; dy$$

$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2 y \cdot 3\cos y \; dy}{\sqrt{9-9\sin^2 y}} =$$

$$= 9\int \frac{\sin^2 y \cdot \cos y \; dy}{\sqrt{1-\sin^2 y}}$$

$$= 9\int \sin^2 y dy = \ldots$$

Example 3 Using the substitution $x = a\sec\theta$

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = ?$$

$$x = \frac{2}{5}\sec y \to dx = \frac{2}{5}\sec y \tan y \ dy$$

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \frac{2}{5}\int \frac{\sec y \tan y \ dy}{\sqrt{4\sec^2 y - 4}} = \frac{1}{5}\int \frac{\sec y \tan y \ dy}{\sqrt{\sec^2 y - 1}}$$

$$= \frac{1}{5}\int \frac{\sec y \tan y \ dy}{\sqrt{\sec^2 y - 1}} = \frac{1}{5}\int \sec y \ dy$$

$$= \frac{1}{5}\ln|\sec y + \tan y| + C = \dots$$

Example 4 Finding the volume of a solid of revolution

$$V = 16\pi \int_0^2 \frac{dx}{\left(x^2+4\right)^2} = ?$$

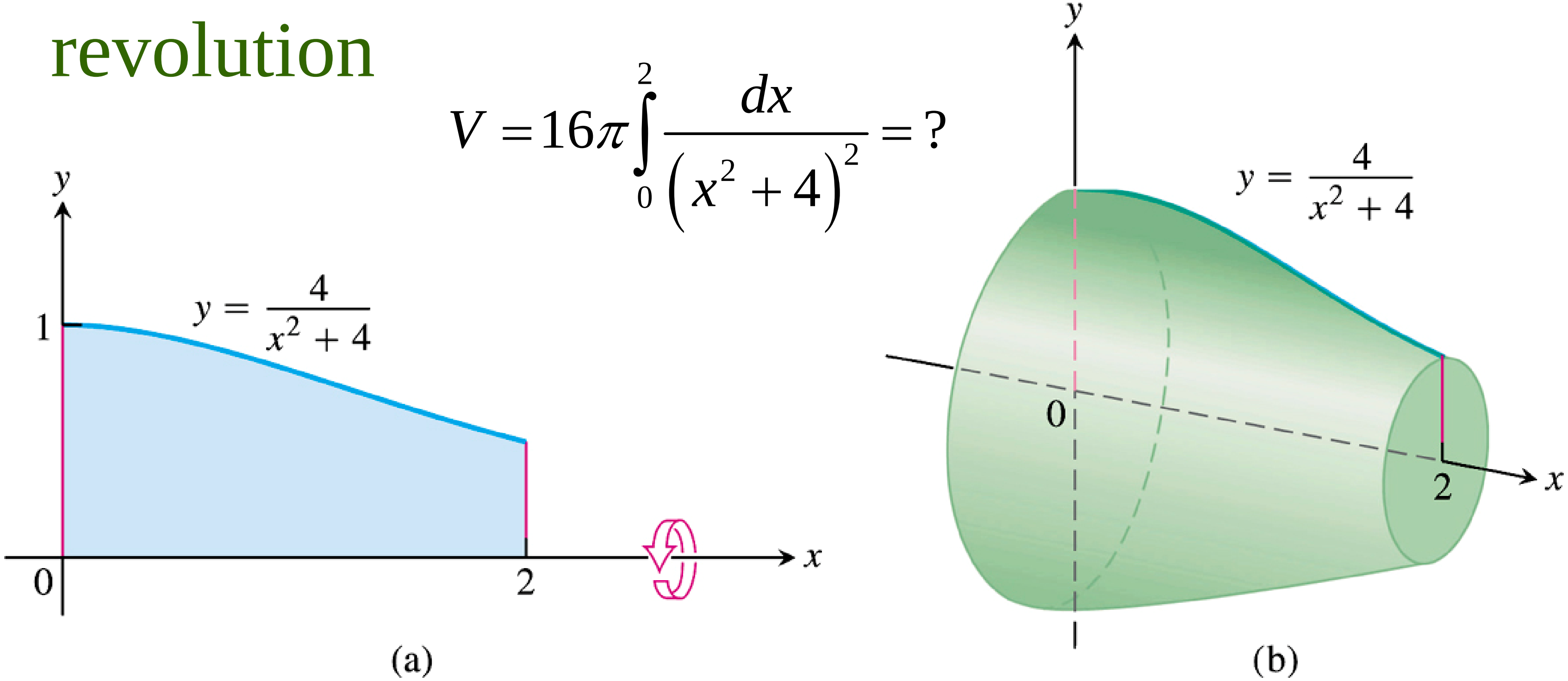


FIGURE 8.7 The region (a) and solid (b) in Example 4.

Solution

$$V = 16\pi \int_0^2 \frac{dx}{\left(x^2+4\right)^2} = ?$$

Let $x = 2\tan y \rightarrow dx = 2\sec^2 y dy$

$$V = \pi \int_0^{\pi/4} \frac{2\sec^2 y dy}{\left(\tan^2 y + 1\right)^2} = \pi \int_0^{\pi/4} \frac{2\sec^2 y dy}{\left(\sec^2 y\right)^2}$$

$$= 2\pi \int_0^{\pi/4} \cos^2 y dy = \ldots$$

8.6

Integral Tables

(1st lecture of week 24/09/07-29/09/07)

Integral tables is provided at the back of Thomas'

- T-4 A brief tables of integrals
- Integration can be evaluated using the tables of integral.

EXAMPLE 1 Find

$$\int x(2x + 5)^{-1}\, dx.$$

Solution We use Formula 8 (not 7, which requires $n \neq -1$):

$$\int x(ax + b)^{-1}\, dx = \frac{x}{a} - \frac{b}{a^2} \ln |ax + b| + C.$$

With $a = 2$ and $b = 5$, we have

$$\int x(2x + 5)^{-1}\, dx = \frac{x}{2} - \frac{5}{4} \ln |2x + 5| + C.$$

EXAMPLE 2 Find

$$\int \frac{dx}{x\sqrt{2x + 4}}.$$

Solution We use Formula 13(b):

$$\int \frac{dx}{x\sqrt{ax + b}} = \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{ax + b} - \sqrt{b}}{\sqrt{ax + b} + \sqrt{b}} \right| + C, \qquad \text{if } b > 0.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x + 4}} = \frac{1}{\sqrt{4}} \ln \left| \frac{\sqrt{2x + 4} - \sqrt{4}}{\sqrt{2x + 4} + \sqrt{4}} \right| + C$$

$$= \frac{1}{2} \ln \left| \frac{\sqrt{2x + 4} - 2}{\sqrt{2x + 4} + 2} \right| + C.$$

EXAMPLE 3 Find

$$\int \frac{dx}{x\sqrt{2x - 4}}.$$

Solution We use Formula 13(a):

$$\int \frac{dx}{x\sqrt{ax - b}} = \frac{2}{\sqrt{b}} \tan^{-1} \sqrt{\frac{ax - b}{b}} + C.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x - 4}} = \frac{2}{\sqrt{4}} \tan^{-1} \sqrt{\frac{2x - 4}{4}} + C = \tan^{-1} \sqrt{\frac{x - 2}{2}} + C.$$

EXAMPLE 4 Find

$$\int \frac{dx}{x^2\sqrt{2x-4}}.$$

Solution We begin with Formula 15:

$$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}} + C.$$

With $a = 2$ and $b = -4$, we have

$$\int \frac{dx}{x^2\sqrt{2x-4}} = -\frac{\sqrt{2x-4}}{-4x} + \frac{2}{2\cdot 4}\int \frac{dx}{x\sqrt{2x-4}} + C.$$

We then use Formula 13(a) to evaluate the integral on the right (Example 3) to obtain

$$\int \frac{dx}{x^2\sqrt{2x-4}} = \frac{\sqrt{2x-4}}{4x} + \frac{1}{4}\tan^{-1}\sqrt{\frac{x-2}{2}} + C.$$

EXAMPLE 5 Find

$$\int x \sin^{-1} x \, dx.$$

Solution We use Formula 99:

$$\int x^n \sin^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \sin^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1} \, dx}{\sqrt{1 - a^2x^2}}, \qquad n \neq -1.$$

With $n = 1$ and $a = 1$, we have

$$\int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2 \, dx}{\sqrt{1 - x^2}}.$$

The integral on the right is found in the table as Formula 33:

$$\int \frac{x^2}{\sqrt{a^2 - x^2}} \, dx = \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) - \frac{1}{2} x \sqrt{a^2 - x^2} + C.$$

With $a = 1$,

$$\int \frac{x^2 \, dx}{\sqrt{1 - x^2}} = \frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1 - x^2} + C.$$

The combined result is

$$\int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \left(\frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1 - x^2} + C \right)$$

$$= \left(\frac{x^2}{2} - \frac{1}{4} \right) \sin^{-1} x + \frac{1}{4} x \sqrt{1 - x^2} + C'.$$

8.8

Improper Integrals

(2nd lecture of week 24/09/07-29/09/07)

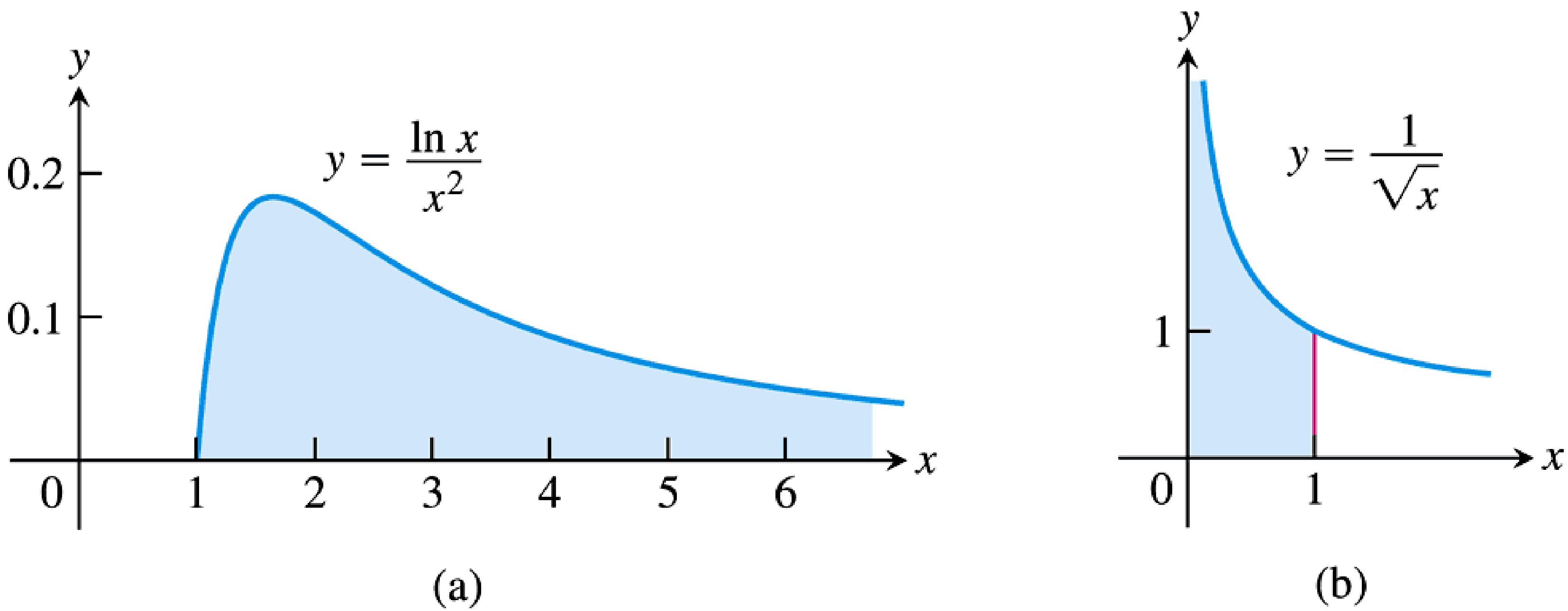


FIGURE 8.17 Are the areas under these infinite curves finite?

Infinite limits of integration

$$A(a) = \lim_{b \to \infty} A(b) = \lim_{b \to \infty} 2 - 2e^{-b/2} = 2$$

$$A(b) = \int_0^b e^{-x/2} dx = \ldots = 2 - 2e^{-b/2}$$

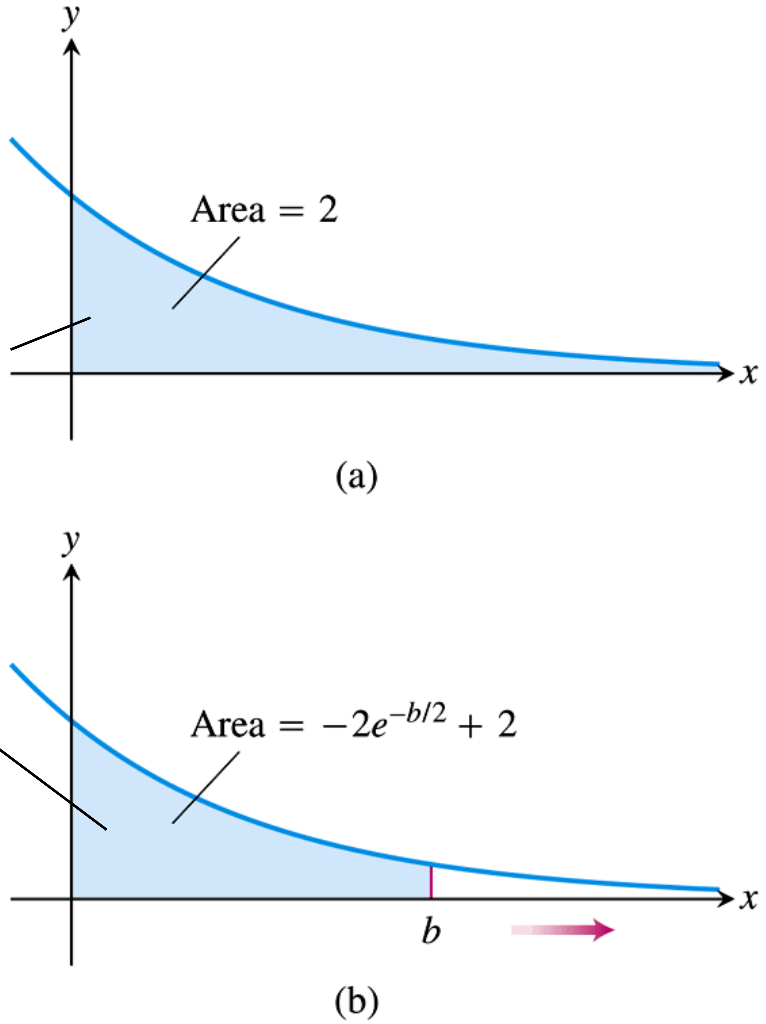


FIGURE 8.18 (a) The area in the first quadrant under the curve $y = e^{-x/2}$ is (b) an improper integral of the first type.

DEFINITION Type I Improper Integrals

Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Example 1 Evaluating an improper integral on [1,∞]

- Is the area under the curve $y=(\ln x)/x^2$ from 1 to ∞ finite? If so, what is it?

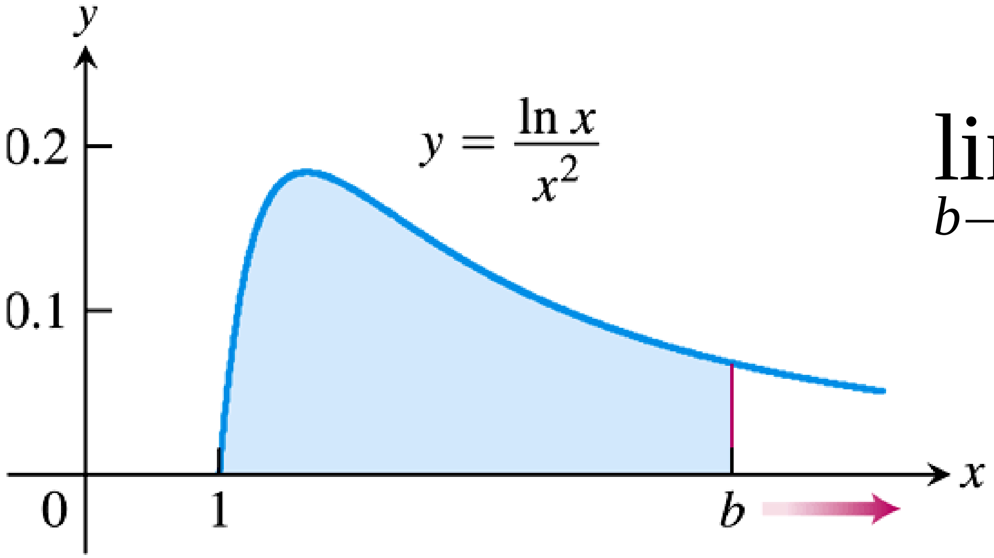


FIGURE 8.19 The area under this curve is an improper integral (Example 1).

$$\lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = ?$$

Solution

$$\int_1^b \frac{\ln x}{x}\frac{dx}{x} = \int_1^b \frac{\ln x}{x} d(\ln x) = \int_{\ln 1}^{\ln b} \frac{u}{e^u} du \qquad ;u = \ln x, x = e^u$$

$$\int_0^{\ln b} u \underbrace{e^{-u} du}_{dw} = u \underbrace{(-e^{-u})}_{w} \Big|_0^{\ln b} - \int_0^{\ln b} \underbrace{(-e^{-u})}_{w} du$$

$$= ue^{-u}\Big|_{\ln b}^{0} + \int_0^{\ln b} e^{-u} du = ue^{-u}\Big|_{\ln b}^{0} - e^{-u}\Big|_0^{\ln b}$$

$$= -\ln b \cdot e^{-\ln b} - (e^{-\ln b} - 1) = -\frac{1}{b}\ln b - \frac{1}{b} + 1$$

$$\lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2} dx = \lim_{b\to\infty} \left[-\frac{1}{b}\ln b - \frac{1}{b} + 1 \right] = 1$$

Example 2 Evaluating an integral on [-∞,∞]

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = ?$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_{-b}^{0} \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2}$$

$$= 2\lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2}$$

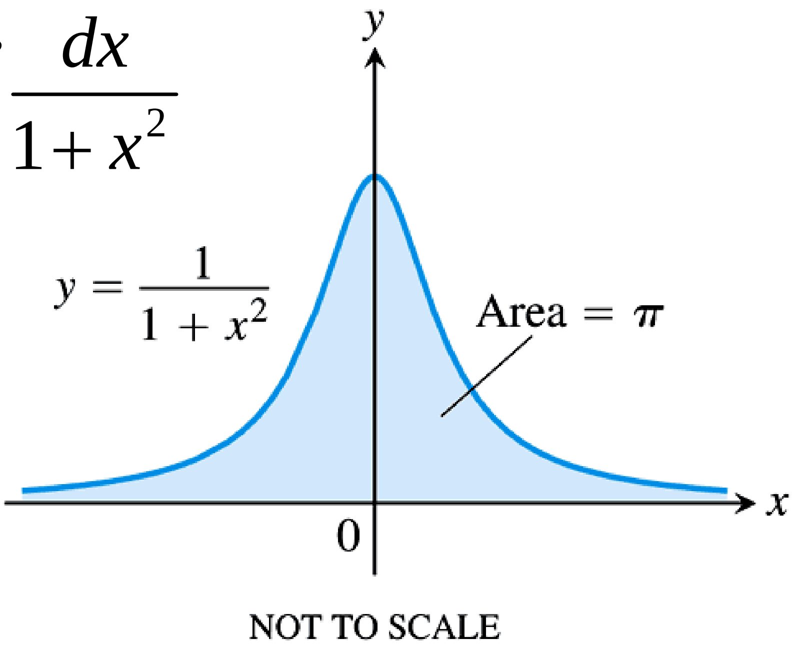


FIGURE 8.20 The area under this curve is finite (Example 2).

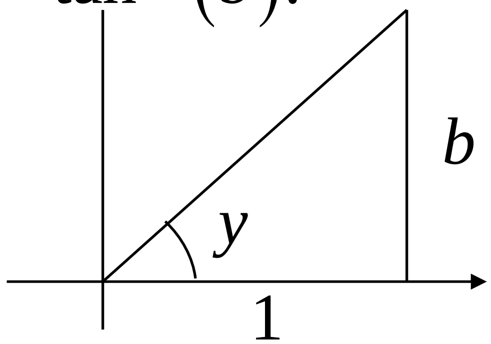
Solution

Using the integral table (Eq. 16)

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int_0^b \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^b = \tan^{-1}(b) - \tan^{-1} 0 = \tan^{-1}(b).$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\lim_{b\to\infty}\tan^{-1} b = 2\cdot\frac{\pi}{2} = \pi$$



$$y = \tan^{-1} b \Rightarrow b = \tan y$$

$$\lim_{b\to\infty}\tan^{-1} b = \frac{\pi}{2}$$

DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

Example 3 Integrands with vertical asymptotes

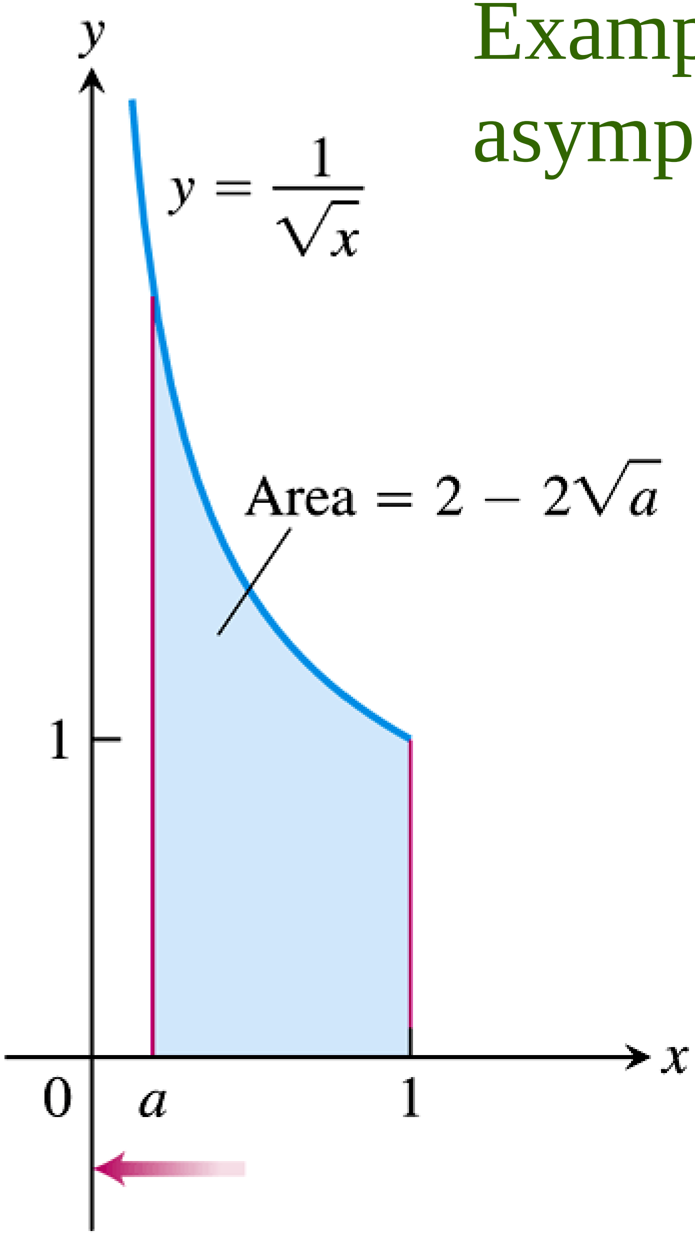
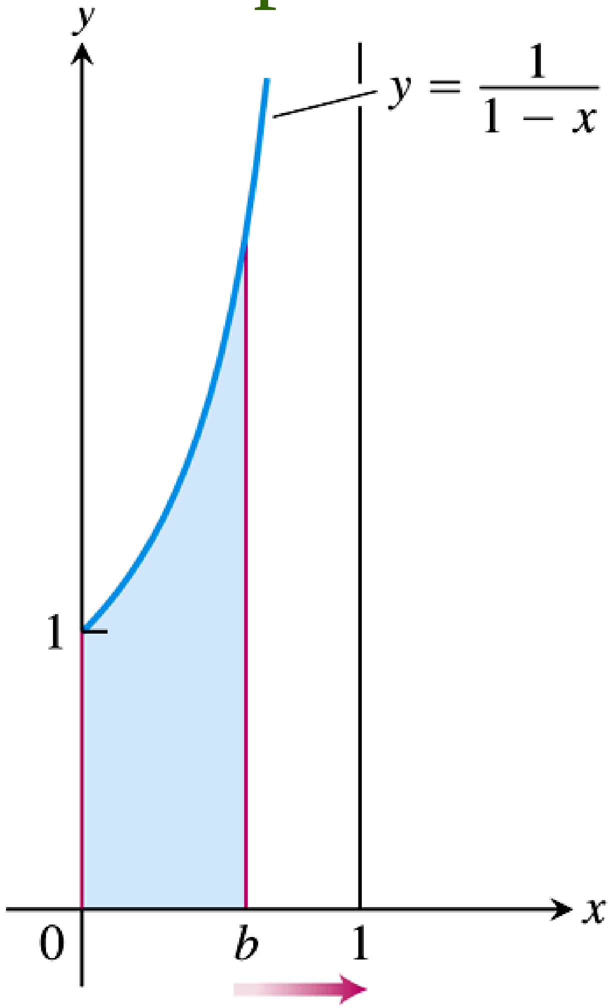


FIGURE 8.21 The area under this curve is

$$\lim_{a \to 0^+} \int_a^1 \left(\frac{1}{\sqrt{x}}\right) dx = 2,$$

an improper integral of the second kind.

Example 4 A divergent improper integral



- Investigate the convergence of $\int_0^1 \frac{dx}{1-x}$

FIGURE 8.22 The limit does not exist:

$$\int_0^1 \left(\frac{1}{1-x}\right) dx = \lim_{b \to 1^-} \int_0^b \frac{1}{1-x} dx = \infty$$

The area beneath the curve and above the x-axis for $[0, 1)$ is not a real number (Example 4).

Solution

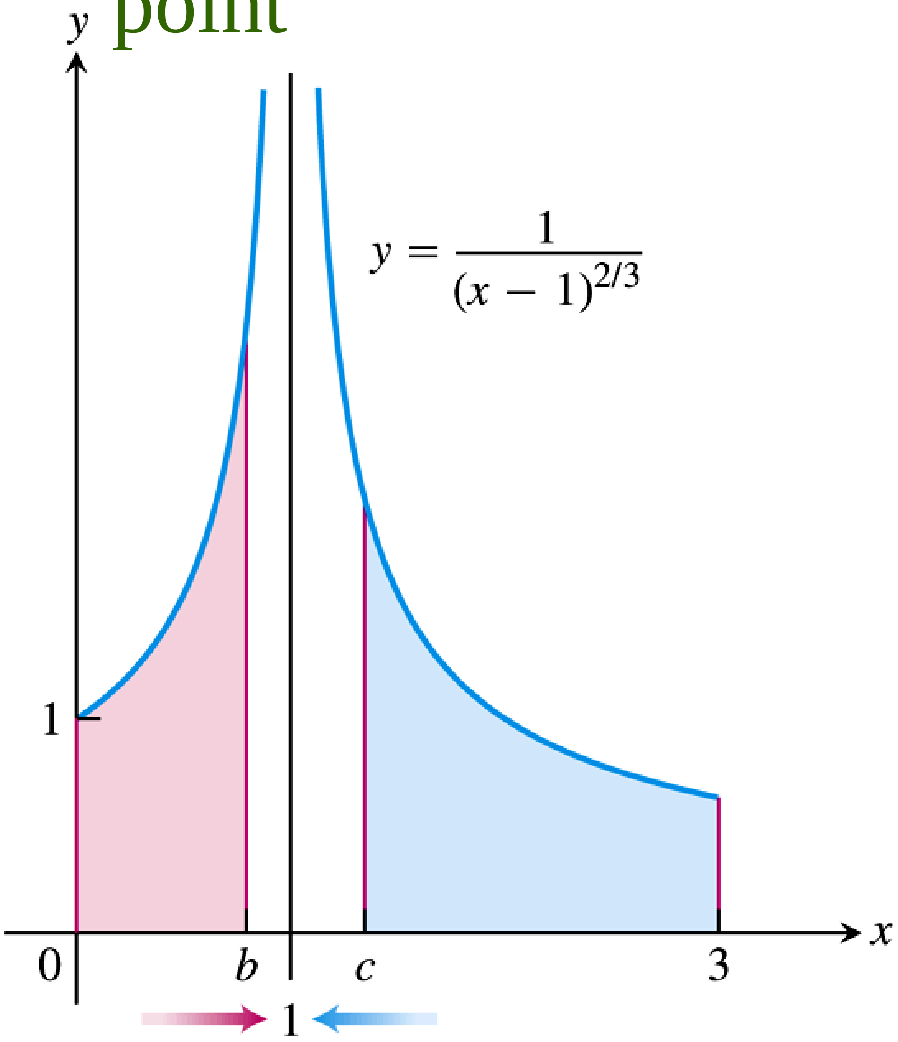
$$\int_0^1 \frac{dx}{1-x} = \lim_{b\to 1^-}\int_0^b \frac{dx}{1-x} = -\lim_{b\to 1^-}\left[\ln|x-1|\right]_0^b$$

$$= -\lim_{b\to 1^-}\left[\ln|b-1| - \ln|0-1|\right]$$

$$= -\lim_{b\to 1^-}\left[\ln|b-1| - \ln|0-1|\right] = \lim_{b\to 1^-}\left[\ln|b-1|^{-1}\right]$$

$$= \lim_{\varepsilon\to 0}\left[\ln\frac{1}{\varepsilon}\right] = \infty$$

Example 5 Vertical asymptote at an interior point



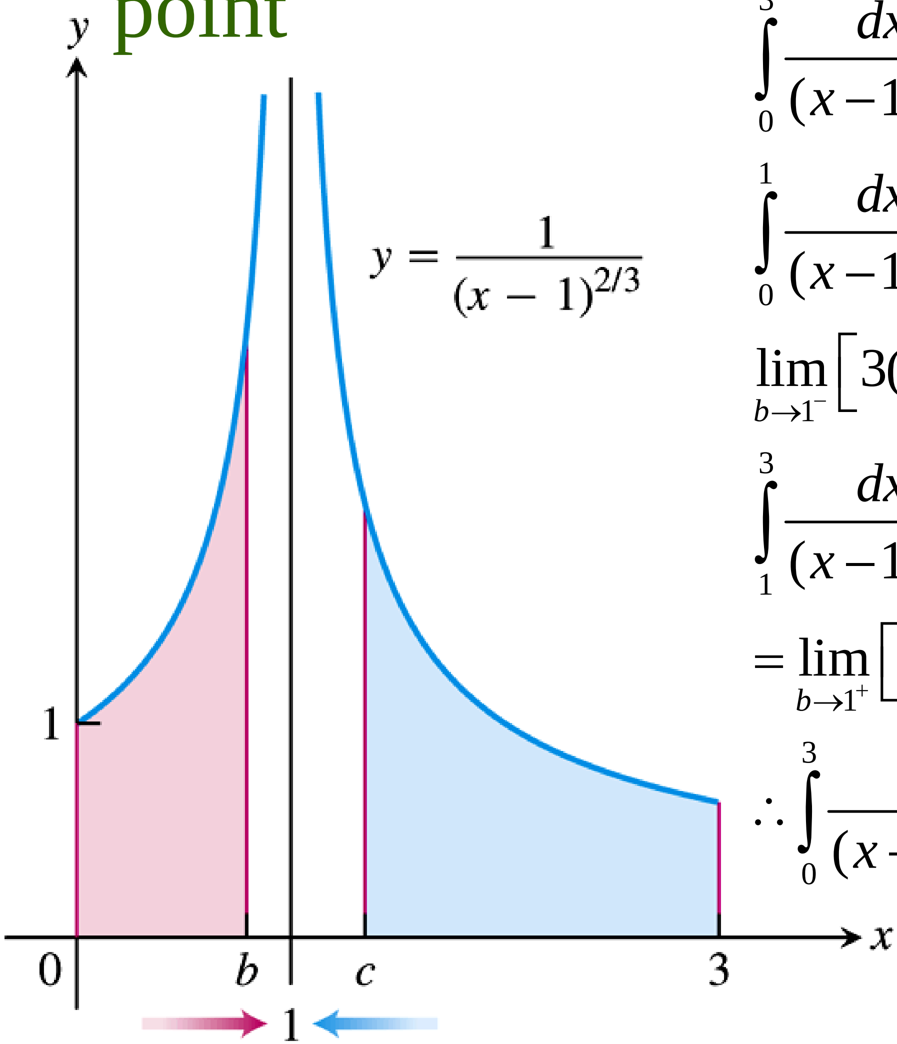
$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = ?$$

FIGURE 8.23 Example 5 shows the convergence of

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = 3 + 3\sqrt[3]{2},$$

so the area under the curve exists (so it is a real number).

Example 5 Vertical asymptote at an interior point



$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \left[3(x-1)^{1/3}\right]_0^b =$$

$$\lim_{b\to 1^-} \left[3(b-1)^{1/3} - 3(-1)^{1/3}\right] = \lim_{b\to 1^-} [0+3] = 3;$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1+} \int_b^3 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^+} \left[3(x-1)^{1/3}\right]_b^3$$

$$= \lim_{b\to 1^+} \left[3(3-1)^{1/3} - 3(b-1)^{1/3}\right] = 3\cdot 2^{2/3}$$

$$\therefore \int_0^3 \frac{dx}{(x-1)^{2/3}} = 3(1+2^{2/3})$$

Example 7 Finding the volume of an infinite solid

- The cross section of the solid in Figure 8.24 perpendicular to the x-axis are circular disks with diameters reaching from the x-axis to the curve $y = e^x$, $-\infty < x < \ln 2$. Find the volume of the horn.

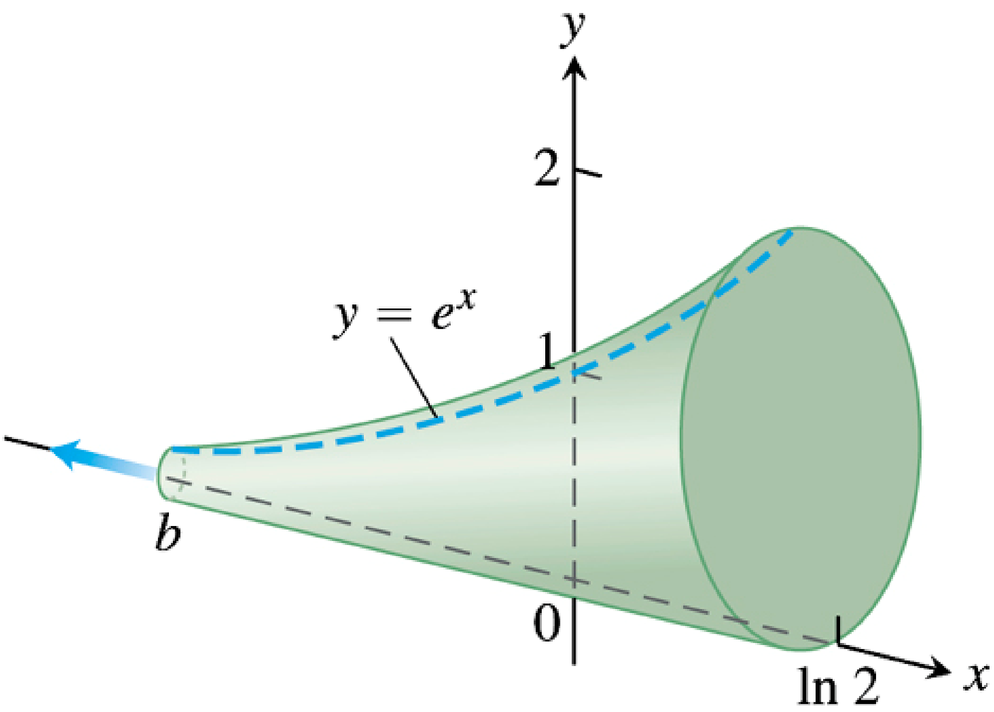


FIGURE 8.24 The calculation in Example 7 shows that this infinite horn has a finite volume.

Example 7 Finding the volume of an infinite solid

volume of a slice of disk of thickness *dx*,diameter *y*

$$V=\int_0^V dV=\frac{1}{4}\lim_{b\to-\infty}\int_b^{\ln 2}\pi y(x)^2\,dx$$

$$=\frac{1}{4}\lim_{b\to-\infty}\int_b^{\ln 2}\pi e^{2x}dx$$

$$=\frac{1}{8}\lim_{b\to-\infty}\left[\pi e^{2x}\right]_b^{\ln 2}$$

$$=\frac{1}{8}\lim_{b\to-\infty}\left[4\pi-\pi e^{2b}\right]$$

$$=\frac{1}{8}\pi\lim_{b\to-\infty}(4-e^{2b})=\frac{\pi}{2}$$

$$dV=\pi(y/2)^2dx$$

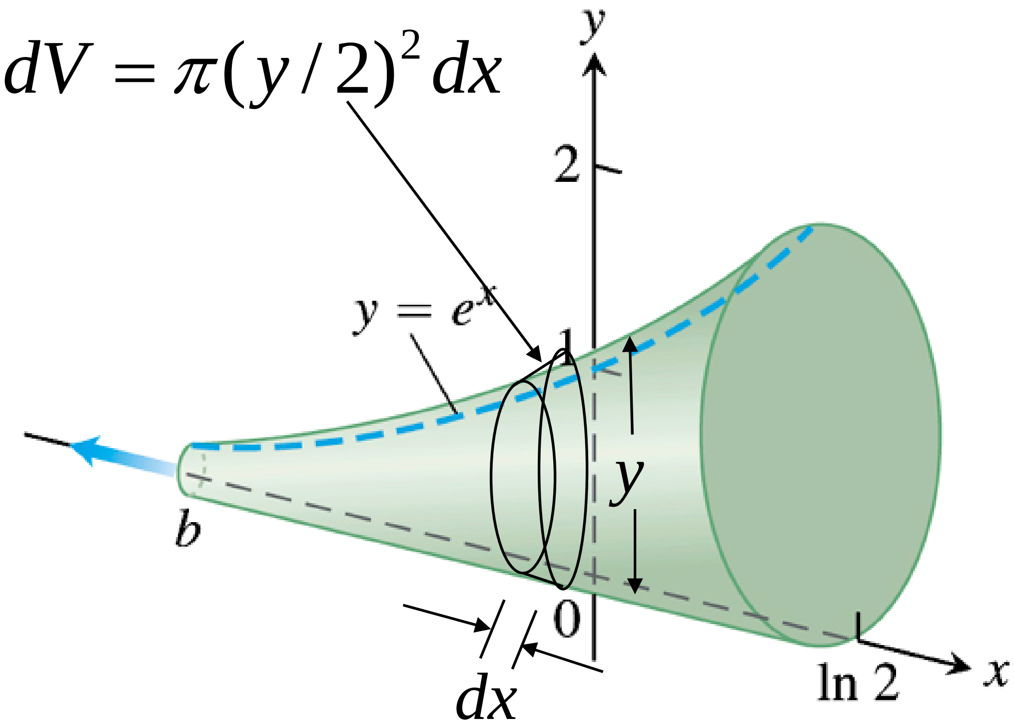


FIGURE 8.24 The calculation in Example 7 shows that this infinite horn has a finite volume.

Chapter 11

Infinite Sequences and Series

11.1

Sequences

(2nd lecture of week 24/09/07-29/09/07)

DEFINITION Infinite Sequence

An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

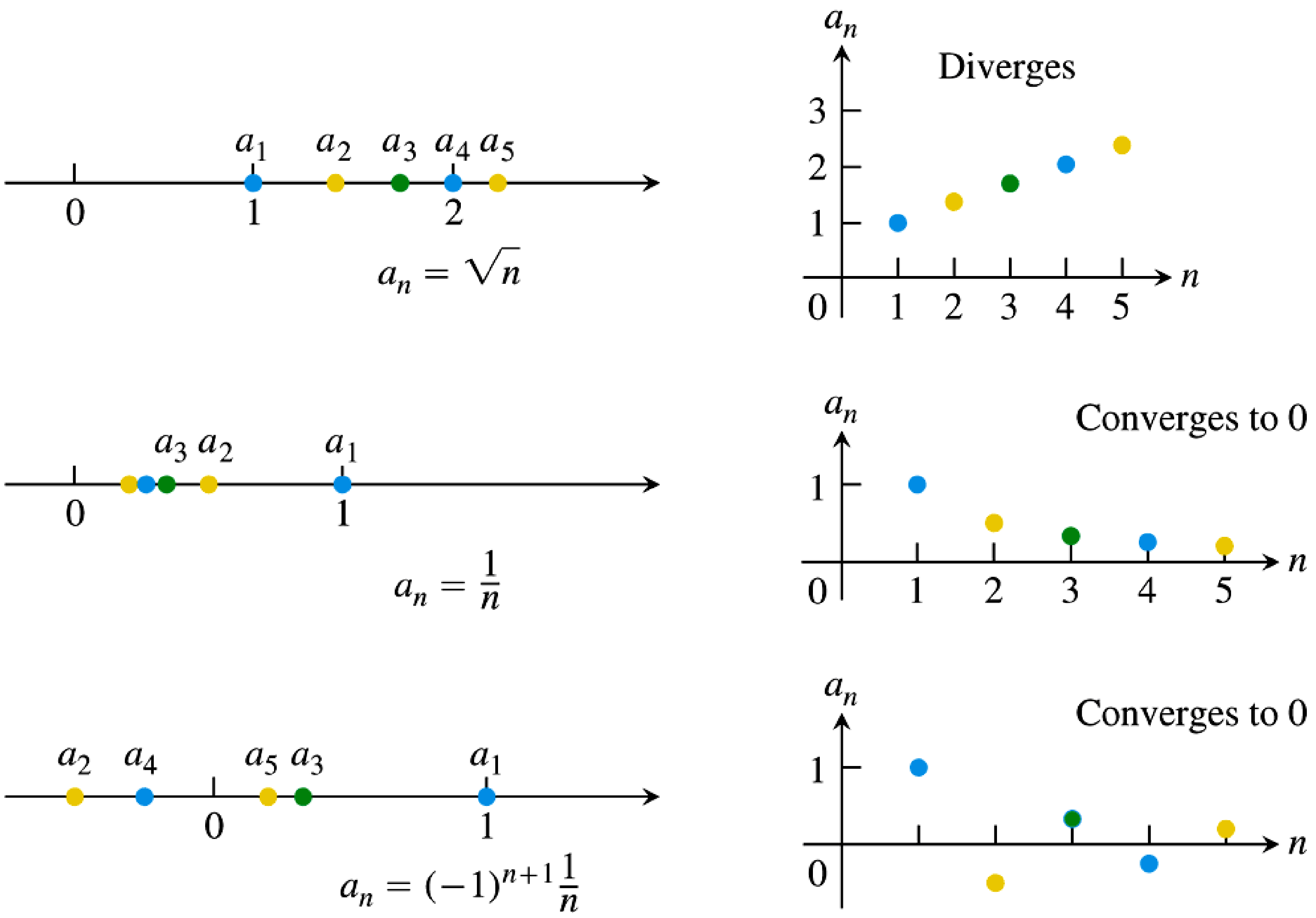


FIGURE 11.1 Sequences can be represented as points on the real line or as points in the plane where the horizontal axis n is the index number of the term and the vertical axis a_n is its value.

DEFINITIONS Converges, Diverges, Limit

The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \qquad \Rightarrow \qquad |a_n - L| < \epsilon .$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Figure 11.2).

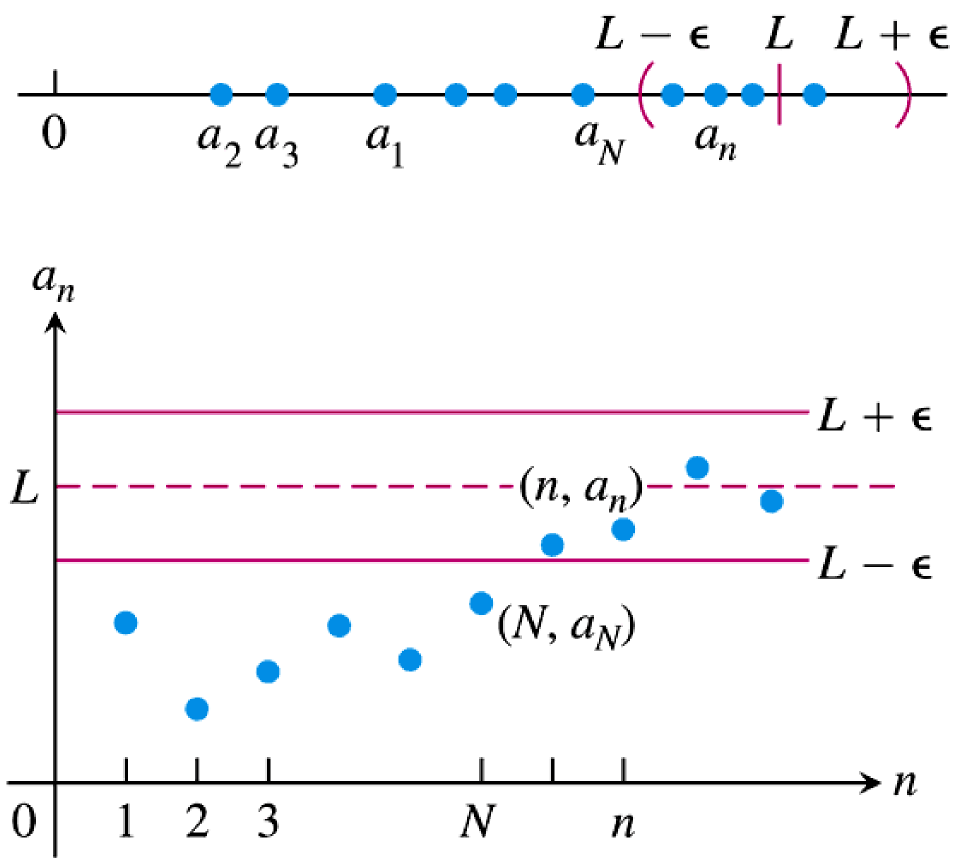


FIGURE 11.2 $a_n \to L$ if $y = L$ is a horizontal asymptote of the sequence of points $\{(n, a_n)\}$. In this figure, all the a_n's after a_N lie within ϵ of L.

DEFINITION **Diverges to Infinity**

The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad a_n \to \infty .$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \qquad \text{or} \qquad a_n \to -\infty .$$

THEOREM 1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$ if $B \neq 0$

EXAMPLE 3 Applying Theorem 1

By combining Theorem 1 with the limits of Example 1, we have:

(a) $$\lim_{n\to\infty}\left(-\frac{1}{n}\right) = -1\cdot\lim_{n\to\infty}\frac{1}{n} = -1\cdot 0 = 0$$ Constant Multiple Rule and Example 1a

(b) $$\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim_{n\to\infty}1 - \lim_{n\to\infty}\frac{1}{n} = 1-0 = 1$$ Difference Rule and Example 1a

(c) $$\lim_{n\to\infty}\frac{5}{n^2} = 5\cdot\lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{1}{n} = 5\cdot 0\cdot 0 = 0$$ Product Rule

(d) $$\lim_{n\to\infty}\frac{4-7n^6}{n^6+3} = \lim_{n\to\infty}\frac{(4/n^6)-7}{1+(3/n^6)} = \frac{0-7}{1+0} = -7.$$ Sum and Quotient Rules

THEOREM 2 The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

EXAMPLE 4 Applying the Sandwich Theorem

Since $1/n \to 0$, we know that

(a) $\dfrac{\cos n}{n} \to 0$ because $-\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n}$;

(b) $\dfrac{1}{2^n} \to 0$ because $0 \le \dfrac{1}{2^n} \le \dfrac{1}{n}$;

(c) $(-1)^n \dfrac{1}{n} \to 0$ because $-\dfrac{1}{n} \le (-1)^n \dfrac{1}{n} \le \dfrac{1}{n}$.

THEOREM 3 The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \rightarrow L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \rightarrow f(L)$.

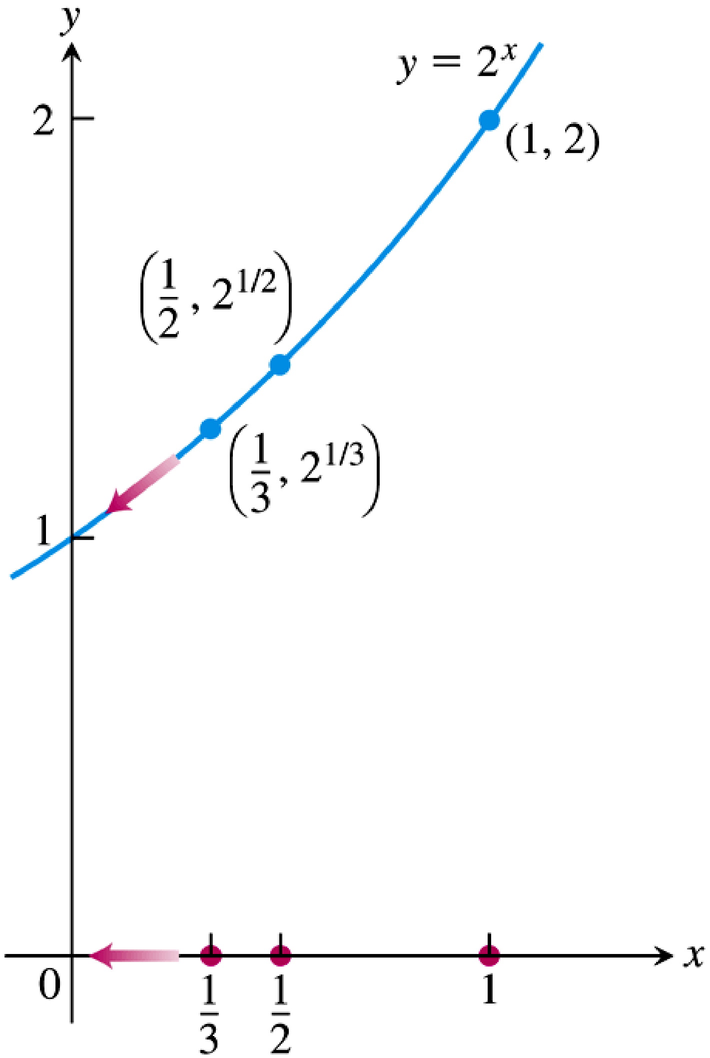


FIGURE 11.3 As $n \to \infty$, $1/n \to 0$ and $2^{1/n} \to 2^0$ (Example 6).

THEOREM 4

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x \to \infty} f(x) = L \qquad \Rightarrow \qquad \lim_{n \to \infty} a_n = L.$$

THEOREM 5

The following six sequences converge to the limits listed below:

1. $$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

2. $$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

3. $$\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$$

4. $$\lim_{n\to\infty} x^{n} = 0 \qquad (|x| < 1)$$

5. $$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{n} = e^{x} \qquad (\text{any } x)$$

6. $$\lim_{n\to\infty} \frac{x^{n}}{n!} = 0 \qquad (\text{any } x)$$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

DEFINITION **Nondecreasing Sequence**

A sequence $\{a_n\}$ with the property that $a_n \le a_{n+1}$ for all n is called a **nondecreasing sequence**.

DEFINITIONS Bounded, Upper Bound, Least Upper Bound

A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

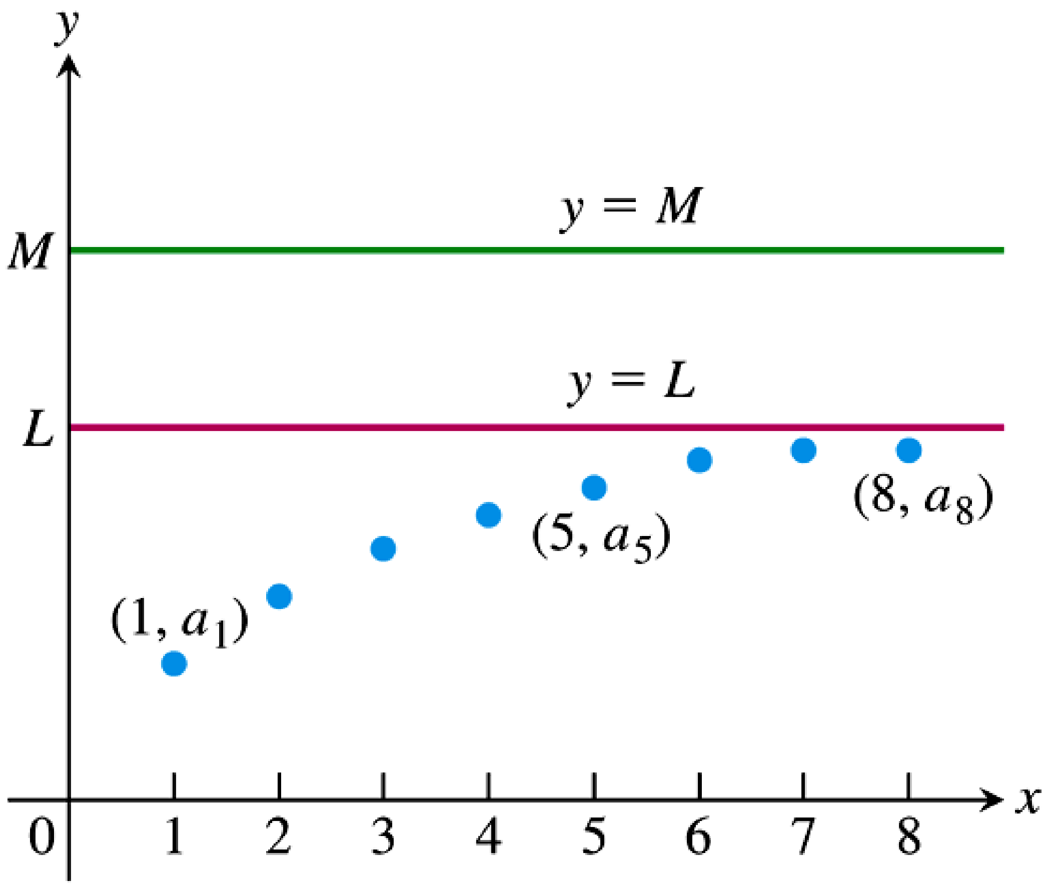


FIGURE 11.4 If the terms of a nondecreasing sequence have an upper bound M, they have a limit $L \leq M$.

THEOREM 6 The Nondecreasing Sequence Theorem

A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

11.2

Infinite Series

(3rd lecture of week 24/09/07-29/09/07)

Partial sum		Suggestive expression for partial sum	Value
First:	$s_1 = 1$	$2 - 1$	1
Second:	$s_2 = 1 + \frac{1}{2}$	$2 - \frac{1}{2}$	$\frac{3}{2}$
Third:	$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$2 - \frac{1}{4}$	$\frac{7}{4}$
⋮	⋮	⋮	⋮
nth:	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$

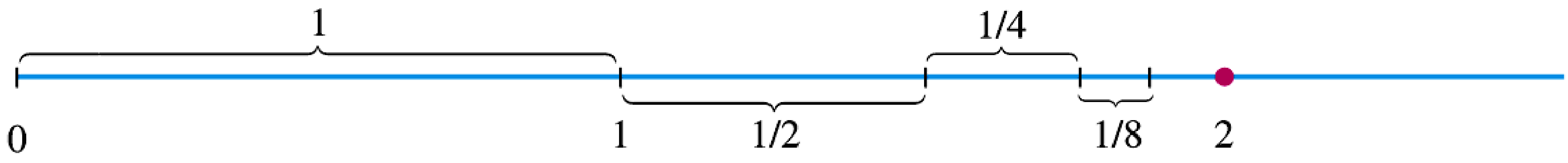


FIGURE 11.5 As the lengths 1, $^1/_2$, $^1/_4$, $^1/_8$, . . . are added one by one, the sum approaches 2.

DEFINITIONS Infinite Series, *n*th Term, Partial Sum, Converges, Sum

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

EXAMPLE 1 Index Starts with $n = 1$

The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1 - (1/3)} = \frac{1}{6}.$$

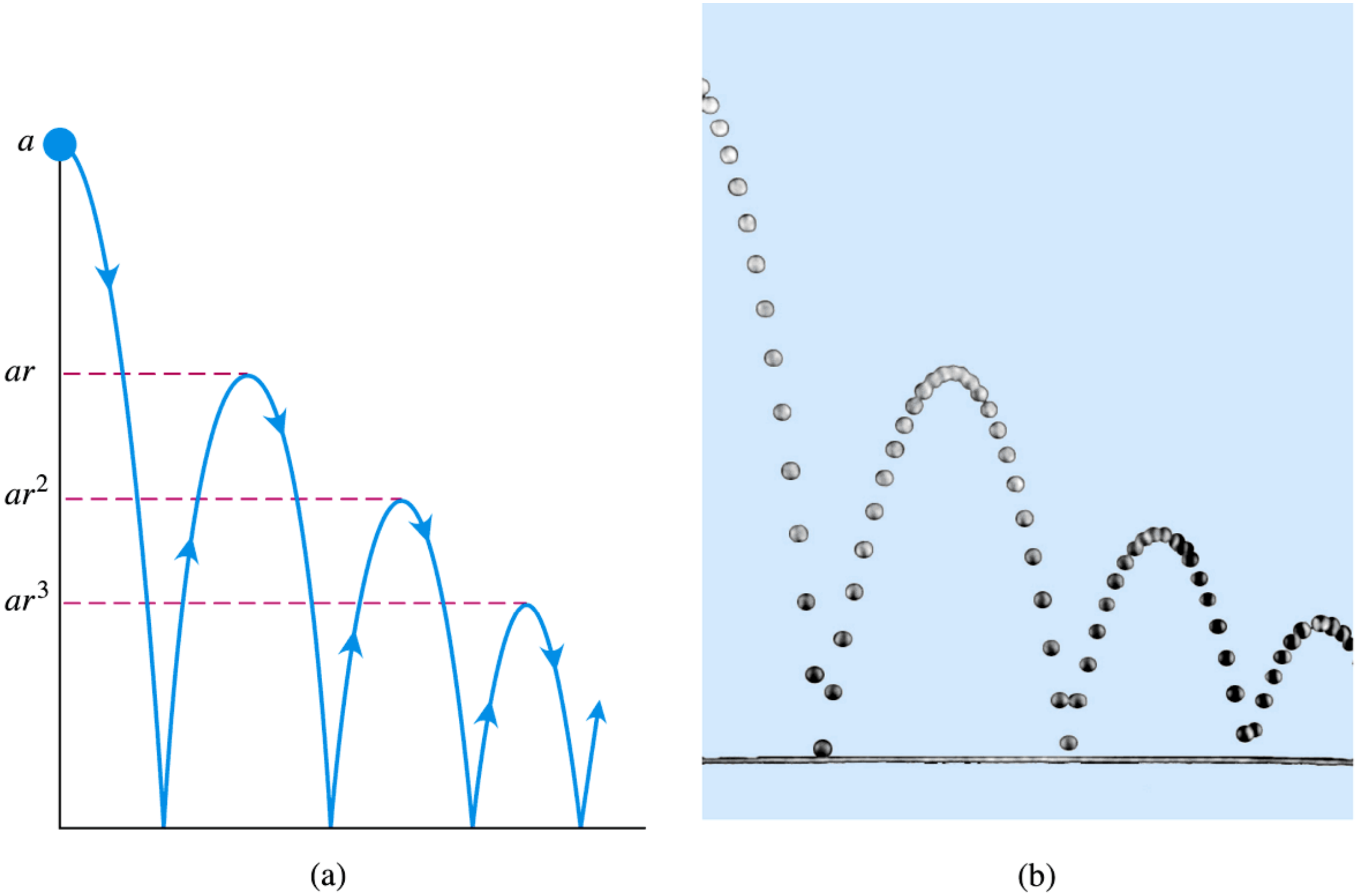


FIGURE 11.6 (a) Example 3 shows how to use a geometric series to calculate the total vertical distance traveled by a bouncing ball if the height of each rebound is reduced by the factor r. (b) A stroboscopic photo of a bouncing ball.

THEOREM 7

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

The *n*th-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

THEOREM 8

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum k a_n = k \sum a_n = kA$ (Any number k).

EXAMPLE 9 Find the sums of the following series.

(a)
$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$ Difference Rule

$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/6)}$$ Geometric series with $a = 1$ and $r = 1/2, 1/6$

$$= 2 - \frac{6}{5}$$

$$= \frac{4}{5}$$

(b)
$$\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n}$$ Constant Multiple Rule

$$= 4\left(\frac{1}{1-(1/2)}\right)$$ Geometric series with $a = 1, r = 1/2$

$$= 8$$

11.3

The Integral Test

(3rd lecture of week 24/09/07-29/09/07)

Corollary of Theorem 6

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

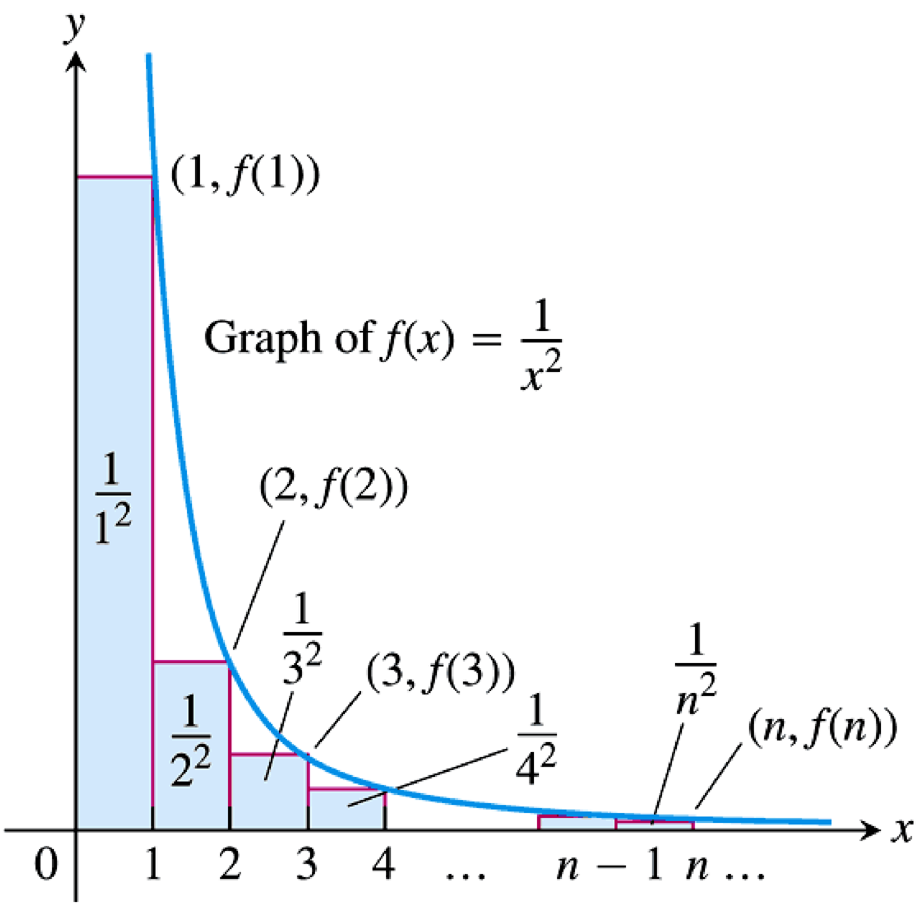


FIGURE 11.7 The sum of the areas of the rectangles under the graph of $f(x) = 1/x^2$ is less than the area under the graph (Example 2).

THEOREM 9 The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

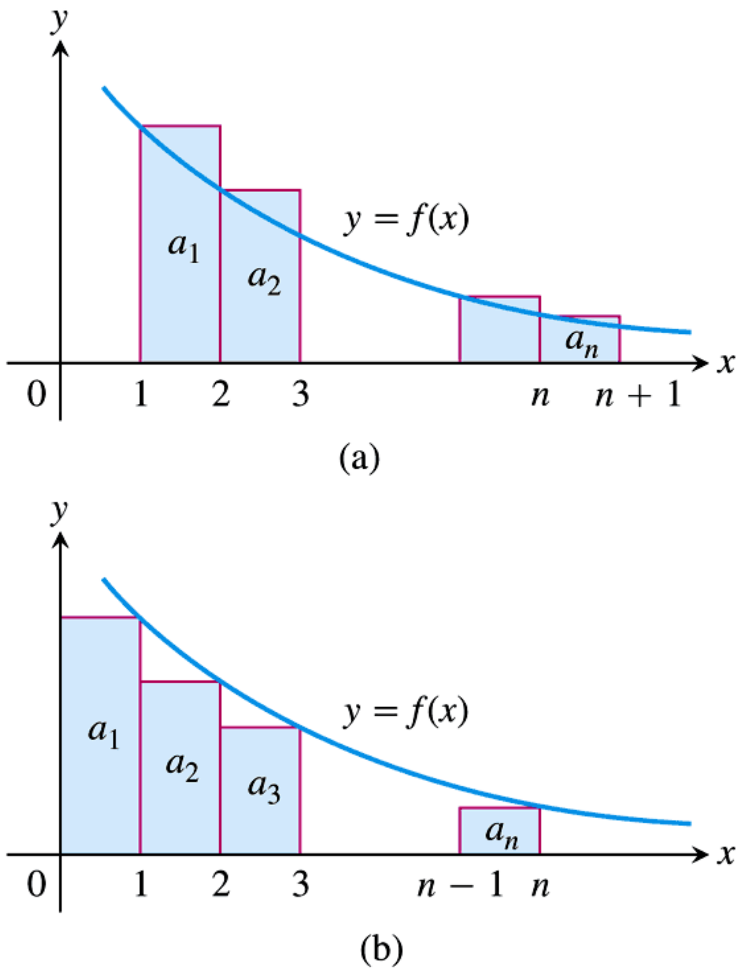


FIGURE 11.8 Subject to the conditions of the Integral Test, the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

EXAMPLE 3 The p-Series

Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

Solution If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x. Since

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \int_1^{\infty} x^{-p}\, dx = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \frac{1}{1-p} \lim_{b\to\infty} \left(\frac{1}{b^{p-1}} - 1\right)$$

$$= \frac{1}{1-p}(0-1) = \frac{1}{p-1},$$

$b^{p-1} \to \infty$ as $b \to \infty$ because $p - 1 > 0$.

the series converges by the Integral Test. We emphasize that the sum of the p-series is *not* $1/(p-1)$. The series converges, but we don't know the value it converges to.

If $p < 1$, then $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \frac{1}{1-p} \lim_{b\to\infty} (b^{1-p} - 1) = \infty.$$

The series diverges by the Integral Test.

If $p = 1$, we have the (divergent) harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

We have convergence for $p > 1$ but divergence for every other value of p.

11.4

Comparison Tests

(1st lecture of week 01/10/07-06/10/07)

THEOREM 10 **The Comparison Test**

Let $\sum a_n$ be a series with no negative terms.

(a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N.

(b) $\sum a_n$ diverges if there is a divergent series of nonnegative terms $\sum d_n$ with $a_n \geq d_n$ for all $n > N$, for some integer N.

EXAMPLE 1 Applying the Comparison Test

(a) The series

$$\sum_{n=1}^{\infty} \frac{5}{5n - 1}$$

diverges because its nth term

$$\frac{5}{5n - 1} = \frac{1}{n - \frac{1}{5}} > \frac{1}{n}$$

is greater than the nth term of the divergent harmonic series.

(b) The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

converges because its terms are all positive and less than or equal to the corresponding terms of

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots .$$

The geometric series on the left converges and we have

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1 - (1/2)} = 3.$$

The fact that 3 is an upper bound for the partial sums of $\sum_{n=0}^{\infty} (1/n!)$ does not mean that the series converges to 3. As we will see in Section 11.9, the series converges to e.

THEOREM 11 Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

3. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

EXAMPLE 2 Using the Limit Comparison Test

Which of the following series converge, and which diverge?

(a) $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

(b) $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

(c) $\frac{1+2\ln 2}{9} + \frac{1+3\ln 3}{14} + \frac{1+4\ln 4}{21} + \cdots = \sum_{n=2}^{\infty} \frac{1+n\ln n}{n^2+5}$

Solution

(a) Let $a_n = (2n+1)/(n^2+2n+1)$. For large n, we expect a_n to behave like $2n/n^2 = 2/n$ since the leading terms dominate for large n, so we let $b_n = 1/n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2,$$

$\sum a_n$ diverges by Part 1 of the Limit Comparison Test. We could just as well have taken $b_n = 2/n$, but $1/n$ is simpler.

Example 2 continued

(b) Let $a_n = 1/(2^n - 1)$. For large n, we expect a_n to behave like $1/2^n$, so we let $b_n = 1/2^n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges}$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2^n}{2^n - 1} \\ &= \lim_{n\to\infty} \frac{1}{1 - (1/2^n)} \\ &= 1,\end{aligned}$$

$\sum a_n$ converges by Part 1 of the Limit Comparison Test.

11.5

The Ratio and Root Tests

(1st lecture of week 01/10/07-06/10/07)

THEOREM 12 The Ratio Test

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho .$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

EXAMPLE 1 Applying the Ratio Test

Investigate the convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$

Solution

(a) For the series $\sum_{n=0}^{\infty} (2^n + 5)/3^n$,

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3} \cdot \left(\frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}}\right) \to \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

The series converges because $\rho = 2/3$ is less than 1. This does *not* mean that $2/3$ is the sum of the series. In fact,

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n} = \frac{1}{1 - (2/3)} + \frac{5}{1 - (1/3)} = \frac{21}{2}.$$

(b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2n + 2)!}{(n + 1)!(n + 1)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{n!n!(2n + 2)(2n + 1)(2n)!}{(n + 1)!(n + 1)!(2n)!}$$

$$= \frac{(2n + 2)(2n + 1)}{(n + 1)(n + 1)} = \frac{4n + 2}{n + 1} \to 4.$$

The series diverges because $\rho = 4$ is greater than 1.

THEOREM 13 The Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

EXAMPLE 3 Applying the Root Test

Which of the following series converges, and which diverges?

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ **(b)** $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ **(c)** $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Solution

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{\left(\sqrt[n]{n}\right)^2}{2} \to \frac{1}{2} < 1.$

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges because $\sqrt[n]{\frac{2^n}{n^2}} = \frac{2}{\left(\sqrt[n]{n}\right)^2} \to \frac{2}{1} > 1.$

11.6

Alternating Series, Absolute and Conditional Convergence

(2nd lecture of week 01/10/07-06/10/07)

THEOREM 14 The Alternating Series Test (Leibniz's Theorem)

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

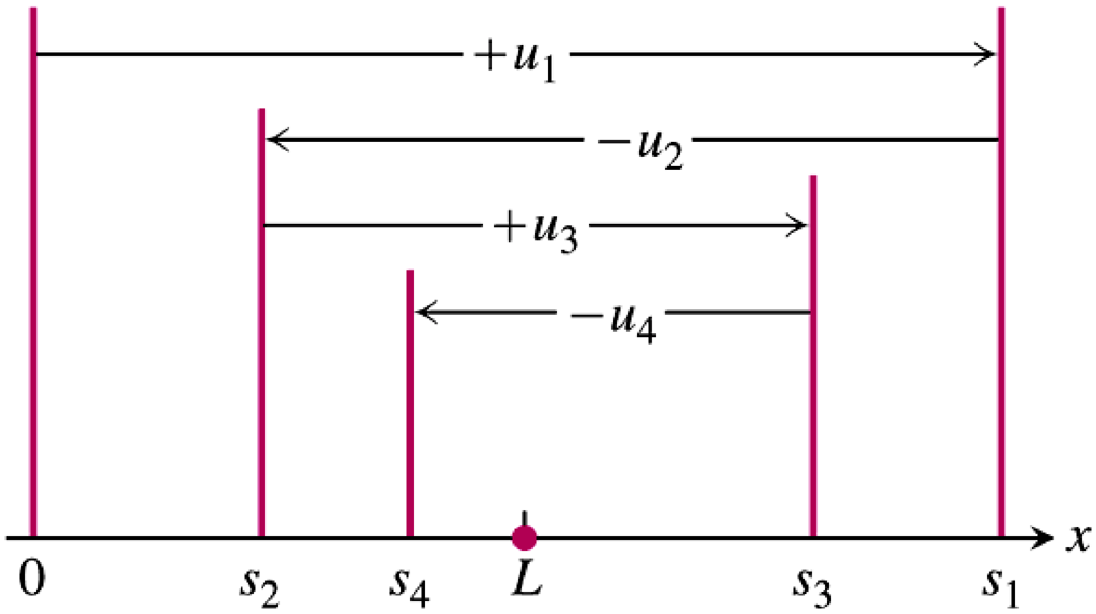


FIGURE 11.9 The partial sums of an alternating series that satisfies the hypotheses of Theorem 14 for $N = 1$ straddle the limit from the beginning.

THEOREM 15 The Alternating Series Estimation Theorem

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 14, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the numerical value of the first unused term. Furthermore, the remainder, $L - s_n$, has the same sign as the first unused term.

EXAMPLE 2 We try Theorem 15 on a series whose sum we know:

$$\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} \,\vdots + \frac{1}{256} - \cdots.$$

The theorem says that if we truncate the series after the eighth term, we throw away a total that is positive and less than $1/256$. The sum of the first eight terms is 0.6640625. The sum of the series is

$$\frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}.$$

The difference, $(2/3) - 0.6640625 = 0.0026041666\ldots$, is positive and less than $(1/256) = 0.00390625$.

DEFINITION **Absolutely Convergent**

A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

DEFINITION Conditionally Convergent

A series that converges but does not converge absolutely **converges conditionally**.

THEOREM 16 The Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

EXAMPLE 3 Applying the Absolute Convergence Test

(a) For $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$, the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots.$$

The original series converges because it converges absolutely.

(b) For $\sum_{n=1}^{\infty}\frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$, the corresponding series of absolute values is

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \cdots,$$

which converges by comparison with $\sum_{n=1}^{\infty}(1/n^2)$ because $|\sin n| \leq 1$ for every n. The original series converges absolutely; therefore it converges.

EXAMPLE 4 Alternating p-Series

If p is a positive constant, the sequence $\{1/n^p\}$ is a decreasing sequence with limit zero. Therefore the alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \qquad p > 0$$

converges.

If $p > 1$, the series converges absolutely. If $0 < p \leq 1$, the series converges conditionally.

$$\text{Conditional convergence:} \qquad 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

$$\text{Absolute convergence:} \qquad 1 - \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} - \frac{1}{4^{3/2}} + \cdots$$

THEOREM 17 The Rearrangement Theorem for Absolutely Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

1. **The *n*th-Term Test:** Unless $a_n \to 0$, the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. ***p*-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge? If yes, so does $\sum a_n$, since absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

11.7

Power Series

(2nd lecture of week 01/10/07-06/10/07)

DEFINITIONS Power Series, Center, Coefficients

A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \tag{1}$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \tag{2}$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

EXAMPLE 1 A Geometric Series

Taking all the coefficients to be 1 in Equation (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This is the geometric series with first term 1 and ratio x. It converges to $1/(1 - x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1. \tag{3}$$

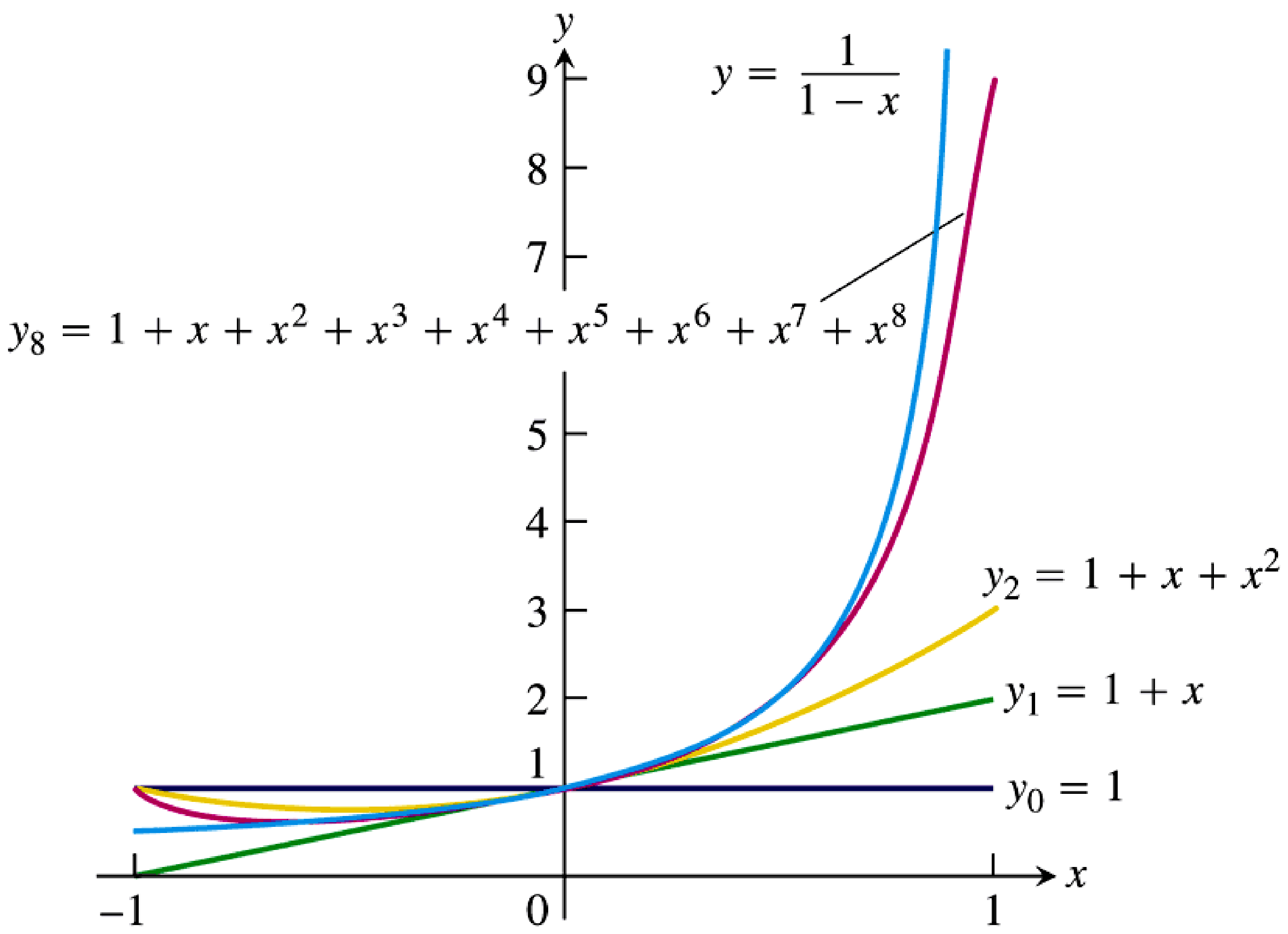


FIGURE 11.10 The graphs of $f(x) = 1/(1 - x)$ and four of its polynomial approximations (Example 1).

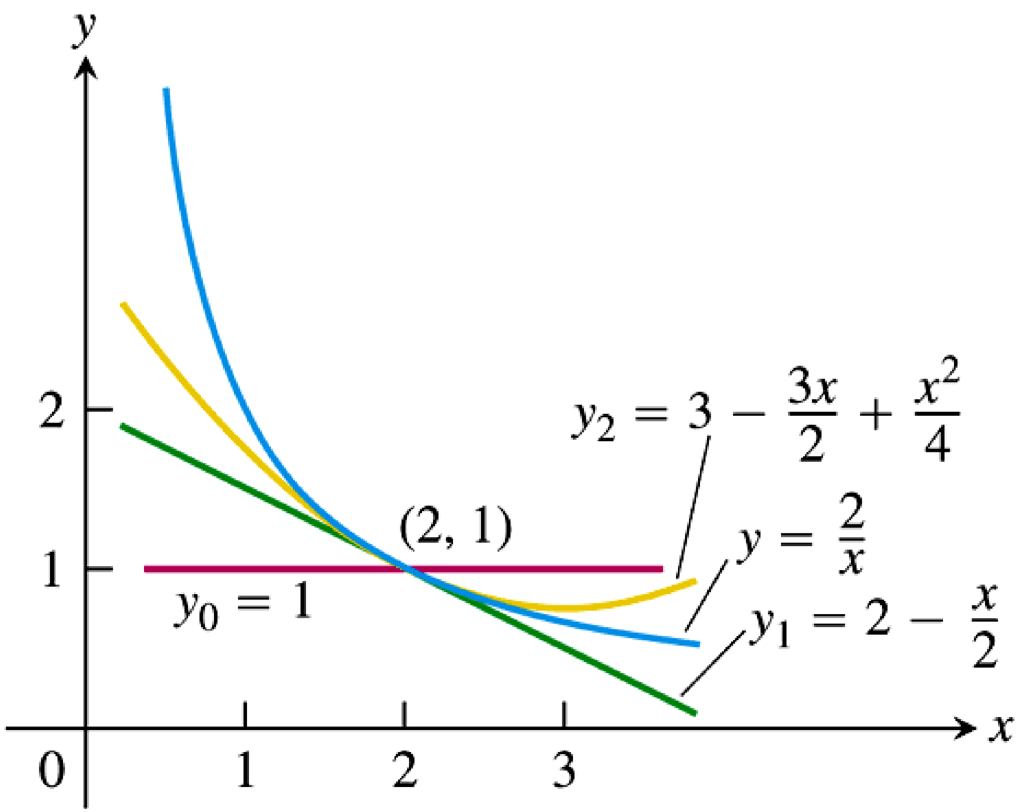


FIGURE 11.11 The graphs of $f(x) = 2/x$ and its first three polynomial approximations (Example 2).

EXAMPLE 3 Testing for Convergence Using the Ratio Test

For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(b) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

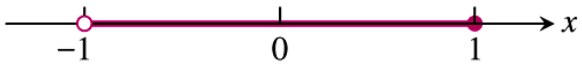
(c) $\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(d) $\sum_{n=0}^{\infty}n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

Solution Apply the Ratio Test to the series $\sum|u_n|$, where u_n is the nth term of the series in question.

(a) $\left|\frac{u_{n+1}}{u_n}\right| = \frac{n}{n+1}|x| \to |x|.$

The series converges absolutely for $|x| < 1$. It diverges if $|x| > 1$ because the nth term does not converge to zero. At $x = 1$, we get the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges. At $x = -1$ we get $-1 - 1/2 - 1/3 - 1/4 - \cdots$, the negative of the harmonic series; it diverges. Series (a) converges for $-1 < x \leq 1$ and diverges elsewhere.



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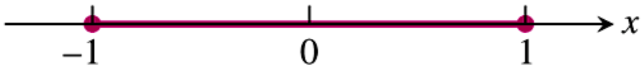
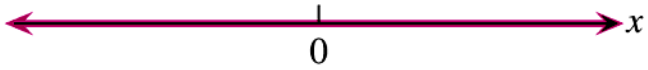
(b) $\left|\frac{u_{n+1}}{u_n}\right| = \frac{2n-1}{2n+1}x^2 \rightarrow x^2.$

The series converges absolutely for $x^2 < 1$. It diverges for $x^2 > 1$ because the nth term does not converge to zero. At $x = 1$ the series becomes $1 - 1/3 + 1/5 - 1/7 + \cdots$, which converges by the Alternating Series Theorem. It also converges at $x = -1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x = -1$ is the negative of the value at $x = 1$. Series (b) converges for $-1 \le x \le 1$ and diverges elsewhere.



(c) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \rightarrow 0$ for every x.

The series converges absolutely for all x.



(d) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \rightarrow \infty$ unless $x = 0$.

The series diverges for all values of x except $x = 0$.

THEOREM 18 The Convergence Theorem for Power Series

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges for $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.

COROLLARY TO THEOREM 18

The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three possibilities:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x $(R = \infty)$.
3. The series converges at $x = a$ and diverges elsewhere $(R = 0)$.

How to Test a Power Series for Convergence

1. *Use the Ratio Test (or nth-Root Test) to find the interval where the series converges absolutely*. Ordinarily, this is an open interval

$$|x - a| < R \qquad \text{or} \qquad a - R < x < a + R.$$

2. *If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint*, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. *If the interval of absolute convergence is* $a - R < x < a + R$, *the series diverges for* $|x - a| > R$ (it does not even converge conditionally), because the nth term does not approach zero for those values of x.

THEOREM 19 The Term-by-Term Differentiation Theorem

If $\sum c_n(x-a)^n$ converges for $a - R < x < a + R$ for some $R > 0$, it defines a function f:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \qquad a - R < x < a + R.$$

Such a function f has derivatives of all orders inside the interval of convergence. We can obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2},$$

and so on. Each of these derived series converges at every interior point of the interval of convergence of the original series.

EXAMPLE 4 Applying Term-by-Term Differentiation

Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

$$= \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1$$

Solution

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1$$

$$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1$$

THEOREM 20 The Term-by-Term Integration Theorem

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$

converges for $a - R < x < a + R \ (R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\, dx = \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1} + C$$

for $a - R < x < a + R$.

EXAMPLE 5 A Series for $\tan^{-1} x$, $-1 \le x \le 1$

Identify the function

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad -1 \le x \le 1.$$

Solution We differentiate the original series term by term and get

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots, \qquad -1 < x < 1.$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1 - (-x^2)} = \frac{1}{1 + x^2}.$$

We can now integrate $f'(x) = 1/(1 + x^2)$ to get

$$\int f'(x)\, dx = \int \frac{dx}{1 + x^2} = \tan^{-1} x + C.$$

The series for $f(x)$ is zero when $x = 0$, so $C = 0$. Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x, \qquad -1 < x < 1. \tag{7}$$

In Section 11.10, we will see that the series also converges to $\tan^{-1} x$ at $x = \pm 1$.

EXAMPLE 6 A Series for $\ln(1 + x)$, $-1 < x \le 1$

The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\ln(1 + x) = \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x \qquad \text{Theorem 20}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x < 1.$$

It can also be shown that the series converges at $x = 1$ to the number $\ln 2$, but that was not guaranteed by the theorem.

THEOREM 21 The Series Multiplication Theorem for Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

EXAMPLE 7 Multiply the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}, \qquad \text{for } |x| < 1,$$

by itself to get a power series for $1/(1-x)^2$, for $|x| < 1$.

Solution Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + \cdots + x^n + \cdots = 1/(1-x)$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = 1 + x + x^2 + \cdots + x^n + \cdots = 1/(1-x)$$

and

$$c_n = \underbrace{a_0 b_n + a_1 b_{n-1} + \cdots + a_k b_{n-k} + \cdots + a_n b_0}_{n+1 \text{ terms}}$$

$$= \underbrace{1 + 1 + \cdots + 1}_{n+1 \text{ ones}} = n + 1.$$

Then, by the Series Multiplication Theorem,

$$A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1) x^n$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots$$

is the series for $1/(1-x)^2$. The series all converge absolutely for $|x| < 1$.

Notice that Example 4 gives the same answer because

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

11.8

Taylor and Maclaurin Series

(3rd lecture of week 01/10/07-06/10/07)

DEFINITIONS Taylor Series, Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

DEFINITION **Taylor Polynomial of Order n**

Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x - a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

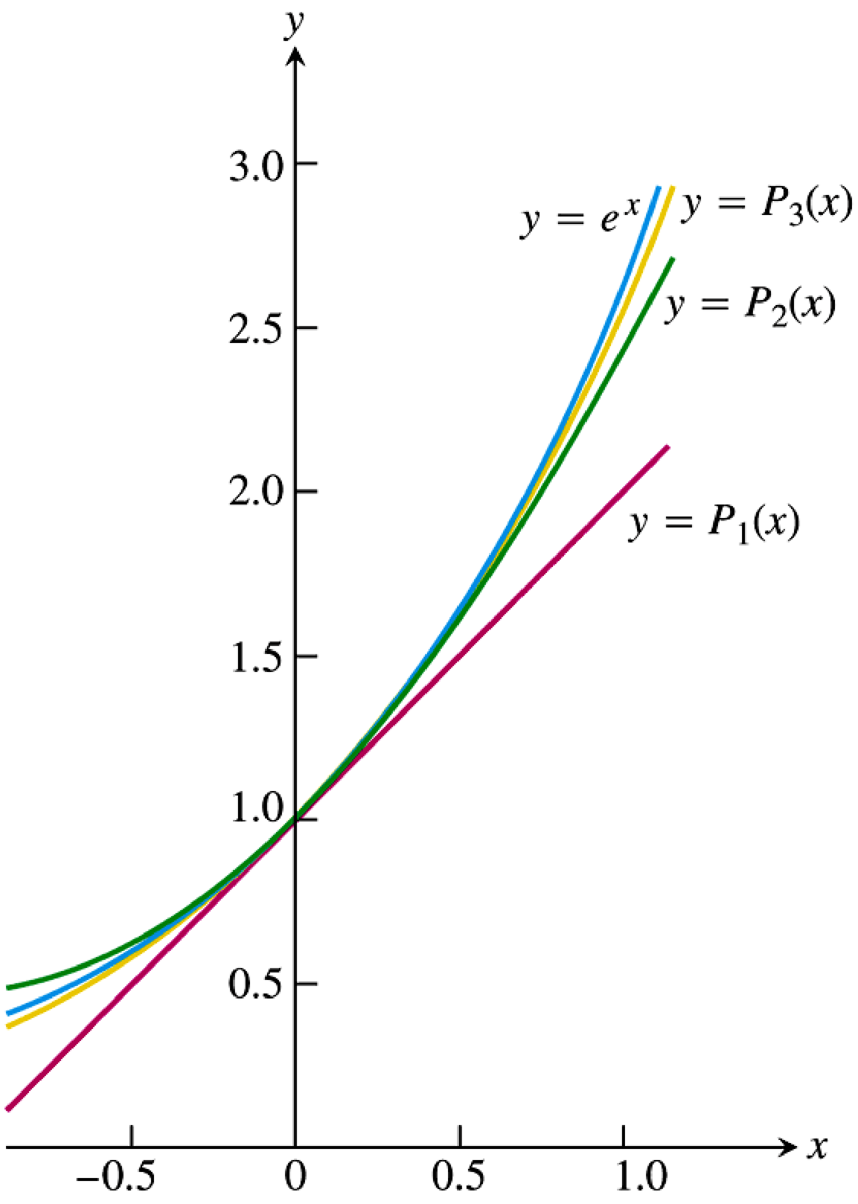


FIGURE 11.12 The graph of $f(x) = e^x$ and its Taylor polynomials

$P_1(x) = 1 + x$

$P_2(x) = 1 + x + (x^2/2!)$

$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!)$.

Notice the very close agreement near the center $x = 0$ (Example 2).

EXAMPLE 3 Finding Taylor Polynomials for cos x

Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

Solution The cosine and its derivatives are

$$\begin{aligned} f(x) &= \cos x, & f'(x) &= -\sin x, \\ f''(x) &= -\cos x, & f^{(3)}(x) &= \sin x, \\ &\vdots & &\vdots \\ f^{(2n)}(x) &= (-1)^n \cos x, & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x. \end{aligned}$$

At $x = 0$, the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0.$$

The Taylor series generated by f at 0 is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ = 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}. \end{aligned}$$

This is also the Maclaurin series for cos x. In Section 11.9, we will see that the series converges to cos x at every x.

Because $f^{(2n+1)}(0) = 0$, the Taylor polynomials of orders $2n$ and $2n + 1$ are identical:

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Figure 11.13 shows how well these polynomials approximate $f(x) = \cos x$ near $x = 0$. Only the right-hand portions of the graphs are given because the graphs are symmetric about the y-axis.

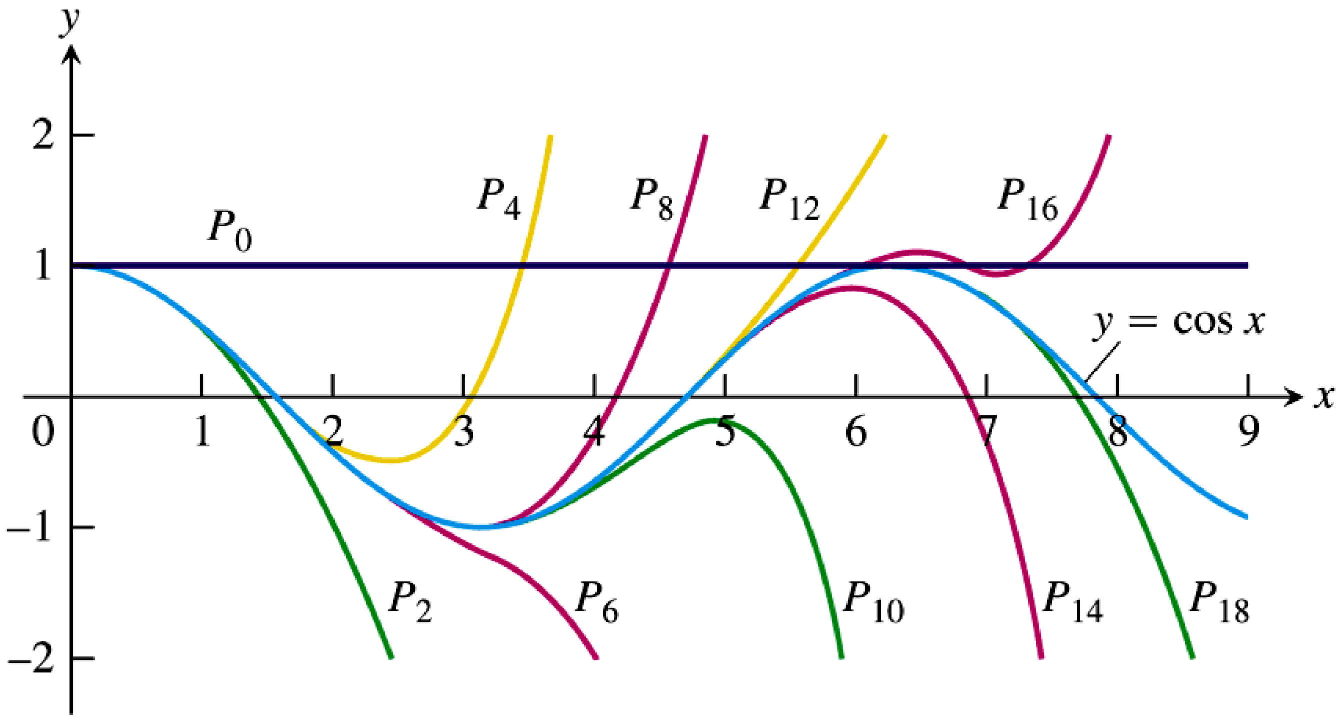


FIGURE 11.13 The polynomials

$$P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$$

converge to $\cos x$ as $n \to \infty$. We can deduce the behavior of $\cos x$ arbitrarily far away solely from knowing the values of the cosine and its derivatives at $x = 0$ (Example 3).

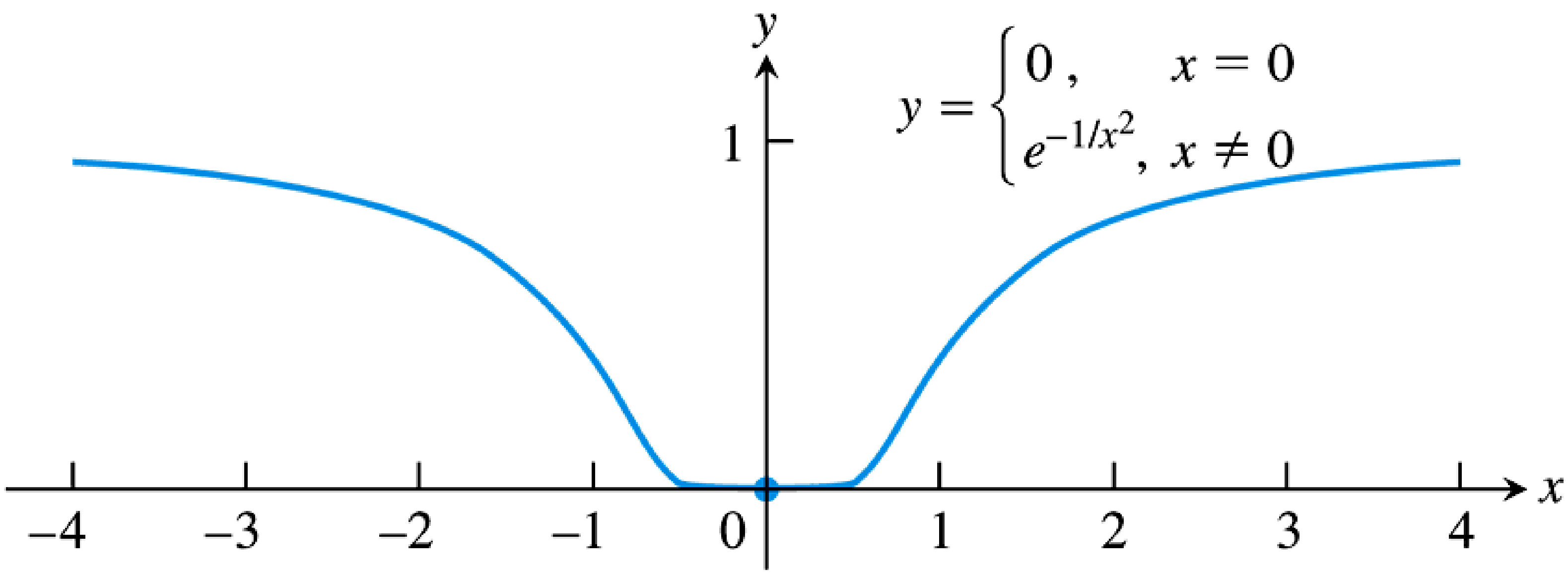


FIGURE 11.14 The graph of the continuous extension of $y = e^{-1/x^2}$ is so flat at the origin that all of its derivatives there are zero (Example 4).

11.9

Convergence of Taylor Series;

Error Estimates

(3rd lecture of week 01/10/07-06/10/07)

THEOREM 22 Taylor's Theorem

If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \cdots$$

$$+ \frac{f^{(n)}(a)}{n!}(b - a)^n + \frac{f^{(n+1)}(c)}{(n + 1)!}(b - a)^{n+1}.$$

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \qquad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \qquad (2)$$

THEOREM 23 The Remainder Estimation Theorem

If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x - a|^{n+1}}{(n + 1)!}.$$

If this condition holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

EXAMPLE 3 The Taylor Series for cos x at $x = 0$ Revisited

Show that the Taylor series for cos x at $x = 0$ converges to cos x for every value of x.

Solution We add the remainder term to the Taylor polynomial for cos x (Section 11.8, Example 3) to obtain Taylor's formula for cos x with $n = 2k$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + R_{2k}(x).$$

Because the derivatives of the cosine have absolute value less than or equal to 1, the Remainder Estimation Theorem with $M = 1$ gives

$$|R_{2k}(x)| \leq 1 \cdot \frac{|x|^{2k+1}}{(2k+1)!}.$$

For every value of x, $R_{2k} \to 0$ as $k \to \infty$. Therefore, the series converges to cos x for every value of x. Thus,

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \quad (5)$$

EXAMPLE 6 Calculate e with an error of less than 10^{-6}.

Solution We can use the result of Example 1 with $x = 1$ to write

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1),$$

with

$$R_n(1) = e^c \frac{1}{(n+1)!} \qquad \text{for some } c \text{ between 0 and 1.}$$

For the purposes of this example, we assume that we know that $e < 3$. Hence, we are certain that

$$\frac{1}{(n+1)!} < R_n(1) < \frac{3}{(n+1)!}$$

because $1 < e^c < 3$ for $0 < c < 1$.

By experiment we find that $1/9! > 10^{-6}$, while $3/10! < 10^{-6}$. Thus we should take $(n + 1)$ to be at least 10, or n to be at least 9. With an error of less than 10^{-6},

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{9!} \approx 2.718282.$$ ■

EXAMPLE 7 For what values of x can we replace $\sin x$ by $x - (x^3/3!)$ with an error of magnitude no greater than 3×10^{-4}?

Solution Here we can take advantage of the fact that the Taylor series for $\sin x$ is an alternating series for every nonzero value of x. According to the Alternating Series Estimation Theorem (Section 11.6), the error in truncating

$$\sin x = x - \frac{x^3}{3!} \;\Big|\; + \frac{x^5}{5!} - \cdots$$

after $(x^3/3!)$ is no greater than

$$\left|\frac{x^5}{5!}\right| = \frac{|x|^5}{120}.$$

Therefore the error will be less than or equal to 3×10^{-4} if

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \qquad \text{or} \qquad |x| < \sqrt[5]{360 \times 10^{-4}} \approx 0.514.$$

Rounded down, to be safe

The Alternating Series Estimation Theorem tells us something that the Remainder Estimation Theorem does not: namely, that the estimate $x - (x^3/3!)$ for $\sin x$ is an underestimate when x is positive because then $x^5/120$ is positive.

Figure 11.15 shows the graph of $\sin x$, along with the graphs of a number of its approximating Taylor polynomials. The graph of $P_3(x) = x - (x^3/3!)$ is almost indistinguishable from the sine curve when $-1 \le x \le 1$.

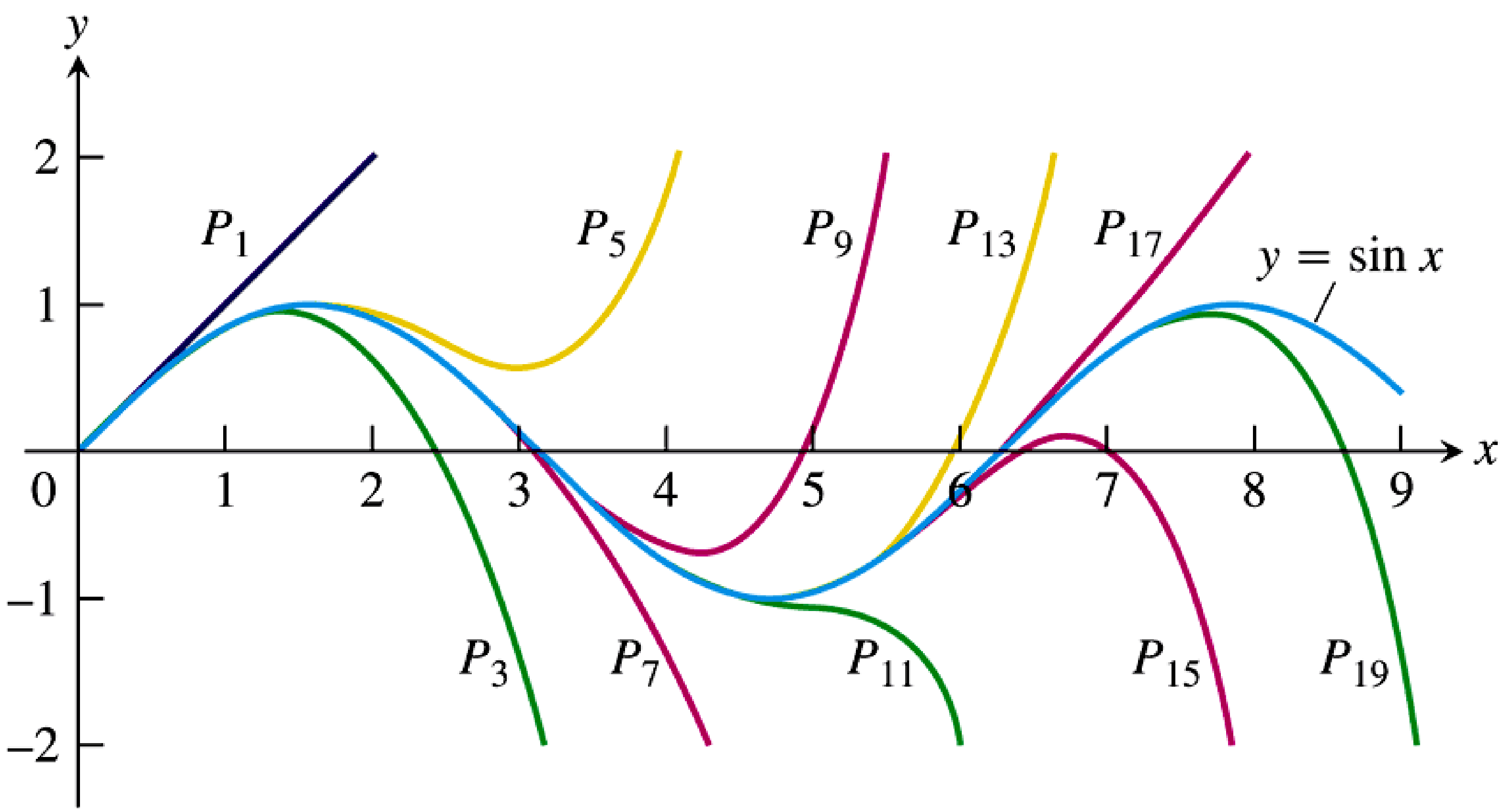


FIGURE 11.15 The polynomials

DEFINITION

For any real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$. (6)

11.10

Applications of Power Series

(1st lecture of week 08/10/07-10/10/07)

The Binomial Series

For $-1 < x < 1$,

$$(1 + x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k \geq 3.$$

EXAMPLE 2 Using the Binomial Series

We know from Section 3.8, Example 1, that $\sqrt{1+x} \approx 1 + (x/2)$ for $|x|$ small. With $m = 1/2$, the binomial series gives quadratic and higher-order approximations as well, along with error estimates that come from the Alternating Series Estimation Theorem:

$$(1+x)^{1/2} = 1 + \frac{x}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots.$$

Substitution for x gives still other approximations. For example,

$$\sqrt{1-x^2} \approx 1 - \frac{x^2}{2} - \frac{x^4}{8} \qquad \text{for } |x^2| \text{ small}$$

$$\sqrt{1-\frac{1}{x}} \approx 1 - \frac{1}{2x} - \frac{1}{8x^2} \qquad \text{for } \left|\frac{1}{x}\right| \text{ small, that is, } |x| \text{ large}.$$

EXAMPLE 5 Express $\int \sin x^2 \, dx$ as a power series.

Solution From the series for $\sin x$ we obtain

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \cdots.$$

Therefore,

$$\int \sin x^2 \, dx = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \frac{x^{10}}{19 \cdot 9!} - \cdots.$$

EXAMPLE 6 Estimating a Definite Integral

Estimate $\int_0^1 \sin x^2 \, dx$ with an error of less than 0.001.

Solution From the indefinite integral in Example 5,

$$\int_0^1 \sin x^2 \, dx = \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \frac{1}{19 \cdot 9!} - \cdots.$$

The series alternates, and we find by experiment that

$$\frac{1}{11 \cdot 5!} \approx 0.00076$$

is the first term to be numerically less than 0.001. The sum of the preceding two terms gives

$$\int_0^1 \sin x^2 \, dx \approx \frac{1}{3} - \frac{1}{42} \approx 0.310.$$

With two more terms we could estimate

$$\int_0^1 \sin x^2 \, dx \approx 0.310268$$

with an error of less than 10^{-6}. With only one term beyond that we have

$$\int_0^1 \sin x^2 \, dx \approx \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \frac{1}{6894720} \approx 0.310268303,$$

with an error of about 1.08×10^{-9}. To guarantee this accuracy with the error formula for the Trapezoidal Rule would require using about 8000 subintervals.

EXAMPLE 7 Limits Using Power Series

Evaluate

$$\lim_{x \to 1} \frac{\ln x}{x - 1}.$$

Solution We represent $\ln x$ as a Taylor series in powers of $x - 1$. This can be accomplished by calculating the Taylor series generated by $\ln x$ at $x = 1$ directly or by replacing x by $x - 1$ in the series for $\ln(1 + x)$ in Section 11.7, Example 6. Either way, we obtain

$$\ln x = (x - 1) - \frac{1}{2}(x - 1)^2 + \cdots,$$

from which we find that

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \left(1 - \frac{1}{2}(x - 1) + \cdots\right) = 1.$$

EXAMPLE 8 Limits Using Power Series

Evaluate

$$\lim_{x \to 0} \frac{\sin x - \tan x}{x^3}.$$

Solution The Taylor series for $\sin x$ and $\tan x$, to terms in x^5, are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$$

Hence,

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \cdots = x^3\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

and

$$\lim_{x \to 0} \frac{\sin x - \tan x}{x^3} = \lim_{x \to 0}\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

$$= -\frac{1}{2}.$$

TABLE 11.1 Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\ln \frac{1+x}{1-x} = 2 \tanh^{-1} x = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$

Binomial Series

$$(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \cdots + \frac{m(m-1)(m-2)\cdots(m-k+1)x^k}{k!} + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k, \qquad |x| < 1,$$

where

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!}, \qquad \binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!} \qquad \text{for } k \ge 3.$$

Note: To write the binomial series compactly, it is customary to define $\binom{m}{0}$ to be 1 and to take $x^0 = 1$ (even in the usually excluded case where $x = 0$), yielding $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$. If m is a *positive integer*, the series terminates at x^m and the result converges for all x.

11.11

Fourier Series

(1st lecture of week 08/10/07-10/10/07)

Suppose we wish to approximate a function f on the interval $[0, 2\pi]$ by a sum of sine and cosine functions,

$$f_n(x) = a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots + (a_n \cos nx + b_n \sin nx)$$

or, in sigma notation,

$$f_n(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx). \tag{1}$$

We would like to choose values for the constants $a_0, a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots, b_n$ that make $f_n(x)$ a "best possible" approximation to $f(x)$. The notion of "best possible" is defined as follows:

1. $f_n(x)$ and $f(x)$ give the same value when integrated from 0 to 2π.

2. $f_n(x) \cos kx$ and $f(x) \cos kx$ give the same value when integrated from 0 to 2π $(k = 1, \ldots, n)$.

3. $f_n(x) \sin kx$ and $f(x) \sin kx$ give the same value when integrated from 0 to 2π $(k = 1, \ldots, n)$.

We chose f_n so that the integrals on the left remain the same when f_n is replaced by f, so we can use these equations to find $a_0, a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots, b_n$ from f:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \qquad (2)$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx, \qquad k = 1, \ldots, n \qquad (3)$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx, \qquad k = 1, \ldots, n \qquad (4)$$

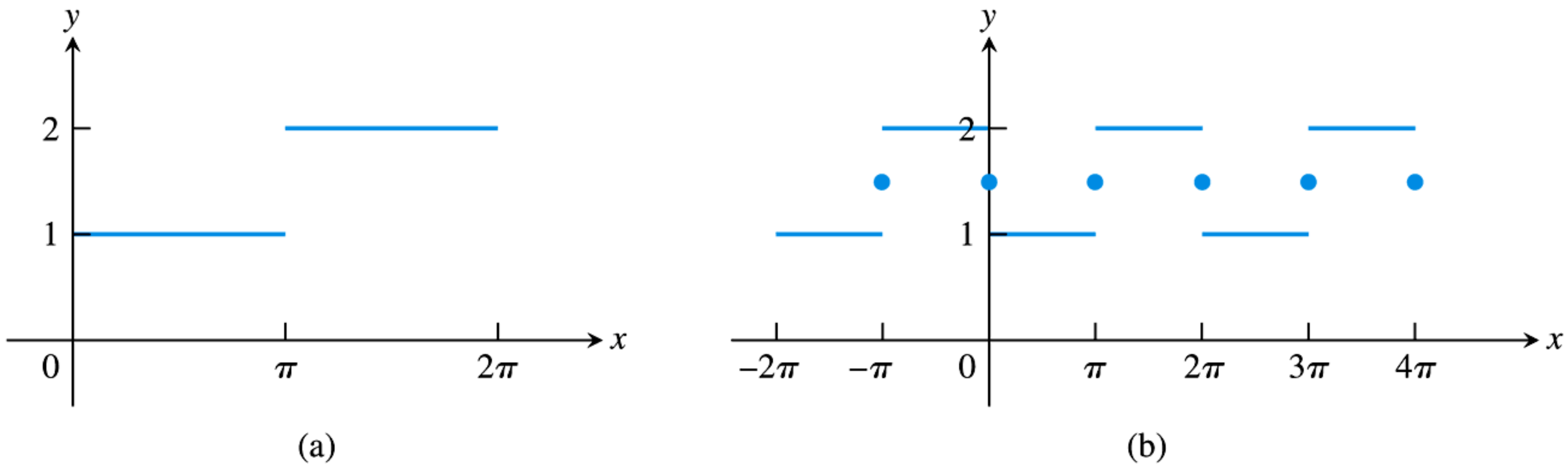


FIGURE 11.16 (a) The step function

$$f(x) = \begin{cases} 1, & 0 \leq x \leq \pi \\ 2, & \pi < x \leq 2\pi \end{cases}$$

(b) The graph of the Fourier series for f is periodic and has the value $3/2$ at each point of discontinuity (Example 1).

EXAMPLE 1 Finding a Fourier Series Expansion

Fourier series can be used to represent some functions that cannot be represented by Taylor series; for example, the step function f shown in Figure 11.16a.

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x \le \pi \\ 2, & \text{if } \pi < x \le 2\pi . \end{cases}$$

The coefficients of the Fourier series of f are computed using Equations (2), (3), and (4).

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \\ &= \frac{1}{2\pi}\left(\int_0^{\pi} 1\,dx + \int_{\pi}^{2\pi} 2\,dx\right) = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx \\ &= \frac{1}{\pi}\left(\int_0^{\pi} \cos kx\,dx + \int_{\pi}^{2\pi} 2\cos kx\,dx\right) \\ &= \frac{1}{\pi}\left(\left[\frac{\sin kx}{k}\right]_0^{\pi} + \left[\frac{2\sin kx}{k}\right]_{\pi}^{2\pi}\right) = 0, \qquad k \ge 1 \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx \\ &= \frac{1}{\pi}\left(\int_0^{\pi} \sin kx\,dx + \int_{\pi}^{2\pi} 2\sin kx\,dx\right) \\ &= \frac{1}{\pi}\left(\left[-\frac{\cos kx}{k}\right]_0^{\pi} + \left[-\frac{2\cos kx}{k}\right]_{\pi}^{2\pi}\right) \\ &= \frac{\cos k\pi - 1}{k\pi} = \frac{(-1)^k - 1}{k\pi}. \end{aligned}$$

So

$$a_0 = \frac{3}{2}, \quad a_1 = a_2 = \cdots = 0,$$

and

$$b_1 = -\frac{2}{\pi}, \quad b_2 = 0, \quad b_3 = -\frac{2}{3\pi}, \quad b_4 = 0, \quad b_5 = -\frac{2}{5\pi}, \quad b_6 = 0, \ldots$$

The Fourier series is

$$\frac{3}{2} - \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

Notice that at $x = \pi$, where the function $f(x)$ jumps from 1 to 2, all the sine terms vanish, leaving $3/2$ as the value of the series. This is not the value of f at π, since $f(\pi) = 1$. The Fourier series also sums to $3/2$ at $x = 0$ and $x = 2\pi$. In fact, all terms in the Fourier series are periodic, of period 2π, and the value of the series at $x + 2\pi$ is the same as its value at x. The series we obtained represents the periodic function graphed in Figure 11.16b, with domain the entire real line and a pattern that repeats over every interval of width 2π. The function jumps discontinuously at $x = n\pi, n = 0, \pm 1, \pm 2, \ldots$ and at these points has value $3/2$, the average value of the one-sided limits from each side. The convergence of the Fourier series of f is indicated in Figure 11.17.

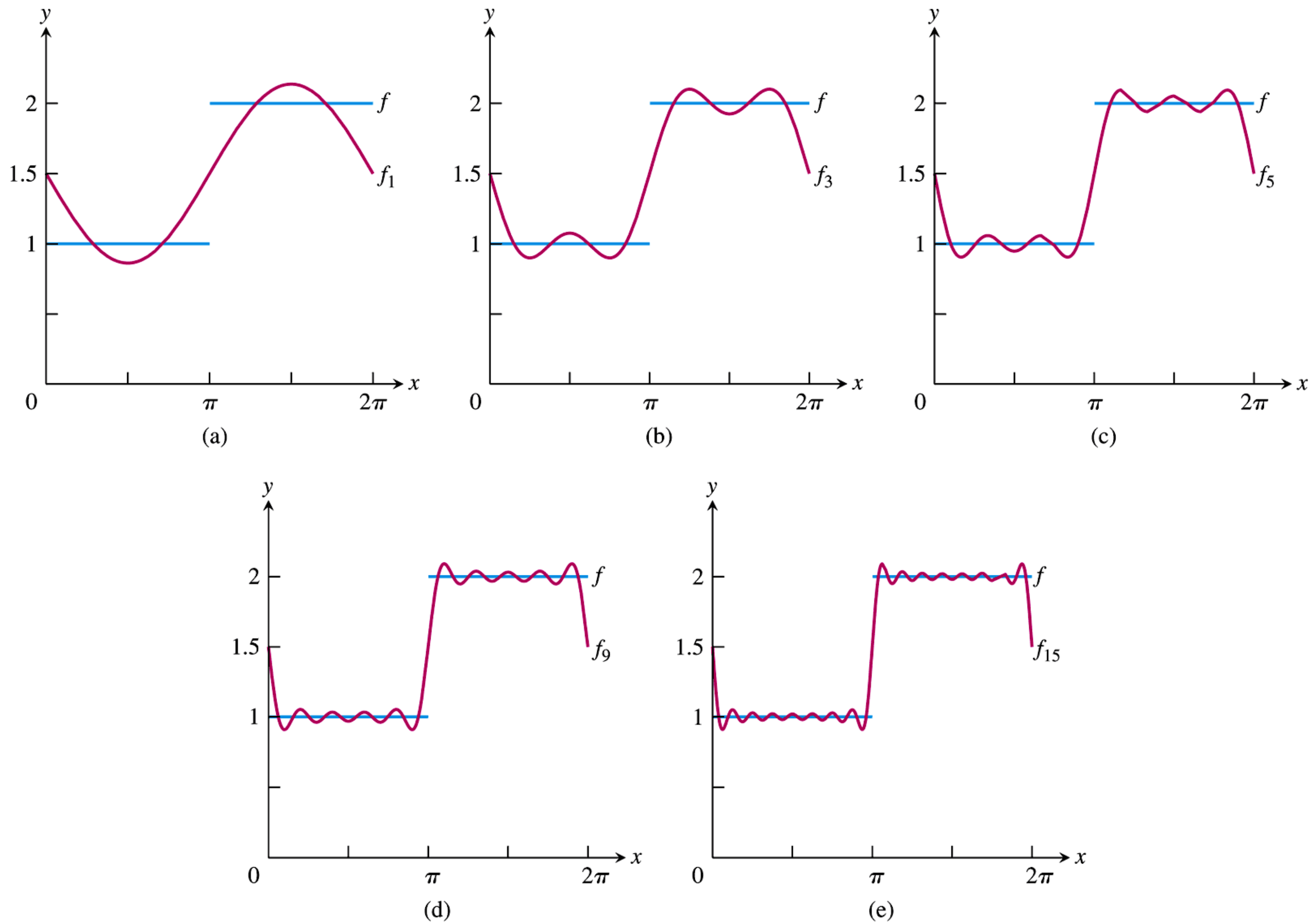


FIGURE 11.17 The Fourier approximation functions f_1, f_3, f_5, f_9, and f_{15} of the function $f(x) = \begin{cases} 1, & 0 \leq x \leq \pi \\ 2, & \pi < x \leq 2\pi \end{cases}$ in Example 1.

THEOREM 24 Let $f(x)$ be a function such that f and f' are piecewise continuous on the interval $[0, 2\pi]$. Then f is equal to its Fourier series at all points where f is continuous. At a point c where f has a discontinuity, the Fourier series converges to

$$\frac{f(c^+) + f(c^-)}{2}$$

where $f(c^+)$ and $f(c^-)$ are the right- and left-hand limits of f at c.