

## Chapter\_10\_Linear EquationsQ

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Consider a system of 3 linear equations in 3 unknown,  $x_1, x_2, x_3$ .

$$x_1 + x_2 + x_3 = 0$$

$$2x_1 + 2x_2 + 2x_3 = 0$$

$$3x_1 + 3x_2 + 3x_3 = 0$$

(i) Give the most trivial solution to the system of equations above. (ii) Give a not-most trivial solution to the system of equations above. (iii) Is the system consistent? (iv) Explain why you say so in (iii). (v) How many solutions are there for the linear equation system?

2. Consider a system of 2 linear equations in 2 unknown,

$$x_1, x_2: x_1 + x_2 = 2, x_1 + x_2 = 1$$

(i) Try sketching the two equations given in the  $x_2$ - $x_1$  plane. (ii) Do these two graphs intersect at all? (iii) Can you find a solution to the linear equation system given? (iv) Is the system consistent? (v) Explain why you say so in (iv).

3. Express the linear equation system in (1) and (2) in (i) matrix form  $AX=H$ . (ii) augmented matrix form  $[A H]$ .

4. Consider the following system of equation:

$$x_1 - x_2 = 1 \quad (\text{Eq. 1})$$

$$x_1 + x_2 = 2 \quad (\text{Eq. 2})$$

(i) Express the equation system above into augmented matrix form.  
 (ii) Perform the following operation: Replace (Eq. 1) by (Eq.1) + (Eq. 2), and call the resultant equation (Eq 1'). At the mean time, leave (Eq. 2) untouched. Write down the resultant equation system. What are the corresponding row

elementary operations that have an equivalent effect on the augmented matrix?

(iii) Now, perform a follow-up operations: Replace (Eq. 2) by (Eq. 2) added with (Eq 1') multiplied by  $(-1/2)$ . Write down the resultant equations. What are the corresponding row elementary operations that have an equivalent effect on the augmented matrix?

(iv) Now, multiply (Eq. 1') by a factor of  $1/2$ . Write down the resultant equations. What are the corresponding row elementary operations that have an equivalent effect on the augmented matrix?

(v) Now, reduce the augmented matrix in (i) into RREF. Do you get the same augmented matrix as resulted in (iv)? (vi) Can you read off the solutions of  $x_1, x_2$  from the resultant RREF matrix by inspection?

5. (i) Express the following linear equation system in augmented matrix form  $[A H]$ .

$$x_1 + x_2 + x_3 = 1; \quad 2x_1 + x_2 + 3x_3 = 2;$$

$$3x_1 + x_2 + x_3 = 0.$$

(ii) Reduce the augmented matrix  $[A H]$  into reduced row echelon form. (iii) Write out the equation system represented by the RREF augmented matrix as obtained in (ii). Read off the solutions by inspection.

6. Choose the correct answer: (i) REF is a (special / general) case of RREF. (ii) The procedure you used in (5), in which the augmented matrix is reduced into RREF, to solve for the solutions, is called (Gaussian /Gaussian-Jordan) elimination.

7. The equation system in

(i) (1) is a (homogeneous/non-homogeneous) system.

(ii) (2) is a (homogeneous/non-homogeneous) system.

(iii) (5) is a (homogeneous/non-homogeneous) system.

8. Refer to theorem III, page 77, Ayers. Answer the following properties of the equation system in (5):

(i) The system in (5) is (homogeneous or non-homogeneous).

(ii) The number of unknowns in (5) is \_\_\_\_\_.

(iii) The number of equation in (5) is \_\_\_\_\_.

(iv) The determinant of  $|A|$  in (5) is (zero / non-zero).

- (v) The solution as obtained in (5) is (unique / not-unique).
10. Given an arbitrary linear equation system of the form  $AX=H$ , there are two possibility on the existence of its solution, i.e. either the solution exists or \_\_\_\_\_. If the solution exists, it could be \_\_\_\_\_ or \_\_\_\_\_, \_\_\_\_\_ or \_\_\_\_\_.
11. Give a simple example of a non-homogeneous system that (i) has a unique solution. What is the unique solution? (ii) has no solution (i.e. not consistent); (iii) has non-unique solutions.
12. Give an example of a 2 by 2 homogeneous system that (i) has a unique solution; (ii) has no solution (i.e. not consistent); (iii) has non-unique solutions.
13. Given a non-homogeneous equation  $AX=H$ , with  $|A| \neq 0$ ,  $A$  an  $n$ -square coefficient matrix,  $X$  column vector of  $n$  variables,  $H$  non-zero column vector of  $n$  components.  
(i) List down all the ways you can think of that can be employed to solve the linear equation systems. I can think of 5, how many can you think of?  
(ii) As an exercise, solve the given equation system using all the methods you have listed. You should get the same answer with all these different methods.
14. Consider an equation system  $AX=H$ , which represent  $m$  equations in  $n$  unknown. What is the sufficient condition that this equation system is consistent? (*Hint: find the answer in page 76 of Ayers "fundamental theorems."*)
15. Consider a homogeneous equation system  $AX=0$ , which has  $n$  equations in  $n$  unknowns.  
(i) Does the system is guaranteed consistent? Explain by referring to your answer in (14).  
(ii) What can you say about the solution  $X$  if the rank of  $A$ ,  $r$ , is equal the number of unknowns,  $n$ ? (Give your answer in terms of its existence, uniqueness and triviality.)  
(iii) What can you say about the solution  $X$  if the rank

of  $A$ ,  $r$ , is less than the number of unknowns,  $n$ ? (Give your answer in terms of its existence, uniqueness and triviality.)

(iv) Consider the statement: ALL homogeneous equation system is consistent. Is this statement true?

16. Consider the homogeneous equation system in (15),  $AX=0$ . (i) If  $A$  is not singular,  $A^{-1}$  exist. What do you get if you operate  $A^{-1}$  on  $AX=0$  from the left-hand-side (LHS)? Try to figure out what will happen to the solution  $X$ . (ii) So, what is your conclusion?

17. Consider (16) again. (i) If  $A$  is singular, i.e.  $A^{-1}$  does not exist, can you claim that  $AX=0$  has no solution? (ii) What's the difference in the solution of a homogeneous equation system of singular coefficient matrix  $A$  and one that is not? Give your answer in terms of its existence, uniqueness and triviality.

18. Consider the equation system

$x_1 - 2x_2 + 3x_3 = 4$ ;  $x_1 + x_2 + 2x_3 = 5$ . (i) Express the system in matrix form. (ii) Is the number of unknowns larger than the number of equation? (iii) So, how many solutions would you expect? (iii) Solve the equation system using Gaussian-Jordan elimination.

19. Refer to solved problems 2, page 79 of Ayers. We will learn how to 'count' the rank of a matrix in this DQ. Consider a homogeneous system of 2 linear equations in 3 unknown,  $x_1, x_2, x_3$ ,  $AX=H$ , where

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 4 & 5 & 3 & 0 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, H = \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$$

- (i) Transform the augmented matrix  $[A \ H]$  into  $[A_1 \ H_1]$ , the RREF form of  $[A \ H]$  using row elementary operations. Express  $[A_1 \ H_1]$  explicitly.
- (ii) Is the rank of  $[A_1 \ H_1]$ ,  $[A \ H]$  equal?
- (iii) Count the number of non-zero row in  $[A_1 \ H_1]$ . The number of non-zero row equals the rank of the augmented matrix  $[A_1 \ H_1]$ . This number is the same as the number of leading 1 in the RREF.

(Note that this simple fact is not mentioned explicitly in Ayers.) Hence, what is the rank of  $[A | H]$ ?

- (iv) From the expression of  $[A_1 | H_1]$ , what is the rank of the matrix  $A_1$ ? What is the rank of matrix  $A$ ?
- (v) By referring to the fundamental theorem I, page 76, Ayers, is the system consistent?

20. Consider the homogeneous equation system of

$$x_1 + 2x_2 + 2x_4 = 0, x_2 + 3x_3 = 0.$$

- (i) Express the system in the matrix form of  $AX=H$ .
- (ii) This is an equation system with \_\_\_\_\_ equations and \_\_\_\_\_ unknowns.
- (iii) Determine the rank of  $A$  and  $[A | 0]$ .
- (iv) Is this system consistent?
- (v) By comparing the rank of  $A$  and the number of unknowns, can you determine whether the system will admit non-trivial solution? State explicitly whether the non-trivial solutions are expected. (*Hint: refer to theorem IV, page 78, Ayers.*)
- (vi) From the answer to (ii), state your expectation whether the solutions will be unique or otherwise.
- (vii) Based on your answer to (v) you know what the rank of the matrix  $A$  is. Hence, how many linearly independent solutions do you expect for the HE system? (*Hint: refer to theorem VI, page 78, Ayers.*)

21. Solve the HE system in the previous DQ. Your solution should agree with the number of linearly independent solutions is as given in (vi) in the same question.

22. You may like to refer to Example 5, page 79, Ayers. Consider a non homogeneous equation system:

$$x_1 - 2x_2 + 3x_3 = 4, x_1 + x_2 + 2x_3 = 5.$$

- (i) Express the system in the matrix form of  $AX=H$ .
- (ii) This is an equation system with \_\_\_\_\_ equations and \_\_\_\_\_ unknowns.
- (iii) Determine the rank of  $[A | H]$  and  $A$ .
- (iv) Is this system consistent?
- (v) From the answer to (ii), state your expectation whether the solutions will be unique or otherwise.
- (vi) Based on your answer to (iii) you know what

the rank of the matrix  $A$  is. Hence, how many linearly independent solutions do you expect for the HE system? (*Hint: refer to theorem VI, page 78, Ayers.*)

(vii) Obtain the solution.

23. You may like to try out the trick you have learnt from the above DQ on the solved problem 1 in page 79, Ayers.