Chapter 1 Matrices

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Give an example of a 3×2 matrix. Call it A.

2. Identify the element a_{13} and a_{23} in A defined in (1).

3. Give any example of a square matrix of order 3. Call it S.

4. List down all the diagonal elements in (3).

5. Calculate the trace of S as defined in 3.

6. Give an example of a pair of equal matrices.

7. Give an example of a zero matrices or order 2×4.

8. Consider a square matrix of order 3, $A = [a_{ij}]$, where a_{ij}=1 for all i,j = 1,2,3, and a square matrix of the same order, $B = [b_{ij}]$, where b_{ij}=2 if i=j, b_{ij}=0 if $i \neq j$. Calculate the matrix $C = A + B$.

9. What is the negative of B, with B defined in (8).

10. If there exist a matrix D such that $A + D = B$, determine D.

11. Given $X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, Y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, find, if possible,

(i) the product *XY*, (ii) the product *YX*.

12. If *K* is a matrix of order 3 by 2, *M* a matrix of order 2 by 3, what is the order of the product (i) *KM*? (ii) *MK*?

13. Give an example of a pair of 3-square matrices *A*, *B* such that (i) $AB \neq BA$, (ii) $AB = BA$.

14. Give an example of a pair of 3-square matrices *A*, *B* such that (i) $AB = 0$ but $A \neq 0$ (ii) $AB = 0$ but $B \neq 0$, (iii) $AB = 0$ with $A=0$, $B=0$.

15. Give an example of a set of 3-square matrices *A*, *B*, *C* such that (i) $AB=AC$ with $B \neq C$, (ii) $AB=AC$ with $B=C$.

Chapter 2 Some types of matrices

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Give an example of a 3×3 (*i*) upper triangular matrix (*ii*) lower triangular matrix. (iii) Give an example of a 3×3 matrix which is both an upper and lower triangular matrix.
   [Ans: (*iii*) any 3×3 diagonal matrix.]

2. Give an example of a (*i*) 2×3 upper triangular matrix (*ii*) 2×3 lower triangular matrix.
   [Ans:

   (i) Any matrix of the form $\begin{pmatrix} x & x & x \\ 0 & x & x \end{pmatrix}$ is a 2×3 upper triangular matrix.

   (ii) Any matrix of the form $\begin{pmatrix} x & 0 \\ x & x \\ x & x \end{pmatrix}$ is a 3×2 lower triangular matrix.

   ]

3. (*i*) Write out explicitly I_4. (*ii*) What is $(I_4)^n$, with n a positive integer.

   [Ans: (*i*) $I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, (*ii*) $(I_4)^n = I_4$]

4. Given a $n \times n$ matrix A with $a_{ii} = \sqrt[n]{n}$, $i = 1, 2, \ldots, n$, where n a positive integer. What is the name for this type of matrix?
   [Ans: a scalar matrix.]

5. Refer to (4), express A explicitly if n=3.

[Ans: $A = \begin{pmatrix} 3^{\frac{1}{3}} & 0 & 0 \\ 0 & 3^{\frac{1}{3}} & 0 \\ 0 & 0 & 3^{\frac{1}{3}} \end{pmatrix}$]

6. Consider the matrix A as defined in (4). Find A^n.

[Ans:

$$A = n^{1/n} I_n \Rightarrow A_n = \left(n^{1/n} I_n\right)^n$$
$$= \left(n^{1/n}\right)^n \left(I_n\right)^n = nI_n$$

]

7. Given B an n-square diagonal matrix, with $b_{ii}= j$, where j = 1, 2,…, n. Let $C = AB$. Let c_{ij} denote the (i, j) element of the matrix C. (i) What is c_{ij} if $i \neq j$? What the element c_{jj}?

[Ans: (i) c_{ij} = 0 if $i \neq j$,

$$(ii)\, c_{jj} = \sum_{k=1}^{n} a_{jk} b_{kj} = a_{jj} b_{jj} = \sqrt[n]{n} \cdot j \backslash$$

]

8. Say A is a matrix of order $n \times n$. Give three simplest matrices you can think of that will commute with A?

[Ans: A, 0, I_n]

9. Give an example of a matrix that anti-commute with I_n.

[Ans: 0].

10. Give two simplest examples you can think of for an idempotent matrix.

[Ans: 0, I_n]

11. (i) Given an $n \times n$ matrix $F = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. What is the period of F? (ii) Construct a 3-square matrix which has a period of 3. *Hint*: you may like to test out some trial matrix in which each row or column has only a single none zero element.

Ans:

(i) $F^3 = F$. $\Rightarrow F^{2+1} = F \Rightarrow F$ has a period of $k=2$.

(ii) $G = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. With $G^3 = G$.

12. If A is nilpotent of order n, what is A^{n+1}? n positive integer.

[Ans: 0]

13. Given an n-square matrix A is the inverse of matrix B, then we write $A = B^{-1}$. (i) Express the inverse of A in term of B. (ii) What is AB? (ii) What is BA?

[Ans: (i) $A^{-1}=B$. (ii,iii) I_n]

14. Prove that $(AB)^{-1} = B^{-1}A^{-1}$.

[Ans: see problem 8, pg 15, Ayers.]

15. Give an easiest example you can think of where the inverse of a matrix is also the matrix itself.

[Ans: I_n]

16. Let Q be a scalar n-square matrix such that $Q = kI_n$, where k an scalar. What is Q^{-1}?

[Ans: $(1/k)\, I_n$]

17. Given the order of matrix A is $m \times n$, what is the order of its transpose, A'? Note that sometimes the transpose of matrix A is written as A^{T} instead of A'.

[Ans: $n \times m$].

18. Consider a 2 by 2 matrix A. (i) what is the sufficient condition that A is also equal to its transpose (i.e. $A'=A$)?

[Ans: the off diagonal element be the same, i.e $a_{21}=a_{21}$]

19. Consider a 3 by 3 non-zero diagonal matrix B. (*i*) Is B symmetric? (*ii*) Could B ever be skew-symmetric?
[Ans: (*i*) YES, (ii), NEVER.]

20. Let A be an n-square matrix. (*i*) What is the symmetry of $A+A^T$ (i.e. is it symmetric or skew-symmetric?). (*i*) What is the symmetry of $A-A^T$?
[(*i*) symmetric, (*ii*) skew-symmetric.]

21. Given the matrix, $C = \begin{pmatrix} w & y \\ x & z \end{pmatrix}$, (*i*) express C as a sum of a symmetric matrix and a skew-symmetric matrix.
[

$$C' = \begin{pmatrix} w & x \\ y & z \end{pmatrix};$$

$$C = \frac{1}{2}(C + C') + \frac{1}{2}(C - C')$$

$$= \frac{\begin{pmatrix} w & y \\ x & z \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix}}{2} + \frac{\begin{pmatrix} w & y \\ x & z \end{pmatrix} - \begin{pmatrix} w & x \\ y & z \end{pmatrix}}{2}$$

$$= \begin{pmatrix} w & \frac{y+x}{2} \\ \frac{y+x}{2} & z \end{pmatrix} + \begin{pmatrix} 0 & \frac{y-x}{2} \\ \frac{x-y}{2} & 0 \end{pmatrix}$$

]

22. (*i*) Express explicitly $\overline{C}$, the conjugate of the matrix C in (21). (*ii*) Express explicitly the conjugate of the matrix kC, where k a real scalar.
[Ans:

(*i*) $\overline{C} = \begin{pmatrix} \overline{w} & \overline{y} \\ \overline{x} & \overline{z} \end{pmatrix}$

(*ii*) $\overline{kC} = \begin{pmatrix} \overline{kw} & \overline{ky} \\ k\overline{x} & \overline{kz} \end{pmatrix} = \overline{k}\begin{pmatrix} \overline{w} & \overline{y} \\ \overline{x} & \overline{z} \end{pmatrix} = k\begin{pmatrix} \overline{w} & \overline{y} \\ \overline{x} & \overline{z} \end{pmatrix}$]

23. (*i*) The transpose of the conjugate of matrix *C* is written as … ? (*ii*) The conjugate of transpose of the matrix *C* is written as …? (*iii*) Are both of these equal each to other?

[Ans: (*i*) $\left(\overline{C}\right)^T$, or $\left(\overline{C}\right)'$, (*ii*) $\overline{\left(C^T\right)}$ or $\overline{\left(C^T\right)'}$, (*iii*) Yes].

24. (*i*) What is *A**? (*ii*) If *A* is a matrix containing no complex element, is there any difference between *A** and A^T? (*iii*) In general, if *A* is a matrix containing complex element, will *A** equal A^T in general?
[Ans: (*i*) *A** is the transpose of the conjugate of *A* = the conjugate of the transpose of *A*, (*ii*) No, because for real matrix, $\overline{A} = A$. (*iii*) No. In general they are not equal.]

25. Is a real, symmetric matrix Hermitian?
[Ans: Yes.]

26. (*i*) By making use of Theorem X in page 13 of Ayers 1982 impression, very quickly give an example of a 2 by 2 Hermitian matrix and an example of a 2 by 2 skew-Hermitian matrix. [Ans: Define any 2 by 2 complex matrix *A*. Then $A + \overline{A}^T$ is automatically Hermitian, whereas $A - \overline{A}^T$ skew-Hermitian.]

27. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, C = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}$. (*i*) What is the order of *S* = diag(*A*, *B*, *C*), the direct sum of *A*, *B*, *C*? (*ii*) What is the order of S^2?
[Ans: $S = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$, number of row = 2 + 2 + 3 = 7, number of column = 2 + 2 +3 = 7, hence the order of *S* is 7 by 7. (*ii*) S^2=diag(A^2, B^2, C^2), dimension is also 7 by 7, since the dimension of an square matrix *Q*, and that of its square, Q^2, are the same.]

Chapter 3 Determinant of a square matrix

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. See if you understand what permutation is.

   (*i*) How many inversions are there in the permutation 1234? (*ii*) 4321? (*iii*) state whether (*i*) is odd or even? (*iv*) state whether (*ii*) is odd or even. (*v*) Give an example of an odd permutation.

   [Ans: (*i*) 0; (*ii*) 6; (iii) even; (iv) even; (v) e.g. 21]

2. See if you can do this: $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -3 & -3 & -3 \end{vmatrix} =$

   [Ans: Use mathematica to solve, give 0]

3. See if you can do this without expanding the determinant:

   (*i*) $\begin{vmatrix} 43 & 88 & 22.34 & 113.8 \\ 65 & 43 & 43 & 24.9 \\ 86 & 443 & 3 & 23 \\ 0 & 0 & 0 & 0 \end{vmatrix} =$

   (*ii*) $\begin{vmatrix} 1 & 0 & 5 \\ 2 & 0 & 6 \\ 4 & 0 & 8 \end{vmatrix} =$

   [Ans: By theorem I, page 21 in Ayers, Det = 0 in both cases.]

4. Can you define the determinant of a matrix that is not square? How?
   [Ans: Yes, but I do not know how.]

5. Given that you know $\begin{vmatrix} 1 & 4 & 1 \\ 2 & 2 & 3 \\ 0 & -3 & -3 \end{vmatrix}$ =21, find $\begin{vmatrix} 1 & 2 & 0 \\ 4 & 2 & -3 \\ 1 & 3 & -3 \end{vmatrix}$ without expansion.

   [Ans: By virtue of theorem II, page 21 of Ayers, $\left|A^{\mathrm{T}}\right| = A$. Hence $\left|A^{\mathrm{T}}\right| = 21$]

6. Given that $\begin{vmatrix} 1 & 2 & 1 \\ 8 & 4 & 4 \\ -3 & 2 & -3 \end{vmatrix} = 32$, find $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ -3 & 2 & -3 \end{vmatrix}$ without expansion.

[Ans: by virtue of theorem III, Ayers, page 22

$$32 = \begin{vmatrix} 1 & 2 & 1 \\ 8 & 4 & 4 \\ -3 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 2(4) & 4 & 4 \\ -3 & 2 & -3 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ -3 & 2 & -3 \end{vmatrix}.$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ -3 & 2 & -3 \end{vmatrix}. = (1/2)\begin{vmatrix} 1 & 2 & 1 \\ 8 & 4 & 4 \\ -3 & 2 & -3 \end{vmatrix} = (1/2)(32) = 16$$
]

7. Given that $\begin{vmatrix} g & j & p \\ y & t & l \\ u & s & m \end{vmatrix} = 53.889$, find

   (*i*) $\begin{vmatrix} y & t & l \\ g & j & p \\ u & s & m \end{vmatrix}$, (*ii*) $\begin{vmatrix} u & s & m \\ y & t & l \\ g & j & p \end{vmatrix}$

   Ans[(*i*) By virtue of theorem IV, page 22, Ayers,

   $$\begin{vmatrix} y & t & l \\ g & j & p \\ u & s & m \end{vmatrix} = -\begin{vmatrix} g & j & p \\ y & t & l \\ u & s & m \end{vmatrix} = -53.889;$$

   (*ii*) By virtue of theorem V, page 22, Ayers,

$$\begin{vmatrix} u & s & m \\ y & t & l \\ g & j & p \end{vmatrix} = -\begin{vmatrix} g & j & p \\ y & t & l \\ u & s & m \end{vmatrix} = -53.889,$$

]

8. Given $|A| = \begin{vmatrix} l & o & v & e \\ s & u & k & a \\ l & i & k & e \\ a & i & 0 & 0 \end{vmatrix}$ =999, what is

$(i)\begin{vmatrix} o & v & l & e \\ u & k & s & a \\ i & k & l & e \\ i & 0 & a & 0 \end{vmatrix}$, $(ii)\begin{vmatrix} o & v & e & l \\ u & k & a & s \\ i & k & e & l \\ i & 0 & 0 & a \end{vmatrix}$

[Ans: (*i*) The matrix of (*i*) is obtained by carrying the {*l*, *s*, *l*, *a*} column in *A* 2 column over to the right, hence by virtue of theorem VI,

$$\begin{vmatrix} o & v & l & e \\ u & k & s & a \\ i & k & l & e \\ i & 0 & a & 0 \end{vmatrix} = (-1)^2 |A| = 999;$$

(*ii*) The matrix of (*ii*) is obtained by carrying the {*l*, *s*, *l*, *a*} column in *A* 3 column over to the right, hence by virtue of theorem VI,

$$\begin{vmatrix} o & v & e & l \\ u & k & a & s \\ i & k & e & l \\ i & 0 & 0 & a \end{vmatrix} = (-1)^3 |A| = -999$$

]

9. How do you convince yourself that

$(i)\ \begin{vmatrix} 0 & 0 & 0 \\ y & t & l \\ g & j & p \end{vmatrix} = \begin{vmatrix} u & s & m \\ 0 & 0 & 0 \\ g & j & p \end{vmatrix} = 0\,?$

$(ii)\ \begin{vmatrix} o & v & 0 & l \\ u & k & 0 & s \\ i & k & 0 & l \\ i & 0 & 0 & a \end{vmatrix} = \begin{vmatrix} o & v & e & l \\ u & k & a & s \\ i & k & e & l \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0\,?$

10. Given $\begin{vmatrix} 2 & t & l \\ 2 & j & p \\ 4 & s & m \end{vmatrix} = 10, \begin{vmatrix} 3 & t & l \\ 4 & j & p \\ 6 & s & m \end{vmatrix} = 9,$ what

is $\begin{vmatrix} 5 & t & l \\ 6 & j & p \\ 10 & s & m \end{vmatrix}$?

[Ans: By virtue of theorem VIII, pg. 22 Ayers,

$$\begin{vmatrix} 5 & t & l \\ 6 & j & p \\ 10 & s & m \end{vmatrix} = \begin{vmatrix} 2+3 & t & l \\ 2+4 & j & p \\ 4+6 & s & m \end{vmatrix}$$

$$= \begin{vmatrix} 2 & t & l \\ 2 & j & p \\ 4 & s & m \end{vmatrix} + \begin{vmatrix} 3 & t & l \\ 4 & j & p \\ 6 & s & m \end{vmatrix} = 10 + 9 = 19$$

]

11. Say $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = x$.

(*i*) Consider the operation "add 2×(second row) to the first row". If this operation is applied to the determinant above, what do you get? Write down the expression explicitly. (*ii*) Without expansion, work out what is the value of the determinant in (*i*).

[Ans:

(*i*)

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \to \begin{vmatrix} 1+2(4) & 2+2(5) & 3+2(6) \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 9 & 12 & 15 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

(*ii*) By virtue of theorem IX, since $\begin{vmatrix} 9 & 12 & 15 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ is obtained from *x* by adding to its first row a scalar multiple of the second row, then both determinant has the same value, i.e. $\begin{vmatrix} 9 & 12 & 15 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = x$.

]

12. Consider the matrix $X = [x_{ij}] = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

(i) What is the first minor of x_{11}? Of x_{23}? Conventionally what symbol you will use to represent these quantities?

(ii) What is the cofactor of x_{11}? Of x_{23}? Conventionally what symbol you will use to represent these quantities?

(iii) What is the matrix of the minors x_{ij}, $[|M_{ij}|]$?

(iv) What is the matrix of the cofactors of x_{ij}, $[\alpha_{ij}]$?

[Ans: (*i*) $|M_{11}| = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3; |M_{23}| = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -6$;

(*ii*)

$$\alpha_{11} = (-1)^{1+1} |M_{11}| = |M_{11}| = -3;$$
$$\alpha_{23} = (-1)^{2+3} |M_{23}| = -|M_{23}| = 6;$$

(*iii*) matrix of the minors x_{ij}, $[|M_{ij}|]$

$$[|M_{ij}|] = \begin{pmatrix} (-1)^{1+1}(-3) & (-1)^{1+2}(-6) & (-1)^{1+3}(-3) \\ (-1)^{2+1}(-6) & (-1)^{2+2}(-12) & (-1)^{1+3}(-6) \\ (-1)^{3+1}(-3) & (-1)^{3+2}(-6) & (-1)^{3+3}(-3) \end{pmatrix}$$
$$= \begin{pmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{pmatrix}$$

See DQ Chapter 3 Q12.nb

(*iv*) Matrix of cofactors of of x_{ij}, $[\alpha_{ij}] =$

$$\begin{pmatrix} (-1)^{1+1}(-3) & (-1)^{1+2}(-6) & (-1)^{1+3}(-3) \\ (-1)^{2+1}(-6) & (-1)^{2+2}(-12) & (-1)^{1+3}(-6) \\ (-1)^{3+1}(-3) & (-1)^{3+2}(-6) & (-1)^{3+3}(-3) \end{pmatrix}$$
]
$$= \begin{pmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{pmatrix}$$

13. Consider the matrix in (12). Calculate

(*i*) $\sum_j^3 x_{1j}\alpha_{1j}$ (*ii*) $\sum_i^3 x_{i3}\alpha_{i3}$

(*ii*) What is the quantity represented by the sum in (*i*) and (*iii*)? Do you get the same value for both (*i*) and (*ii*)?
(*iv*) So, what is the determinant of *X*?
[Ans:

(*i*) $\sum_j^3 x_{1j}\alpha_{1j}$ =1(-3)+2(6)+3(-3) = 0;

(*ii*) $\sum_i^3 x_{i3}\alpha_{i3}$ =3(-3)+6(6)+9(-3)=0

(*iii*) YES (*iv*) 0
]

14. If you were to manually find the determinant of $\begin{vmatrix} 44.1 & 0 & 52.3 \\ 33.6 & 1.98 & 7.1 \\ 4.2 & 0 & 1.6 \end{vmatrix}$, along which column or row would you like to follow to calculate the determinant?
[Ans: along the j=2 column.]

Chapter 4 Evaluation of determinants

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

> Consider the operation "add to row j by ($k \times$ row i)", $j \neq i$, where k is a non-zero scalar. We will conveniently represent the above operation by $R_j^i(k)$, and we have $A \xrightarrow{R_j^i(k)} A' = R_j^i(k)A$.
>
> Sometimes this operation is symbolized by $\{j\} \rightarrow \{j\} + k\{i\}$, meaning: "replace row j by (row j + k times row i)". In Ayer's notation, this operation is denoted by $H_{ij}(k)$. We will adopt the $R_j^i(k)$ notation for the rest of the course. There is also a similar operation that acts on the columns, denoted by $C_j^i(k)$.

1. Given A= $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 9 \\ 8 & 10 & 6 \end{pmatrix}$, what is $A' = R_3^2(-2)A$?

Ans:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 9 \\ 8 & 10 & 6 \end{pmatrix} \xrightarrow{R_3^2(-2)}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 9 \\ 8-2(4) & 10-2(5) & 6-2(9) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 9 \\ 0 & 0 & -12 \end{pmatrix}$$
$= A'$

2. (i) What is the determinant of A in (1)? (ii) That of A'? (iii) What is your conclusion?

Ans:

(i,ii) $|A|$=$|A'|$=-12 $-12\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -12(5-8) = 36$;

(iii) If two matrices are related by an operation of $R_j^i(k)$, their determinant equals.

3. If you understand the idea intended to be conveyed in (1), what is your strategy if you were asked to evaluate $\begin{vmatrix} 1 & 66 & 54 \\ 4 & 11 & 9 \\ 8 & 10 & 43 \end{vmatrix}$? Inspect the values of the elements then think of the best strategy possible.

Ans:
One would first try to see if there is any way to reduce a row in the determinant to as many zeroes as possible in $\begin{vmatrix} 1 & 66 & 54 \\ 4 & 11 & 9 \\ 8 & 10 & 43 \end{vmatrix}$. One may like to operate $R_1^2(-6)$ so that the second row becomes (-23, 0, 0). Then the determinant becomes easily evaluated.

4. How about evaluating $\begin{vmatrix} 1 & 4 & 8 \\ 7 & 11 & 10 \\ 7 & 8 & 16 \end{vmatrix}$?

Ans:
Try to transform either column 2 or 3 of the determinant into one containing two zeroes. E.g., carry out the operation $C_3^2(-2)$ so that the third column becomes $(0, -12, 0)^T$. Then the transformed determinant $\begin{vmatrix} 1 & 4 & 0 \\ 7 & 11 & -12 \\ 7 & 8 & 0 \end{vmatrix}$ becomes easily evaluated:

$$\begin{vmatrix} 1 & 4 & 0 \\ 7 & 11 & -12 \\ 7 & 8 & 0 \end{vmatrix} = (-1)^{2+3}(-12)\begin{vmatrix} 1 & 4 \\ 7 & 8 \end{vmatrix} = 12(-20) = -240$$

5. Given $A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$. Carry out successive operations of $R_j^i(k)$ to transform A into a matrix which has as many zero elements as possible in row 3. Call the resultant matrix B. (i) What is your resultant matrix, B? (ii) Is A and B equivalent? (iii) Does A and B have the determinant? Justify your answer. (iv) Evaluate $|A|$.

Ans:

(*i*)

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\xrightarrow{R_3^1(-7)}$$

$$\begin{pmatrix}1&2&3\\4&5&6\\7-7(1)&8-7(2)&9-7(3)\end{pmatrix}=\begin{pmatrix}1&2&3\\4&5&6\\0&-6&-12\end{pmatrix}$$

$$\xrightarrow{R_2^1(-4)}$$

$$\begin{pmatrix}1&2&3\\4-4(1)&5-4(2)&6-4(3)\\0&-6&-12\end{pmatrix}=\begin{pmatrix}1&2&3\\0&-3&-6\\0&-6&-12\end{pmatrix}$$

$$\xrightarrow{R_3^2(-2)}$$

$$\begin{pmatrix}1&2&3\\0&-3&-6\\0-2(0)&-6-2(-3)&-12-2(-6)\end{pmatrix}=\begin{pmatrix}1&2&3\\0&-3&-6\\0&0&0\end{pmatrix}.$$

(ii) Yes. (iii) One of the theorems in chapter 3 on the properties of determinant says so. (iv) Clearly, $|B|=0$ due to the last row of 3 zeros. Hence, $|A|=|B|=0$.

Chapter 5 Equivalence

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Consider an *n*-square non-zero matrix *A*. (*i*) What is the highest possible rank of matrix *A*? (*ii*) What is the smallest possible rank of *A*? (*iii*) What is the condition on *A* for it to assume the highest possible rank? (*vi*) What kind of matrix *A* is if it fulfils the condition in (*iii*)?

Ans:
(*i*) *n*; (*ii*) 0. (*iii*) when $|A| \neq 0$. (*iii*) non-singular.

2. In general, for an *n*-square non-zero matrix *A*, its first minors* are (*i*) ___-square minors (*ii*) Can you recognize what the "*n*-square minors" of *A* is?
(*We will refer "first minors" simply as "minors" in the future unless specify otherwise.)

Ans:
(*i*) (*n*-1)-square minors; (*ii*) the "*n*-square minor" of *A* is non other than the determinant of *A*, $|A|$.

3. Consider the 3-square matrix $Y=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 8\end{pmatrix}$. (*i*) The minors of *Y* are ____-square minors. (*ii*) What is the "3-square minor" of *Y*? (*iii*) What is the rank of *Y*? (*iv*) What is the 4-square minor of *Y*?

Ans:
(*i*) The minors of *Y* are "2-square" minors.
(*ii*) The "3-square minor" of *Y* is $|Y|=3\neq 0$. (*iii*) The rank of *Y* is 3 (highest possible rank for a matrix of order 3). (*iv*) Since the order of *A* is *n*=3, the 4-square minor of *A* is not defined.

4. You may refer to Chapter 3, designed question (12).
Consider the 3-square matrix $X=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$. (*i*) What is the value of the "3-square minor" of *X*? (*ii*) How many "2-square minors" are there in *X*? (*iii*) Is ALL of the "2-square minors" zero? (*iv*) What is the rank of *X*?

Ans: (*i*) $|X|= 0$. (*ii*) There are 9 of them. (*iii*) The matrix of "2-square minors" of *Y* is $[M_{ij}]=$ $\begin{pmatrix}-3 & -6 & -3\\ -6 & -12 & -6\\ -3 & -6 & -3\end{pmatrix}$. Not all of them are zero. (*iv*) The rank of *X* is $r = 2$.

5. Consider the 3-square matrix $Z=\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}$. (*i*) What is the value of the "3-square minor" of *Z*? (*ii*) How many "2-square minors" are there in *Z*? (*iii*) Is ALL of the "2-square minors" zero? (*iv*) How many "1-square minors" are there in *Z*? (*v*) Is ALL of the "1-square minors" zero? (*vi*)What is the rank of *X*?

Ans:
(*i*) $|Z|=0$.
(*ii*) There are 9 of them.
(*iii*) The matrix of "2-square minors" of *Y* is $[M_{ij}]=$ $\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$. ALL of the 2-square minors are zero.
(*iv*) There are 9 "1-square minors" in *Z*. (*v*) The matrix of "1-square minors" of *Z* is $\begin{pmatrix}|1| & |2| & |3|\\ |1| & |2| & |3|\\ |1| & |2| & |3|\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}$.
(*v*) Not all of the "1-square minors" of Z is zero. (*vi*) rank of *Z* is $r = 1$.

6. Consider matrix X as defined in (4). What is X', the image of X when it is transformed under the operations

(*i*) R_2^1 (*ii*) C_2^1 (*iii*) $R_2(3)$ (*iv*) $C_2(3)$

(*v*) $R_2^1(3)$ (*vi*) $C_2^1(3)$

7. (*i*) What do you call the operations in (6)?
(*ii*) What is the determinant of X' in each case in (6)?
(*iii*) What is the order of X' in each case in (6)?
(*iv*) What is the rank of X' in each case in (6)?

Ans:

(*i*) Elementary transformations.
(*ii*, *iii*, *iv*) Determinant, order and rank of a matrix remain unchanged under elementary transformations.

8. Given two examples of equivalent matrices to X as defined in (6).

Ans:
All of the examples in 6(*i*-*vi*) are equivalent matrices to X.

9. Let $A=\begin{pmatrix}1&3&0\\0&2&0\\0&0&1\end{pmatrix}$. Find a single row elementary transformation (let's call it $E_{A\to B}$) that transforms A into a 3-square diagonal matrix, B? We will use the notation $B=E_{A\to B}A$.

Ans:

$$A=\begin{pmatrix}1&3&0\\0&2&0\\0&0&1\end{pmatrix}\xrightarrow{R_1^2(-3/2)}\begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}=B$$

10. Find a single row elementary transformation (let's call it $E_{B\to I_3}$) that transforms B in (9) into the unit matrix I_3?

We will denote $I_3=E_{B\to I_3}A$.

Ans:

$$B=\begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}\xrightarrow{R_2(1/2)}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=I_3$$

11. If we define the operation $U=E_{B\to I_3}E_{A\to B}$, what will you get if U is operated on A? We will denote the resultant matrix as $V=UA$.

Ans:

$V=UA=E_{B\to I_3}(E_{A\to B}A)=E_{B\to I_3}B=I_3$.

12. Let $A=\begin{pmatrix}1&3&2\\0&2&0\\5&0&1\end{pmatrix}$. Transforms A into a 3-square diagonal matrix, call it D, by successively applying row elementary operations. I can achieve a 3-square diagonal matrix in three steps (I call it $R_{stp}=R_{stp3}R_{stp2}R_{stp1}$, so that $R_{stp}A$ is diagonal matrix.) How many steps you need, and what is your that R_{stp}?

Ans:

$$A=\begin{pmatrix}1&3&2\\0&2&0\\5&0&1\end{pmatrix}\xrightarrow{R_1^3(-2)}\begin{pmatrix}-9&3&0\\0&2&0\\5&0&1\end{pmatrix}\xrightarrow{R_1^2(-3/2)}\begin{pmatrix}-9&0&0\\0&2&0\\5&0&1\end{pmatrix}\xrightarrow{R_3^1(5/9)}\begin{pmatrix}-9&0&0\\0&2&0\\0&0&1\end{pmatrix}=D.$$

Hence $R_{stp}=R_{stp3}R_{stp2}R_{stp1}=R_3^1(5/9)\,R_1^2(-3/2)\,R_1^3(-2)$.

13. Reduce the resultant diagonal matrix in (12) into a unit matrix by applying a Sequence of row elementary transformations. Call this operation R_d. What is your R_d?

Ans:

$$D=\begin{pmatrix}-9&0&0\\0&2&0\\0&0&1\end{pmatrix}\xrightarrow{R_1(-1/9)}\begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}\xrightarrow{R_2(1/2)}$$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Hence, $R_d = R_2(1/2)\, R_1(-1/9)$

14. Now, I will call the operation that transforms A, as in (12), into a 3-square matrix identity matrix $E_{A\to I_3}$.

    With this notation, we have $E_{A\to I_3} A = I_3$. Write down the form of $E_{A\to I_3}$ in terms of the operations R_{stp} and R_d as obtained in (12) and (13).

Ans:

$E_{A\to I_3} = R_d R_{stp}$

$= R_2(1/2)\, R_1(-1/9)\, R_3^1(5/9)\, R_1^2(-3/2)\, R_1^3(-2)$

15. "Canonical matrix" as mentioned in page 40, Ayers, is a general form of matrix that fulfills the set of properties (a)-(d) stated in the same page. A special case of canonical matrices are matrices in Row Reduce Echelon form (RREF). These are matrices that have the following properties:

1. Rows of all zeros, if there are any, appear at the bottom of the matrix.
2. The first nonzero entry of a nonzero row is 1. This is called a leading 1.
3. For each nonzero row, the leading 1 appears to the right and below any leading 1's in preceding rows.
4. Any column in which a leading 1 appears has zeros in every other entry.

A matrix in RREF appears as a staircase pattern of leading 1's descending from the upper left corner of the matrix. The columns of the leading 1's are columns of an identity matrix. A matrix is in row echelon form (REF) if properties 1, 2, and 3 above are satisfied.

Now, see if you can demonstrate your understanding on what RREF is by answering the following questions:

Given the following matrices, state whether they are in RREF, REF or neither.

(i) $\begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 0 & 1 & -6 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 5 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (iv) $\begin{pmatrix} 0 & 0 & 1 & -6 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$

(v) $\begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 2 & 5 \end{pmatrix}$.

Ans: (i) RREF (ii) RREF (iii) REF (iv)REF (v) neither. However, under $R_3(1/2)$, matrix (v) will be reduced to REF.

16. Convert the above non-RREF matrices in to RREF via elementary row transformations.

    Ans:

(iii) $R_1^2(-5)\begin{pmatrix} 1 & 5 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(iv) $R_1^2(-2)\begin{pmatrix} 0 & 0 & 1 & -6 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix} =$

$$\begin{pmatrix} 0 & 0 & 1 & -6 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

(v)

$$\begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 2 & 5 \end{pmatrix} \xrightarrow{R_2^3(-5/2)} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 0 & -13/2 \\ 0 & 0 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow{R_1^3(1/2)} \begin{pmatrix} 1 & 2 & 0 & 3/2 \\ 0 & 1 & 0 & -13/2 \\ 0 & 0 & 2 & 5 \end{pmatrix} \xrightarrow{R_1^2(-2)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 29/2 \\ 0 & 1 & 0 & -13/2 \\ 0 & 0 & 2 & 5 \end{pmatrix} \xrightarrow{R_3(1/2)} \begin{pmatrix} 1 & 0 & 0 & 29/2 \\ 0 & 1 & 0 & -13/2 \\ 0 & 0 & 1 & 5/2 \end{pmatrix}$$

[See mathematica file for verification].

17. Consider $B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$. (*i*) Transform B into a row echelon form via elementary row transformations. (*ii*) What is the reduced row echelon form of B? (*iii*) Is the REF as obtained in (*ii*) the same as your friend's? (*iv*) Is the RREF as obtained in (*iii*) the same as your friend's? (*v*) What can you conclude from (*iv*) and (*v*)? After this exercise you should have learnt the trick of reducing any generic matrix into RREF form.

Ans:

(*i*) (non-unique)

(*ii*) Using mathematica,

$$\texttt{RowReduce[B]} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(*iii*) No. (*iv*) Yes. (*v*) REF is not unique but RREF is.

Chapter 6 The adjoint of a square matrix

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Given $A=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$. (*i*) Obtain the cofactor α_{ij} for all *i*, *j*. (*ii*) Form the matrix of the cofactors of *A*, and call it *X*. (*iii*) How is *X* be related to adj *A*? (*iv*) What is |*A*|? (You should have been very familiar with *A*, see DQ in Chapter 5.) (*v*) What is adj*A*·*A*? Try to relate the answer in (*v*) to theorem II, page 50, Ayers.

Ans:

$\alpha_{11}=\begin{vmatrix}5 & 6\\ 8 & 9\end{vmatrix}$=45-48=-3; $\alpha_{12}=-\begin{vmatrix}4 & 6\\ 7 & 9\end{vmatrix}$=-(36-42)= 6

$\alpha_{13}=\begin{vmatrix}4 & 5\\ 7 & 8\end{vmatrix}$= 32-35=-3; $\alpha_{21}=-\begin{vmatrix}2 & 3\\ 8 & 9\end{vmatrix}$= -(18-24)= 6;

$\alpha_{22}=\begin{vmatrix}1 & 3\\ 7 & 9\end{vmatrix}$= (9-21)=-12; $\alpha_{23}=-\begin{vmatrix}1 & 2\\ 7 & 8\end{vmatrix}$= -(8-14)=6;

$\alpha_{31}=\begin{vmatrix}2 & 3\\ 5 & 6\end{vmatrix}$=12-15=-3; $\alpha_{32}=-\begin{vmatrix}1 & 3\\ 4 & 6\end{vmatrix}$= -(6-12)=6;

$\alpha_{33}=\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix}$=5-8=-3; $X=\begin{pmatrix}-3 & 6 & -3\\ 6 & -12 & 6\\ -3 & 6 & -3\end{pmatrix}$

(*iii*) *X* = adj *A*; (*iv*) |*A*|=0; (*v*) adj*A*·*A*=0, this is due to the fact that *A* being square and singular.

2. How would you convince yourself that indeed adj*A*·*A*= *A*·adj*A*=diag(|*A*, *A*, *A*, …,|*A*|)?

Ans: See (6.2), page 49, Ayers.

3. Say *A*, *B* are two matrices conformable for a product in the order of *AB*. What is |*A*||*B*|?

Ans:
See (4.2), page 33, Ayers.

4. How would you convince yourself that |*A*| |adj*A*| = $|A|^n$? [*Hint*: use DQ (2).]

Ans:
Eq. (6.2), page 49, Ayers.

5. Consider this statement: If *X* a square matrix and singular, then |adj*X*|=0. Is this statement true?

Ans: I don't know. If one can show a counter example of the existence an |adj*X*| ≠ 0 yet |*A*|=0, that means the statement is false.

6. Try to relate this question with what you have learnt in the secondary school. Given $G=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$, (*i*) what is Adj *G*? You should be able to write down the answer by inspection. (*ii*) What is |*G*|? (*ii*) Let *H* = Adj *G* /|*G*|, work out what is *HG*. (*iii*) Based on the answer of (*ii*), what is the product *GH*? (*iv*) So, what can you conclude from the above exercise?

Ans:

(i) Adj *G* = $\begin{pmatrix}4 & -3\\ -2 & 1\end{pmatrix}^T=\begin{pmatrix}4 & -2\\ -3 & 1\end{pmatrix}$; (*ii*) |*G*|=-2;

(ii) *HG* =(-1/2) $\begin{pmatrix}4 & -2\\ -3 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$=

$$(-1/2)\begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

(iii) Since *HG*=*I*, that means *H* is the inverse of *G*. Hence *GH*=*HG*=*I*.

(*iv*) Adj *G* /|*G*| is simply the inverse of *G*.

Chapter 7 The inverse of a matrix

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. For the following two questions, please refer to the designed questions (12), (13), (14), in Chapter 5. There, $E_{A\to I_3}$ A= I_3. Now, what would you get when if $E_{A\to I_3}$ is operated on I_3 instead of on A? In other words, ask yourself, what is $E_{A\to I_3}$ I_3?

Ans:

Comparing I_3= $E_{A\to I_3}$ A with the definition of the inverse of A, i.e. I_3= $A^{-1}A$, we conclude that $E_{A\to I_3} \equiv A^{-1}$. Hence,

$E_{A\to I_3}$ I_3= $A^{-1}I_3$= A^{-1}.

2. So, by now, have you learnt how to find the inverse of a matrix? Find A^{-1}, where A is as defined in DQ (12), Chapter 5, $A=\begin{pmatrix}1 & 3 & 2\\ 0 & 2 & 0\\ 5 & 0 & 1\end{pmatrix}$.

Ans:

A^{-1}= $E_{A\to I_3}$ I_3=

$$R_2(1/2)\ R_1(-1/9)\ R_3^1(5/9)\ R_1^2(-3/2)\ R_1^3(-2)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$= R_2(1/2)\ R_1(-1/9)\ R_3^1(5/9)\ R_1^2(-3/2)\begin{pmatrix}1 & 0 & -2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$= R_2(1/2)\ R_1(-1/9)\ R_3^1(5/9)\begin{pmatrix}1 & -3/2 & -2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$= R_2(1/2)\ R_1(-1/9)\begin{pmatrix}1 & -3/2 & -2\\ 0 & 1 & 0\\ 5/9 & -5/6 & -1/9\end{pmatrix}$$

$$= R_2(1/2)\begin{pmatrix}-1/9 & 1/6 & 2/9\\ 0 & 1 & 0\\ 5/9 & -5/6 & -1/9\end{pmatrix} =$$

$$\begin{pmatrix}-1/9 & 1/6 & 2/9\\ 0 & 1/2 & 0\\ 5/9 & -5/6 & -1/9\end{pmatrix} = A^{-1}$$

3. Now carry out the $E_{A\to I_3}$ A= and $E_{A\to I_3}$ I_3 operations in a "two-in-one" manner, i.e. if the augmented matrix form. First form the augmented matrix of the form $(A|I)=\left(\begin{array}{ccc|ccc}1 & 3 & 2 & 1 & 0 & 0\\ 0 & 2 & 0 & 0 & 1 & 0\\ 5 & 0 & 1 & 0 & 0 & 1\end{array}\right)$. Carrying out $E_{A\to I_3}$ on both sides to arrive at $(I|A^{-1})$.

4. In the designed questions (12), (13), (14) of Chapter 5, you are asked to find a sequence of elementary transformations that transform a generic matrix A into a unit matrix. Now, if A were our old friend $\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$, what would happen if you were to attempt to reduce it into an identity matrix via a sequence of elementary transformations? Explain.

Ans:

Since our old friend is a singular square matrix with no inverse, it is not possible to reduce it into a unit matrix via a finite sequence of elementary transformation. If we were able to achieve that, that means there exists a finite sequence of operation similar to $E_{A\to I_3}$ such that $E_{A\to I_3}$ I_3

gives us the inverse of A, which is a contradiction. Hence, it is not possible to obtain a finite sequence of elementary transformation to reduce out old friend to identity matrix.

5. Deduce A^{-1}, where A is as defined (2), using A^{-1}= adj$A$/$|A|$. Do you get the same answer as in (2) where the inverse is obtained using row reduced echelon form method?

Ans: Of course yes.

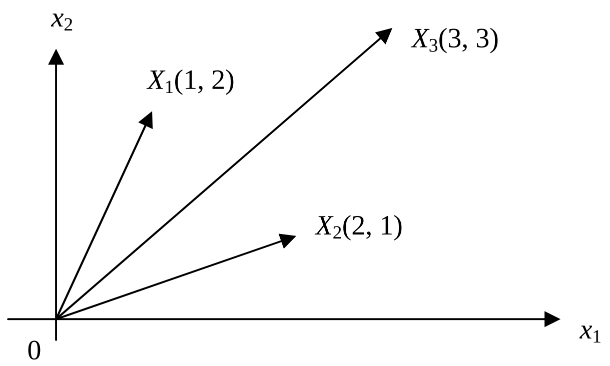
Chapter 9 Linear dependence of vectors and forms

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Give examples of two distinct, non-zero 2-vectors in row vector form. Call them X_1 and X_2. (*ii*) What is their sum, $X_3 = X_1 + X_2$? Write it down explicitly. (*iii*) Sketch a picture representing these vectors in a 2-dimensional space. Label your drawing properly (including the vectors and the axes).

Ans:

(*i*) $X_1 = [1, 2]$; (*i*) $X_2 = [2, 1]$; $X_3 = X_1 + X_2 = [3, 3]$.



2. (*i*) Repeat (*i*) and (*ii*) of the above question but for 2-vectors in *column* form (i.e. giving examples, writing down their sum). (*ii*) If you were to sketch a picture representing these 2-dimensional column vectors in a 2-dimensional plane, as you did in (1)(*iii*), will the drawing be the same as in (1)(*iii*)?

Ans:
$X = [x_1, x_2]^T$;
(*i*) $X_1 = [1, 2]^T$; (*i*) $X_2 = [2, 1]^T$; $X_3 = X_1 + X_2 = [3, 3]^T$, or,

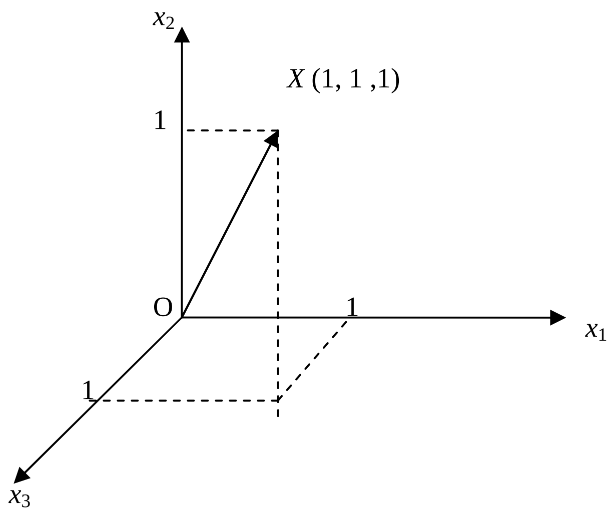
equivalently, $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; (*i*) $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$; $X_3 = X_1 + X_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ (*ii*)

YES.

3. (*i*) Give an example of a simplest possible 3-vector in column vector form, with all positive, non-zero components you can think of. Call it X. (*iii*) Sketch a picture representing this vector in a 3-dimensional space. Label your drawing properly (including the vectors and the axes).

Ans:

(*i*) $X = [1, 1, 1]^T$.



4. (*i*) Give an example of a simplest possible 4-vector in column vector form, with all positive, non-zero components you can think of. Call it X. (*ii*) Can you possibly sketch a picture to represent this vector in the similar manner as you do for (2) and (3)? Explain.

Ans:
(*i*)$X = [1, 1, 1, 1]^T$. (*ii*) Not possible. Since we live in a 3-D world, we can only draw vectors up to 3-D but not higher than that.

5. What is the dimensionality of the vectors in (1), (2), (3) and (4)?

Ans:
For (1), (2), the vectors are 2-demensional; For (3), 3-dimensional; For (4), 4-dimensional.

6. Consider the 3-vectors pair $X_1 = [1, 2, 3]$, $X_2 = [-1, -2, -3]$. (*i*) Find any possible values of k_1 and k_2, with $\{k_1\ k_2\} \neq \{0, 0\}$, such that $k_1X_b + k_1X_2 = 0$. (*ii*) Are the vectors linearly

independent?

Ans: (*i*) Any arbitrary value of $k_1 = k_2 = k \neq 0$ will do. (*ii*). They are linearly dependent, since there exist values of $\{k_1, k_2\}$, not all zero, such that $k_1 X_b + k_1 X_2 = 0$.

7. Explain why is that a zero 3-vector is always linearly dependent with any 3-vector?

Ans: [0, 0, 0] is linearly dependent with any 3-vector X since there always exist some $\{k_1, k_2\} \neq \{0, 0\}$ such that $k_1 0 + k_2 X = 0$ is satisfied, e.g. $k_1 = 101$, $k_2 = 0$.

8. Consider $X_1=[a, b, c]$ and $X_2=s[a, b, c]$, where s a non-zero scalar. (*i*) Are these 3-vectors linearly independent? (*ii*) Explain why you say so.

Ans:
$k_1X_1+k_2X_2= k_1X_1+ sk_2X_1= X_1(k_1+sk_2)$. For $k_1X_1+k_2X_2=X_1(k_1+sk_2) = 0$, any values of k_2 and k_1 satisfying the condition $k_1 + sk_2= 0$ will do the job, e.g. $k_1 = 0$, $k_2= 1$. In other words, there always exist NOT ALL ZERO coefficients $\{k_1, k_2\}$ that satisfy $k_1X_1+k_2X_2=0$. Hence, X_1, X_2 are NOT linearly independent.

9. Consider $X_1=[1, 2, 3]$ and $X_2=[4, 5, 6]$. (*i*) Are they linearly independent? (*ii*) Explain why you say so.

Ans:
For $k_1X_1+ k_2X_2=[0, 0, 0]$, we need $k_1 + 4k_2 = 0$, $2k_1 + 5k_2 = 0$, $3k_1+6k_2= 0$. The only possible solution is k_1, k_2 both being zero. This means that for $k_1X_1+ k_2X_2=[0, 0, 0]$, the coefficients $\{k_1, k_2\}$ *must be all zero*. This prove the linearly independence of X_1 and X_2.

10. Give a set of three distinct, non-zero 2-vectors, X_1, X_2, X_3 that are linearly independent.

Ans:
This is not possible. Such a set must necessarily be linearly dependent.

11. (*i*) Consider 3 distinct, non-zero 2-vectors. These vectors must be (linearly dependent / linearly independent). (*ii*), Consider 2 distinct, non-zero 3-vectors. Furthermore, these vectors are not in the form of $X_1=sX_2$ (in other words, they are not parallel nor anti-parallel). These vectors must be (linearly dependent / linearly independent).

Ans:
(*i*) linearly dependent; (*ii*) linearly independent.

12. Refer (9.5) in page 69, Ayers. Given a set of m vectors, we want to know whether they are linearly independent or otherwise. What is the easiest way (or one of the easier ways) to determine the linear independence of such a set of vectors?

Ans: Use row elementary operations to reduce the matrix A formed by these vectors to RREF. The number of non-zero row in the RREF of A is the rank of the matrix A, r. The rank, r, also tells us how many linearly independent vectors are there in the set of m vectors.
If $r = m$, then the set of this m vectors is linearly independent.
If $r < m$, then the set of m vectors is linearly dependent.
In such a case, there are exactly r vectors of the set which are linearly independent while each of the remaining m-r vectors can be expressed as a linear combination of these r vectors.

13. Consider the set S containing the following 4 3-vectors: $K_1=[1,1,1]^T$, $K_2=[1,3,5]^T$, $K_3=[1,5, 3]^T$, $K_4=[5, 3, 1]^T$; $S=\{K_1, K_1, K_3\ K_4\}$. (*i*) Form the matrix A whose rows are made up of the vectors K_i^T , i=1, 2, 3, 4. (*ii*) Reduce A into RREF. (*iii*) What is the rank of A? (*iv*) How many linearly independent vectors are there in the set S? (*v*) Are the vectors in set S linearly independent?

Ans:

(*i*) $A = (K_1^T, K_2^T, K_3^T, K_4^T) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \\ 5 & 3 & 1 \end{pmatrix}$

(*ii*) $A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (*iii*) $r = 3$

(*iv*) There are r=3 linearly independent vectors in the set S.
(*v*) Since the number of vectors in S, m = 4 is larger than the number of linearly independent vectors, r = 3, the set of vectors in S is not linearly independent. They are linearly dependent, by the virtue of theorem V, page 69, Ayers.

14. There is another way to prove the linearly independence of a set of vectors. Consider a set of three vectors K_1= (1, 4, 7)^T, K_2= (2, 5, 8)^T, K_3= (3, 6, 9)^T. Let's find out whether they are linearly independent or otherwise. If the set of these vectors is linearly independent, then the only solution to the homogeneous equation system

$$x_1K_1+x_2K_2+x_3K_3=0$$

is the trivial solution, i.e. $X=(x_1, x_2, x_3)^T=(0, 0, 0)^T$.

(*i*) If we write the homogeneous equation system in the matrix form of $KX = 0$. What is the matrix K? (*ii*) Reduce K into RREF to determine rank(K). (*iii*) How many unknowns are there in the HE system? (*iv*) By comparing your answer in (*ii*) and (*iii*) what can you say about the solution X? (*iv*) Is the set of three vectors linearly independent?

Ans:

(*i*) $x_1K_1+x_2K_2+x_3K_3 = (K_1, K_2, K_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = K\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Hence,

$$K= (K_1, K_2, K_3)=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

(*ii*) $K\sim\text{RREF}(K) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Hence, rank ($K$)=3.

(*iii*) Number of unknown = 3.

(*iv*) Since rank (K) = n = number of unknown, the HE system has only the trivial solution, $X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

see Ayers, page 78, the statement before theorem IV.

(*v*) Since the only solution to $x_1K_1+x_2K_2+x_3K_3=0$ is the trivial solution, by definition, the set of vectors is linearly independent.

15.

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$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \\ 5 & 3 & 1 \end{pmatrix} \xrightarrow{R_4^1(-5)} \begin{matrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \\ 0 & -2 & -4 \end{matrix} \xrightarrow{R_4^1(2)} \begin{matrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \\ 2 & 0 & -2 \end{matrix}$$

(wrong way)

$$\Big\downarrow R_1^2(-1)$$ [This makes another 0 in the first column].

$$\begin{matrix} 0 & -2 & -4 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \\ 0 & -2 & -4 \end{matrix} \xrightarrow{R_4^1(-1)} \begin{matrix} 0 & -2 & -4 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \\ 0 & 0 & 0 \end{matrix} \xrightarrow{R_3^1} \begin{matrix} 1 & 5 & 3 \\ 1 & 3 & 5 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{matrix}$$

$$\xrightarrow{R_2^1(-1)} \begin{matrix} 1 & 5 & 3 \\ 0 & -2 & 2 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{matrix} \xrightarrow{R_3^2(-1)} \begin{matrix} 1 & 5 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{matrix}$$

not possible to make them into zero

$$\xrightarrow[R_2(-1/2)]{R_3(-1/6)} \begin{pmatrix} 1 & 5 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

REF

$$\Big\downarrow R_1^2(-5)$$

$$\begin{matrix} 1 & 0 & 8 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \xrightarrow{R_1^3(-8)} \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \xrightarrow{R_2^3(1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Finally, RREF!

Chapter_10_Linear EquationsQ

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Consider a system of 3 linear equations in 3 unknown, x_1, x_2, x_3.

$$x_1 + x_2 + x_3 = 0$$

$$2x_1 + 2x_2 + 2x_3 = 0$$

$$3x_1 + 3x_2 + 3x_3 = 0$$

(*i*) Give the most trivial solution to the system of equations above. (*ii*) Give a not-most trivial solution to the system of equations above. (*iii*) Is the system consistent? (*iv*) Explain why you say so in (*iii*). (*v*) How many solutions are there for the linear equation system?

2. Consider a system of 2 linear equations in 2 unknown, x_1, x_2: $x_1 + x_2 = 2$, $x_1 + x_2 = 1$

   (*i*) Try sketching the two equations given in the x_2-x_1 plane. (*ii*) Do these two graphs intersect at all? (*iii*) Can you find a solution to the linear equation system given? (*iv*) Is the system consistent? (*v*) Explain why you say so in (*iv*).

3. Express the linear equation system in (1) and (2) in (*i*) matrix form *AX*=*H*. (*ii*) augmented matrix form [*A H*].

4. Consider the following system of equation:

$$x_1 - x_2 = 1 \qquad \text{(Eq. 1)}$$

$$x_1 + x_2 = 2 \qquad \text{(Eq. 2)}$$

(*i*) Express the equation system above into augmented matrix form.
(*ii*) Perform the following operation: Replace (Eq. 1) by (Eq.1) + (Eq. 2), and call the resultant equation (Eq 1’). At the mean time, leave (Eq. 2) untouched. Write down the resultant equation system. What are the corresponding row

elementary operations that have an equivalent effect on the augmented matrix?

(*iii*) Now, perform a follow-up operations: Replace (Eq. 2) by (Eq. 2) added with (Eq 1') multiplied by (-1/2). Write down the resultant equations. What are the corresponding row elementary operations that have an equivalent effect on the augmented matrix?

(*iv*) Now, multiply (Eq. 1') by a factor of 1/2. Write down the resultant equations. What are the corresponding row elementary operations that have an equivalent effect on the augmented matrix?

(*v*) Now, reduce the augmented matrix in (*i*) into RREF. Do you get the same augmented matrix as resulted in (*iv*)? (*vi*) Can you read off the solutions of x_1, x_2 from the resultant RREF matrix by inspection?

5. (*i*) Express the following linear equation system in augmented matrix form [*A H*].

   $$x_1 + x_2 + x_3 = 1;\ \ 2x_1 + x_2 + 3x_3 = 2;$$

   $$3x_1 + x_2 + x_3 = 0.$$

   (*ii*) Reduce the augmented matrix [*A H*] into reduced row echelon form. (*iii*) Write out the equation system represented by the RREF augmented matrix as obtained in (*ii*). Read off the solutions by inspection.

6. Choose the correct answer: (*i*) REF is a (special / general) case of RREF. (*ii*) The procedure you used in (5), in which the augmented matrix is reduced into RREF, to solve for the solutions, is called (Gaussian /Gaussian-Jordan) elimination.

7. The equation system in

(*i*) (1) is a (homogeneous/non-homogeneous) system.

(*ii*) (2) is a (homogeneous/non-homogeneous) system.

(*iii*) (5) is a (homogeneous/non-homogeneous) system.

8. Refer to theorem III, page 77, Ayers. Answer the following properties of the equation system in (5):

   (*i*) The system in (5) is (homogeneous or non-homogeneous).

   (*ii*) The number of unknowns in (5) is _______.

   (*iii*) The number of equation in (5) is _______.

   (*iv*) The determinant of |*A*| in (5) is (zero / non-zero).

(*v*) The solution as obtained in (5) is (unique / not-unique).

10. Given an arbitrary linear equation system of the form $AX=H$, there are two possibility on the existence of its solution, i.e. either the solution exists or __________. If the solution exists, it could be ___________or ____________, ___________ or _____________.

11. Give a simple example of a non-homogeneous system that (*i*) has a unique solution. What is the unique solution? (*ii*) has no solution (i.e. not consistent); (*iii*) has non-unique solutions.

12. Give an example of a 2 by 2 homogeneous system that (*i*) has a unique solution; (*ii*) has no solution (i.e. not consistent); (*iii*) has non-unique solutions.

13. Given a non-homogeneous equation $AX=H$, with $|A| \neq 0$, A an n-square coefficient matrix, X column vector of n variables, H non-zero column vector of n components.
(*i*) List down all the ways you can think of that can be employed to solve the linear equation systems. I can think of 5, how many can you think of?
(*ii*) As an exercise, solve the given equation system using all the methods you have listed. You should get the same answer with all these different methods.

14. Consider an equation system $AX=H$, which represent m equations in n unknown. What is the sufficient condition that this equation system is consistent? (*Hint*: *find the answer in page 76 of Ayers "fundamental theorems."*)

15. Consider a homogeneous equation system $AX=0$, which has n equations in n unknowns.
(*i*) Does the system is guaranteed consistent? Explain by referring to your answer in (14).
(*ii*) What can you say about the solution X if the rank of A, r, is equal the number of unknowns, n? (Give your answer in terms of its existence, uniqueness and triviality.)
(*iii*) What can you say about the solution X if the rank

of A, r, is less than the number of unknowns, n? (Give your answer in terms of its existence, uniqueness and triviality.)

(*iv*) Consider the statement: ALL homogeneous equation system is consistent. Is this statement true?

16. Consider the homogeneous equation system in (15), AX=0. (*i*) If A is not singular, A^{-1} exist. What do you get if you operate A^{-1} on AX=0 from the left-hand-side (LHS)? Try to figure out what will happen to the solution X. (*ii*) So, what is your conclusion?

17. Consider (16) again. (*i*) If A is singular, i.e. A^{-1} does not exist, can you claim that AX=0 has no solution? (*ii*) What's the difference in the solution of a homogeneous equation system of singular coefficient matrix A and one that is not? Give your answer in terms of its existence, uniqueness and triviality.

18. Consider the equation system

$x_1 - 2x_2 + 3x_3 = 4$; $x_1 + x_2 + 2x_3 = 5$. (*i*) Express the system in matrix form. (*ii*) Is the number of unknowns larger than the number of equation? (*iii*) So, how many solutions would you expect? (*iii*) Solve the equation system using Gaussian-Jordan elimination.

19. Refer to solved problems 2, page 79 of Ayers. We will learn how to 'count' the rank of a matrix in this DQ. Consider a homogeneous system of 2 linear equations in 3 unknown, x_1, x_2, x_3, $AX=H$, where

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 4 & 5 & 3 & 0 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad H = \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$$

(i) Transform the augmented matrix [A H] into [A_1 | H_1], the RREF form of [A | H] using row elementary operations. Express [A_1 | H_1] explicitly.

(ii) Is the rank of [A_1 | H_1], [A | H] equal?

(iii) Count the number of non-zero row in [A_1 | H_1]. The number of non-zero row equals the rank of the augmented matrix [A_1 | H_1]. This number is the same as the number of leading 1 in the RREF.

(Note that this simple fact is not mentioned explicitly in Ayers.) Hence, what is the rank of $[A \mid H]$?

(iv) From the expression of $[A_1 \mid H_1]$, what is the rank of the matrix A_1? What is the rank of matrix A?

(v) By referring to the fundamental theorem I, page 76, Ayers, is the system consistent?

20. Consider the homogeneous equation system of

$$x_1 + 2x_2 + 2x_4 = 0 \text{ , } x_2 + 3x_3 = 0 \text{ .}$$

(i) Express the system in the matrix form of *AX=H*.

(ii) This is an equation system with _____equations and _______ unknowns.

(iii) Determine the rank of A and $[A \mid 0]$.

(iv) Is this system consistent?

(v) By comparing the rank of A and the number of unknowns, can you determine whether the system will admit non-trivial solution? State explicitly whether the non-trivial solutions are expected. (*Hint: refer to theorem IV, page 78, Ayers.*)

(vi) From the answer to (*ii*), state your expectation whether the solutions will be unique or otherwise.

(vii) Based on your answer to (*v*) you know what the rank of the matrix A is. Hence, how many linearly independent solutions do you expect for the HE system? (*Hint: refer to theorem VI, page 78, Ayers.*)

21. Solve the HE system in the previous DQ. Your solution should agree with the number of linearly independent solutions is as given in (*vi*) in the same question.

22. You may like to refer to Example 5, page 79, Ayers. Consider a non homogeneous equation system:

$$x_1 - 2x_2 + 3x_3 = 4 \text{ , } \quad x_1 + x_2 + 2x_3 = 5 \text{ .}$$

(i) Express the system in the matrix form of *AX=H*.

(ii) This is an equation system with _____equations and _______ unknowns.

(iii) Determine the rank of $[A \mid H]$ and A.

(iv) Is this system consistent?

(v) From the answer to (*ii*), state your expectation whether the solutions will be unique or otherwise.

(vi) Based on your answer to (*iii*) you know what

the rank of the matrix A is. Hence, how many linearly independent solutions do you expect for the HE system? (*Hint: refer to theorem VI, page 78, Ayers.*)

(vii) Obtain the solution.

23. You may like to try out the trick you have learnt from the above DQ on the solved problem 1 in page 79, Ayers.

1. **Definition:** n-vector
   An vector $\boldsymbol{a}$ with n-component is an n-tuple of real numbers, $\boldsymbol{a} = \{a_1, a_2, \ldots, a_n\}$. We call this an n-vector. a_i, i=1, 2, …,n are the components of $\boldsymbol{a}$. It has n components.

2. As an special example, for n=3, $\boldsymbol{a}=\{a_1, a_2, a_3\}$. $\boldsymbol{a}$ can be imagined as a point in 3-space, the 3-dimensional space we human resides in. For example, the 3-vector $\boldsymbol{a}=\{0,0,0\}$ represents a point with spatial coordinates {0 , 0, 0}.

3. Imagine the collection of all possible 3-vectors into a set V containing all points in the 3-space. We call the set of all 3-vectors, (or in other words, all points in the 3-D space), R^3. Each vector in R^3 is equivalent to a point in the 3-space.

4. Similarly, R^2 is the set of all 2-vectors. R^2 is the set of all points in 2-space.

5. R, the set of all real number, is the set of all ‘1-vector’ (‘1-vector’ is just the real scalar we all familiar with). The collection of all ‘points’ in the 1-space is equivalent to the set of all points in a 1-dimensional ‘real-number line’.

6. For the 2-vectors and 3-vectors, we know that we can add and do scalar multiplication on them according to well-defined rules of vector addition and scalar multiplication. As an illustration, consider this: Given two 3-vectors in R^3, $\boldsymbol{a}=\{a_1, a_2, a_3\}$ and $\boldsymbol{b}=\{b_1, b_2, b_3\}$, the vector addition $\boldsymbol{a} + \boldsymbol{b}$ is defined as a new 3-vector, $\boldsymbol{c} = \{a_1+b_1, a_2+b_2, a_3+b_3\}$. Similarly, the scalar multiplication between a scalar k and a vector $\boldsymbol{a}$ is defined as a new vector $\boldsymbol{d}=\{ka_1, ka_2, ka_3\}$.

7. **Definition:** Consider a set V containing some elements on which operations of **vector addition** and **scalar multiplication** are defined. The set V is called a **vector space** if the following ten properties are satisfied:

DEFINITION 7.5 Vector Space

Let V be a set of elements on which two operations called **vector addition** and **scalar multiplication** are defined. Then V is said to be a **vector space** if the following ten properties are satisfied.

Axioms for Vector Addition

(*i*) If $\mathbf{x}$ and $\mathbf{y}$ are in V, then $\mathbf{x}+\mathbf{y}$ is in V.
(*ii*) For all $\mathbf{x}, \mathbf{y}$ in V, $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$. (commutative law)
(*iii*) For all $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in V, $\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}$. (associative law)
(*iv*) There is a unique vector $\mathbf{0}$ in V such that $\mathbf{0}+\mathbf{x}=\mathbf{x}+\mathbf{0}=\mathbf{0}$. (zero vector)
(*v*) For each $\mathbf{x}$ in V, there exists a vector $-\mathbf{x}$ such that $\mathbf{x}+(-\mathbf{x})=(-\mathbf{x})+\mathbf{x}=\mathbf{0}$. (negative of a vector)

Axioms for Scalar Multiplication

(*vi*) If k is any scalar and $\mathbf{x}$ is in V, then $k\mathbf{x}$ is in V.
(*vii*) $k(\mathbf{x}+\mathbf{y})=k\mathbf{x}+k\mathbf{y}$
(*viii*) $(k_1+k_2)\mathbf{x}=k_1\mathbf{x}+k_2\mathbf{x}$ (distributive laws)
(*ix*) $k_1(k_2\mathbf{x})=(k_1k_2)\mathbf{x}$
(*x*) $1\mathbf{x}=\mathbf{x}$

8. Consider the 3-space, R^3. As mentioned, this a vector space. Can you justify this claim by referring to the definition as given?

Ans:

This is a vector space because (*i*) vector addition and scalar multiplication are well defined on all of the 3-vectors, the elements in R^3, (*ii*) all of the 3-vectors, the elements in R^3, fulfill the 10 axioms. In particular, all 3-vectors are closed under vector addition and closed under scalar multiplication.

9. Explain what do you understand by (*i*) ‘closure under vector addition’. (*ii*) ‘closure under scalar multiplication’.

10. Consider R^2. Is it also a vector space? How about the set of all real number, the 1-space, R? How do you convince yourself that they are indeed also vector space?

11. **Definition:** A set of vectors V_s from a vector space V is a **subspace** of V if V_s is closed under addition and scalar multiplication.

Example: The set containing only the element 0, $V_s=\{0\}$, is a subspace of the vector space R, since the $\{0\}$ is

(i) a element vector from R,

(ii) closed under scalar multiplication:

$k \cdot 0 = 0 \in V_s$,

(iii) closed under vector addition:

$0 + 0 = 0 \in V_s$.

Note that the subspace $\{0\}$ has only a single element. The criteria of being closed under addition are fulfilled: “if x and y are element is V_s, then $x + y$ is also an element in V_s”. Here, x=0, y=0, because there is no any other element in V_s other than 0. In other words, ‘any element’ in {0} (the x), when vectorially added to ‘any element’ in {0} (the y) will result in $x+ y = 0$, an element of V_s.

12. Every vector space V has at least two subspaces. One of if is the zero subspace, {**0**}, which is illustrated above. Can you think of what’s the other one?

13. **Definition:** Consider a set S containing vectors $\mathbf{x}_1$, $\mathbf{x}_2$, …$\mathbf{x}_m$ in a vector space V. (To help you visualize better, think of V as the vector space of R^3 that contains an infinite number of 3-vectors. Think of S as a set containing, say, m=3 vectors selected from R^3.) We form linear combinations of these m vectors in the form of $k_1\mathbf{x}_1+ k_2\mathbf{x}_2+ \dots k_m\mathbf{x}_m$, where k_i are scalars. The set of all linear combinations of the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, …$\mathbf{x}_m$ is called the **span** of the vectors, and is written as **Span(*S*)**.

14. Span(S) is a subspace of V. Span(S) is said to be a subspace spanned by the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, …$\mathbf{x}_n$.

15. If every vector in the vector space V can be written as a linear combination of the vectors in S, then S is called a **spanning set** for V.

Example: Let V be the vector space containing all 3-vectors, R^3. Consider the set S={**i**, **j**, **k**} containing the three rectangular unit vectors. The set of all linear combination $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, where a, b, c are scalar, is the span of the vectors {**i**, **j**, **k**}, Span(S). Span(S) is a subspace in R^3 spanned by **i**, **j**, **k**.

16. We say ‘the set S={**i**, **j**, **k**} is a spanning set for R^3’. Think of Span(S) in terms of the set of all possible linear combination in terms of **i**, **j**, **k**, $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$. Can you imagine what does Span(S) represent? *Hint*: Imagine the point at the tip of the 3-vector $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$. Imagine the pervasive cloud form by the tip of $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ when a, b, c vary continuously.

17. Can you think of any other spanning set for R^3?

Ans: e.g. {**r**, θ, ϕ}.

18. Is the set {**i**, **j**, **k**, **i**+**j**, **k**+**i**, **i** + **k**} also spanning set for R^3?

19. Is {**i**, **j**} a spanning set for R^3? Explain your answer.

20. Consider the set S containing the following 4 3-vectors: $K_1=[1,1,1]^T$, $K_2=[1,3,5]^T$, $K_3=[1,5,3]^T$, $K_4=[5,3,1]^T$; $S=\{K_1, K_1, K_3\ K_4\}$. How would you prove that the S is the spanning set of R^3 (or in other words, S span R^3)? *Hint:* To prove that the set of vectors in S span R^3, one needs to prove the existence of the solution

$X=[x_1, x_2, x_3, x_4]^T$ for the non-homogeneous equation

system $A = x_1K_1 + x_2K_2 + x_3K_3 + x_4K_4$, given an

arbitrary 3-vector $A=(a,b,c)^T$ from R^3**.** If the solution X

exists, then S spans R^3, otherwise it doesn't. The reasoning is: If the solution X exists, this means that any arbitrary vector A from R^3 can always be expressed as a unique linear combination in the form

of $x_1K_1 + x_2K_2 + x_3K_3 + x_4K_4$. Hence, by definition, if

the set of vectors in S is a spanning set of R^3.

21. In general, given a set of m n-vectors, $K_i=(k_1, k_2,\ldots, k_n)^T$, $i=1,2,\ldots m$, we can determines whether they span a vector space R^n, the vector space containing the set of all n-vector by looking for the existence proof of solution X to the non-homogeneous system. The procedure is as followed:

22. Let $K=(K_1, K_2,..., K_m)$, an n by m matrix,

$X=(x_1, x_2,..., x_m)^T$, an m by 1 column vector,

$A=(a_1, a_2,..., a_n)^T$, an arbitrary n-vector in R^n. Consider

the NH system $A = x_1K_1 + x_2K_2 + \cdots + x_mK_m = KX$. If

the solution for the NH systems does not exist, i.e. rank$[K] \neq$ rank $[K \mid A]$, then the set of vectors K_i does not span R^n. Otherwise, they do.

23. In (20), we see that the set $\{K_1, K_2, K_3, K_4\}$ comprises of 4 3-vector spans R^3. Can we span R^3 with less than 4 3-vectors (e.g., say, 3 or even 2 3-vectors)? In general, for a vector space V containing elements made up of n-vectors, we want to know what is the smallest number of linearly independent n-vector that spans the vector space V.

24. In fact, out of the four 3-vectors in the set S in (20), only three are linearly independent (refer DQ 13, Chapter 9), namely K_1, K_2, K_3, whereas K_4 can be expressed as a linear combination of the other three vectors.

    (*i*) Prove the linearly independence of the vector set K_1, K_2, K_3. (*Hint:* Refer to DQ 12, 13, 14 in Chapter 9.)

    (*ii*) Prove, using the procedure mentioned in (22) above, that this set of vectors K_1, K_2, K_3 spans R^3.

25. Now, we ask: can any of the 2 vectors (which are necessarily linearly independent) form the set $\{K_1, K_2, K_3\}$ span R^3? The answer can be proven to be negative. (Prove this). So, it appears that the minimum number of linearly independent 3-vectors to span R^3 is 3, not 2.

26. **Definition:** The minimum number of linearly independent vectors that is required to span a vector space is called the **dimension** of the vector space. In the above example, the dimension of the vector space R^3 is 3 since the minimum number of linearly independent vectors in R^3 is 3.

27. **Definition:** Consider a vector space V with dimension r. A set of r linearly independent vectors in V is called the **basis** (or basis set) of the vector space. It happens that given any set of r vectors, which are linearly independent, from V, they (*i*) will form a basis set for V, and (*ii*) any vector in V can be expressed as a unique linear combination in this set of r vectors.

28. Let's consider the vector space R^3. We know that the dimension of it is r=3.

(i) If I simply pick any three vectors in R^3, say $X_1 = (a, b, c)$, $X_2 = (d, e, f)$, $X_3 = (g, h, i)$, in general, will the set $\{X_1, X_2, X_3\}$ form a basis for R^3?

(ii) Is the basis set of R^3 unique?

(iii) How many basis set can R^3 possibly has?

29. Consider the set of three vectors in R^3, $S=\{E_1, E_2, E_3\}$, where $E_1= [1, 0, 0]^T$, $E_2= [0, 1, 0]^T$, $E_3= [0, 0, 1]^T$.

(i) Are the vectors in S linearly independent (you should be able to answer this simply question by visual inspection)?

(ii) Do the vectors in the set S form a basis set for R^3?

(iii) Do the vectors in the set S span R^3?

(iv) Can every vector in R^3 be expressed as linear combination of E_1, E_2, E_3?

(v) What's the name of these E-vectors? (*Hint: see page 88 of Ayers).* (Note: we will refer this basis set by the name 'the E-basis').

30. You may like to refer to Ayers page 88. Say I have an arbitrary vector in R^3, $X=(a, b, c)^T$.

(i) Write X as a linear combination of the unit vectors, E_i, defined in (30).

(ii) What are the components (or referred to as 'coordinates') of X relative to the E-basis? Write these components in the form of a column vector and call it 'the component vector of X relative to the E-basis', denoted by X_E.

31. In the previous question, we have an arbitrary vector in R^3, X. Let's say that the vector X when expressed in the E-basis is represented by the component vectors $X_E=(1, 2, 3)^T$. Normally, a vector is by default expressed in the E-basis. In general, other than the E-basis, we can also represent a vector in other basis set. To illustrate this point, let's consider another basis set $Z = \{Z_1, Z_2, Z_3\}$ ('the Z-basis'), where $Z_1= [2, -1, 3]^T$, $Z_2= [1, 2, -1]^T$, $Z_3= [1, -1, -1]^T$. What is the component vector of X relative to the Z-basis, X_Z? [*Hint:* In order to obtain X_Z, you need

to express X as a linear combination of $\{Z_1, Z_2, Z_3\}$: $X_E = a_1Z_1 + a_2Z_2 + a_3Z_3$. Then the component vector of X in the Z-basis is simply $X_Z = (a_1, a_2, a_3)^T$.]

32. Refer to Example 5, page 88 Ayers. Now, see if you can do things another way round: If the component vector of X is given in the Z representation, i.e. $X_Z = (1,2,3)^T$ is known. What is component vector of X in the E-basis? In other words, what is X_E? *Hint*: Follow the procedure as described in (32), then try to find a similar relation that relates X_E to X_Z in the form of

$$X_E = [\text{some matrix}] \cdot X_Z$$

Linear transformation and the change of basis

Abstract

This short note supplements the Linear algebra part of ZCA 110. In particular it discusses in understandable language (i) the idea of linear transformation involving different bases, as discussed in Chapter 12, Ayers, and (i) the idea of bases and coordinates, page 88-89, Ayers.

1 Going from one basis to another

Consider a generic $n-$vector, X, in $V_n(R)$. The vector X can be represented in different basis (as a simile: think of the appearance of an actor viewed through different coloured glasses by different autdience. Despite it is the same actor, to different audience the actor appears differently.) As an illustration, we will discuss how the vector be represented in two different basis. Let's agree to call these two generic basis the $W-$basis and the $Z-$basis.

The W-basis consists of a set of n $n-$vector, namely $\{W_1, W_2, \cdots, W_n\}$, where each of the W_i, $i = 1, 2, \cdots n$ is an n by 1 column vector, $W_i = (w_1, w_2, \cdots, w_n)^T$. Similarly, the $Z-$basis consists of the set $\{Z_1, Z_2, \cdots, Z_n\}$, $Z_i = (z_1, z_2, \cdots, z_n)^T$, $i = 1, 2, \cdots n$. The connection between the two bases can be worked out via the following consideration:

In the $W-$basis, the vector X presents itself as a linear combination in terms of W_i's, i.e.

$$X = a_1 W_1 + a_2 W_2 + \cdots a_n W_n. \tag{1}$$

a_i are scalars called the components of X in the $W-$basis. By definition, the components vector of X in the $W-$ basis is the column vector that contains all of the components, or coordinate, of vector X in the $W-$basis. It is denoted by $X_W = (a_1, a_2, \cdots, a_n)^T$. Let us arrange all of the basis vector W_i column-by-colum into the matrix $W = (W_1, W_2, \cdots, W_n)$, where W is a $n \times n$ matrix. Now, Eq. (1) can be compactly written in the form of

$$X = a_1 W_1 + a_2 W_2 + \cdots a_n W_n = (W_1, W_2, W_3, \cdots, W_n)(a_1, a_2, \cdots, a_n)^T = W \cdot X_W. \tag{2}$$

X_W is the coordinate vector of X relative to the W-basis.

Similarly, if the vector X were to be represented in the $Z-$basis,

$$X = b_1 Z_1 + b_2 Z_2 + \cdots b_n Z_n = (Z_1, Z_2, Z_3, \cdots, Z_n)(b_1, b_2, \cdots, b_n)^T = Z \cdot X_Z. \tag{3}$$

$X_Z = (b_1, b_2, \cdots, b_n)^T$ is the coordinate vector of X relative to the W-basis.

The vector X is the same vector irrespective of its basis representation, hence

$$X \quad = \quad W \cdot X_W = Z \cdot X_Z \tag{4}$$

Eq. (4) relates the coordinate vector of vector X represented in the Z-basis to that in the W-basis. The coodinate vector in one basis can be determined if the coordinate vector in the other is known, and vice versa.

For example, if we know X_W, we can determine X_Z by making use of Eq. (4): We form the matrix

$$P = Z^{-1} \cdot W, \tag{5}$$

operate it to X_W from the left to obtain

$$X_Z = P \cdot X_W = (Z^{-1} \cdot W) \cdot X_W. \tag{6}$$

Conversely, we can obtain X_W if X_Z is known via

$$X_W = P^{-1} X_Z. \tag{7}$$

2 Linear Transformation

In simple language, a transformation is an operation that operates onto a vector to make it into another vector. Normally the operation is realised via matrix multiplication. Say X is a vector to be transformed into anoter vector, call it Y. To implement the transformation, we will operate a matrix A onto X to make it into Y. Symbolically, $X \overset{A}{\rightarrow} Y$; operationally, $Y = AX$. The transformation matrix A contains the information (instruction) of how the vector is to be transformed (e.g. to rotate X about the origin by $+90$ degree, to reflect the vector X about the origin, etc.). Y, the resultant vector under the transformation, is called the image of X under transformation A.

A transformation can be carried out in any basis. Consider a vector X is tranformed into vector Y. Such a transformation can be represented in both the W-basis and the Z-basis. In each basis the transformation takes on different forms. Say A is the transformation matrix in the W-basis representation, whereas B is the correspoinding transformation in the Z-basis representation. The linear transformations in both bases are given by:

W-basis	Z-basis
$W = (W_1, W_2, \cdots, W_n)$	$Z = (Z_1, Z_2, \cdots, Z_n)$
$X \overset{A}{\rightarrow} Y$	$X \overset{B}{\rightarrow} Y$
In component form	In component form
$X_W \overset{A}{\rightarrow} Y_W$	$X_Z \overset{B}{\rightarrow} Y_Z$
In matrix form	In matrix form
$Y_W = AX_W$	$Y_Z = BX_Z$

Question: What is the definition of 'linear transformation'? (*Hint:* Refer to Ayers, page 94.)

Now, we shall prove that: If

$$\begin{aligned} X_W &\overset{A}{\rightarrow} Y_W \text{ in the } W\text{-basis} \\ X_Z &\overset{B}{\rightarrow} Y_Z \text{ in the } Z\text{-basis,} \end{aligned}$$

then the two transformation A, B are similar, i.e.,

$$B = Q^{-1}AQ,$$

where $Q = P^{-1} = (Z^{-1}W)^{-1} = W^{-1}Z$.

Question: What does it mean, mathematically, when it is said that matrix A and matrix B are 'similar'?

The proof is as followed: We begin with

$$Y_Z = BX_Z. \tag{8}$$

In Eq. (8), the LHS, i.e. Y_Z is related to Y_W via $Y_Z = PY_W$ [see Eq. (6)], whereas X_Z in the RHS is related to X_W via $X_Z = PX_W$ [see Eq. (7)]. Hence, Eq. (8) can be written as

$$\begin{aligned} & PY_W = B(PX_W) \\ \Rightarrow \quad & Y_W = (P^{-1}BP)X_W. \end{aligned} \tag{9}$$

Eq. (9) is just the tranformation of X_W into Y_W by A (in the W-basis), i.e.

$$Y_W = (P^{-1}BP)X_W \equiv AX_W. \tag{10}$$

Hence, we can identify

$$A = P^{-1}BP,$$

or

$$B = PAP^{-1} = Q^{-1}AQ,$$

where P is given by Eq. (5).

Questions: (*i*) Attempt example 2, page 96, Ayers, yourself. (*ii*) Attempt Solved Problem 1, in the same page, yourself. Try to make yourself proud by solving these problems without reading the solutions.