Lecture notes

ZCA 110/4 (for Class B)

Calculus and Linear Algebra

This set of lecture note is prepared by Assoc. Prof. Dr. Rosy Teh Chooi Gim, who is in charged of ZCA 110 (for class A).

The lecturer in charged of ZCA 110 (for Class B) is Yoon Tiem Leong. Both classes (A and B) will adopt the same set of lecture notes.

Course webpage:
http://www2.fizik.usm.my/tlyoon/teaching/ZCA110_0708/

TEXT BOOK:

1. Schaum's Outline of Theory and Problems of Matrices SI (Metric) Edition by Frank Ayres, McGraw-Hill (1974).

2. Thomas's Calculus, Tenth Edition, by George B. Thomas, Ross L. Finney, Maurice D. Weir, and Frank R. Giordano, Addison-Wesley Publishing.(2001).

REFERENCE:

1. S.L. Salas, E. Hille, and G.J. Etgen, Calculus, John Wiley and Sons, New York, 9th Edition, 2003, John Wiley and Sons.
2. Edwards and Penny, Calculus, 6th Edition, 2002, Prentice Hall.
3. Gerald L. Bradley and Karl J. Smith, Calculus, 2nd Edition, 1999, Prentice Hall.
4. Seymour Lipschutz and Marc Lipson, Schaums Outlines, Linear Algebra, 3rd Edition, 2001, McGraw-Hill.
5. Introductory Linear Algebra with Application by Bernard Kolman and David R. Hill, 7th Edition, 2001, Prentice Hall.

Sequence of lectures to be covered is as followed:

LINEAR ALGEBRA (WEEK 1 – WEEK 3)

3 Lectures
1 Matrix
1.1 Matrix Algebra
1.2 Type of Matrices:
Identity Matrix, Special Square Matrices, Inverse of a Matrix, Transpose of a Matrix, Symmetric Matrices, Conjugate of a Matrix, Hermitian Matrices, Direct Sums.

2 Lectures
1.3 Determinant of a Square Matrix:
Determinants of orders 2 and 3, Properties of Determinants, Minors and Cofactors, Adjoint of a Square Matrix, Evaluation of Determinant, The Inverse of a Matrix, Elementary Transformation.

2 Lectures
1.4 System of Linear Equations:
Solution using a Matrix, Fundamental Theorems, Homogeneous Equations.

2 Lectures
2 Vector Spaces
2.1 Vector Spaces: Subspace
2.2 Basis and Dimension: Basis and Coordinate 2.3 Linear Transformation: Definition, Basic Theorems, Change of Basis

CALCULUS (WEEK 4 – WEEK 14)

2 Lectures
Preliminaries
0.1 Functions and Graphs
0.2 Exponential Functions
0.3 Inverse Functions and Logarithms:
One-to-One Functions, Inverses, Logarithm Functions
0.4 Trigonometric Functions and Their Inverses

2 Lectures
1. Limits and Continuity
1.1 Rates of Change and Limits
1.2 Limits and One-Sided Limits:
Properties of Limits, One-Sided Limits

2 Lectures
1.3 Limits Involving Infinity
1.4 Continuity

2 Lectures
2 Derivatives
2.1 The Derivative as a Function
2.2 The Derivative as a Rate of Change
2.3 Derivatives of Products, Quotients, and Negative Powers
2.4 Derivatives of Trigonometric Functions
2.5 The Chain Rule

2 Lectures
2.6 Implicit Differentiation
3 Applications of the Derivatives
3.1 Extreme Values of Functions

4 Lectures
3.2 The Mean Value Theorem
3.3 The Shape of a Graph
3.4 Optimization
4 Integration
4.1 Indefinite Integrals
4.2 Integration by Substitution
4.3 Definite Integrals
4.4 The Mean Value and Fundamental Theorems

3 Lectures
4.5 Substitution in Indefinite Integrals
5 Applications of Integrals
5.1 Length of Plane Curves
6 Transcendental Functions
6.1 Logarithms
6.2 Exponential Functions
6.3 Derivatives

3 Lectures
6.4 Hyperbolic Functions
7 Integration Techniques
7.1 Basic Integration Formulas
7.2 Integration by Parts
7.3 Partial Fractions

<u>2 Lectures</u>
7.4 Trigonometric Substitutions
7.5 Integral Tables
7.6 L' Hˆopital's Rule

<u>1 Lecture</u>
7.7 Improper Integrals

<u>3 Lectures</u>
8 Infinite Series
8.1 Limits of Sequences of Numbers
8.2 Bounded Sequences
8.3 Infinite Series Geometric Series
8.4 Series of Nonnegtive Terms

<u>1 Lecture</u>
8.5 Alternating Series
8.6 Power Series

<u>2 Lectures</u>
8.6 Power Series
8.7 Taylor and Maclaurin Series
8.8 Binomial Series

<u>2 Lectures</u>
8.9 Fourier Series
8.10 Fourier Cosine and Sine Series

<u>1 Lecture</u>
A Set
A.1 Set
A.2 Real Numbers
B Complex Numbers

Chapter 1

Matrices

1.1 Algebra of Matrices

A matrix is a rectangular array of numbers.

Example 1.1

$$C = \begin{bmatrix} 7 & 3 & 2 \\ 1 & 5 & -1 \end{bmatrix} \tag{1.1}$$

C is a 2×3 matrix (2 rows and 3 columns).

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In general, an $m \times n$ **matrix** A, is a rectangular array of numbers with m rows and n columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \tag{1.2}$$

where a_{ij} is an **element** of the matrix A, with $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. The element can be a number or a function. The matrix A is then of **order** $m \times n$ and can also be written as

$$A = [a_{ij}]. \tag{1.3}$$

Example 1.2

$$\text{The matrix} \qquad C = \begin{bmatrix} 7 & 3 & 2 \\ 1 & 5 & -1 \end{bmatrix},$$

can be the **coefficient matrix** of a system of homogeneous linear equations

$$7x + 3y + 2z = 0$$

$$x + 5y - z = 0 \tag{1.4}$$

or the **augmented matrix** of a system of non-homogeneous linear equations

$$\begin{aligned} 7x + 3y &= 2 \\ x + 5y &= -1. \end{aligned} \tag{1.5}$$

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Square Matrix

When $m = n$, matrix A from Eq.(1.2) becomes a **square matrix** of order n, or a n-square matrix. The elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are the **diagonal elements**. The sum of the diagonal elements of a matrix A is the **trace of A**.

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are the **same** if and only if (iff) they have the same order and

$$a_{ij} = b_{ij}, \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n. \tag{1.6}$$

A matrix where all its elements are zero is a **zero matrix** .

Sum of Matrices

If two matrices, $A = [a_{ij}]$ and $B = [b_{ij}]$, are matrices of order $m \times n$, then their sum is given by

$$\begin{aligned} A + B &= [a_{ij}] + [b_{ij}] \\ &= [a_{ij} + b_{ij}] \end{aligned} \tag{1.7}$$

and their **difference** is given by

$$\begin{aligned} A - B &= [a_{ij}] - [b_{ij}] \\ &= [a_{ij} - b_{ij}]. \end{aligned} \tag{1.8}$$

The matrices A and B are **conformable** to addition and subtraction if they are of the same order, that is $m \times n$.

Example 1.3

$$\text{Given} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 5 & 6 \\ 0 & 3 & 2 \end{bmatrix}.$$

$$\text{Then} \quad A+B=\begin{bmatrix} 0 & 7 & 9 \\ 4 & 8 & 8 \end{bmatrix}, \quad A-B=\begin{bmatrix} 2 & -3 & -3 \\ 4 & 2 & 4 \end{bmatrix}. \tag{1.9}$$

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Multiplication of Scalar with Matrices

If k is any scalar and the matrix $A=[a_{ij}]$, then

$$\begin{aligned} kA &= k[a_{ij}]=[ka_{ij}]=[a_{ij}k] \\ &= [a_{ij}]k=Ak. \end{aligned} \tag{1.10}$$

Example 1.4

$$k=3, \quad A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix},$$

$$kA=3\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}. \tag{1.11}$$

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The negative of a matrix A is $-A$ and it is the same as -1 times with A. $A+(-A)=0 \equiv$ a zero matrix of the same order as A.

If the matrices A, B, and C are conformable to addition, then:

$$A+B = B+A, \quad \text{Commutative Law} \tag{1.12}$$

$$A+(B+C) = (A+B)+C, \quad \text{Associative Law} \tag{1.13}$$

$$k(A+B) = kA+kB=(A+B)k, \quad k=\text{scalar}. \tag{1.14}$$

Hence conformable matrices obey the same addition laws as the elements of these matrices.

Multiplication of Matrices

If A is a $1\times m$ matrix and B a $m\times 1$ matrix, then,

$$A=\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1m} \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}$$

$$\Longrightarrow \quad AB=a_{11}b_{11}+a_{12}b_{21}+\ldots+a_{1m}b_{m1}=C \tag{1.15}$$

where C is a 1×1 matrix, that is, a number. The product operation is **row by column**.

If $A = [a_{ij}]$ is a $m \times p$ matrix and $B = [b_{ij}]$ is a $p \times n$ matrix, then the product AB in that order is $AB = C = [c_{ij}]$ is a $m \times n$ matrix,

$$\begin{aligned} \text{where} \quad c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj} \\ &= \sum_{k=1}^{p} a_{ik}b_{kj}, \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n. \end{aligned} \tag{1.16}$$

NOTE: Each row of A is multiplied once and only once into each column of B. The elemen c_{ij} is a product of the ith row of matrix A and the jth column of matrix B.

Example 1.5

If

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

then

$$\begin{aligned} C &= AB = \begin{bmatrix} 1\times 4+2\times 1 & 1\times 5+2\times 2 & 1\times 6+2\times 3 \\ 3\times 4+4\times 1 & 3\times 5+4\times 2 & 3\times 6+4\times 3 \\ 5\times 4+6\times 1 & 5\times 5+6\times 2 & 5\times 6+6\times 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 9 & 12 \\ 16 & 23 & 30 \\ 26 & 37 & 48 \end{bmatrix}. \end{aligned} \tag{1.17}$$

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If the product AB is defined, then A is conformable to B for multiplication. Hence the number of colmns of A is equal to the number of rows of B. However it is not necessarily true that BA must be defined. Assuming that A, B, and C are conformable to the below multiplication and addition, then we have

$$A(B+C) = AB + AC, \quad \text{First Distributive Law} \tag{1.18}$$

$$(A+B)C = AC + BC, \quad \text{Second Distributive Law} \tag{1.19}$$

$$A(BC) = (AB)C, \quad \text{Associative Law.} \tag{1.20}$$

However,

(a) $AB \neq BA$, in general, (1.21)

(b) $AB = 0$, does not necessarily imply that $A = 0$ or $B = 0$, (1.22)

(c) $AB = AC$, does not necessarily imply that $B = C$. (1.23)

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<u>Example 1.6</u>

$$\text{Given} \quad A = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$AB = 1 \times (-2) + 2 \times 1 = -2 + 2 = 0$$

$$BA = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}.$$

Hence $AB = 0$ does not necessariliy mean that $A = 0$ or $B = 0$. Also $AB \neq BA$ in this case.

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<u>Example 1.7</u>

$$\text{Given} \quad A = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad , \quad B = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$AC = 1 \times 2 + 2 \times (-1) = 0, \quad AB = 0 \quad \text{(from Example 1.6)}$$

$$AB = AC = 0 \quad \text{but} \quad B \neq C.$$

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1.2 Types of Matrices

1.2.1 Identity Matrix

A square matrix with elements $a_{ij} = 0$ for $i > j$ is called an **upper triangular matrix** and a matrix A with elements $a_{ij} = 0$ for $i < j$ is called a **lower triangular matrix**. Hence,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ 0 & a_{22} & a_{23} & \ldots & a_{2n} \\ 0 & 0 & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & a_{nn} \end{bmatrix} \quad \text{is an upper triangular matrix,} \qquad (1.24)$$

and

$$B = \begin{bmatrix} b_{11} & 0 & 0 & \ldots & 0 \\ b_{21} & b_{22} & 0 & \ldots & 0 \\ b_{31} & b_{32} & b_{33} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \ldots & b_{nn} \end{bmatrix} \quad \text{is a lower triangular matrix.} \qquad (1.25)$$

If a matrix C is both upper and lower triangular, then C is a **diagonal matrix**.

Hence,

$$C = \begin{bmatrix} c_{11} & 0 & 0 & \dots & 0 \\ 0 & c_{22} & 0 & \dots & 0 \\ 0 & 0 & c_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & c_{nn} \end{bmatrix} \quad \text{is a diagonal matrix.} \tag{1.26}$$

C can also be written as

$$C = \text{diag}(c_{11}, c_{22}, c_{33}, \dots\dots, c_{nn}).$$

If the diagonal matrix C possesses elements,

$$c_{11} = c_{22} = c_{33} = \dots\dots = c_{nn} = k,$$

C is called a **scalar matrix**. If $k = 1$, matrix C is called an **identity matrix**, and the symbol used is I_n or I only.

Example 1.8

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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For p terms,

$$\underbrace{I_n + I_n + \dots + I_n}_{p \text{ terms}} = pI_n = \text{diag}(p, p, \dots, p) \tag{1.27}$$

$$I^p = I. \tag{1.28}$$

If A is 2×3 matrix, then,

$$I_2 A = A I_3 = I_2 A I_3 = A. \tag{1.29}$$

1.2.2 Special Square Matrices

If A and B are square matrices and

$$AB = BA, \tag{1.30}$$

then A and B are **commutative** or A commute with B. If A is a n-square matrix, then

$$AI_n = I_n A. \tag{1.31}$$

Matrices A and B are **anti-commutative**, if

$$AB = -BA. \tag{1.32}$$

If $A^{k+1} = A$ where k is a positive integer, then the matrix A is said to be **periodic**. If k is the smallest possible positive integer where $A^{k+1} = A$, then the matrix A has **period k**. If $k = 1$, then $A^2 = A$, and the matrix A is **idempotent**. An example of an idempotent matrix is

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}.$$

If $A^p = 0$, where p is a positive integer, then A is **nilpotent**. If p is the smallest positive integer where $A^p = 0$, then A is said to be **nilpotent of index p**.

1.2.3 The Inverse of a Matrix

If A and B are square matrices and $AB = BA = I$, then B is the **inverse** of matrix A, and we can write, $B = A^{-1}$. Also A is the inverse of matrix B and $A = B^{-1}$.

Example 1.9

Please refer to section 1.3.7 for the definition of elementary transformations.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{then} \quad B = A^{-1}.$$

$$[A \;\; I_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2^1(-1)} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1^2(-2)} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right] = [I_2 \;\; A^{-1}].$$

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Example 1.10

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1/2 & 0 & 0 \\ -1/4 & 0 & 1/2 \\ 1/8 & 1/2 & -1/4 \end{bmatrix},$$

$$AB = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ -1/4 & 0 & 1/2 \\ 1/8 & 1/2 & -1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$[A \quad I_3] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1(1/2)} \xrightarrow{R_3^2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2^1(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 2 & 0 & -1/2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2(1/2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/4 & 0 & 1/2 \\ 0 & 1 & 2 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3^2(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/4 & 0 & 1/2 \\ 0 & 0 & 2 & 1/4 & 1 & -1/2 \end{array}\right]$$

$$\xrightarrow{R_3(1/2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/4 & 0 & 1/2 \\ 0 & 0 & 1 & 1/8 & 1/2 & -1/4 \end{array}\right] = [I_3 \quad A^{-1}].$$

****

Not all matrices possess an inverse. If the matrix A possesses an inverse, then the inverse is unique. If A and B are square matrices of the same order with inverse A^{-1} and B^{-1} respectively, then

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{1.33}$$

If matrix A is such that, $A^2 = I$, then A is said to be **involutary**. Hence involutary matrices are matrices of period 2. I is involutary as $I^2 = I$ and an involutary matrix is the inverse of itself, that is, $A^{-1} = A$.

If a matrix A has a zero row or column, then A is **singular**. A singular matrix does not possess an inverse.

1.2.4 The Transpose of a Matrix

A matrix of order $n \times m$ that is obtained by interchanging the rows and columns of a matrix A of order $m \times n$, is called the **transpose** of A and is denoted by A^T or A transpose.

Example 1.11

$$\text{Given} \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \text{then,} \quad A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}.$$

****

If k is a scalar, then,

$$(A^T)^T = A \tag{1.34}$$

$$(kA)^T = kA^T \tag{1.35}$$

$$(A+B)^T = A^T + B^T \tag{1.36}$$

$$(AB)^T = B^T A^T \tag{1.37}$$

$$A = [a_{ij}], \quad A^T = [a_{ij}^T] = [a_{ji}]. \tag{1.38}$$

1.2.5 Symmetric Matrices

A square matrix A with the property, $A^T = A$, is said to be **symmetric**. So if $A = [a_{ij}]$, then,

$$[a_{ij}^T] = [a_{ji}] = [a_{ij}], \quad \forall \ i, \ j. \tag{1.39}$$

Example 1.12

$$\text{The matrix,} \quad A = \begin{bmatrix} 1 & 6 & 7 \\ 6 & 2 & 5 \\ 7 & 5 & 3 \end{bmatrix} \quad \text{is symmetric.}$$

****

If A is symmetric, then kA is also symmetric (k is a scalar). If A is a n-square matrix, then $(A + A^T)$ is symmetric. A square matrix A, such that $A^T = -A$ is said to be **skew-symmetric**, that is,

$$[a_{ij}^T] = [a_{ji}] = [-a_{ij}]. \tag{1.40}$$

Hence the diagonal elements of a skew-symmetric matrix are zeros.

Example 1.13

The matrix, $A = \begin{bmatrix} 0 & -4 & 3 \\ 4 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$ is skew-symmetric.

****

If A is skew-symmetric, then kA is also skew-symmetric (k is a scalar). If A is a n-square matrix, then $(A - A^T)$ is skew-symmetric. Every square matrix A can be written as the sum of the symmetric matrix $B = \frac{1}{2}(A + A^T)$ and the skew-symmetric matrix $C = \frac{1}{2}(A - A^T)$; that is $A = B + C$.

1.2.6 The Conjugate of a Matrix

If a and b are real numbers and $i = \sqrt{-1}$, then $z = a + bi$ is a **complex** number. The conjugate of z is given by $\bar{z} = a - bi$. The conjugate of $\bar{z}$ is $\bar{\bar{z}} = a + bi = z$. Hence,

$$\bar{\bar{z}} = z. \tag{1.41}$$

If $z_1 = a + bi$ and $z_2 = c + di$, then,

$$\begin{aligned} z_1 + z_2 &= (a + c) + (b + d)i \\ \overline{z_1 + z_2} &= (a + c) - (b + d)i = (a - bi) + (c - di) \\ \overline{z_1 + z_2} &= \bar{z}_1 + \bar{z}_2. \end{aligned} \tag{1.42}$$

$$\begin{aligned} z_1 z_2 &= (ac - bd) + (ad + bc)i \\ \overline{z_1 z_2} &= (ac - bd) - (ad + bc)i \\ &= (a - bi)(c - di) = \bar{z}_1 \bar{z}_2 \\ \overline{z_1 z_2} &= \bar{z}_1 \bar{z}_2. \end{aligned} \tag{1.43}$$

If A is a matrix with complex number elements, then the matrix obtained from A by replacing all its elements with the conjugate of its elements is called the **conjugate** of A and is denoted by $\bar{A}$, or A conjugate.

Example 1.14

$$\text{Given} \quad A = \begin{bmatrix} 2+i & -i \\ 4 & 3-2i \end{bmatrix} \text{ then,} \quad \bar{A} = \begin{bmatrix} 2-i & i \\ 4 & 3+2i \end{bmatrix}.$$

****

If k is a given scalar, then,

$$\overline{kA} = \bar{k}\bar{A} \tag{1.44}$$

$$\bar{\bar{A}} = A \tag{1.45}$$

$$\overline{A+B} = \bar{A}+\bar{B} \tag{1.46}$$

$$\overline{AB} = \bar{A}\ \bar{B} \quad \text{(in the same order)} \tag{1.47}$$

$$\bar{A}^T = \overline{A^T} = A^* \tag{1.48}$$

where $A^* = \bar{A}^T$ is A conjugate transpose.

1.2.7 Hermitian Matrices

A square matrix $A = [a_{ij}]$ such that $\bar{A}^T = A$ is called **Hermitian**. Hence,

$$[a_{ij}] = [\overline{a}_{ji}]. \tag{1.49}$$

Hence the diagonal elements of a Hermitian matrix are real numbers.

Example 1.15

$$\text{Given the matrix} \quad A = \begin{bmatrix} 2 & 1+i & 1 \\ 1-i & 4 & -i \\ 1 & i & 6 \end{bmatrix} \quad \text{is Hermitian.}$$

****

If k is a scalar, and A is Hermitian, then,

$$\overline{kA}^T = \bar{k}\bar{A}^T = \bar{k}A = kA, \tag{1.50}$$

$\Rightarrow \quad \bar{k} = k \quad \Rightarrow \quad k$ is a real number if kA is Hermitian.

A square matrix $A = [a_{ij}]$ such that $\bar{A}^T = -A$ is called **skew-Hermitian**. Hence,

$$[a_{ij}] = [-\bar{a}_{ji}], \tag{1.51}$$

and the diagonal elements of a skew-Hermitian matrix are purely imaginary.

Example 1.16

Given the matrix $A = \begin{bmatrix} 2i & 1+i & 3 \\ -1+i & 0 & 3i \\ -3 & 3i & i \end{bmatrix}$ is skew-Hermitian.

****

If k is a scalar and the matrix A is skew-Hermitian, then,

$$\overline{kA}^T = \bar{k}\bar{A}^T = -\bar{k}A = -kA. \tag{1.52}$$

Hence kA is skew-Hermitian if k is a real number. If k is an **imaginary number**, then kA becomes a Hermitian matrix.

If A is any n-square matrix, then $(A + \bar{A}^T)$ is Hermitian and $(A - \bar{A}^T)$ is skew-Hermitian,

$$\overline{(A + \bar{A}^T)}^T = (\bar{A} + A^T)^T = \bar{A}^T + A = A + \bar{A}^T, \tag{1.53}$$

$$\overline{(A - \bar{A}^T)}^T = (\bar{A} - A^T)^T = \bar{A}^T - A = -(A - \bar{A}^T). \tag{1.54}$$

Any matrix A can be written as the sum of a Hermitian matrix B and a skew-Hermitian matrix C,

$$\begin{aligned} A &= B + C \\ B &= \frac{1}{2}(A + \bar{A}^T), \quad \text{Hermitian} \\ C &= \frac{1}{2}(A - \bar{A}^T), \quad \text{skew-Hermitian.} \end{aligned} \tag{1.55}$$

1.2.8 Direct Sum

If $A_1, A_2, \ldots, A_s$ are square matrices of order $m_1, m_2, \ldots, m_s$ respectively. Then the diagonal matrix

$$A = \begin{bmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & A_s \end{bmatrix} = \text{diag}(A_1, A_2, \ldots, A_s) \tag{1.56}$$

is called the **direct sum** of the matrices A_i.

Example 1.17

$$\text{Given the matrices} \quad A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix},$$

$$\text{then the direct sum is} \quad A = \text{diag}(A_1, A_2) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 2 & 5 & 8 \\ 0 & 0 & 3 & 6 & 9 \end{bmatrix}.$$

****

If A_i and B_i are of the same order for $i = 1, 2, \ldots, s$, then,

$$\begin{aligned} AB &= \text{diag}(A_1B_1, A_2B_2, \ldots, A_sB_s), \quad \text{where} \\ A &= \text{diag}(A_1, A_2, \ldots, A_s) \\ B &= \text{diag}(B_1, B_2, \ldots, B_s). \end{aligned} \tag{1.57}$$

1.3 Determinant of a Square Matrix

In a given **permutation**, if there exist a larger integer preceding a smaller integer, then we have an **inversion**. If the number of inversions in the permutation is even (odd), then the permutation is said to be **even (odd)**.

Example 1.18

Given 4 5 2 1 3. There are 7 inversions, hence the permutation is odd.

****

1.3.1 Determinant of Matrices of Order Two or Three

The determinant of a matrix A is given by

$$|A| = \sum_{\rho} \epsilon_{j_1 j_2 \ldots j_n} a_{1j_1} a_{2j_2} \ldots a_{nj_n} \tag{1.58}$$

where the sum is over $\rho = n!$ permutations of $j_1 j_2 \ldots j_n$ of the integers $1, 2, \ldots, n$.

$$\text{Hence} \quad |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \epsilon_{12} a_{11} a_{22} + \epsilon_{21} a_{12} a_{21} = a_{11} a_{22} - a_{12} a_{21}$$

$$
\begin{aligned}
|A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\
&+ a_{13}(a_{21}a_{32} - a_{22}a_{31}). \qquad (1.59)
\end{aligned}
$$

<u>Example 1.19</u>

$$
\begin{aligned}
\begin{vmatrix} 1 & 2 & -1 \\ -3 & 1 & 1 \\ 1 & 4 & 3 \end{vmatrix} &= 1(3-4) - 2(-9-1) - 1(-12-1) \\
&= -1 + 20 + 13 = 32.
\end{aligned}
$$

or

$$
\begin{aligned}
\left|\begin{array}{ccc|cc} 1 \searrow & 2 \searrow & -1 \searrow & 1 & 2 \\ -3 & 1 \searrow & 1 \searrow & -3 \searrow & 1 \\ 1 & 4 & 3 & 1 & 4 \end{array}\right|_{+}^{-} &= +(3+2+12) - (-1+4-18) \\
&= 32.
\end{aligned}
$$

NOTE: The second method in used Example 1.19 is only true for matrices of order 3.

****

1.3.2 Properties of Determinants

(i) If all the elements in a certain row (column) of a square matrix A are zero, then $|A| = 0$.

(ii) If A is a square matrix, then

$$
|A^T| = |A|. \qquad (1.60)
$$

(iii)

$$
\begin{vmatrix} a_{11} & ka_{12} & a_{13} \\ a_{21} & ka_{22} & a_{23} \\ a_{31} & ka_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (1.61)
$$

(iv) If $A \xrightarrow{R^i_{i+1}} B$, then $|B| = -|A|$. (If $A \xrightarrow{C^i_{i+1}} B$, then $|B| = -|A|$.)

(v) If $A \xrightarrow{R^i_{i+p}} B$, then $|B| = (-1)^p|A|$. (If $A \xrightarrow{C^i_{i+p}} B$, then $|B| = (-1)^p|A|$.)

(vi) If two rows or columns of the matrix A are the same, then $|A| = 0$.

(vii) If A is a singular matrix, then $|A| = 0$.

(viii)

$$\begin{vmatrix} b_{11}+c_{11} & b_{12}+c_{12} & b_{13}+c_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & b_{13}+c_{13} \\ a_{21} & a_{22} & b_{23}+c_{23} \\ a_{31} & a_{32} & b_{33}+c_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & b_{13} \\ a_{21} & a_{22} & b_{23} \\ a_{31} & a_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & c_{13} \\ a_{21} & a_{22} & c_{23} \\ a_{31} & a_{32} & c_{33} \end{vmatrix} \tag{1.62}$$

(ix)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13}+ka_{12} \\ a_{21} & a_{22} & a_{23}+ka_{22} \\ a_{31} & a_{32} & a_{33}+ka_{32} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{1.63}$$

Hence if $A \xrightarrow{R_j^i(k)} B$, then $|A| = |B|$ and

$A \xrightarrow{C_j^i(k)} B$, then $|A| = |B|$.

(x) If M is an upper (lower) triangular matrix with the diagonal blocks, $A_1, A_2, \ldots, A_s$, then

$$|M| = |A_1| \cdot |A_2| \cdot \ldots |A_s.|$$

If A, B, C, and D are square matrices and

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

$$\text{then} \quad |M| \neq |A||D| - |B||C| \quad \text{(in general)}.$$

(xi) If A and B are n-square matrices, then $|AB| = |A| \cdot |B|$.

1.3.3 First Minors and Cofactors

Given that A is a n-square matrix. Matrix M_{ij} is the matrix A with the ith row and the jth column removed. The determinant $|M_{ij}|$ is called the **first minor** or **minor** of A. The minor with sign $(-1)^{i+j}|M_{ij}|$ is called the **cofactor** of the matrix $[a_{ij}]$ and is denoted by α_{ij}. Hence

$$\alpha_{ij} = (-1)^{i+j}|M_{ij}|. \tag{1.64}$$

Example 1.20

Given $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then $|M_{21}| = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$, $|M_{22}| = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$,

$$|M_{23}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}. \tag{1.65}$$

$$\begin{aligned} \text{Hence} \quad |A| &= a_{21}\alpha_{21} + a_{22}\alpha_{22} + a_{23}\alpha_{23} \\ &= a_{21}(-1)^{2+1}|M_{21}| + a_{22}(-1)^{2+2}|M_{22}| + a_{23}(-1)^{2+3}|M_{23}| \\ &= -a_{21}|M_{21}| + a_{22}|M_{22}| - a_{23}|M_{23}| \ . \end{aligned} \tag{1.66}$$

****

$$\begin{aligned} |A| &= a_{i1}\alpha_{i1} + a_{i2}\alpha_{i2} + \ldots + a_{in}\alpha_{in} = \sum_{k=1}^{n} a_{ik}\alpha_{ik} \\ i &= 1, 2, \ldots, n. \end{aligned} \tag{1.67}$$

$$\begin{aligned} |A| &= a_{1j}\alpha_{1j} + a_{2j}\alpha_{2j} + \ldots + a_{nj}\alpha_{nj} = \sum_{k=1}^{n} a_{kj}\alpha_{kj}, \\ j &= 1, 2, \ldots, n. \end{aligned} \tag{1.68}$$

$$\begin{aligned} \sum_{k=1}^{n} a_{lk}\alpha_{mk} &= |A| \quad \text{if} \quad l = m \\ &= 0 \quad \text{if} \quad l \neq m. \end{aligned} \tag{1.69}$$

$$\begin{aligned} \sum_{k=1}^{n} a_{kl}\alpha_{km} &= |A| \quad \text{if} \quad l = m \\ &= 0 \quad \text{if} \quad l \neq m. \end{aligned} \tag{1.70}$$

Example 1.21

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$\begin{aligned} a_{11}\alpha_{11} + a_{12}\alpha_{12} + a_{13}\alpha_{13} &= |A| \\ a_{11}\alpha_{31} + a_{12}\alpha_{32} + a_{13}\alpha_{33} &= 0 \quad (1.71) \\ a_{11}\alpha_{11} + a_{21}\alpha_{21} + a_{31}\alpha_{31} &= |A| \\ a_{11}\alpha_{12} + a_{21}\alpha_{22} + a_{31}\alpha_{32} &= 0. \quad (1.72) \end{aligned}$$

****

A square matrix A is singular if $|A| = 0$, if $|A| \neq 0$, A is non singular.

1.3.4 The Adjoint of a Square Matrix

The **adjoint** of the square matrix A, is denoted by

$$\text{adjoint } A = \text{adj } A = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n2} \\ \vdots & \vdots & \dots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{nn} \end{bmatrix}. \quad (1.73)$$

$$\begin{aligned} A(\text{adj } A) &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n2} \\ \vdots & \vdots & \dots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{nn} \end{bmatrix} \\ &= \text{diag}(|A|, |A|, |A|, \dots, |A|) \\ &= |A| \cdot I_n = (\text{adj } A)A. \quad (1.74) \end{aligned}$$

Taking determinant of the above equation, we get

$$|A||\text{adj } A| = |A|^n = |\text{adj } A||A|. \quad (1.75)$$

If A is non singular, then

$$|\text{adj } A| = |A|^{n-1}. \quad (1.76)$$

If A is singular, then

$$A(\text{adj } A) = (\text{adj } A)A = 0 \quad \text{(zero matrix)}. \tag{1.77}$$

If A and B are n-square matrices, then

$$\text{adj}(AB) = (\text{adj } B)(\text{adj } A). \tag{1.78}$$

NOTE: If C and A are n-square matrices, k a scalar and $C = kA$, then the determinant

$$|C| = k^n|A|.$$

Hence

$$\begin{aligned} |A(\text{adj } A)| &= ||A| \cdot I_n| \\ |A||\text{adj } A| &= |A|^n|I_n| \\ &= |A|^n, \qquad \text{where} \quad |I_n| = 1. \end{aligned}$$

1.3.5 Evaluation of Determinant

Procedure for evaluating determinant

Use the elementary transformation $R_j^i(k)$ or $C_j^i(k)$ repeatedly over the matrix A to get the matrix $B = [b_{ij}]$ whose elements, except one, of a row (column) are zero. If b_{pq} is the non zero element and β_{pq} its cofactor, then

$$|A| = |B| = b_{pq}\ \beta_{pq} = (-1)^{p+q}\ b_{pq} \times \text{minor of } \ b_{pq} \tag{1.79}$$

Example 1.22

$$\begin{vmatrix} 1 & 2 & -1 \\ -3 & 1 & 1 \\ 1 & 4 & 3 \end{vmatrix} \xrightarrow{R_2^1(1)} \begin{vmatrix} 1 & 2 & -1 \\ -2 & 3 & 0 \\ 1 & 4 & 3 \end{vmatrix} \xrightarrow{R_3^1(3)} \begin{vmatrix} 1 & 2 & -1 \\ -2 & 3 & 0 \\ 4 & 10 & 0 \end{vmatrix} \xrightarrow{R_3^2(2)}$$

$$\begin{vmatrix} 1 & 2 & -1 \\ -2 & 3 & 0 \\ 0 & 16 & 0 \end{vmatrix} = (-1)^{3+2}16 \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -16(0-2) = 32.$$

****

If A and B are n-square matrices, then

$$|AB| = |A| \cdot |B|. \tag{1.80}$$

Differentiation of Determinant

Example 1.23

$$\frac{d}{dx}\begin{vmatrix} x^3 & x^2+3 & 5 \\ 2 & 2x^3-5 & x^4 \\ x & -3 & 0 \end{vmatrix} = \begin{vmatrix} 3x^2 & 2x & 0 \\ 2 & 2x^3-5 & x^4 \\ x & -3 & 0 \end{vmatrix} + \begin{vmatrix} x^3 & x^2+3 & 5 \\ 0 & 6x^2 & 4x^3 \\ x & -3 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} x^3 & x^2+3 & 5 \\ 2 & 2x^3-5 & x^4 \\ 1 & 0 & 0 \end{vmatrix}.$$

****

Example 1.24

$$\begin{aligned} \frac{d}{dx}\begin{vmatrix} f_1(x) & f_2(x) \\ f_3(x) & f_4(x) \end{vmatrix} &= \begin{vmatrix} df_1/dx & df_2/dx \\ f_3 & f_4 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 \\ df_3/dx & df_4/dx \end{vmatrix} \\ &= \frac{d}{dx}(f_1 f_4 - f_2 f_3). \end{aligned} \tag{1.81}$$

The above procedure can also be applied to the columns.

$$\begin{aligned} \frac{d}{dx}\begin{vmatrix} f_1(x) & f_2(x) \\ f_3(x) & f_4(x) \end{vmatrix} &= \begin{vmatrix} df_1/dx & f_2 \\ df_3/dx & f_4 \end{vmatrix} + \begin{vmatrix} f_1 & df_2/dx \\ f_3 & df_4/dx \end{vmatrix} \\ &= \frac{d}{dx}(f_1 f_4 - f_2 f_3). \end{aligned}$$

****

1.3.6 The Inverse of a Matrix

If $AB = BA = I$, then $B = A^{-1}$.

(i) The square matrix A has an inverse iff, A is non singular.

(ii) If A is non singular, then

$$AB = AC \quad \Longrightarrow \quad B = C, \tag{1.82}$$

$\Longrightarrow$ the inverse is unique.

(iii) The inverse of the diagonal matrix, $\mathrm{diag}(k_1, k_2, \ldots, k_n)$ is $\mathrm{diag}(1/k_1, 1/k_2, \ldots, 1/k_n)$; $k_1, k_2, \ldots, k_n \neq 0$.

(iv) From Eq.(1.74), the inverse of matrix A can be written as,

$$A^{-1} = \frac{\mathrm{adj}A}{|A|}. \tag{1.83}$$

1.3.7 Elementary Transformations

1. R^i_j = interchange the ith row and the jth row.

   C^i_j = interchange the ith column and the jth column.

2. $R_i(k)$ = Multiplying every elements of the ith row with the non zero scalar k.

   $C_i(k)$ = Multiplying every elements of the ith column with the non zero scalar k.

3. $R^i_j(k)$ = Multiplying every elements of the ith row with the non zero scalar k and then adding them to the elements of the jth row.

   $C^i_j(k)$ = Multiplying every elements of the ith column with the non zero scalar k and then adding them to the elements of the jth column.

The transformation R is called the **elementary row transformations** and the transformation C is called the **elementary column transformations**.
Two matrices A and B are said to be **equivalent**, if B is obtained from A through a sequence of elementary transformations.
Equivalent matrices have the same order and **rank**.

(a) A matrix A is of **rank** r if there exist at least a r-square minor that is non zero while every $(r+1)$-square minors are zero. A zero matrix has rank $r = 0$.
(b) For a non singular matrix A (that is $|A| \neq 0$), the rank $r = n$ (order)
(c) For a singular matrix, the rank $r < n$.

Example 1.25

(a)

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 1 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

$$[A\ I] = \left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 6 & 1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2^1(-1)} \left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & -1 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1^2(-1)} \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 2 & -1 & 0 \\ 3 & 0 & 1 & -1 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3^1(-2)} \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 2 & -1 & 0 \\ 3 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & -4 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3^1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & 1 \\ 3 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_2^1(-3)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & 11 & -5 & -3 \\ 0 & 1 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3^2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & 1 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 11 & -5 & -3 \end{array}\right] \xrightarrow{R_2^3(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & 1 \\ 0 & 1 & 0 & -9 & 4 & 3 \\ 0 & 0 & 1 & 11 & -5 & -3 \end{array}\right]$$

$$= [I\ A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -4 & 2 & 1 \\ -9 & 4 & 3 \\ 11 & -5 & -3 \end{bmatrix}.$$

What is the rank of the matrix A?

(b)

$$B = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 1 & 2 & 2 \end{bmatrix}$$

$$|B| = 0; \quad \text{Minor of } B, \quad |M_{11}| = \begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} = 4 - 8 = -4.$$

What is the rank of the matrix B?

****

Example 1.26

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \end{bmatrix}$$

$$[A\ I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & -2 & 0 & 1 & 0 \\ 4 & -3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2^1(-3)} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -5 & -5 & -3 & 1 & 0 \\ 4 & -3 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2(-1/5)} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3/5 & -1/5 & 0 \\ 4 & -3 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3^2(3)} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3/5 & -1/5 & 0 \\ 4 & 0 & 2 & 9/5 & -3/5 & 1 \end{array}\right]$$

$$\xrightarrow{R_1^2(-2)} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1/5 & 2/5 & 0 \\ 0 & 1 & 1 & 3/5 & -1/5 & 0 \\ 4 & 0 & 2 & 9/5 & -3/5 & 1 \end{array}\right]$$

$$\xrightarrow{R_3^1(2)} \xrightarrow{R_3(1/6)} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1/5 & 2/5 & 0 \\ 0 & 1 & 1 & 3/5 & -1/5 & 0 \\ 1 & 0 & 0 & 7/30 & 1/30 & 1/6 \end{array}\right]$$

$$\xrightarrow{R_3^1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/30 & 1/30 & 1/6 \\ 0 & 1 & 1 & 3/5 & -1/5 & 0 \\ 1 & 0 & -1 & -1/5 & 2/5 & 0 \end{array}\right]$$

$$\xrightarrow{R_3^1(-1)} \xrightarrow{R_3(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/30 & 1/30 & 1/6 \\ 0 & 1 & 1 & 3/5 & -1/5 & 0 \\ 0 & 0 & 1 & 13/30 & -11/30 & 1/6 \end{array}\right]$$

$$\xrightarrow{R_2^3(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/30 & 1/30 & 1/6 \\ 0 & 1 & 0 & 1/6 & 1/6 & -1/6 \\ 0 & 0 & 1 & 13/30 & -11/30 & 1/6 \end{array}\right]$$

$$= [I\ A^{-1}]$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 7/5 & 1/5 & 1 \\ 1 & 1 & -1 \\ 13/5 & -11/5 & 1 \end{bmatrix}.$$

What is the rank of the matrix A?

1.3.8 Echelon Forms

A matrix is in **reduced row echelon form (RREF)** if

1. Rows of all zeros, if there are any, appear at the bottom of the matrix.
2. The first nonzero entry of a nonzero row is 1. This is called a **leading 1**.
3. For each nonzero row, the leading 1 appears to the right and below any leading 1's in preceding rows.
4. Any column in which a leading 1 appears has zeros in every other entry.

A matrix in RREF appears as a staircase pattern of leading 1's descending from the upper left corner of the matrix. The columns of the leading 1's are columns of an identity matrix.

A matrix is in **row echelon form (REF)** if properties 1, 2, and 3 above are satisfied.

Example 1.27

Determine which of the following matrices are in upper triangular form, REF, RREF or none of these forms.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{upper triangular and REF.}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{upper triangular, REF and RREF.}$$

$$C = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \text{none of these forms.}$$

$$D = \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{upper triangular.}$$

****

1.4 System of Linear Equations

A system of m linear equations is given by

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = h_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = h_2 \\ \vdots \quad \vdots \quad \vdots \quad \ldots \quad \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = h_m \end{array} . \tag{1.84}$$

If the system of m linear equations possesses solutions, it is said to be **consistent**, if not it is said to be **inconsistent**. A consistent system possesses either only one solution or infinitely many solutions.

Two systems of linear equations are said to be **equivalent** if every solution of one of the systems is the same as the solution of the other system. An equivalent system can be obtained from a system A by applying the elementary transformations over it.

1.4.1 Solution Using a Matrix

In matrix notation, the system of equations, Eq. (1.84), can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{bmatrix}$$

$$AX = H \tag{1.85}$$

where $A = [a_{ij}]$ is the **coefficient matrix**, $X = [x_1, x_2, \ldots, x_n]^T$, and $H = [h_1, h_2, \ldots, h_m]^T$. X is a n-**vector** and H is a m-**vector**. The **augmented matrix** is given by

$$[A\ H] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & | & h_1 \\ a_{21} & a_{22} & \dots & a_{2n} & | & h_2 \\ \vdots & \vdots & \dots & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & | & h_m \end{bmatrix} \tag{1.86}$$

Example 1.28

Solved the following non-homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 3x_1 + x_2 - 2x_3 &= 1 \\ 4x_1 - 3x_2 - x_3 &= 1. \end{aligned}$$

$$[A\ H] = \begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 3 & 1 & -2 & | & 1 \\ 4 & -3 & -1 & | & 1 \end{bmatrix} \xrightarrow{R_2^1(-3)} \begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 0 & -5 & -5 & | & -5 \\ 4 & -3 & -1 & | & 1 \end{bmatrix}$$

$$\xrightarrow{R_2(-1/5)} \begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 0 & 1 & 1 & | & 1 \\ 4 & -3 & -1 & | & 1 \end{bmatrix} \xrightarrow{R_3^2(3)} \begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 0 & 1 & 1 & | & 1 \\ 4 & 0 & 2 & | & 4 \end{bmatrix}$$

$$\xrightarrow{R_1^2(-2)} \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 1 & | & 1 \\ 4 & 0 & 2 & | & 4 \end{bmatrix} \xrightarrow[R_3(1/6)]{R_3^1(2)} \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 1 & | & 1 \\ 1 & 0 & 0 & | & 2/3 \end{bmatrix}$$

$$\xrightarrow{R_3^1} \begin{bmatrix} 1 & 0 & 0 & | & 2/3 \\ 0 & 1 & 1 & | & 1 \\ 1 & 0 & -1 & | & 0 \end{bmatrix} \xrightarrow[R_3(-1)]{R_3^1(-1)} \begin{bmatrix} 1 & 0 & 0 & | & 2/3 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 1 & | & 2/3 \end{bmatrix}$$

$$\xrightarrow{R_2^3(-1)} \begin{bmatrix} 1 & 0 & 0 & | & 2/3 \\ 0 & 1 & 0 & | & 1/3 \\ 0 & 0 & 1 & | & 2/3 \end{bmatrix}$$

$$x_1 = x_3 = \frac{2}{3}, \quad x_2 = \frac{1}{3}.$$

****

1.4.2 Fundamental Theorems

1. A system of linear equations $AX = H$ is consistent iff the coefficient matrix $[A]$ and the augmented matrix $[AH]$ have the same rank.

2. If H is not a zero vector, then the system, $AX = H$, is called a **non-homogeneous system**. A system of n non-homogeneous equations with n unknowns possesses a unique solution provided that the matrix A is non singular, that is $|A| \neq 0$.

This system, $AX = H$, with n equations and n unknowns can be solved by two methods:

(a) Method 1 (**Cramer's Rule**, provided $|A| \neq 0$.)

If $A_i, \quad i = 1, 2, \ldots, n$ is a matrix obtained from A by replacing the ith column with the constant column H, then the unique solution of the system is the given by

$$x_i = \frac{|A_i|}{|A|}, \qquad i = 1, 2, \ldots, n. \tag{1.87}$$

Example 1.29

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 3x_1 + x_2 - 2x_3 &= 1 \\ 4x_1 - 3x_2 - x_3 &= 1 \end{aligned}$$

$$\begin{aligned} |A| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \end{vmatrix} &= 1(-1-6) - 2(-3+8) + 1(-9-4) \\ &= -7 - 10 - 13 = -30 \end{aligned}$$

$$\begin{aligned} |A_1| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 1 & -2 \\ 1 & -3 & -1 \end{vmatrix} &= 2(-1-6) - 2(-1+2) + 1(-3-1) \\ &= -14 - 2 - 4 = -20 \end{aligned}$$

$$\begin{aligned} |A_2| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & 1 & -1 \end{vmatrix} &= 1(-1+2) - 2(-3+8) + 1(3-4) \\ &= 1 - 10 - 1 = -10 \end{aligned}$$

$$|A_3| = \begin{vmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & -3 & 1 \end{vmatrix} = 1(1+3) - 2(3-4) + 2(-9-4)$$
$$= 4 + 2 - 26 = -20$$

$$x_1 = x_3 = \frac{-20}{-30} = \frac{2}{3}, \quad x_2 = \frac{-10}{-30} = \frac{1}{3}.$$

****

(b) Method 2 (By using the inverse A^{-1}.)

$$\begin{aligned} AX &= H \\ A^{-1}AX &= A^{-1}H \\ X &= A^{-1}H. \end{aligned} \quad (1.88)$$

Example 1.30

From Example 1.26

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \end{bmatrix}, \quad A^{-1} = \frac{1}{6}\begin{bmatrix} 7/5 & 1/5 & 1 \\ 1 & 1 & -1 \\ 13/5 & -11/5 & 1 \end{bmatrix}.$$

$$\begin{aligned} X &= A^{-1}H \\ &= \frac{1}{6}\begin{bmatrix} 7/5 & 1/5 & 1 \\ 1 & 1 & -1 \\ 13/5 & -11/5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 14/5 + 1/5 + 1 \\ 2 + 1 - 1 \\ 26/5 - 11/5 + 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}. \end{aligned}$$

****

1.4.3 Homogeneous Equations

A system of linear equations, $AX = 0$, is a **homogeneous system of equations**. For this system, the rank of the coefficient matrix A and the augmented matrix $[A\ 0]$ is the same. Hence the system is always consistent. $X = 0$ is a solution of the system and it is the **trivial solution**. If the rank of A is n, then the unique solution is the zero solution, $X = 0$. If the rank of A is $r < n$, then non-trivial solutions exist.

1. A necessary and sufficient condition for a homogeneous system to possess non-trivial solution is that the rank r of the coefficient matrix A must be less than n, that is $r < n$.

2. If the system $AX = H$ is consistent, the complete solution of the system is given by the complete solution of the homogeneous system $AX = 0$ plus the particular solution of the non-homogeneous system $AX = H$.

If given X_p is the particular solution of $AX = H$ and X_h is complete solution of the homogeneous system $AX = 0$, then the complete solution of $AX = H$ is given by

$$X_c = X_p + X_h. \tag{1.89}$$

$$\begin{aligned} \text{Proof:} \quad AX_c &= A(X_p + X_h) \\ &= AX_p + AX_h = H + 0 \\ \implies \quad AX_c &= H. \end{aligned}$$

Hence X_c is also a solution of the system $AX = H$.

3. If the rank of the coefficient matrix A of the homogeneous system, $AX = 0$, is $r < n$, then the system possesses exactly $(n - r)$ linearly independent solutions. Every given solution is a linear combination of the $(n - r)$ linearly independent solutions and every linear combination of solutions is also a solution.

Example 1.31

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 3x_1 + x_2 - 2x_3 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ 3x_1 + x_2 &= 1, \end{aligned} \quad \text{Fixed} \quad x_3 = 0.$$

$$\begin{aligned}
|A| &= \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1-6=-5, \quad \text{(Cramer's Rule)} \\
|A_1| &= \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 2-2=0 \\
|A_2| &= \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1-6=-5
\end{aligned}$$

$$\begin{aligned}
x_1 &= 0, \quad x_2 = \frac{-5}{-5} = 1 \\
\text{Particular solution,} \quad X_p &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
x_1 + 2x_2 + x_3 &= 0 \\
3x_1 + x_2 - 2x_3 &= 0
\end{aligned}$$

$$\begin{aligned}
x_1 + 2x_2 &= -a \\
3x_1 + x_2 &= 2a
\end{aligned} \quad \text{Say} \quad x_3 = a.$$

$$\begin{aligned}
|A| &= \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1-6=-5, \quad \text{(Cramer's Rule)} \\
|A_1| &= \begin{vmatrix} -a & 2 \\ 2a & 1 \end{vmatrix} = -a-4a=-5a \\
|A_2| &= \begin{vmatrix} 1 & -a \\ 3 & 2a \end{vmatrix} = 2a+3a=5a \\
x_1 &= a, \quad x_2 = -a \\
\text{The complete solution for, } AX=0, \text{ is} \quad X_h &= \begin{bmatrix} a \\ -a \\ a \end{bmatrix}. \\
\text{The complete solution for, } AX=H, \text{ is} \quad X_c &= \begin{bmatrix} a+0 \\ -a+1 \\ a+0 \end{bmatrix} = \begin{bmatrix} a \\ -a+1 \\ a \end{bmatrix}.
\end{aligned}$$

****

Example 1.32

$$\begin{aligned}
x_1 + 2x_2 + 2x_4 &= 0 \\
x_2 + 3x_3 &= 0 \\
\text{Hence} \quad AX &= 0\ ,
\end{aligned}$$

$$
\begin{aligned}
\text{where the coefficient matrix } A &= \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\text{rank } r &= 2 < n = 4, \\
\text{If given } x_3 &= a \quad \text{and} \quad x_4 = b, \\
x_1 + 2x_2 &= -2b \\
x_2 &= -3a \\
\Rightarrow \quad x_1 &= 6a - 2b, \qquad \Rightarrow \\
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} 6a - 2b \\ -3a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 6 \\ -3 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

Hence, the homogeneous system of equations, $AX = 0$, possesses two linearly independent solutions , that is, $[6, -3, 1, 0]^T$ and $[-2, 0, 0, 1]^T$ and all linear combinations of these two solutions is also a solution of the system.

Chapter 2

Vector Spaces

2.1 Vector Spaces

Assume that that all vectors are column vectors , that is

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1, x_2, \ldots, x_n]^T . \tag{2.1}$$

A collection or a set S of two or more elements, together with two operations, that is addition $(+)$ and multiplication $(\cdot)$ is called a **field** F.

If $\quad a, b, c, \ldots \in F$, then

1. $a + b \in F$.
2. $a + b = b + a, \quad a + (b + c) = (a + b) + c$.
3. For every $a \in F$, there exist $0 \in F$, where $\quad a + 0 = 0 + a = a$.
4. For every $a \in F$, there exist $-a \in F$, where $\quad a + (-a) = 0$.
5. $a \cdot b \in F$.
6. $ab = ba, \quad (ab)c = a(bc)$.
7. For every $a \in F$, there exist $1(\neq 0) \in F$ where $\quad 1 \cdot a = a \cdot 1 = a$.
8. For every $a \in F$, there exist $a^{-1} \in F$ where $\quad aa^{-1} = a^{-1}a = 1$.
9. $a(b + c) = ab + ac, \quad (a + b)c = ac + bc$.

A set of n-vector over F is said to be **closed under addition** if the sum of two vectors from F is also a vector from F. This set is said to be **closed**

under scalar multiplication, if every scalar multiple of a vector from the set is also a vector of the set.

Example 2.1

The set of all vectors $[a, b, c]^T$ of ordinary 3-space is closed under addition and scalar multiplication.

****

A set of n-vectors over F which is closed under addition and scalar multiplication is called a **vector space** or a vector space of n-dimension.

If given the n-vectors $X_1, X_2, \ldots, X_m \in F$, then the set of all the linear combinations $k_1X_1 + k_2X_2 + \ldots + k_mX_m$, (where $k_1, k_2, \ldots, k_m$ are scalars) is a linear vector space over F.

The set of all the vectors $[a, b, c]^T$ from Example 2.1 is a vector space.

2.1.1 Subspace

A set of vectors V from the vector space $V_n(F)$ is a **subspace** of $V_n(F)$ if V is closed under addition and scalar multiplication. Hence the zero n-vector from $V_n(F)$ is a subspace of $V_n(F)$.

If given the n-vectors $X_1, X_2, \ldots, X_m \in F$, then the space of all linear combinations of the vectors is a subspace of $V_n(F)$.

The vector space V is said to be **spanned** by the n-vectors $X_1, X_2, \ldots, X_m$ if $X_1, X_2, \ldots, X_m \in F$ and every vector in V is their linear combination. However, $X_1, X_2, \ldots, X_m$ are not necessarily **linearly independent**.

Example 2.2

The vectors

$$K_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix},$$

are from $S = V_3(R)$ where R is the field of real numbers. Any vector $[a,\ b,\ c]^T$

from S can be written as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x_1K_1 + x_2K_2 + x_3K_3 + x_4K_4$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 + 5x_4 \\ x_1 + 3x_2 + 5x_3 + 3x_4 \\ x_1 + 5x_2 + 3x_3 + x_4 \end{bmatrix}.$$

This system of equations is consistent. Hence the vectors K_1, K_2, K_3, K_4 span S.

The vectors K_1 and K_2 is linearly independent and span a subspace of planes π of S which contains all the vectors $pK_1 + qK_2$ where $p, q \in R$.

The vector K_4 spans the subspace of lines L of S which contains all the vectors pK_4, where $p \in R$.

****

2.2 Basis and Dimension

The **dimension** of a vector space V is the maximum number of linearly independent vectors in V or the minimum number of linearly independent vectors that is required to span the space V. Ordinary space is of linear dimension 3, the space of planes is of dimension 2 and the space of lines is of dimension 1.

A n-vectors space of dimension r over F is denoted by $V_n^r(F)$. When $r = n$, we write $V_n(F)$. A set of r linearly independent vectors from $V_n^r(F)$ is called the **basis** of that space. Every vector from the space is a unique linear combination of the basis vectors.

All bases from $V_n^r(F)$ have the same number of vectors and r linearly independent vectors from that space form a basis.

<u>Example 2.3</u>

From Example 2.2, the given vectors are

$$K_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}.$$

The vectors K_1, K_2, and K_3 span S because any vectors can be writen as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x_1K_1 + x_2K_2 + x_3K_3 = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + 3x_2 + 5x_3 \\ x_1 + 5x_2 + 3x_3 \end{bmatrix}.$$

and the system of equations has a unique solution. Hence K_1, K_2, and K_3 is a basis of the space.

<u>Proof:</u>

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \end{bmatrix} \underset{R_3^1(-1)}{\overset{R_2^1(-1)}{\longrightarrow}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{bmatrix} \underset{R_3(1/2)}{\overset{R_2(1/2)}{\longrightarrow}}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \underset{R_3(-1/3)}{\overset{R_3^2(-2)}{\longrightarrow}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \underset{R_1^3(-1)}{\overset{R_2^3(-2)}{\longrightarrow}}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{R_1^2(-1)}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The unit vectors, $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$ is a basis of the space S.

The vectors K_1, K_2, and K_4 is not a basis of the space S.

<u>Proof:</u>

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 3 & 3 \\ 1 & 5 & 1 \end{bmatrix} \underset{R_3^1(-1)}{\overset{R_2^1(-1)}{\longrightarrow}} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 2 & -2 \\ 0 & 4 & -4 \end{bmatrix} \underset{R_2(1/2)}{\overset{R_3^2(-2)}{\longrightarrow}}$$

$$\begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1^2(-1)}{\longrightarrow} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$e_1 = [1, 0, 0]^T$ and $e_2 = [0, 1, 0]^T$ spans the subspace π of planes. Hence K_1, K_2 is a basis of the subspace π.

****

1. If $X_1, X_2, \ldots, X_n$ are linearly independent n-vectors of the vector space $V_n(F)$, then the set $X_1, X_2, \ldots, X_n$ is a basis of $V_n(F)$.

2. If $X_1, X_2, \ldots, X_n$ are linearly independent n-vectors over F, then vectors

$$Y_i = \sum_{j=1}^{n} a_{ij} X_j, \quad i = 1, 2, \ldots, n \tag{2.2}$$

are linearly independent iff the matrix $[a_{ij}]$ is non singular. The set $Y_i, \quad i = 1, 2, \ldots, n$ is also a basis of $V_n(F)$.

2.2.1 Bases and Coordinates

The n-vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{2.3}$$

are called the **elementary vectors** or **unit vectors** over F.

$e_j =$ the jth unit vector.

$e_1, e_2, \ldots, e_n$ is an important basis of $V_n(F)$.

For every vector $X \in V_n(F)$, X can be uniquely written as,

$$X = \sum_{i=1}^{n} x_i e_i = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n, \tag{2.4}$$

where $(x_1, x_2, \ldots, x_n)$ is the coordinate of X relative to the e-basis.

Given that $Z_1, Z_2, \ldots, Z_n$ is a basis of $V_n(F)$, then there exist the unique scalars $a_1, a_2, \ldots, a_n \in F$ such that

$$X = \sum_{i=1}^{n} a_i Z_i = a_1 Z_1 + a_2 Z_2 + \ldots + a_n Z_n$$

$$X = [Z_1, Z_2, \ldots, Z_n] \cdot X_Z = Z \cdot X_Z \tag{2.5}$$

where $X_Z = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, is the coordinate of X relative to the Z-basis.

If given that $W_1, W_2, \ldots, W_n$ is also a basis of $V_n(F)$, so that,

$$X = [W_1, W_2, \ldots, W_n] \cdot X_W = W \cdot X_W, \tag{2.6}$$

then

$$\begin{aligned} X &= Z \cdot X_Z = W \cdot X_W \\ X_W &= W^{-1} \cdot Z \cdot X_Z = P X_Z \end{aligned} \tag{2.7}$$

where $P = W^{-1} \cdot Z$ is a non singular matrix and X_W is the coordinate of X relative to the W-basis.

2.3 Linear Transformations

2.3.1 Definition

Given the vectors, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in V_n(F)$, are of the same basis and

$$\begin{array}{l} y_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ y_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \;\vdots \qquad \vdots \qquad \vdots \qquad \ldots \qquad \vdots \\ y_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{array}$$

$$\Longrightarrow \qquad Y = AX, \tag{2.8}$$

where $A = [a_{ij}]$ is over F. Then Eq.(2.8) is a transformation that takes the vector $X \in V_n(F)$ to the vector $Y \in V_n(F)$. Y is called the **image** of X.

If $Y_1 = AX_1$ and $Y_2 = AX_2$, then for every scalar k, a, and b,

$$kY_1 = A(kX_1), \tag{2.9}$$

$$aY_1 + bY_2 = A(aX_1 + bX_2). \tag{2.10}$$

This transformation is then said to be **linear**.

2.3.2 Basic Theorems

If $X = e_j$ and $A = [a_{ij}]$, then

$$Y = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} \quad \text{where} \quad Y = AX. \tag{2.11}$$

1. The transformation $Y = AX$ is non singular iff the matrix A is non singular.

2. A non singular linear transformation takes linearly independent (dependent) vectors into linearly independent (dependent) vectors.

3. If the transformation, $Y = AX$, is non singular, then it maps the vector space $V_n(F)$ onto itself.

4. The elementary vectors e_i of $V_n(F)$ can be transformed into any set of linearly independent n-vectors by a non singular linear transformation and conversely.

5. For any given two sets of linearly independent n-vectors, there exist a non singular linear transformation that can take the vectors of one of the sets into the vectors of the other.

2.3.3 Change of Basis

If given two matrices A and B, and there exist a non singular matrix Q such that

$$B = Q^{-1}AQ, \tag{2.12}$$

then the matrices A and B are said to be **similar**.

If $Y_Z = AX_Z$ is a linear transformation over $V_n(F)$ relative to the Z-basis and $Y_W = BX_W$ is another linear transformation relative to the W-basis, then A and B are similar.

Proof: From Eq. (2.7), $X_W = PX_Z$, we let $Q = P^{-1}$, then

$$X_Z = QX_W \quad \text{and} \quad Y_Z = QY_W$$

$$
\begin{aligned}
Y_W &= Q^{-1}Y_Z = Q^{-1}(AX_Z) = Q^{-1}A(QX_W) \\
Y_W &= \underbrace{(Q^{-1}AQ)}_{B} X_W \\
B &= Q^{-1}AQ.
\end{aligned}
$$

Hence A and B are similar.

<u>Example 2.4</u>

$$
A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \end{bmatrix}
$$

From Example 2.3, A is non singular. Given that $Y = AX$ is a linear transformation with respect to the e-basis and the vectors,

$$
W_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix},
$$

formed a new basis. If $X = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, find the coordinate of the its image relative to the W-basis by answering the following questions:

(a) First find the coordinate of X relative to the W-basis.

(b) Find the linear transformation $Y_W = BX_W$ corresponding to $Y = AX$.

(c) Find the image Y_W of X_W.

<u>Solution</u>

(a) From Example 1.26,

$$
W = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \end{bmatrix}, \quad W^{-1} = \frac{1}{30} \begin{bmatrix} 7 & 1 & 5 \\ 5 & 5 & -5 \\ 13 & -11 & 5 \end{bmatrix}
$$

$$
\text{Eq. (2.7)} \quad \Longrightarrow \quad X = WX_W \quad \Longrightarrow \quad X_W = W^{-1}X.
$$

$$
X_W = \frac{1}{30} \begin{bmatrix} 7 & 1 & 5 \\ 5 & 5 & -5 \\ 13 & -11 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{30} \begin{bmatrix} 7+10 \\ 5-10 \\ 13+10 \end{bmatrix} = \frac{1}{30} \begin{bmatrix} 17 \\ -5 \\ 23 \end{bmatrix}
$$

(b) $\quad Y_W = BX_W \quad$ and $\quad Y = AX$

$$Y = WY_W \quad \Longrightarrow \quad Y_W = W^{-1}Y = W^{-1}AX = \underbrace{(W^{-1}AW)}_{B} X_W$$

$$\begin{aligned}
B &= W^{-1}AW \\
&= \frac{1}{30}\begin{bmatrix} 7 & 1 & 5 \\ 5 & 5 & -5 \\ 13 & -11 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \end{bmatrix} \\
&= \frac{1}{15}\begin{bmatrix} 113 & -10 & -42 \\ 25 & -20 & 0 \\ -43 & 50 & 12 \end{bmatrix} \\
B &= \begin{bmatrix} 113/15 & -2/3 & -14/5 \\ 5/3 & -4/3 & 0 \\ -43/15 & 10/3 & 4/5 \end{bmatrix}
\end{aligned}$$

(c)

$$\begin{aligned}
Y_W = BX_W &= \begin{bmatrix} 113/15 & -2/3 & -14/5 \\ 5/3 & -4/3 & 0 \\ -43/15 & 10/3 & 4/5 \end{bmatrix} \times \frac{1}{30}\begin{bmatrix} 17 \\ -5 \\ 23 \end{bmatrix} \\
&= \begin{bmatrix} 4847/450 \\ 53/6 \\ -47/30 \end{bmatrix}.
\end{aligned}$$

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Chapter 3

Tutorials

3.1 Tutorial 1

1. The matrices given are:

$$A = \begin{bmatrix} 3 & 4 & p^2+1 \\ 2 & 3 & 1 \\ 1 & 4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -5 & 1 \\ 2 & 1 & -3 \\ 4 & -4 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 1 & 2 \\ 4 & 0 & 3 \\ 2 & 4 & -1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 3 & q^2 & 10 \\ 2 & 3 & 1 \\ 1 & 4 & -1 \end{bmatrix}.$$

(a) If $A = A_1$, find all the possible values of p and q.

(b) Find the matrix $D_1 = AC + B$ and

(c) the matrix $D_2 = CA + B$. Is the matrix D_1 the same as D_2. If not, why? Give reasons.

(d) Find the determinant of the matrices A, B, and C and

(e) their respective adjoints.

(f) From part (d) and (e), find the inverse of the matrices A, B, and C.

(g) Find the inverse of the matrix A again by using the elementary row transformations of matrices.

2. The matrices given are:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

(a) If A is an upper triangular matrix, and its determinant $|A|$ is equal to one, what is the matrix A?

(b) If $B = A^T$, find the matrix B? What type of matrix is B if the elements b_{21}, b_{31} and b_{32} are non zero?

(c) What type of matrix is B if the elements b_{21}, b_{31} and b_{32} are zero? Given the matrix $C = kB$, where k is a scalar. What type of matrix is C?

3.

(a) Given the matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is an idempotent matrix. If $b = 2$, what is the matrix A? (That is find the value of a, c, and d.) Is the matrix A singular?

(b) Find an idempotent matrix that is non singular.

4.

(a) Show that ALL nilpotent matrices are singular.

(b) Show that the given matrix is nilpotent,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) Given $B = I - A$. Show that the matrix B is non singular and find the inverse of matrix B.

5. Find the matrix $(AB)^{-1}$, if

$$A^{-1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & -2 \\ 1 & 1 & -1 \end{bmatrix}.$$

6. Given the matrix,

$$A = \begin{bmatrix} 3i & 3+2i & 1-3i \\ 3-2i & 5-i & 3-2i \\ 1-2i & 1+2i & 2 \end{bmatrix}.$$

Construct from matrix A, the following matrices:

(a) B, a Hermitian matrix;

(b) C, a skew-Hermitian matrix;

(c) D, a symmetric matrix; and

(d) E, a skew-symmetric matrix.

(e) Show that $F = iC$ and $G = \bar{B}$ are Hermitian matrices.

7. Show that the following matrices are periodic,

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find the period of these matrices.

3.2 Tutorial 2

1. (a) Write the following system of linear equations,

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ 2x_1 - x_2 - 3x_3 &= 1 \\ x_1 + x_2 - 2x_3 &= 5. \end{aligned}$$

in matrix notation. What is the coefficient matrix of this system of linear equations?.

(b) Find the inverse of the coefficient matrix by using the elementary row transformation.

(c) Find the adjoint of the coefficient matrix.

(d) Find the inverse of the coefficient matrix by using the answer from part (c) above. Make sure that the answer is the same as the answer for part (b).

(e) Find all possible solutions of the system by using the inverse of the coefficient matrix.

(f) Make sure that the answers for part (e) are correct by using Cramer's Rule.

2. Given the following system of linear equations:

$$\begin{aligned} x_1 + 3x_2 + x_3 - 2x_4 &= 2 \\ 2x_1 - x_2 - 3x_3 &= 1 \\ x_1 + 2x_2 - 2x_3 &= 2 \\ x_2 + 3x_4 &= 0. \end{aligned}$$

Find the solution for this system of linear equations by using Cramer's Rule. Is the solution unique?

3. Find all the possible solutions for the following system of linear equations:

$$\begin{aligned} 3x_1 + x_2 + 2x_3 + x_4 &= -2 \\ 2x_1 + x_2 + 3x_3 - x_4 &= 3 \\ x_1 + x_3 + 2x_4 - x_5 &= 4. \end{aligned}$$

4. Given the matrix,

$$A = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 1 & -2 \\ 3 & 3 & -6 \end{bmatrix},$$

find the matrix, B, of rank two such that $AB = 0$. What is the rank of matrix A?

5. A given vector space is spanned by the following vectors;

$$K_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

By using the elementary row transformation, choose a suitable set of basis from the above vectors for this vector space. What is the dimension of this vector space?

6. Given that X_Z dan X_W are coordinates with respect to a pair of bases,

$$Z_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad Z_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad \text{and}$$

$$W_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

respectively. The vector, $X = [2, 7, 1]^T$. Find the coordinates, X_Z and X_W, and the matrix P, such that, $X_W = PX_Z$.

7. The matrix equation,

$$Y = AX = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} X,$$

is a linear transformation relative to the e-basis. A new basis chosen is

$$Z_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad Z_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Relative to the e-basis,

$$X = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}.$$

Find:

(a) the image Y of X for the given linear transformation,

(b) the inverse Z^{-1} of the basis matrix Z,

(c) the coordinate X_Z of X and the coordinate Y_Z of Y with respect to the Z-basis, and

(d) the matrix R of the new transformation $Y_Z = RX_Z$.