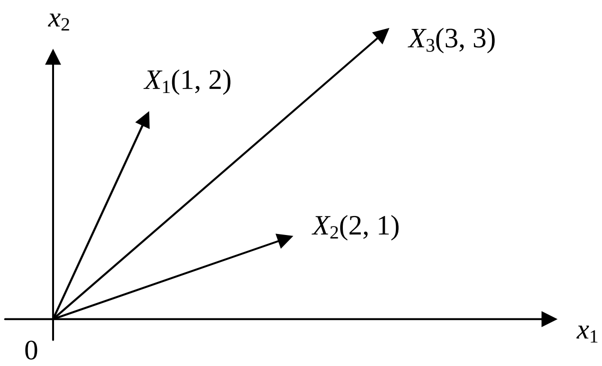
Chapter 9 Linear dependence of vectors and forms

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Give examples of two distinct, non-zero 2-vectors in row vector form. Call them X_1 and X_2. (*ii*) What is their sum, $X_3 = X_1 + X_2$? Write it down explicitly. (*iii*) Sketch a picture representing these vectors in a 2-dimensional space. Label your drawing properly (including the vectors and the axes).

Ans:

(*i*) $X_1 = [1, 2]$; (*i*) $X_2 = [2, 1]$; $X_3 = X_1 + X_2 = [3, 3]$.



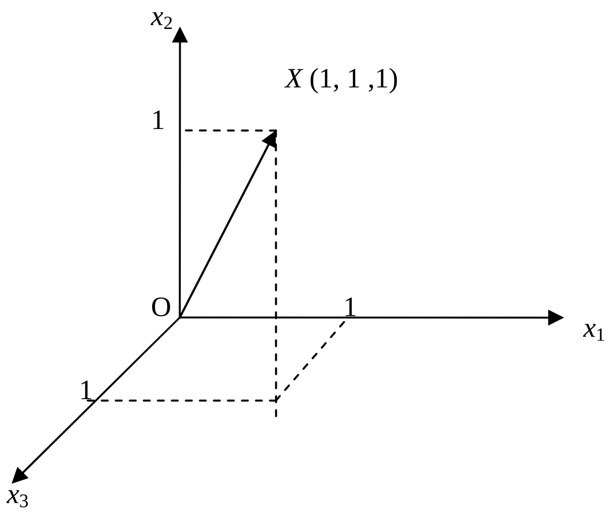
2. (*i*) Repeat (*i*) and (*ii*) of the above question but for 2-vectors in *column* form (i.e. giving examples, writing down their sum). (*ii*) If you were to sketch a picture representing these 2-dimensional column vectors in a 2-dimensional plane, as you did in (1)(*iii*), will the drawing be the same as in (1)(*iii*)?

Ans:
$X = [x_1, x_2]^T$;
(*i*) $X_1 = [1, 2]^T$; (*i*) $X_2 = [2, 1]^T$; $X_3 = X_1 + X_2 = [3, 3]^T$, or, equivalently, $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; (*i*) $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$; $X_3 = X_1 + X_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ (*ii*) YES.

3. (*i*) Give an example of a simplest possible 3-vector in column vector form, with all positive, non-zero components you can think of. Call it X. (*iii*) Sketch a picture representing this vector in a 3-dimensional space. Label your drawing properly (including the vectors and the axes).

Ans:

(*i*) $X = [1, 1, 1]^T$.



4. (*i*) Give an example of a simplest possible 4-vector in column vector form, with all positive, non-zero components you can think of. Call it X. (*ii*) Can you possibly sketch a picture to represent this vector in the similar manner as you do for (2) and (3)? Explain.

Ans:
(*i*)$X = [1, 1, 1, 1]^T$. (*ii*) Not possible. Since we live in a 3-D world, we can only draw vectors up to 3-D but not higher than that.

5. What is the dimensionality of the vectors in (1), (2), (3) and (4)?

Ans:
For (1), (2), the vectors are 2-demensional; For (3), 3-dimensional; For (4), 4-dimensional.

6. Consider the 3-vectors pair $X_1 = [1, 2, 3]$, $X_2 = [-1, -2, -3]$. (*i*) Find any possible values of k_1 and k_2, with $\{k_1\ k_2\} \neq \{0, 0\}$, such that $k_1X_b + k_1X_2 = 0$. (*ii*) Are the vectors linearly

independent?

Ans: (*i*) Any arbitrary value of $k_1= k_2 = k \neq 0$ will do. (*ii*). They are linearly dependent, since there exist values of $\{k_1, k_2\}$, not all zero, such that $k_1X_b+ k_1X_2= 0$.

7. Explain why is that a zero 3-vector is always linearly dependent with any 3-vector?

Ans: [0, 0, 0] is linearly dependent with any 3-vector X since there always exist some $\{k_1, k_2\}\neq\{0, 0\}$ such that $k_10 + k_2X = 0$ is satisfied, e.g. $k_1= 101$, $k_2= 0$.

8. Consider $X_1=[a, b, c]$ and $X_2=s[a, b, c]$, where s a non-zero scalar. (*i*) Are these 3-vectors linearly independent? (*ii*) Explain why you say so.

Ans:
$k_1X_1+k_2X_2= k_1X_1+ sk_2X_1= X_1(k_1+sk_2)$. For $k_1X_1+k_2X_2=X_1(k_1+sk_2) = 0$, any values of k_2 and k_1 satisfying the condition $k_1 + sk_2= 0$ will do the job, e.g. $k_1 = 0$, $k_2= 1$. In other words, there always exist NOT ALL ZERO coefficients $\{k_1, k_2\}$ that satisfy $k_1X_1+k_2X_2=0$. Hence, X_1, X_2 are NOT linearly independent.

9. Consider $X_1=[1, 2, 3]$ and $X_2=[4, 5, 6]$. (*i*) Are they linearly independent? (*ii*) Explain why you say so.

Ans:
For $k_1X_1+ k_2X_2=[0, 0, 0]$, we need $k_1 + 4k_2 = 0$, $2k_1 + 5k_2 = 0$, $3k_1+6k_2= 0$. The only possible solution is k_1, k_2 both being zero. This means that for $k_1X_1+ k_2X_2=[0, 0, 0]$, the coefficients $\{k_1, k_2\}$ *must be all zero*. This prove the linearly independence of X_1 and X_2.

10. Give a set of three distinct, non-zero 2-vectors, X_1, X_2, X_3 that are linearly independent.

Ans:
This is not possible. Such a set must necessarily be linearly dependent.

11. (*i*) Consider 3 distinct, non-zero 2-vectors. These vectors must be (linearly dependent / linearly independent). (*ii*), Consider 2 distinct, non-zero 3-vectors. Furthermore, these vectors are not in the form of $X_1=sX_2$ (in other words, they are not parallel nor anti-parallel). These vectors must be (linearly dependent / linearly independent).

Ans:
(*i*) linearly dependent; (*ii*) linearly independent.

12. Refer (9.5) in page 69, Ayers. Given a set of m vectors, we want to know whether they are linearly independent or otherwise. What is the easiest way (or one of the easier ways) to determine the linear independence of such a set of vectors?

Ans: Use row elementary operations to reduce the matrix A formed by these vectors to RREF. The number of non-zero row in the RREF of A is the rank of the matrix A, r. The rank, r, also tells us how many linearly independent vectors are there in the set of m vectors.
If $r = m$, then the set of this m vectors is linearly independent.
If $r < m$, then the set of m vectors is linearly dependent.
In such a case, there are exactly r vectors of the set which are linearly independent while each of the remaining m-r vectors can be expressed as a linear combination of these r vectors.

13. Consider the set S containing the following 4 3-vectors: $K_1=[1,1,1]^T$, $K_2=[1,3,5]^T$, $K_3=[1,5, 3]^T$, $K_4=[5, 3, 1]^T$; $S=\{K_1, K_1, K_3\ K_4\}$. (*i*) Form the matrix A whose rows are made up of the vectors K_i^T , i=1, 2, 3, 4. (*ii*) Reduce A into RREF. (*iii*) What is the rank of A? (*iv*) How many linearly independent vectors are there in the set S? (*v*) Are the vectors in set S linearly independent?

Ans:

(*i*) $A = \begin{pmatrix} K_1^T \\ K_2^T \\ K_3^T \\ K_4^T \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \\ 5 & 3 & 1 \end{pmatrix}$

(*ii*) $A\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (*iii*) $r = 3$

(iv) There are r=3 linearly independent vectors in the set S.

(v) Since the number of vectors in S, m = 4 is larger than the number of linearly independent vectors, r = 3, the set of vectors in S is not linearly independent. They are linearly dependent, by the virtue of theorem V, page 69, Ayers.

14. There is another way to prove the linearly independence of a set of vectors. Consider a set of three vectors K_1= (1, 4, 7)^T, K_2= (2, 5, 8)^T, K_3= (3, 6, 9)^T. Let's find out whether they are linearly independent or otherwise. If the set of these vectors is linearly independent, then the only solution to the homogeneous equation system

$$x_1K_1+x_2K_2+x_3K_3=0$$

is the trivial solution, i.e. $X=(x_1, x_2, x_3)^T=(0, 0, 0)^T$.

(i) If we write the homogeneous equation system in the matrix form of KX = 0. What is the matrix K? (ii) Reduce K into RREF to determine rank(K). (iii) How many unknowns are there in the HE system? (iv) By comparing your answer in (ii) and (iii) what can you say about the solution X? (iv) Is the set of three vectors linearly independent?

Ans:

(i) $x_1K_1+x_2K_2+x_3K_3 = (K_1, K_2, K_3)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = K\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$. Hence,

$$K=(K_1, K_2, K_3)=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}.$$

(ii) $K\sim\text{RREF}(K)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$. Hence, rank ($K$)=3.

(iii) Number of unknown = 3.

(iv) Since rank (K) = n = number of unknown, the HE system has only the trivial solution, $X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, see Ayers, page 78, the statement before theorem IV.

(v) Since the only solution to $x_1K_1+x_2K_2+x_3K_3=0$ is the trivial solution, by definition, the set of vectors is linearly independent.

15.

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$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \\ 5 & 3 & 1 \end{pmatrix} \xrightarrow{R_4^1(-5)} \begin{matrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \\ 0 & -2 & -4 \end{matrix} \xrightarrow{R_4^1(2)} \begin{matrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \\ 2 & 0 & -2 \end{matrix}$$

(wrong way)

$\downarrow R_1^2(-1)$ [This makes another 0 in the first column].

$$\begin{matrix} 0 & -2 & -4 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \\ 0 & -2 & -4 \end{matrix} \xrightarrow{R_4^1(-1)} \begin{matrix} 0 & -2 & -4 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \\ 0 & 0 & 0 \end{matrix} \xrightarrow{R_3^1} \begin{matrix} 1 & 5 & 3 \\ 1 & 3 & 5 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{matrix}$$

$$\xrightarrow{R_2^1(-1)} \begin{matrix} 1 & 5 & 3 \\ 0 & -2 & 2 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{matrix} \xrightarrow{R_3^2(-1)} \begin{matrix} 1 & 5 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{matrix}$$

not possible to make them into zero

$$\xrightarrow[R_2(-1/2)]{R_3(-1/6)} \begin{pmatrix} 1 & 5 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

REF

$\downarrow R_1^2(-5)$

$$\begin{matrix} 1 & 0 & 8 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \xrightarrow{R_1^3(-8)} \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \xrightarrow{R_2^3(1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Finally, RREF!