Page 331 - 337.

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In the preceding sections we were dealing with points and vectors in 2- and 3-space. Mathematicians in the nineteenth century, notably the English mathematicians Arthur Cayley (1821–1895) and James Joseph Sylvester (1814–1897) and the Irish mathematician William Rowan Hamilton (1805–1865), realized that the concepts of point and vector could be generalized, that vectors could be described, or defined, by analytic rather than geometric properties. This was a truly significant breakthrough in the history of mathematics. There is no need to stop with three dimensions. Ordered quadruples $\langle a_1, a_2, a_3, a_4\rangle$, quintuples $\langle a_1, a_2, a_3, a_4, a_5\rangle$, and n-tuples $\langle a_1, a_2, \ldots, a_n\rangle$ of real numbers can be thought of as vectors just as well as ordered pairs $\langle a_1, a_2\rangle$ and ordered triples $\langle a_1, a_2, a_3\rangle$, the only difference being that we lose our visualization of directed line segments or arrows in four-dimensional, five-dimensional, or n-dimensional space.

In formal terms, a **vector in n-space** is any ordered n-tuple $\langle a_1, a_2, \ldots, a_n\rangle$ of real numbers called the **components** of **a**. The set of all vectors in n-space is denoted by R^n. The concepts of vector addition, scalar multiplication, equality, and so on listed in Definition 7.2 carry over to R^n in a natural way. For example, if $\mathbf{a} = \langle a_1, a_2, \ldots, a_n\rangle$ and $\mathbf{b} = \langle b_1, b_2, \ldots, b_n\rangle$, then addition and scalar multiplication in n-space are defined by

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n\rangle \quad \text{and} \quad k\mathbf{a} = \langle ka_1, ka_2, \cdots, ka_n\rangle. \tag{1}$$

The zero vector in R^n is $\langle 0, 0, \ldots, 0\rangle$. The notion of **magnitude** or **length** of a vector $\mathbf{a} = \langle a_1, a_2, \ldots, a_n\rangle$ in n-space is just an extension of that concept in 2- and 3-space:

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

A vector **a** in R^n is a **unit vector** if $\|\mathbf{a}\| = 1$. The **dot product** of two n-vectors $\mathbf{a} = \langle a_1, a_2, \ldots, a_n\rangle$ and $\mathbf{b} = \langle b_1, b_2, \ldots, b_n\rangle$ is the real number defined by

$$\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2, \ldots, a_n\rangle \cdot \langle b_1, b_2, \ldots, b_n\rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n. \tag{2}$$

Two nonzero vectors **a** and **b** in R^n are **orthogonal** if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Vector Space We can even go beyond the notion of a vector as an ordered n-tuple in R^n. A **vector** can be defined as anything we want it to be: an ordered n-tuple, a number, or even a function. But we are particularly interested in vectors that are elements in a special kind of set called a **vector space.** Fundamental to this notion of vector space are two kinds of objects, vectors and scalars, and two algebraic operations analogous to those given in (1). For a set of vectors we want to be able to add two vectors in this set and get another vector in the same set, and we want to multiply a vector by a scalar and obtain a vector in the same set. Whether a set of objects is a vector space depends on whether the set possesses these two algebraic operations along with certain other properties. These properties, the axioms of a vector space, are given next.

DEFINITION 7.5 Vector Space

Let V be a set of elements on which two operations called **vector addition** and **scalar multiplication** are defined. Then V is said to be a **vector space** if the following ten properties are satisfied.

Axioms for Vector Addition

(*i*) If $\mathbf{x}$ and $\mathbf{y}$ are in V, then $\mathbf{x} + \mathbf{y}$ is in V.
(*ii*) For all $\mathbf{x}$, $\mathbf{y}$ in V, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. (commutative law)
(*iii*) For all $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in V, $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$. (associative law)
(*iv*) There is a unique vector $\mathbf{0}$ in V such that $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{0}$. (zero vector)
(*v*) For each $\mathbf{x}$ in V, there exists a vector $-\mathbf{x}$ such that $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$. (negative of a vector)

Axioms for Scalar Multiplication

(*vi*) If k is any scalar and $\mathbf{x}$ is in V, then $k\mathbf{x}$ is in V.
(*vii*) $k(\mathbf{x} + \mathbf{y}) = k\mathbf{x} + k\mathbf{y}$
(*viii*) $(k_1 + k_2)\mathbf{x} = k_1\mathbf{x} + k_2\mathbf{x}$ (distributive laws)
(*ix*) $k_1(k_2\mathbf{x}) = (k_1k_2)\mathbf{x}$
(*x*) $1\mathbf{x} = \mathbf{x}$

In this abbreviated introduction to abstract vectors, we shall take the scalars in Definition 7.5 to be real numbers. In this case V is referred to as a *real* vector space, although we shall not belabor this term. When the scalars are allowed to be complex numbers, we obtain a *complex* vector space. Since properties (*i*)–(*viii*) on page 300 are the prototypes for the axioms in Definition 7.5, it is clear that R^2 is a vector space. Moreover, since vectors in R^3 and R^n have these same properties, we conclude that R^3 and R^n are also vector spaces. Axioms (*i*) and (*vi*) are called the **closure** axioms; we say that a vector space V is closed under vector addition and scalar multiplication. Note too that concepts such as length and the dot product are not part of the axiomatic structure of a vector space.

Example 1 **Checking the Closure Axioms**

Determine whether the sets (a) $V = \{1\}$ and (b) $V = \{0\}$ under ordinary addition and multiplication by real numbers are vector spaces.

SOLUTION (a) For this system consisting of one element, many axioms are violated. In particular, axioms (*i*) and (*vi*) of closure are not satisfied. Neither the sum $1 + 1 = 2$ nor the scalar multiple $k \cdot 1 = k$, for $k \neq 1$, is in V. Hence V is not a vector space.

(b) In this case the closure axioms are satisfied, since $0 + 0 = 0$ and $k \cdot 0 = 0$ for any real number k. The commutative and associative axioms are satisfied, since $0 + 0 = 0 + 0$ and $0 + (0 + 0) = (0 + 0) + 0$. In this manner it is easy to verify that the remaining axioms are also satisfied. Hence V is a vector space. □

The vector space $V = \{\mathbf{0}\}$ is often called the **trivial** or **zero vector space.**

Readers should not take the names vector *addition* and scalar *multiplication* too literally. These operations are *defined,* and the student must accept them at face value even though these operations may not bear any resemblance to the usual understanding of ordinary addition and multiplication in, say, R, R^2, R^3, or R^n. For example, the addition of two vectors $\mathbf{x}$ and $\mathbf{y}$ could be $\mathbf{x} - \mathbf{y}$. With this forewarning consider the next example.

Example 2 **An Example of a Vector Space**

Consider the set V of positive real numbers where addition is defined by

$$\mathbf{x} + \mathbf{y} = xy$$

and scalar multiplication is defined by

$$k\mathbf{x} = x^k.$$

Determine whether V is a vector space.

SOLUTION We shall go through all ten axioms.

(*i*) For $\mathbf{x} = x > 0$ and $\mathbf{y} = y > 0$, $\mathbf{x} + \mathbf{y} = xy > 0$. Thus, the sum $\mathbf{x} + \mathbf{y}$ is in V; V is closed under addition.

(*ii*) Since multiplication of positive real numbers is commutative, we have for all $\mathbf{x} = x$ and $\mathbf{y} = y$ in V, $\mathbf{x} + \mathbf{y} = xy = yx = \mathbf{y} + \mathbf{x}$. Thus, addition is commutative.

(*iii*) For all $\mathbf{x} = x$, $\mathbf{y} = y$, $\mathbf{z} = z$ in V, $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = x(yz) = (xy)z = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$. Thus, addition is associative.

(*iv*) Since $\mathbf{1} + \mathbf{x} = 1x = x = \mathbf{x}$ and $\mathbf{x} + \mathbf{1} = x1 = x = \mathbf{x}$, the zero vector $\mathbf{0}$ is $\mathbf{1} = 1$.

(*v*) If we define $-\mathbf{x} = \dfrac{1}{x}$, then $\mathbf{x} + (-\mathbf{x}) = x\dfrac{1}{x} = 1 = \mathbf{1} = \mathbf{0}$ and $(-\mathbf{x}) + \mathbf{x} = \dfrac{1}{x}x = 1 = \mathbf{1} = \mathbf{0}$. The negative of a vector is its reciprocal.

(*vi*) If k is any scalar and $\mathbf{x} = x > 0$ is any vector, then $k\mathbf{x} = x^k > 0$. Hence V is closed under scalar multiplication.

(*vii*) If k is any scalar, then $k(\mathbf{x} + \mathbf{y}) = (xy)^k = x^k y^k = k\mathbf{x} + k\mathbf{y}$.

(*viii*) For scalars k_1 and k_2, $(k_1 + k_2)\mathbf{x} = x^{(k_1+k_2)} = x^{k_1}x^{k_2} = k_1\mathbf{x} + k_2\mathbf{x}$.

(*ix*) For scalars k_1 and k_2, $k_1(k_2\mathbf{x}) = (x^{k_2})^{k_1} = x^{k_1k_2} = (k_1k_2)\mathbf{x}$.

(*x*) $1\mathbf{x} = x^1 = x = \mathbf{x}$.

Since all the axioms of Definition 7.5 are satisfied, we conclude that V is a vector space. ❑

Following are some important vector spaces. The operations of vector addition and scalar multiplication are the usual operations associated with the set.

- The set R of real numbers
- The set P_n of polynomials of degree less than or equal to n
- The set of real-valued functions f defined on the entire real line
- The set $C[a, b]$ of real-valued functions f continuous on the closed interval $a \le x \le b$
- The set $C(-\infty, \infty)$ of real-valued functions f continuous on the entire real line
- The set $C^n[a, b]$ of all real-valued functions f for which $f, f', \ldots, f^{(n)}$ exist and are continuous on the interval $[a, b]$

Subspace It may happen that a subset of vectors W of a vector space V is itself a vector space.

> **DEFINITION 7.6 Subspace of a Vector Space**
>
> If a subset W of a vector space V is itself a vector space under the operations of vector addition and scalar multiplication defined on V, then W is called a **subspace** of V.

Every vector space V has at least two subspaces: V itself and the zero subspace $\{\mathbf{0}\}$; $\{\mathbf{0}\}$ is a subspace since the zero vector must be an element in every vector space.

To show that a subset W of a vector space V is a subspace it is not necessary to demonstrate that all ten axioms are satisfied. Since all the vectors in W are also in V, these vectors must satisfy axioms such as (*ii*) and (*iii*). In other words, W inherits most of the properties of a vector space from V. Indeed, to show that a subset W is a subspace we need only check the two closure axioms.

> **THEOREM 7.4 Criteria for a Subspace**
>
> A nonempty subset W of a vector space V is a subspace of V if and only if W is closed under vector addition and scalar multiplication defined on V:
>
> (*i*) If $\mathbf{x}$ and $\mathbf{y}$ are in W, then $\mathbf{x} + \mathbf{y}$ is in W.
>
> (*ii*) If $\mathbf{x}$ is in W and k is any scalar, then $k\mathbf{x}$ is W.

Example 3 **A Subspace**

Suppose $\mathbf{f}$ and $\mathbf{g}$ are continuous real-valued functions defined on the entire real line. Then we know from calculus that $\mathbf{f} + \mathbf{g}$ and $k\mathbf{f}$, for any real number k, are continuous and real-valued functions. From this we can conclude that $C(-\infty, \infty)$ is a subspace of the vector space of real-valued functions defined on the entire real line. □

It is always a good idea to have concrete visualizations of vector spaces and subspaces. The subspaces of the vector space R^3 of three-dimensional vectors can be easily visualized by thinking of a vector as point (a_1, a_2, a_3). Of course, R^3 and $\{\mathbf{0}\}$ are subspaces; other subspaces are all lines passing through the origin and all planes passing through the origin. The lines and planes must pass through the origin since the zero vector $\mathbf{0} = (0, 0, 0)$ must be an element in each subspace.

Similar to Definition 3.1, we can define linearly independent vectors:

> **DEFINITION 7.7 Linear Independence**
>
> A set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is said to be **linearly independent** if the only constants satisfying the equation
>
> $$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_n\mathbf{x}_n = \mathbf{0} \tag{3}$$
>
> are $k_1 = k_2 = \cdots = k_n = 0$. If the set of vectors is not linearly independent, then it is said to be **linearly dependent**.

In R^3, the vectors $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$ are linearly independent, since the equation $k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k} = 0$ is the same as

$$k_1\langle 1, 0, 0\rangle + k_2\langle 0, 1, 0\rangle + k_3\langle 0, 0, 1\rangle = \langle 0, 0, 0\rangle \quad \text{or} \quad \langle k_1, k_2, k_3\rangle = \langle 0, 0, 0\rangle.$$

By equality of vectors, (*ii*) of Definition 7.2, we conclude that $k_1 = 0$, $k_2 = 0$, and $k_3 = 0$. In Definition 7.7, linear dependence means that there are constants $k_1, k_2, \ldots, k_n$ *not all zero* such that $k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_n\mathbf{x}_n = \mathbf{0}$. Hence the vectors $\mathbf{a} = \langle 1, 1, 1\rangle$, $\mathbf{b} = \langle 2, -1, 4\rangle$, and $\mathbf{c} = \langle 3, 0, 5\rangle$ are linearly dependent since $\mathbf{a} + \mathbf{b} - \mathbf{c} = \mathbf{0}$; that is, (3) is satisfied when $k_1 = k_2 = 1$ and $k_3 = -1$. We observe that two vectors are linearly independent if neither is a constant multiple of the other.

Basis Any vector in R^3 can be written as a linear combination of the linearly independent vectors **i, j,** and **k.** In Section 7.2 we said that these vectors form a **basis** for the system of three-dimensional vectors.

DEFINITION 7.8 Basis for a Vector Space

Consider a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ in a vector space V. If this set of vectors is linearly independent and if every vector in V can be expressed as a linear combination of these vectors, then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is said to be a **basis** for V.

Although we cannot prove it in this course, every vector space has a basis. The vector space P_n of all polynomials of degree less than or equal to n has the basis $1, x, x^2, \ldots, x^n$, since any vector (polynomial) $p(x)$ of degree n or less can be written as the linear combination $p(x) = c_nx^n + \cdots + c_2x^2 + c_1x + c_0$. A vector space may have many bases. We mentioned previously that the vectors **i, j, k** are a basis for R^3. But it can be proved that $\mathbf{u}_1 = \langle 1, 0, 0\rangle$, $\mathbf{u}_2 = \langle 1, 1, 0\rangle$, and $\mathbf{u}_3 = \langle 1, 1, 1\rangle$ are linearly independent (see Problem 23 in Exercises 7.6) and, furthermore, every vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ can be expressed as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$:

$$\mathbf{a} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3.$$

Hence, the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are another basis for R^3. Indeed, any set of *three* linearly independent vectors in R^3 forms a basis for that space. However, the vectors **i, j, k** are referred to as the **standard basis.** For the space P_n the standard basis is $1, x, x^2, \ldots, x^n$; for the vector space R^n the standard basis consists of the n vectors $\mathbf{e}_1 = \langle 1, 0, 0, \ldots, 0\rangle$, $\mathbf{e}_2 = \langle 0, 1, 0, \ldots, 0\rangle, \ldots, \mathbf{e}_n = \langle 0, 0, 0, \ldots, 1\rangle$.

DEFINITION 7.9 Dimension of a Vector Space

The number of vectors in a basis for a vector space is said to be the **dimension** of the space.

Example 4 **Dimensions of Some Vector Spaces**

In agreement with our intuition, the dimensions of the vector spaces R, R^2, R^3, and R^n are, in turn, 1, 2, 3, and n. Since there are $n + 1$ vectors in the basis $1, x, x^2, \ldots, x^n$, the dimension of the vector space P_n of polynomials of degree less than or equal to n is $n + 1$. ❑

If a vector space has dimension n, then every basis for that space must contain n vectors. If the basis of a vector space contains a finite number of vectors, then we say that the vector space is **finite dimensional;** otherwise it is **infinite dimensional.** The function space $C^n(I)$ of n times continuously differentiable functions on an interval I is an example of an infinite-dimensional vector space. The zero vector space $\{\mathbf{0}\}$ is given special consideration. This space contains only $\mathbf{0}$ and since $\mathbf{0}$ is linearly dependent, it is not a basis. In this case it is customary to take the empty set as the basis and to define the dimension of $\{\mathbf{0}\}$ as zero.

Linear Differential Equations Consider the homogeneous linear nth-order differential equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{4}$$

on an interval I on which the coefficients are continuous and $a_n(x) \neq 0$ for every x in the interval. A solution y_1 of (4) is necessarily a vector in the vector space $C^n(I)$. In addition, we know from the theory examined in Section 3.1 that if y_1 and y_2 are solutions of (4), then the sum $y_1 + y_2$ and any constant multiple ky_1 are also solutions. Since the solution set is closed under addition and scalar multiplication, it follows from Theorem 7.4 that the solution set of (4) is a subspace of $C^n(I)$. Hence the solution set of (4) deserves to be called the **solution space** of the differential equation. We also know that if $y_1, y_2, \ldots, y_n$ are linearly independent solutions of (4), then its general solution is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x).$$

Recall that any solution of the equation can be found from this general solution by specialization of the constants $c_1, c_2, \ldots, c_n$. Hence the set of linearly independent solutions $y_1, y_2, \ldots, y_n$ comprise a basis for the solution space. The dimension of this solution space is n.

Example 5 **Dimension of a Solution Space**

The general solution of the homogeneous linear second-order differential equation $y'' + 25y = 0$ is $y = c_1 \cos 5x + c_2 \sin 5x$. A basis for the solution space consists of the linearly independent vectors $\cos 5x$ and $\sin 5x$. The solution space is two-dimensional. □

The set of solutions of a nonhomogeneous linear differential equation is not a vector space. Several axioms of a vector space are not satisfied; most notably the set of solutions does not contain a zero vector. In other words, $y = 0$ is not a solution of a nonhomogeneous linear differential equation.

Span If S denotes any set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ in a vector space V, then a sum of the form $k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \ldots + k_n\mathbf{x}_n$, where the k_i, $i = 1, \ldots, n$ are scalars, is called a **linear combination** of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. The set of *all* linear combinations of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is called the **span** of the vectors and written Span(S) or Span($\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$). It is left as an exercise to show that Span(S) is a subspace of the vector space V. See Problem 33 in Exercises 7.6. Span(S) is said to be a subspace *spanned* by the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. If every vector in the vector space V can be written as a linear combination of the vectors in S, then S is called a **spanning set** for V. For example, each of the three sets

$$\mathbf{i}, \mathbf{j}, \mathbf{k}, \quad \text{and} \quad \mathbf{i}, \mathbf{i}+\mathbf{j}, \mathbf{i}+\mathbf{j}+\mathbf{k}, \quad \text{and} \quad \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{i}+\mathbf{j}, \mathbf{i}+\mathbf{j}+\mathbf{k}$$

is a spanning set for the vector space R^3. But note that the first two sets are linearly independent, whereas the third set is dependent. With these new concepts we can rephrase Definitions 7.8 and 7.9 in the following manner:

A set S of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ *in a vector space V is a* **basis** *for V if S is a spanning set for V and S is linearly independent. The number of vectors in this spanning set S is the* **dimension** *of the space V.*

EXERCISES 7.6

Answers to odd-numbered problems begin on page A-42.

In Problems 1–10, determine whether the given set is a vector space. If not, give at least one axiom that is not satisfied. Unless stated to the contrary, assume that vector addition and scalar multiplication are the ordinary operations defined on that set.

1. The set of vectors $\langle a_1, a_2\rangle$, where $a_1 \geq 0$, $a_2 \geq 0$

2. The set of vectors $\langle a_1, a_2\rangle$, where $a_2 = 3a_1 + 1$

3. The set of vectors $\langle a_1, a_2\rangle$, scalar multiplication defined by $k\langle a_1, a_2\rangle = \langle ka_1, 0\rangle$

4. The set of vectors $\langle a_1, a_2\rangle$, where $a_1 + a_2 = 0$

5. The set of vectors $\langle a_1, a_2, 0\rangle$

6. The set of vectors $\langle a_1, a_2\rangle$, addition and scalar multiplication defined by

$$\langle a_1, a_2\rangle + \langle b_1, b_2\rangle = \langle a_1 + b_1 + 1, a_2 + b_2 + 1\rangle$$

$$k\langle a_1, a_2\rangle = \langle ka_1 + k - 1, ka_2 + k - 1\rangle$$

7. The set of real numbers, addition defined by $\mathbf{x} + \mathbf{y} = x - y$

8. The set of complex numbers $a + bi$, where $i^2 = -1$, addition and scalar multiplication defined by

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$$

$$k(a + bi) = ka + kbi, \quad k \text{ a real number}$$

9. The set of arrays of real numbers

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$, addition and scalar multiplication defined by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix} = \begin{pmatrix} a_{12} + b_{12} & a_{11} + b_{11} \\ a_{22} + b_{22} & a_{21} + b_{21}\end{pmatrix}$$

$$k\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22}\end{pmatrix}$$

10. The set of all polynomials of degree 2

In Problems 11–16, determine whether or not the given set is a subspace of the vector space $C(-\infty, \infty)$.

11. All functions f such that $f(1) = 0$

12. All functions f such that $f(0) = 1$

13. All nonnegative functions f

14. All functions f such that $f(-x) = f(x)$

15. All differentiable functions f

16. All functions f of the form $f(x) = c_1 e^x + c_2 x e^x$

In Problems 17–20, determine whether or not the given set is a subspace of the indicated vector space.

17. Polynomials of the form $p(x) = c_3 x^3 + c_1 x$; P_3

18. Polynomials p that are divisible by $x - 2$; P_2

19. All unit vectors; R^3

20. Functions f such that $\int_a^b f(x)\,dx = 0$; $C[a, b]$

21. In 3-space a line through the origin can be written as $S = \{(x, y, z) \mid x = at,\ y = bt,\ z = ct,\ a, b, c \text{ real numbers}\}$. With addition and scalar multiplication the same as for vectors $\langle x, y, z\rangle$ show that S is a subspace of R^3.

22. In 3-space a plane through the origin can be written as $S = \{(x, y, z) \mid ax + by + cz = 0,\ a, b, c \text{ real numbers}\}$. Show that S is a subspace of R^3.

23. The vectors $\mathbf{u}_1 = \langle 1, 0, 0\rangle$, $\mathbf{u}_2 = \langle 1, 1, 0\rangle$, and $\mathbf{u}_3 = \langle 1, 1, 1\rangle$ form a basis for the vector space R^3.

(a) Show that $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent.

(b) Express the vector $\mathbf{a} = \langle 3, -4, 8\rangle$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

24. The vectors $p_1(x) = x + 1$, $p_2(x) = x - 1$ form a basis for the vector space P_1.

(a) Show that $p_1(x)$ and $p_2(x)$ are linearly independent.

(b) Express the vector $p(x) = 5x + 2$ as a linear combination of $p_1(x)$ and $p_2(x)$.