*SCHAUM'S OUTLINE OF*

# THEORY AND PROBLEMS

OF

# MATRICES

## SI (METRIC) EDITION

BY

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**SCHAUM'S OUTLINE SERIES**

*McGraw-Hill International Book Company*
*Singapore*

ASIAN STUDENT EDITION

FIRST IMPRESSION JUNE 1982

ISBN 0-07-99035-2

Printed and Bound in Singapore by KIN KEONG PRINTING CO. PTE. LTD.

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# Preface

Elementary matrix algebra has now become an integral part of the mathematical background necessary for such diverse fields as electrical engineering and education, chemistry and sociology, as well as for statistics and pure mathematics. This book, in presenting the more essential material, is designed primarily to serve as a useful supplement to current texts and as a handy reference book for those working in the several fields which require some knowledge of matrix theory. Moreover, the statements of theory and principle are sufficiently complete that the book could be used as a text by itself.

The material has been divided into twenty-six chapters, since the logical arrangement is thereby not disturbed while the usefulness as a reference book is increased. This also permits a separation of the treatment of real matrices, with which the majority of readers will be concerned, from that of matrices with complex elements. Each chapter contains a statement of pertinent definitions, principles, and theorems, fully illustrated by examples. These, in turn, are followed by a carefully selected set of solved problems and a considerable number of supplementary exercises.

The student new to matrix algebra soon finds that the solutions of numerical exercises are disarmingly simple. Difficulties are likely to arise from the constant round of definition, theorem, proof. The trouble here is essentially a matter of lack of mathematical maturity, and normally to be expected, since usually the student's previous work in mathematics has been concerned with the solution of numerical problems while precise statements of principles and proofs of theorems have in large part been deferred for later courses. The aim of the present book is to enable the reader, if he persists through the introductory paragraphs and solved problems in any chapter, to develop a reasonable degree of self-assurance about the material.

The solved problems, in addition to giving more variety to the examples illustrating the theorems, contain most of the proofs of any considerable length together with representative shorter proofs. The supplementary problems call both for the solution of numerical exercises and for proofs. Some of the latter require only proper modifications of proofs given earlier; more important, however, are the many theorems whose proofs require but a few lines. Some are of the type frequently misnamed "obvious" while others will be found to call for considerable ingenuity. None should be treated lightly, however, for it is due precisely to the abundance of such theorems that elementary matrix algebra becomes a natural first course for those seeking to attain a degree of mathematical maturity. While the large number of these problems in any chapter makes it impractical to solve all of them before moving to the next, special attention is directed to the supplementary problems of the first two chapters. A mastery of these will do much to give the reader confidence to stand on his own feet thereafter.

The author wishes to take this opportunity to express his gratitude to the staff of the Schaum Publishing Company for their splendid cooperation.

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Carlisle, Pa.
October, 1962

# Chapter 1

## Matrices

**A RECTANGULAR ARRAY OF NUMBERS** enclosed by a pair of brackets, such as

$$(a)\ \begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \quad \text{and} \quad (b)\ \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 4 & 7 & 6 \end{bmatrix},$$

and subject to certain rules of operations given below is called a **matrix**. The matrix $(a)$ could be considered as the **coefficient matrix** of the system of homogeneous linear equations $\begin{cases} 2x+3y+7z = 0 \\ x - y + 5z = 0 \end{cases}$ or as the **augmented** matrix of the system of non-homogeneous linear equations $\begin{cases} 2x+3y = 7 \\ x - y = 5 \end{cases}$ Later, we shall see how the matrix may be used to obtain solutions of these systems. The matrix $(b)$ could be given a similar interpretation or we might consider its rows as simply the coordinates of the points $(1,3,1)$, $(2,1,4)$, and $(4,7,6)$ in ordinary space. The matrix will be used later to settle such questions as whether or not the three points lie in the same plane with the origin or on the same line through the origin.

In the matrix

$$\textbf{(1.1)} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots\cdots & a_{mn} \end{bmatrix}$$

the numbers or functions $a_{ij}$ are called its **elements.** In the double subscript notation, the first subscript indicates the row and the second subscript indicates the column in which the element stands. Thus, all elements in the second row have 2 as first subscript and all the elements in the fifth column have 5 as second subscript. A matrix of $m$ rows and $n$ columns is said to be of order "$m$ by $n$" or $m \times n$.

(In indicating a matrix pairs of parentheses, ( ), and double bars, ‖ ‖, are sometimes used. We shall use the double bracket notation throughout.)

At times the matrix **(1.1)** will be called "the $m \times n$ matrix $[a_{ij}]$" or "the $m \times n$ matrix $A = [a_{ij}]$". When the order has been established, we shall write simply "the matrix $A$".

**SQUARE MATRICES.** When $m = n$, **(1.1)** is square and will be called a **square matrix** of order $n$ or an $n$-square matrix.

In a square matrix, the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are called its **diagonal elements.**

The sum of the diagonal elements of a square matrix $A$ is called the **trace** of $A$.

**EQUAL MATRICES.** Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be **equal** $(A = B)$ if and only if they have the same order and each element of one is equal to the corresponding element of the other, that is, if and only if

$$a_{ij} = b_{ij}, \qquad (i = 1, 2, \ldots, m;\; j = 1, 2, \ldots, n)$$

Thus, two matrices are equal if and only if one is a duplicate of the other.

**ZERO MATRIX.** A matrix, every element of which is zero, is called a **zero matrix**. When $A$ is a zero matrix and there can be no confusion as to its order, we shall write $A = 0$ instead of the $m \times n$ array of zero elements.

**SUMS OF MATRICES.** If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $m \times n$ matrices, their sum (difference), $A \pm B$, is defined as the $m \times n$ matrix $C = [c_{ij}]$, where each element of $C$ is the sum (difference) of the corresponding elements of $A$ and $B$. Thus, $A \pm B = [a_{ij} \pm b_{ij}]$.

**Example 1.** If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 1+2 & 2+3 & 3+0 \\ 0+(-1) & 1+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix}$$

and

$$A - B = \begin{bmatrix} 1-2 & 2-3 & 3-0 \\ 0-(-1) & 1-2 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Two matrices of the same order are said to be **conformable** for addition or subtraction. *Two matrices of different orders cannot be added or subtracted.* For example, the matrices (*a*) and (*b*) above are non-conformable for addition and subtraction.

The sum of $k$ matrices $A$ is a matrix of the same order as $A$ and each of its elements is $k$ times the corresponding element of $A$. We define: If $k$ is any **scalar** (we call $k$ a scalar to distinguish it from $[k]$ which is a $1 \times 1$ matrix) then by $kA = Ak$ is meant the matrix obtained from $A$ by multiplying each of its elements by $k$.

**Example 2.** If $A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$, then

$$A + A + A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 6 & 9 \end{bmatrix} = 3A = A \cdot 3$$

and

$$-5A = \begin{bmatrix} -5(1) & -5(-2) \\ -5(2) & -5(3) \end{bmatrix} = \begin{bmatrix} -5 & 10 \\ -10 & -15 \end{bmatrix}$$

In particular, by $-A$, called the **negative** of $A$, is meant the matrix obtained from $A$ by multiplying each of its elements by $-1$ or by simply changing the sign of **all** of its elements. For every $A$, we have $A + (-A) = 0$, where 0 indicates the zero matrix of the same order as $A$.

Assuming that the matrices $A, B, C$ are conformable for addition, we state:

(*a*) $A + B = B + A$ (commutative law)
(*b*) $A + (B + C) = (A + B) + C$ (associative law)
(*c*) $k(A + B) = kA + kB = (A + B)k$, $k$ a scalar
(*d*) There exists a matrix $D$ such that $A + D = B$.

These laws are a result of the laws of elementary algebra governing the addition of numbers and polynomials. They show, moreover,

1. Conformable matrices obey the same laws of addition as the elements of these matrices.

**MULTIPLICATION.** By the product $AB$ **in that order** of the $1 \times m$ matrix $A = [a_{11}\ a_{12}\ a_{13}\ \ldots\ a_{1m}]$ and the $m \times 1$ matrix $B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ \vdots \\ b_{m1} \end{bmatrix}$ is meant the $1 \times 1$ matrix $C = [a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1}]$.

That is,

$$[a_{11}\ a_{12}\ \ldots\ a_{1m}] \cdot \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix} = [a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1}] = \left[\sum_{k=1}^{m} a_{1k}b_{k1}\right].$$

Note that the operation is **row by column;** each element of the row is multiplied into the corresponding element of the column and then the products are summed.

**Example 3.** (a) $[2\ \ 3\ \ 4]\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = [2(1)+3(-1)+4(2)] = [7]$

(b) $[3\ \ -1\ \ 4]\begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix} = [-6-6+12] = 0$

By the product $AB$ **in that order** of the $m \times p$ matrix $A = [a_{ij}]$ and the $p \times n$ matrix $B = [b_{ij}]$ is meant the $m \times n$ matrix $C = [c_{ij}]$ where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}, \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n).$$

Think of $A$ as consisting of $m$ rows and $B$ as consisting of $n$ columns. In forming $C = AB$ each row of $A$ is multiplied once and only once into each column of $B$. The element $c_{ij}$ of $C$ is then the product of the $i$th row of $A$ and the $j$th column of $B$.

**Example 4.**

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}$$

3x2 2x2 3x2

The product $AB$ is defined or $A$ is **conformable** to $B$ for multiplication only when the number of columns of $A$ is equal to the number of rows of $B$. If $A$ is conformable to $B$ for multiplication ($AB$ is defined), $B$ is not necessarily conformable to $A$ for multiplication ($BA$ may or may not be defined).

See Problems 3-4.

Assuming that $A, B, C$ are conformable for the indicated sums and products, we have

(e) $A(B+C) = AB + AC$ (first distributive law)

(f) $(A+B)C = AC + BC$ (second distributive law)

(g) $A(BC) = (AB)C$ (associative law)

However,

(h) $AB \neq BA$, generally,

(i) $AB = 0$ does not necessarily imply $A = 0$ or $B = 0$,

(j) $AB = AC$ does not necessarily imply $B = C$.

See Problems 3-

**PRODUCTS BY PARTITIONING.** Let $A = [a_{ij}]$ be of order $m \times p$ and $B = [b_{ij}]$ be of order $p \times n$. In

NOT TO BE COVERED

[illegible] and $B$ [illegible] parti-

[illegible]

$$A = \begin{bmatrix} (m_1 \times p_1) & (m_1 \times p_2) & (m_1 \times p_3) \\ (m_2 \times p_1) & (m_2 \times p_2) & (m_2 \times p_3) \end{bmatrix}, \qquad B = \begin{bmatrix} (p_1 \times n_1) & (p_1 \times n_2) \\ (p_2 \times n_1) & (p_2 \times n_2) \\ (p_3 \times n_1) & (p_3 \times n_2) \end{bmatrix}$$

or

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$$

In any such partitioning, it is necessary that the columns of $A$ and the rows of $B$ be partitioned in exactly the same way; however $m_1, m_2, n_1, n_2$ may be any non-negative (including 0) integers such that $m_1 + m_2 = m$ and $n_1 + n_2 = n$. Then

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} & A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} & A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = C$$

**Example 5.** Compute $AB$, given $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 3 & 1 & 2 \end{bmatrix}$

Partitioning so that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 3 & 2 & 0 \\ \hline 1 & 0 & 1 \end{array}\right] \quad \text{and} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ \hline 2 & 3 & 1 & 2 \end{array}\right],$$

we have

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}[2 \;\; 3 \;\; 1] & \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}[2] \\ [1 \;\; 0]\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} + [1][2 \;\; 3 \;\; 1] & [1 \;\; 0]\begin{bmatrix} 0 \\ 0 \end{bmatrix} + [1][2] \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 4 & 3 & 3 \\ 7 & 5 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ [1 \;\; 1 \;\; 1] + [2 \;\; 3 \;\; 1] & [0] + [2] \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 4 & 3 & 3 \\ 7 & 5 & 5 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ [3 \;\; 4 \;\; 2] & [2] \end{bmatrix} = \begin{bmatrix} 4 & 3 & 3 & 0 \\ 7 & 5 & 5 & 0 \\ 3 & 4 & 2 & 2 \end{bmatrix}$$

See also Problem 9.

Let $A, B, C, \ldots$ be $n$-square matrices. Let $A$ be partitioned into matrices of the indicated orders

$$\begin{bmatrix} (p_1 \times p_1) & (p_1 \times p_2) & \ldots & (p_1 \times p_s) \\ (p_2 \times p_1) & (p_2 \times p_2) & \ldots & (p_2 \times p_s) \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ (p_s \times p_1) & (p_s \times p_2) & \ldots & (p_s \times p_s) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1s} \\ A_{21} & A_{22} & \ldots & A_{2s} \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ A_{s1} & A_{s2} & \ldots & A_{ss} \end{bmatrix}$$

and let $B, C, \ldots$ be partitioned in exactly the same manner. Then sums, differences, and products may be formed using the matrices $A_{11}, A_{12}, \ldots; B_{11}, B_{12}, \ldots; C_{11}, C_{12}, \ldots$.

# SOLVED PROBLEMS

**1.** (a) $$\begin{bmatrix}1&2&-1&0\\4&0&2&1\\2&-5&1&2\end{bmatrix}+\begin{bmatrix}3&-4&1&2\\1&5&0&3\\2&-2&3&-1\end{bmatrix}=\begin{bmatrix}1+3&2+(-4)&-1+1&0+2\\4+1&0+5&2+0&1+3\\2+2&-5+(-2)&1+3&2+(-1)\end{bmatrix}=\begin{bmatrix}4&-2&0&2\\5&5&2&4\\4&-7&4&1\end{bmatrix}$$

(b) $$\begin{bmatrix}1&2&-1&0\\4&0&2&1\\2&-5&1&2\end{bmatrix}-\begin{bmatrix}3&-4&1&2\\1&5&0&3\\2&-2&3&-1\end{bmatrix}=\begin{bmatrix}1-3&2+4&-1-1&0-2\\4-1&0-5&2-0&1-3\\2-2&-5+2&1-3&2+1\end{bmatrix}=\begin{bmatrix}-2&6&-2&-2\\3&-5&2&-2\\0&-3&-2&3\end{bmatrix}$$

(c) $$3\begin{bmatrix}1&2&-1&0\\4&0&2&1\\2&-5&1&2\end{bmatrix}=\begin{bmatrix}3&6&-3&0\\12&0&6&3\\6&-15&3&6\end{bmatrix}$$

(d) $$-\begin{bmatrix}1&2&-1&0\\4&0&2&1\\2&-5&1&2\end{bmatrix}=\begin{bmatrix}-1&-2&1&0\\-4&0&-2&-1\\-2&5&-1&-2\end{bmatrix}$$

**2.** If $A=\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}$ and $B=\begin{bmatrix}-3&-2\\1&-5\\4&3\end{bmatrix}$, find $D=\begin{bmatrix}p&q\\r&s\\t&u\end{bmatrix}$ such that $A+B-D=0$.

If $A+B-D=\begin{bmatrix}1-3-p&2-2-q\\3+1-r&4-5-s\\5+4-t&6+3-u\end{bmatrix}=\begin{bmatrix}-2-p&-q\\4-r&-1-s\\9-t&9-u\end{bmatrix}=\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}$, $-2-p=0$ and $p=-2$, $4-r=0$ and $r=4$, ..... . Then $D=\begin{bmatrix}-2&0\\4&-1\\9&9\end{bmatrix}=A+B$.

**3.** (a) $$[4\;\;5\;\;6]\begin{bmatrix}2\\3\\-1\end{bmatrix}=[4(2)+5(3)+6(-1)]=[17]$$

(b) $$\begin{bmatrix}2\\3\\-1\end{bmatrix}[4\;\;5\;\;6]=\begin{bmatrix}2(4)&2(5)&2(6)\\3(4)&3(5)&3(6)\\-1(4)&-1(5)&-1(6)\end{bmatrix}=\begin{bmatrix}8&10&12\\12&15&18\\-4&-5&-6\end{bmatrix}$$

(c) $$[1\;\;2\;\;3]\begin{bmatrix}4&-6&9&6\\0&-7&10&7\\5&8&-11&-8\end{bmatrix}$$
$$=[1(4)+2(0)+3(5)\quad 1(-6)+2(-7)+3(8)\quad 1(9)+2(10)+3(-11)\quad 1(6)+2(7)+3(-8)]$$
$$=[19\;\;4\;\;-4\;\;-4]$$

(d) $$\begin{bmatrix}2&3&4\\1&5&6\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}2(1)+3(2)+4(3)\\1(1)+5(2)+6(3)\end{bmatrix}=\begin{bmatrix}20\\29\end{bmatrix}$$

(e) $$\begin{bmatrix}1&2&1\\4&0&2\end{bmatrix}\begin{bmatrix}3&-4\\1&5\\-2&2\end{bmatrix}=\begin{bmatrix}1(3)+2(1)+1(-2)&1(-4)+2(5)+1(2)\\4(3)+0(1)+2(-2)&4(-4)+0(5)+2(2)\end{bmatrix}=\begin{bmatrix}3&8\\8&-12\end{bmatrix}$$

**4.** Let $A=\begin{bmatrix}2&-1&1\\0&1&2\\1&0&1\end{bmatrix}$. Then

$$A^2=\begin{bmatrix}2&-1&1\\0&1&2\\1&0&1\end{bmatrix}\begin{bmatrix}2&-1&1\\0&1&2\\1&0&1\end{bmatrix}=\begin{bmatrix}5&-3&1\\2&1&4\\3&-1&2\end{bmatrix}\quad\text{and}\quad A^3=A^2\cdot A=\begin{bmatrix}5&-3&1\\2&1&4\\3&-1&2\end{bmatrix}\begin{bmatrix}2&-1&1\\0&1&2\\1&0&1\end{bmatrix}=\begin{bmatrix}11&-8&0\\8&-1&8\\8&-4&3\end{bmatrix}$$

The reader will show that $A^3=A\cdot A^2$ and $A^2\cdot A^3=A^3\cdot A^2$.

5. Show that:

(a) $\sum_{k=1}^{2} a_{ik}(b_{kj}+c_{kj}) = \sum_{k=1}^{2} a_{ik}b_{kj} + \sum_{k=1}^{2} a_{ik}c_{kj}$,

(b) $\sum_{i=1}^{2}\sum_{j=1}^{3} a_{ij} = \sum_{j=1}^{3}\sum_{i=1}^{2} a_{ij}$,

(c) $\sum_{k=1}^{2} a_{ik}(\sum_{h=1}^{3} b_{kh}c_{hj}) = \sum_{h=1}^{3}(\sum_{k=1}^{2} a_{ik}b_{kh})c_{hj}$.

(a) $$\sum_{k=1}^{2} a_{ik}(b_{kj}+c_{kj}) = a_{i1}(b_{1j}+c_{1j}) + a_{i2}(b_{2j}+c_{2j}) = (a_{i1}b_{1j}+a_{i2}b_{2j}) + (a_{i1}c_{1j}+a_{i2}c_{2j})$$
$$= \sum_{k=1}^{2} a_{ik}b_{kj} + \sum_{k=1}^{2} a_{ik}c_{kj}.$$

(b) $$\sum_{i=1}^{2}\sum_{j=1}^{3} a_{ij} = \sum_{i=1}^{2}(a_{i1}+a_{i2}+a_{i3}) = (a_{11}+a_{12}+a_{13}) + (a_{21}+a_{22}+a_{23})$$
$$= (a_{11}+a_{21}) + (a_{12}+a_{22}) + (a_{13}+a_{23})$$
$$= \sum_{i=1}^{2} a_{i1} + \sum_{i=1}^{2} a_{i2} + \sum_{i=1}^{2} a_{i3} = \sum_{j=1}^{3}\sum_{i=1}^{2} a_{ij}.$$

This is simply the statement that in summing all of the elements of a matrix, one may sum first the elements of each row or the elements of each column.

(c) $$\sum_{k=1}^{2} a_{ik}(\sum_{h=1}^{3} b_{kh}c_{hj}) = \sum_{k=1}^{2} a_{ik}(b_{k1}c_{1j} + b_{k2}c_{2j} + b_{k3}c_{3j})$$
$$= a_{i1}(b_{11}c_{1j} + b_{12}c_{2j} + b_{13}c_{3j}) + a_{i2}(b_{21}c_{1j} + b_{22}c_{2j} + b_{23}c_{3j})$$
$$= (a_{i1}b_{11} + a_{i2}b_{21})c_{1j} + (a_{i1}b_{12} + a_{i2}b_{22})c_{2j} + (a_{i1}b_{13} + a_{i2}b_{23})c_{3j}$$
$$= (\sum_{k=1}^{2} a_{ik}b_{k1})c_{1j} + (\sum_{k=1}^{2} a_{ik}b_{k2})c_{2j} + (\sum_{k=1}^{2} a_{ik}b_{k3})c_{3j}$$
$$= \sum_{h=1}^{3}(\sum_{k=1}^{2} a_{ik}b_{kh})c_{hj}.$$

6. Prove: If $A = [a_{ij}]$ is of order $m\times n$ and if $B = [b_{ij}]$ and $C = [c_{ij}]$ are of order $n\times p$, then $A(B+C) = AB + AC$.

The elements of the $i$th row of $A$ are $a_{i1}, a_{i2}, \ldots, a_{in}$ and the elements of the $j$th column of $B+C$ are $b_{1j}+c_{1j},\ b_{2j}+c_{2j},\ \ldots,\ b_{nj}+c_{nj}$. Then the element standing in the $i$th row and $j$th column of $A(B+C)$ is $a_{i1}(b_{1j}+c_{1j}) + a_{i2}(b_{2j}+c_{2j}) + \ldots + a_{in}(b_{nj}+c_{nj}) = \sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$, the sum of the elements standing in the $i$th row and $j$th column of $AB$ and $AC$.

7. Prove: If $A = [a_{ij}]$ is of order $m\times n$, if $B = [b_{ij}]$ is of order $n\times p$, and if $C = [c_{ij}]$ is of order $p\times q$, then $A(BC) = (AB)C$.

The elements of the $i$th row of $A$ are $a_{i1}, a_{i2}, \ldots, a_{in}$ and the elements of the $j$th column of $BC$ are $\sum_{h=1}^{p} b_{1h}c_{hj}$, $\sum_{h=1}^{p} b_{2h}c_{hj}$, ..., $\sum_{h=1}^{p} b_{nh}c_{hj}$; hence the element standing in the $i$th row and $j$th column of $A(BC)$ is

$$a_{i1}\sum_{h=1}^{p} b_{1h}c_{hj} + a_{i2}\sum_{h=1}^{p} b_{2h}c_{hj} + \ldots + a_{in}\sum_{h=1}^{p} b_{nh}c_{hj} = \sum_{k=1}^{n} a_{ik}(\sum_{h=1}^{p} b_{kh}c_{hj})$$
$$= \sum_{h=1}^{p}(\sum_{k=1}^{n} a_{ik}b_{kh})c_{hj} = (\sum_{k=1}^{n} a_{ik}b_{k1})c_{1j} + (\sum_{k=1}^{n} a_{ik}b_{k2})c_{2j} + \ldots + (\sum_{k=1}^{n} a_{ik}b_{kp})c_{pj}$$

This is the element standing in the $i$th row and $j$th column of $(AB)C$; hence, $A(BC) = (AB)C$.

8. Assuming $A, B, C, D$ conformable, show in two ways that $(A+B)(C+D) = AC + AD + BC + BD$.

Using (e) and then (f), $(A+B)(C+D) = (A+B)C + (A+B)D = AC + BC + AD + BD$.

Using (f) and then (e), $(A+B)(C+D) = A(C+D) + B(C+D) = AC + AD + BC + BD$
$= AC + BC + AD + BD$.

9. (a) $\begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}[3\ \ 1\ \ 2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 9 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 3 & 4 \\ 9 & 3 & 7 \end{bmatrix}$

(b) $$\left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{array}\right]\left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right] = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & [0] & [0] \\ [0] & \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} & [0] \\ [0] & [0] & \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 18 \end{bmatrix}$$

(c) $$\left[\begin{array}{cc|ccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]\left[\begin{array}{cc|ccc|c} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 \\ \hline 8 & 7 & 6 & 5 & 4 & 1 \end{array}\right] = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\cdot\begin{bmatrix} 6 \\ 7 \end{bmatrix} \\ \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 4 & 5 \\ 9 & 8 \end{bmatrix} & \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 6 & 7 \\ 6 & 7 & 8 \\ 7 & 6 & 5 \end{bmatrix} & \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}\cdot\begin{bmatrix} 8 \\ 9 \\ 4 \end{bmatrix} \\ [1]\cdot[8\ \ 7] & [1]\cdot[6\ \ 5\ \ 4] & [1]\cdot[1] \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix} & \begin{bmatrix} 7 & 9 & 11 \\ 10 & 13 & 16 \end{bmatrix} & \begin{bmatrix} 13 \\ 19 \end{bmatrix} \\ \begin{bmatrix} 31 & 33 \\ 20 & 22 \\ 13 & 13 \end{bmatrix} & \begin{bmatrix} 35 & 37 & 39 \\ 24 & 26 & 28 \\ 13 & 13 & 13 \end{bmatrix} & \begin{bmatrix} 41 \\ 30 \\ 13 \end{bmatrix} \\ [8\ \ 7] & [6\ \ 5\ \ 4] & [1] \end{bmatrix} = \begin{bmatrix} 3 & 5 & 7 & 9 & 11 & 13 \\ 4 & 7 & 10 & 13 & 16 & 19 \\ 31 & 33 & 35 & 37 & 39 & 41 \\ 20 & 22 & 24 & 26 & 28 & 30 \\ 13 & 13 & 13 & 13 & 13 & 13 \\ 8 & 7 & 6 & 5 & 4 & 1 \end{bmatrix}$$

10. Let $\begin{cases} x_1 = a_{11}y_1 + a_{12}y_2 \\ x_2 = a_{21}y_1 + a_{22}y_2 \\ x_3 = a_{31}y_1 + a_{32}y_2 \end{cases}$ be three linear forms in $y_1$ and $y_2$ and let $\begin{cases} y_1 = b_{11}z_1 + b_{12}z_2 \\ y_2 = b_{21}z_1 + b_{22}z_2 \end{cases}$ be a linear transformation of the coordinates $(y_1, y_2)$ into new coordinates $(z_1, z_2)$. The result of applying the transformation to the given forms is the set of forms

$$\begin{cases} x_1 = (a_{11}b_{11} + a_{12}b_{21})z_1 + (a_{11}b_{12} + a_{12}b_{22})z_2 \\ x_2 = (a_{21}b_{11} + a_{22}b_{21})z_1 + (a_{21}b_{12} + a_{22}b_{22})z_2 \\ x_3 = (a_{31}b_{11} + a_{32}b_{21})z_1 + (a_{31}b_{12} + a_{32}b_{22})z_2 \end{cases}$$

Using matrix notation, we have the three forms $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and the transformation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$. The result of applying the transformation is the set of three forms

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Thus, when a set of $m$ linear forms in $n$ variables with matrix $A$ is subjected to a linear transformation of the variables with matrix $B$, there results a set of $m$ linear forms with matrix $C = AB$

## SUPPLEMENTARY PROBLEMS

**11.** Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$,

(*a*) Compute: $A+B = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$, $A-C = \begin{bmatrix} -3 & 1 & -5 \\ 5 & -3 & 0 \\ 0 & 1 & -2 \end{bmatrix}$

(*b*) Compute: $-2A = \begin{bmatrix} -2 & -4 & 6 \\ -10 & 0 & -4 \\ -2 & 2 & -2 \end{bmatrix}$, $0 \cdot B = 0$

(*c*) Verify: $A + (B-C) = (A+B) - C$.

(*d*) Find the matrix $D$ such that $A+D = B$. Verify that $D = B - A = -(A-B)$.

**12.** Given $A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, compute $AB = 0$ and $BA = \begin{bmatrix} -11 & 6 & -1 \\ -22 & 12 & -2 \\ -11 & 6 & -1 \end{bmatrix}$. Hence, $AB \neq BA$ generally.

**13.** Given $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$, show that $AB = AC$. Thus, $AB = AC$ does not necessarily imply $B = C$.

**14.** Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$, show that $(AB)C = A(BC)$.

**15.** Using the matrices of Problem 11, show that $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$.

**16.** Explain why, in general, $(A \pm B)^2 \neq A^2 \pm 2AB + B^2$ and $A^2 - B^2 \neq (A-B)(A+B)$.

**17.** Given $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$,

(*a*) show that $AB = BA = 0$, $AC = A$, $CA = C$.

(*b*) use the results of (*a*) to show that $ACB = CBA$, $A^2 - B^2 = (A-B)(A+B)$, $(A \pm B)^2 = A^2 + B^2$.

**18.** Given $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, where $i^2 = -1$, derive a formula for the positive integral powers of $A$.

*Ans.* $A^n = I, A, -I, -A$ according as $n = 4p, 4p+1, 4p+2, 4p+3$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

**19.** Show that the product of any two or more matrices of the set $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$, $\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ is a matrix of the set.

**20.** Given the matrices $A$ of order $m \times n$, $B$ of order $n \times p$, and $C$ of order $r \times q$, under what conditions on $p$, $q$, and $r$ would the matrices be conformable for finding the products and what is the order of each: (*a*) $ABC$, (*b*) $ACB$, (*c*) $A(B+C)$?

*Ans.* (*a*) $p = r$; $m \times q$ (*b*) $r = n = q$; $m \times p$ (*c*) $r = n$, $p = q$; $m \times q$

21. Compute $AB$, given:

(a) $A = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline 0 & 0 & 1 \end{array}\right]$ and $B = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 1 & 0 \end{array}\right]$ *Ans.* $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

(b) $A = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$ and $B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline -1 & 2 \end{array}\right]$ *Ans.* $\begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix}$

(c) $A = \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right]$ and $B = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$ *Ans.* $\begin{bmatrix} 0 & 0 & 4 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix}$

22. Prove: (a) trace $(A+B)$ = trace $A$ + trace $B$, (b) trace $(kA) = k$ trace $A$.

23. If $\begin{cases} x_1 = y_1 - 2y_2 + y_3 \\ x_2 = 2y_1 + y_2 - 3y_3 \end{cases}$ and $\begin{cases} y_1 = z_1 + 2z_2 \\ y_2 = 2z_1 - z_2 \\ y_3 = 2z_1 + 3z_2 \end{cases}$, verify $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$
$= \begin{bmatrix} -z_1 + 7z_2 \\ -2z_1 - 6z_2 \end{bmatrix}$.

24. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are of order $m \times n$ and if $C = [c_{ij}]$ is of order $n \times p$, show that $(A+B)C = AC + BC$.

25. Let $A = [a_{ij}]$ and $B = [b_{jk}]$, where $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p;\ k = 1, 2, \ldots, n)$. Denote by $\beta_j$ the sum of the elements of the $j$th row of $B$, that is, let $\beta_j = \sum_{k=1}^{n} b_{jk}$. Show that the element in the $i$th row of $A \cdot \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix}$ is the sum of the elements lying in the $i$th row of $AB$. Use this procedure to check the products formed in Problems 12 and 13.

26. A relation (such as parallelism, congruency) between mathematical entities possessing the following properties:

(i) Determination — Either $a$ is in the relation to $b$ or $a$ is not in the relation to $b$.
(ii) Reflexivity — $a$ is in the relation to $a$, for all $a$.
(iii) Symmetry — If $a$ is in the relation to $b$ then $b$ is in the relation to $a$.
(iv) Transivity — If $a$ is in the relation to $b$ and $b$ is in the relation to $c$ then $a$ is in the relation to $c$.

is called an **equivalence relation**.

Show that the parallelism of lines, similarity of triangles, and equality of matrices are equivalence relations. Show that perpendicularity of lines is not an equivalence relation.

27. Show that conformability for addition of matrices is an equivalence relation while conformability for multiplication is not.

28. Prove: If $A, B, C$ are matrices such that $AC = CA$ and $BC = CB$, then $(AB \pm BA)C = C(AB \pm BA)$.

# Chapter 2

## Some Types of Matrices

**THE IDENTITY MATRIX.** A square matrix $A$ whose elements $a_{ij} = 0$ for $i > j$ is called **upper triangular**; a square matrix $A$ whose elements $a_{ij} = 0$ for $i < j$ is called **lower triangular**. Thus

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \quad \text{is upper triangular and}$$

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \quad \text{is lower triangular.}$$

The matrix $D = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$, which is both upper and lower triangular, is called a **diagonal matrix**. It will frequently be written as

$$D = \text{diag}(a_{11}, a_{22}, a_{33}, \ldots, a_{nn})$$

See Problem 1.

If in the diagonal matrix $D$ above, $a_{11} = a_{22} = \ldots = a_{nn} = k$, $D$ is called **a scalar matrix**; if, in addition, $k = 1$, the matrix is called the **identity matrix** and is denoted by $I_n$. For example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When the order is evident or immaterial, an identity matrix will be denoted by $I$. Clearly, $I_n + I_n + \ldots$ to $p$ terms $= p \cdot I_n = \text{diag}(p, p, p, \ldots, p)$ and $I^p = I \cdot I \ldots$ to $p$ factors $= I$. Identity matrices have some of the properties of the integer 1. For example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $I_2 \cdot A = A \cdot I_3 = I_2 A I_3 = A$, as the reader may readily show.

**SPECIAL SQUARE MATRICES.** If $A$ and $B$ are square matrices such that $AB = BA$, then $A$ and $B$ are called **commutative** or are said to **commute**. It is a simple matter to show that if $A$ is any $n$-square matrix, it commutes with itself and also with $I_n$.

See Problem 2.

If $A$ and $B$ are such that $AB = -BA$, the matrices $A$ and $B$ are said to **anti-commute**.

A matrix $A$ for which $A^{k+1} = A$, where $k$ is a positive integer, is called **periodic**. If $k$ is the least positive integer for which $A^{k+1} = A$, then $A$ is said to be of **period** $k$.

If $k = 1$, so that $A^2 = A$, then $A$ is called **idempotent**.

See Problems 3-4.

A matrix $A$ for which $A^p = 0$, where $p$ is a positive integer, is called **nilpotent**. If $p$ is the least positive integer for which $A^p = 0$, then $A$ is said to be nilpotent of **index** $p$.

See Problems 5-6.

**THE INVERSE OF A MATRIX.** If $A$ and $B$ are square matrices such that $AB = BA = I$, then $B$ is called the **inverse** of $A$ and we write $B = A^{-1}$ ($B$ equals $A$ inverse). The matrix $B$ also has $A$ as its inverse and we may write $A = B^{-1}$.

**Example 1.** Since $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$, each matrix in the product is the inverse of the other.

We shall find later (Chapter 7) that not every square matrix has an inverse. We can show here, however, that if $A$ has an inverse then that inverse is unique.

See Problem 7.

If $A$ and $B$ are square matrices of the same order with inverses $A^{-1}$ and $B^{-1}$ respectively, then $(AB)^{-1} = B^{-1}\cdot A^{-1}$, that is,

**I.** The inverse of the product of two matrices, having inverses, is the product **in reverse order** of these inverses.

See Problem 8.

A matrix $A$ such that $A^2 = I$ is called **involutory**. An identity matrix, for example, is involutory. An involutory matrix is its own inverse.

See Problem 9.

**THE TRANSPOSE OF A MATRIX.** The matrix of order $n\times m$ obtained by interchanging the rows and columns of an $m\times n$ matrix $A$ is called the **transpose** of $A$ and is denoted by $A'$ ($A$ transpose). For example, the transpose of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is $A' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$. Note that the element $a_{ij}$ in the $i$th row and $j$th column of $A$ stands in the $j$th row and $i$th column of $A'$.

If $A'$ and $B'$ are transposes respectively of $A$ and $B$, and if $k$ is a scalar, we have immediately

$$(a)\ (A')' = A \qquad \text{and} \qquad (b)\ (kA)' = kA'$$

In Problems 10 and 11, we prove:

**II.** The transpose of the sum of two matrices is the sum of their transposes, i.e.,

$$(A+B)' = A' + B'$$

and

**III**. The transpose of the product of two matrices is the product **in reverse order** of their transposes, i.e.,

$$(AB)' = B' \cdot A'$$

See Problems 10-12.

**SYMMETRIC MATRICES**. A square matrix $A$ such that $A' = A$ is called **symmetric**. Thus, a square matrix $A = [a_{ij}]$ is symmetric provided $a_{ij} = a_{ji}$, for all values of $i$ and $j$. For example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric and so also is $kA$ for any scalar $k$.

In Problem 13, we prove

**IV**. If $A$ is an $n$-square matrix, then $A + A'$ is symmetric.

A square matrix $A$ such that $A' = -A$ is called **skew-symmetric**. Thus, a square matrix $A$ is skew-symmetric provided $a_{ij} = -a_{ji}$ for all values of $i$ and $j$. Clearly, the diagonal elements are zeros. For example, $A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$ is skew-symmetric and so also is $kA$ for any scalar $k$.

With only minor changes in Problem 13, we can prove

**V**. If $A$ is any $n$-square matrix, then $A - A'$ is skew-symmetric.

From Theorems **IV** and **V** follows

**VI**. Every square matrix $A$ can be written as the sum of a symmetric matrix $B = \frac{1}{2}(A+A')$ and a skew-symmetric matrix $C = \frac{1}{2}(A-A')$.

See Problems 14-15.

**THE CONJUGATE OF A MATRIX**. Let $a$ and $b$ be real numbers and let $i = \sqrt{-1}$; then, $z = a+bi$ is called a **complex number**. The complex numbers $a+bi$ and $a-bi$ are called **conjugates**, each being the **conjugate** of the other. If $z = a+bi$, its conjugate is denoted by $\bar{z} = \overline{a+bi}$.

If $z_1 = a+bi$ and $z_2 = \bar{z}_1 = a-bi$, then $\bar{z}_2 = \bar{\bar{z}}_1 = \overline{a-bi} = a+bi$, that is, the conjugate of the conjugate of a complex number $z$ is $z$ **itself**.

If $z_1 = a+bi$ and $z_2 = c+di$, then

(i) $z_1 + z_2 = (a+c) + (b+d)i$ and $\overline{z_1 + z_2} = (a+c) - (b+d)i = (a-bi) + (c-di) = \bar{z}_1 + \bar{z}_2$,

that is, the conjugate of the sum of two complex numbers is the sum of their conjugates.

(ii) $z_1 \cdot z_2 = (ac-bd) + (ad+bc)i$ and $\overline{z_1 \cdot z_2} = (ac-bd) - (ad+bc)i = (a-bi)(c-di) = \bar{z}_1 \cdot \bar{z}_2$,

that is, the conjugate of the product of two complex numbers is the product of their conjugates.

When $A$ is a matrix having complex numbers as elements, the matrix obtained from $A$ by replacing each element by its conjugate is called the **conjugate** of $A$ and is denoted by $\bar{A}$ ($A$ conjugate).

**Example 2**. When $A = \begin{bmatrix} 1+2i & i \\ 3 & 2-3i \end{bmatrix}$ then $\bar{A} = \begin{bmatrix} 1-2i & -i \\ 3 & 2+3i \end{bmatrix}$.

If $\bar{A}$ and $\bar{B}$ are the conjugates of the matrices $A$ and $B$ and if $k$ is any scalar, we have readily

$$(c)\quad (\bar{\bar{A}}) = A \qquad \text{and} \qquad (d)\quad (\overline{kA}) = \bar{k} \cdot \bar{A}$$

Using (i) and (ii) above, we may prove

**VII**. The conjugate of the sum of two matrices is the sum of their conjugates, i.e., $\overline{(A+B)} = \bar{A} + \bar{B}$.

**VIII**. The conjugate of the product of two matrices is the product, **in the same order**, of their conjugates, i.e., $\overline{(AB)} = \bar{A}\cdot\bar{B}$.

The transpose of $\bar{A}$ is denoted by $\bar{A}'$ ($A$ conjugate transpose). It is sometimes written as $A^*$. We have

**IX**. The transpose of the conjugate of $A$ is equal to the conjugate of the transpose of $A$, i.e., $(\bar{A})' = \overline{(A')}$.

**Example 3**. From Example 2

$$(\bar{A})' = \begin{bmatrix} 1-2i & 3 \\ -i & 2+3i \end{bmatrix} \quad \text{while} \quad A' = \begin{bmatrix} 1+2i & 3 \\ i & 2-3i \end{bmatrix} \quad \text{and} \quad \overline{(A')} = \begin{bmatrix} 1-2i & 3 \\ -i & 2+3i \end{bmatrix} = (\bar{A})'$$

**HERMITIAN MATRICES**. A square matrix $A = [a_{ij}]$ such that $\bar{A}' = A$ is called **Hermitian**. Thus, $A$ is Hermitian provided $a_{ij} = \bar{a}_{ji}$ for all values of $i$ and $j$. Clearly, the diagonal elements of an Hermitian matrix are real numbers.

**Example 4.** The matrix $A = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$ is Hermitian.

Is $kA$ Hermitian if $k$ is any real number? any complex number?

A square matrix $A = [a_{ij}]$ such that $\bar{A}' = -A$ is called **skew-Hermitian**. Thus, $A$ is skew-Hermitian provided $a_{ij} = -\bar{a}_{ji}$ for all values of $i$ and $j$. Clearly, the diagonal elements of a skew-Hermitian matrix are either zeros or pure imaginaries.

**Example 5**. The matrix $A = \begin{bmatrix} i & 1-i & 2 \\ -1-i & 3i & i \\ -2 & i & 0 \end{bmatrix}$ is skew-Hermitian. Is $kA$ skew-Hermitian if $k$ is any real number? any complex number? any pure imaginary?

By making minor changes in Problem 13, we may prove

**X**. If $A$ is an $n$-square matrix then $A + \bar{A}'$ is Hermitian and $A - \bar{A}'$ is skew-Hermitian.

From Theorem **X** follows

**XI**. Every square matrix $A$ with complex elements can be written as the sum of an Hermitian matrix $B = \frac{1}{2}(A + \bar{A}')$ and a skew-Hermitian matrix $C = \frac{1}{2}(A - \bar{A}')$.

**DIRECT SUM**. Let $A_1, A_2, \ldots, A_s$ be square matrices of respective orders $m_1, m_2, \ldots, m_s$. The generalization

$$A = \begin{bmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A_s \end{bmatrix} = \operatorname{diag}(A_1, A_2, \ldots, A_s)$$

of the diagonal matrix is called the **direct sum** of the $A_i$.

**Example 6.** Let $A_1 = [2]$, $A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and $A_3 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 4 & 1 & -2 \end{bmatrix}$.

The direct sum of $A_1, A_2, A_3$ is $\text{diag}(A_1, A_2, A_3) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 4 & 1 & -2 \end{bmatrix}$

Problem 9(*b*), Chapter 1, illustrates

**XII.** If $A = \text{diag}(A_1, A_2, \ldots, A_s)$ and $B = \text{diag}(B_1, B_2, \ldots, B_s)$, where $A_i$ and $B_i$ have the same order for $(i = 1, 2, \ldots, s)$, then $AB = \text{diag}(A_1B_1, A_2B_2, \ldots, A_sB_s)$.

## SOLVED PROBLEMS

1\. Since $\begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & & a_{mm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{m1} & b_{m2} & \ldots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \ldots & a_{11}b_{1n} \\ a_{22}b_{21} & a_{22}b_{22} & \ldots & a_{22}b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{mm}b_{m1} & a_{mm}b_{m2} & \ldots & a_{mm}b_{mn} \end{bmatrix}$, the product $AB$ of an $m$-square diagonal matrix $A = \text{diag}(a_{11}, a_{22}, \ldots, a_{mm})$ and any $m \times n$ matrix $B$ is obtained by multiplying the first row of $B$ by $a_{11}$, the second row of $B$ by $a_{22}$, and so on.

2\. Show that the matrices $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $\begin{bmatrix} c & d \\ d & c \end{bmatrix}$ commute for all values of $a, b, c, d$.

This follows from $\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} c & d \\ d & c \end{bmatrix} = \begin{bmatrix} ac+bd & ad+bc \\ bc+ad & bd+ac \end{bmatrix} = \begin{bmatrix} c & d \\ d & c \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

3\. Show that $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

4\. Show that if $AB = A$ and $BA = B$, then $A$ and $B$ are idempotent.

$ABA = (AB)A = A \cdot A = A^2$ and $ABA = A(BA) = AB = A$; then $A^2 = A$ and $A$ is idempotent. Use $BAB$ to show that $B$ is idempotent.

5. Show that $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent of order 3.

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \quad \text{and} \quad A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = 0$$

6. If $A$ is nilpotent of index 2, show that $A(I \pm A)^n = A$ for $n$ any positive integer.

Since $A^2 = 0$, $A^3 = A^4 = \ldots = A^n = 0$. Then $A(I \pm A)^n = A(I \pm nA) = A \pm nA^2 = A$.

7. Let $A, B, C$ be square matrices such that $AB = I$ and $CA = I$. Then $(CA)B = C(AB)$ so that $B = C$. Thus, $B = C = A^{-1}$ is the **unique** inverse of $A$. (What is $B^{-1}$?)

8. Prove: $(AB)^{-1} = B^{-1} \cdot A^{-1}$.

By definition $(AB)^{-1}(AB) = (AB)(AB)^{-1} = I$. Now

$$(B^{-1} \cdot A^{-1})AB = B^{-1}(A^{-1} \cdot A)B = B^{-1} \cdot I \cdot B = B^{-1} \cdot B = I$$

and

$$AB(B^{-1} \cdot A^{-1}) = A(B \cdot B^{-1})A^{-1} = A \cdot A^{-1} = I$$

By Problem 7, $(AB)^{-1}$ is unique; hence, $(AB)^{-1} = B^{-1} \cdot A^{-1}$.

9. Prove: A matrix $A$ is involutory if and only if $(I - A)(I + A) = 0$.

Suppose $(I-A)(I+A) = I - A^2 = 0$; then $A^2 = I$ and $A$ is involutory.

Suppose $A$ is involutory; then $A^2 = I$ and $(I-A)(I+A) = I - A^2 = I - I = 0$.

10. Prove: $(A + B)' = A' + B'$.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$. We need only check that the element in the $i$th row and $j$th column of $A'$, $B'$, and $(A+B)'$ are respectively $a_{ji}$, $b_{ji}$, and $a_{ji} + b_{ji}$.

11. Prove: $(AB)' = B'A'$.

Let $A = [a_{ij}]$ be of order $m \times n$, $B = [b_{ij}]$ be of order $n \times p$; then $C = AB = [c_{ij}]$ is of order $m \times p$. The element standing in the $i$th row and $j$th column of $AB$ is $c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$ and this is also the element standing in the $j$th row and $i$th column of $(AB)'$.

The elements of the $j$th row of $B'$ are $b_{1j}, b_{2j}, \ldots, b_{nj}$ and the elements of the $i$th column of $A'$ are $a_{i1}$, $a_{i2}, \ldots, a_{in}$. Then the element in the $j$th row and $i$th column of $B'A'$ is

$$\sum_{k=1}^{n} b_{kj} \cdot a_{ik} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} = c_{ij}$$

Thus, $(AB)' = B'A'$.

12. Prove: $(ABC)' = C'B'A'$.

Write $ABC = (AB)C$. Then, by Problem 11, $(ABC)' = \{(AB)C\}' = C'(AB)' = C'B'A'$.

**13.** Show that if $A = [a_{ij}]$ is $n$-square, then $B = [b_{ij}] = A + A'$ is symmetric.

**First Proof.**

The element in the $i$th row and $j$th column of $A$ is $a_{ij}$ and the corresponding element of $A'$ is $a_{ji}$; hence, $b_{ij} = a_{ij} + a_{ji}$. The element in the $j$th row and $i$th column of $A$ is $a_{ji}$ and the corresponding element of $A'$ is $a_{ij}$; hence, $b_{ji} = a_{ji} + a_{ij}$. Thus, $b_{ij} = b_{ji}$ and $B$ is symmetric.

**Second Proof.**

By Problem 10, $(A+A')' = A' + (A')' = A' + A = A + A'$ and $(A+A')$ is symmetric.

**14.** Prove: If $A$ and $B$ are $n$-square symmetric matrices then $AB$ is symmetric if and only if $A$ and $B$ commute.

Suppose $A$ and $B$ commute so that $AB = BA$. Then $(AB)' = B'A' = BA = AB$ and $AB$ is symmetric.

Suppose $AB$ is symmetric so that $(AB)' = AB$. Now $(AB)' = B'A' = BA$; hence, $AB = BA$ and the matrices $A$ and $B$ commute.

**15.** Prove: If the $m$-square matrix $A$ is symmetric (skew-symmetric) and if $P$ is of order $m \times n$ then $B = P'AP$ is symmetric (skew-symmetric).

If $A$ is symmetric then (see Problem 12) $B' = (P'AP)' = P'A'(P')' = P'A'P = P'AP$ and $B$ is symmetric.

If $A$ is skew-symmetric then $B' = (P'AP)' = -P'AP$ and $B$ is skew-symmetric.

**16.** Prove: If $A$ and $B$ are $n$-square matrices then $A$ and $B$ commute if and only if $A - kI$ and $B - kI$ commute for every scalar $k$.

Suppose $A$ and $B$ commute; then $AB = BA$ and

$$\begin{aligned}(A-kI)(B-kI) &= AB - k(A+B) + k^2I \\ &= BA - k(A+B) + k^2I = (B-kI)(A-kI)\end{aligned}$$

Thus, $A - kI$ and $B - kI$ commute.

Suppose $A - kI$ and $B - kI$ commute; then

$$\begin{aligned}(A-kI)(B-kI) &= AB - k(A+B) + k^2I \\ &= BA - k(A+B) + k^2I = (B-kI)(A-kI)\end{aligned}$$

$AB = BA$, and $A$ and $B$ commute.

## SUPPLEMENTARY PROBLEMS

**17.** Show that the product of two upper (lower) triangular matrices is upper (lower) triangular.

**18.** Derive a rule for forming the product $BA$ of an $m\times n$ matrix $B$ and $A = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$.
*Hint.* See Problem 1.

**19.** Show that the scalar matrix with diagonal element $k$ can be written as $kI$ and that $kA = kIA = \text{diag}(k, k, \ldots, k)\, A$, where the order of $I$ is the row order of $A$.

**20.** If $A$ is $n$-square, show that $A^p \cdot A^q = A^q \cdot A^p$ where $p$ and $q$ are positive integers.

**21.** (*a*) Show that $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ are idempotent.

(*b*) Using $A$ and $B$, show that the converse of Problem 4 does not hold.

**22.** If $A$ is idempotent, show that $B = I - A$ is idempotent and that $AB = BA = 0$.

**23.** (*a*) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A - 5I = 0$.

(*b*) If $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, show that $A^3 - 2A^2 - 9A = 0$, but $A^2 - 2A - 9I \neq 0$.

**24.** Show that $\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}^4 = I$.

**25.** Show that $A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$ is periodic, of period 2.

**26.** Show that $\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ is nilpotent.

**27.** Show that (*a*) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -1 & -6 \\ 3 & 2 & 9 \\ -1 & -1 & -4 \end{bmatrix}$ commute,

(*b*) $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ -1 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2/3 & 0 & -1/3 \\ -3/5 & 2/5 & 1/5 \\ 7/15 & -1/5 & 1/15 \end{bmatrix}$ commute.

**28.** Show that $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$ anti-commute and $(A+B)^2 = A^2 + B^2$.

**29.** Show that each of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ anti-commutes with the others.

**30.** Prove: The only matrices which commute with every $n$-square matrix are the $n$-square scalar matrices.

**31.** (*a*) Find all matrices which commute with diag(1, 2, 3).
(*b*) Find all matrices which commute with $\text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$.
*Ans.* (*a*) diag($a, b, c$) where $a, b, c$ are arbitrary.

**32.** Show that (a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ -2 & 3 & 1 & 1 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 8 & -1 & -1 & 1 \end{bmatrix}$.

**33.** Set $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ to find the inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. *Ans.* $\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

**34.** Show that the inverse of a diagonal matrix $A$, all of whose diagonal elements are different from zero, is a diagonal matrix whose diagonal elements are the inverses of those of $A$ and in the same order. Thus, the inverse of $I_n$ is $I_n$.

**35.** Show that $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ are involutory.

**36.** Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & -1 & 0 \\ c & d & 0 & -1 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ A_{21} & -I_2 \end{bmatrix}$ by partitioning. Show that $A^2 = \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \end{bmatrix} = I_4$.

**37.** Prove: (a) $(A')' = A$, (b) $(kA)' = kA'$, (c) $(A^p)' = (A')^p$ for $p$ a positive integer.

**38.** Prove: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. *Hint.* Write $ABC = (AB)C$.

**39.** Prove: (a) $(A^{-1})^{-1} = A$, (b) $(kA)^{-1} = \frac{1}{k}A^{-1}$, (c) $(A^p)^{-1} = (A^{-1})^p$ for $p$ a positive integer.

**40.** Show that every real symmetric matrix is Hermitian.

**41.** Prove: (a) $\overline{(\bar{A})} = A$, (b) $\overline{(A+B)} = \bar{A} + \bar{B}$, (c) $\overline{(kA)} = \bar{k}\,\bar{A}$, (d) $\overline{(AB)} = \bar{A}\,\bar{B}$.

**42.** Show: (a) $A = \begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 0 \end{bmatrix}$ is Hermitian,

(b) $B = \begin{bmatrix} i & 1+i & 2-3i \\ -1+i & 2i & 1 \\ -2-3i & -1 & 0 \end{bmatrix}$ is skew-Hermitian,

(c) $iB$ is Hermitian,

(d) $\bar{A}$ is Hermitian and $\bar{B}$ is skew-Hermitian.

**43.** If $A$ is $n$-square, show that (a) $AA'$ and $A'A$ are symmetric, (b) $A+\bar{A}'$, $A\bar{A}'$, and $\bar{A}'A$ are Hermitian.

**44.** Prove: If $H$ is Hermitian and $A$ is any conformable matrix then $(\bar{A})'HA$ is Hermitian.

**45.** Prove: Every Hermitian matrix $A$ can be written as $B+iC$ where $B$ is real and symmetric and $C$ is real and skew-symmetric.

**46.** Prove: (a) Every skew-Hermitian matrix $A$ can be written as $A = B+iC$ where $B$ is real and skew-symmetric and $C$ is real and symmetric. (b) $\bar{A}'A$ is real if and only if $B$ and $C$ anti-commute.

**47.** Prove: If $A$ and $B$ commute so also do $A^{-1}$ and $B^{-1}$, $A'$ and $B'$, and $\bar{A}'$ and $\bar{B}'$.

**48.** Show that for $m$ and $n$ positive integers, $A^m$ and $B^n$ commute if $A$ and $B$ commute.

49. Show (a) $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$. (b) $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & \frac{1}{2}n(n-1)\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}$

50. Prove: If $A$ is symmetric or skew-symmetric then $AA' = A'A$ and $A^2$ are symmetric.

51. Prove: If $A$ is symmetric so also is $aA^p + bA^{p-1} + \ldots + gI$ where $a, b, \ldots, g$ are scalars and $p$ is a positive integer.

52. Prove: Every square matrix $A$ can be written as $A = B + C$ where $B$ is Hermitian and $C$ is skew-Hermitian.

53. Prove: If $A$ is real and skew-symmetric or if $A$ is complex and skew-Hermitian then $\pm iA$ are Hermitian.

54. Show that the theorem of Problem 52 can be stated:
Every square matrix $A$ can be written as $A = B + iC$ where $B$ and $C$ are Hermitian.

55. Prove: If $A$ and $B$ are such that $AB = A$ and $BA = B$ then (a) $B'A' = A'$ and $A'B' = B'$, (b) $A'$ and $B'$ are idempotent, (c) $A = B = I$ if $A$ has an inverse.

56. If $A$ is involutory, show that $\frac{1}{2}(I+A)$ and $\frac{1}{2}(I-A)$ are idempotent and $\frac{1}{2}(I+A) \cdot \frac{1}{2}(I-A) = 0$.

57. If the $n$-square matrix $A$ has an inverse $A^{-1}$, show:
(a) $(A^{-1})' = (A')^{-1}$, (b) $(\bar{A})^{-1} = \overline{A^{-1}}$, (c) $(\bar{A}')^{-1} = \overline{(A^{-1})'}$
Hint. (a) From the transpose of $AA^{-1} = I$, obtain $(A^{-1})'$ as the inverse of $A'$.

58. Find all matrices which commute with (a) diag(1, 1, 2, 3), (b) diag(1, 1, 2, 2).
*Ans.* (a) diag($A, b, c$), (b) diag($A, B$) where $A$ and $B$ are 2-square matrices with arbitrary elements and $b$, $c$ are scalars.

59. If $A_1, A_2, \ldots, A_s$ are scalar matrices of respective orders $m_1, m_2, \ldots, m_s$, find all matrices which commute with diag$(A_1, A_2, \ldots, A_s)$.
*Ans.* diag$(B_1, B_2, \ldots, B_s)$ where $B_1, B_2, \ldots, B_s$ are of respective orders $m_1, m_2, \ldots, m_s$ with arbitrary elements.

60. If $AB = 0$, where $A$ and $B$ are non-zero $n$-square matrices, then $A$ and $B$ are called **divisors of zero**. Show that the matrices $A$ and $B$ of Problem 21 are divisors of zero.

61. If $A = \text{diag}(A_1, A_2, \ldots, A_s)$ and $B = \text{diag}(B_1, B_2, \ldots, B_s)$ where $A_i$ and $B_i$ are of the same order, $(i = 1, 2, \ldots, s)$, show that
(a) $A + B = \text{diag}(A_1+B_1, A_2+B_2, \ldots, A_s+B_s)$
(b) $AB = \text{diag}(A_1B_1, A_2B_2, \ldots, A_sB_s)$
(c) trace $AB$ = trace $A_1B_1$ + trace $A_2B_2$ + ... + trace $A_sB_s$.

62. Prove: If $A$ and $B$ are $n$-square skew-symmetric matrices then $AB$ is symmetric if and only if $A$ and $B$ commute.

63. Prove: If $A$ is $n$-square and $B = rA + sI$, where $r$ and $s$ are scalars, then $A$ and $B$ commute.

64. Let $A$ and $B$ be $n$-square matrices and let $r_1, r_2, s_1, s_2$ be scalars such that $r_1s_2 \neq r_2s_1$. Prove that $C_1 = r_1A + s_1B$, $C_2 = r_2A + s_2B$ commute if and only if $A$ and $B$ commute.

65. Show that the $n$-square matrix $A$ will not have an inverse when (a) $A$ has a row (column) of zero elements or (b) $A$ has two identical rows (columns) or (c) $A$ has a row (column) which is the sum of two other rows (columns).

66. If $A$ and $B$ are $n$-square matrices and $A$ has an inverse, show that
$$(A+B)A^{-1}(A-B) = (A-B)A^{-1}(A+B)$$

# Chapter 3

## Determinant of a Square Matrix

**PERMUTATIONS.** Consider the $3! = 6$ permutations of the integers $1, 2, 3$ taken together

(3.1) $$123 \quad 132 \quad 213 \quad 231 \quad 312 \quad 321$$

and eight of the $4! = 24$ permutations of the integers $1, 2, 3, 4$ taken together

(3.2) $$\begin{array}{cccc} 1234 & 2134 & 3124 & 4123 \\ 1324 & 2314 & 3214 & 4213 \end{array}$$

If in a given permutation a larger integer precedes a smaller one, we say that there is an **inversion**. If in a given permutation the number of inversions is even (odd), the permutation is called **even (odd)**. For example, in **(3.1)** the permutation 123 is even since there is no inversion, the permutation 132 is odd since in it 3 precedes 2, the permutation 312 is even since in it 3 precedes 1 and 3 precedes 2. In **(3.2)** the permutation 4213 is even since in it 4 precedes 2, 4 precedes 1, 4 precedes 3, and 2 precedes 1.

**THE DETERMINANT OF A SQUARE MATRIX.** Consider the $n$-square matrix

(3.3) $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

and a product

(3.4) $$a_{1j_1}\, a_{2j_2}\, a_{3j_3} \cdots a_{nj_n}$$

of $n$ of its elements, selected so that one and only one element comes from any row and one and only one element comes from any column. In **(3.4)**, as a matter of convenience, the factors have been arranged so that the sequence of first subscripts is the natural order $1, 2, \ldots, n$; the sequence $j_1, j_2, \ldots, j_n$ of second subscripts is then some one of the $n!$ permutations of the integers $1, 2, \ldots, n$. (Facility will be gained if the reader will parallel the work of this section beginning with a product arranged so that the sequence of second subscripts is in natural order.)

For a given permutation $j_1, j_2, \ldots, j_n$ of the second subscripts, define $\epsilon_{j_1 j_2 \ldots j_n} = +1$ or $-1$ according as the permutation is even or odd and form the signed product

(3.5) $$\epsilon_{j_1 j_2 \ldots j_n}\, a_{1j_1}\, a_{2j_2} \cdots a_{nj_n}$$

By the **determinant** of $A$, denoted by $|A|$, is meant the sum of all the different signed products of the form **(3.5)**, called **terms** of $|A|$, which can be formed from the elements of $A$; thus,

(3.6) $$|A| = \sum_{\rho} \epsilon_{j_1 j_2 \ldots j_n}\, a_{1j_1}\, a_{2j_2} \cdots a_{nj_n}$$

where the summation extends over $\rho = n!$ permutations $j_1 j_2 \cdots j_n$ of the integers $1, 2, \ldots, n$.

The determinant of a square matrix of order $n$ is called a determinant of order $n$.

**DETERMINANTS OF ORDER TWO AND THREE.** From (3.6) we have for $n=2$ and $n=3$,

$$(3.7)\qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \epsilon_{12}\, a_{11}a_{22} + \epsilon_{21}\, a_{12}a_{21} = a_{11}a_{22} - a_{12}a_{21}$$

and

$$(3.8)\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \epsilon_{123}\, a_{11}a_{22}a_{33} + \epsilon_{132}\, a_{11}a_{23}a_{32} + \epsilon_{213}\, a_{12}a_{21}a_{33} + \epsilon_{231}\, a_{12}a_{23}a_{31} + \epsilon_{312}\, a_{13}a_{21}a_{32} + \epsilon_{321}\, a_{13}a_{22}a_{31}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

**Example 1.**

(a) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 3 = 4 - 6 = -2$

(b) $\begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} = 2\cdot 0 - (-1)3 = 0 + 3 = 3$

(c) $\begin{vmatrix} 2 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$

$= 2(0\cdot 0 - 1\cdot 1) - 3(1\cdot 0 - 1\cdot 2) + 5(1\cdot 1 - 0\cdot 2) = 2(-1) - 3(-2) + 5(1) = 9$

(d) $\begin{vmatrix} 2 & -3 & -4 \\ 1 & 0 & -2 \\ 0 & -5 & -6 \end{vmatrix} = 2\{0(-6) - (-2)(-5)\} - (-3)\{1(-6) - (-2)0\} + (-4)\{1(-5) - 0\cdot 0\}$

$= -20 - 18 + 20 = -18$

See Problem 1.

**PROPERTIES OF DETERMINANTS.** Throughout this section, $A$ is the square matrix whose determinant $|A|$ is given by (3.6).

Suppose that every element of the $i$th row (every element of the $j$th column) is zero. Since every term of (3.6) contains one element from this row (column), every term in the sum is zero and we have

**I.** If every element of a row (column) of a square matrix $A$ is zero, then $|A| = 0$.

Consider the transpose $A'$ of $A$. It can be seen readily that every term of (3.6) can be obtained from $A'$ by choosing properly the factors in order from the first, second, ... columns. Thus,

**II.** If $A$ is a square matrix then $|A'| = |A|$; that is, for every theorem concerning the rows of a determinant there is a corresponding theorem concerning the columns and vice versa.

Denote by $B$ the matrix obtained by multiplying each of the elements of the $i$th row of $A$ by a scalar $k$. Since each term in the expansion of $|B|$ contains one and only one element from its $i$th row, that is, one and only one element having $k$ as a factor,

$$|B| = k \sum_{p} \{\epsilon_{j_1 j_2 \ldots j_n} a_{1j_1} a_{2j_2} \cdots a_{nj_n}\} = k\,|A|$$

Thus,

**III.** If every element of a row (column) of a determinant $|A|$ is multiplied by a scalar $k$, the determinant is multiplied by $k$; if every element of a row (column) of a determinant $|A|$ has $k$ as a factor then $k$ may be factored from $|A|$. For example,

$$\begin{vmatrix} a_{11} & ka_{12} & a_{13} \\ a_{21} & ka_{22} & a_{23} \\ a_{31} & ka_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix}$$

Let $B$ denote the matrix obtained from $A$ by interchanging its $i$th and $(i+1)$st rows. Each product in **(3.6)** of $|A|$ is a product of $|B|$, and vice versa; hence, except possibly for signs, **(3.6)** is the expansion of $|B|$. In counting the inversions in subscripts of any term of **(3.6)** as a term of $|B|$, $i$ before $i+1$ in the row subscripts is an inversion; thus, each product of **(3.6)** with its sign changed is a term of $|B|$ and $|B| = -|A|$. Hence,

**IV.** If $B$ is obtained from $A$ by interchanging any two adjacent rows (columns), then $|B| = -|A|$.

As a consequence of Theorem **IV**, we have

**V.** If $B$ is obtained from $A$ by interchanging **any** two of its rows (columns), then $|B| = -|A|$.

**VI.** If $B$ is obtained from $A$ by carrying its $i$th row (column) over $p$ rows (columns), then $|B| = (-1)^p |A|$.

**VII.** If two rows (columns) of $A$ are identical, then $|A| = 0$.

Suppose that each element of the first row of $A$ is expressed as a binomial $a_{1j} = b_{1j} + c_{1j}$, $(j = 1, 2, \ldots, n)$. Then

$$\begin{aligned} |A| &= \sum_\rho \epsilon_{j_1 j_2 \cdots j_n} (b_{1j_1} + c_{1j_1}) a_{2j_2} a_{3j_3} \cdots a_{nj_n} \\ &= \sum_\rho \epsilon_{j_1 j_2 \cdots j_n} b_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n} + \sum_\rho \epsilon_{j_1 j_2 \cdots j_n} c_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n} \\ &= \begin{vmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \end{aligned}$$

In general,

**VIII.** If every element of the $i$th row (column) of $A$ is the sum of $p$ terms, then $|A|$ can be expressed as the sum of $p$ determinants. The elements in the $i$th rows (columns) of these $p$ determinants are respectively the first, second, ..., $p$th terms of the sums and all other rows (columns) are those of $A$.

The most useful theorem is

**IX.** If $B$ is obtained from $A$ by adding to the elements of its $i$th row (column), a scalar multiple of the corresponding elements of another row (column), then $|B| = |A|$. For example,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+ka_{13} & a_{12} & a_{13} \\ a_{21}+ka_{23} & a_{22} & a_{23} \\ a_{31}+ka_{33} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+ka_{21} & a_{32}+ka_{22} & a_{33}+ka_{23} \end{vmatrix}$$

See Problems 2-7.

**FIRST MINORS AND COFACTORS.** Let $A$ be the $n$-square matrix **(3.3)** whose determinant $|A|$ is given by **(3.6)**. When from $A$ the elements of its $i$th row and $j$th column are removed, the determinant of the remaining $(n-1)$-square matrix is called a **first minor** of $A$ or of $|A|$ and denoted by $|M_{ij}|$.

More frequently, it is called **the minor** of $a_{ij}$. The signed minor, $(-1)^{i+j}|M_{ij}|$ is called the **cofactor** of $a_{ij}$ and is denoted by $\alpha_{ij}$.

**Example 2.** If $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$,

$$|M_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad |M_{12}| = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad |M_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

and

$$\alpha_{11} = (-1)^{1+1}|M_{11}| = |M_{11}|, \quad \alpha_{12} = (-1)^{1+2}|M_{12}| = -|M_{12}|,$$
$$\alpha_{13} = (-1)^{1+3}|M_{13}| = |M_{13}|$$

Then (3.8) is

$$\begin{aligned} |A| &= a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| \\ &= a_{11}\alpha_{11} + a_{12}\alpha_{12} + a_{13}\alpha_{13} \end{aligned}$$

In Problem 9, we prove

**X.** The value of the determinant $|A|$, where $A$ is the matrix of (3.3), is the sum of the products obtained by multiplying each element of a ro (column) of $|A|$ by its cofactor, i.e.,

**(3.9)** $$|A| = a_{i1}\alpha_{i1} + a_{i2}\alpha_{i2} + \cdots + a_{in}\alpha_{in} = \sum_{k=1}^{n} a_{ik}\alpha_{ik}$$

**(3.10)** $$|A| = a_{1j}\alpha_{1j} + a_{2j}\alpha_{2j} + \cdots + a_{nj}\alpha_{nj} = \sum_{k=1}^{n} a_{kj}\alpha_{kj} \qquad (i, j, = 1, 2, \ldots, n)$$

Using Theorem **VII**, we can prove

**XI.** The sum of the products formed by multiplying the elements of a row (column) of an $n$-square matrix $A$ by the corresponding cofactors of **another** row (column) of $A$ is zero.

**Example 3.** If $A$ is the matrix of Example 2, we have

$$a_{31}\alpha_{31} + a_{32}\alpha_{32} + a_{33}\alpha_{33} = |A|$$

and

$$a_{12}\alpha_{12} + a_{22}\alpha_{22} + a_{32}\alpha_{32} = |A|$$

while

$$a_{31}\alpha_{21} + a_{32}\alpha_{22} + a_{33}\alpha_{23} = 0$$

and

$$a_{12}\alpha_{13} + a_{22}\alpha_{23} + a_{32}\alpha_{33} = 0$$

See Problems 10-11.

**MINORS AND ALGEBRAIC COMPLEMENTS.** Consider the matrix (3.3). Let $i_1, i_2, \ldots, i_m$, arranged in order of magnitude, be $m$, $(1 \leq m < n)$, of the row indices $1, 2, \ldots, n$ and let $j_1, j_2, \ldots, j_m$ arranged in order of magnitude, be $m$ of the column indices. Let the remaining row and column indices, arranged in order of magnitude, be respectively $i_{m+1}, i_{m+2}, \ldots, i_n$ and $j_{m+1}, j_{m+2}, \ldots, j_n$. Such a separation of the row and column indices determines uniquely two matrices

**(3.11)** $$A^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_m} = \begin{bmatrix} a_{i_1, j_1} & a_{i_1, j_2} & \cdots & a_{i_1, j_m} \\ a_{i_2, j_1} & a_{i_2, j_2} & \cdots & a_{i_2, j_m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_m, j_1} & a_{i_m, j_2} & \cdots & a_{i_m, j_m} \end{bmatrix}$$

and

$$(3.12) \qquad A^{j_{m+1},j_{m+2},\ldots,j_n}_{i_{m+1},i_{m+2},\ldots,i_n} = \begin{bmatrix} a_{i_{m+1},j_{m+1}} & a_{i_{m+1},j_{m+2}} & \ldots & a_{i_{m+1},j_n} \\ a_{i_{m+2},j_{m+1}} & a_{i_{m+2},j_{m+2}} & \ldots & a_{i_{m+2},j_n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_n,j_{m+1}} & a_{i_n,j_{m+2}} & \ldots & a_{i_n,j_n} \end{bmatrix}$$

called sub-matrices of $A$.

The determinant of each of these sub-matrices is called a **minor** of $A$ and the pair of minors

$$\left| A^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m} \right| \quad \text{and} \quad \left| A^{j_{m+1},j_{m+2},\ldots,j_n}_{i_{m+1},i_{m+2},\ldots,i_n} \right|$$

are called **complementary minors** of $A$, each being the complement of the other.

**Example 3.** For the 5-square matrix $A = \left[a_{ij}\right]$,

$$\left| A^{1,3}_{2,5} \right| = \begin{vmatrix} a_{21} & a_{23} \\ a_{51} & a_{53} \end{vmatrix} \quad \text{and} \quad \left| A^{2,4,5}_{1,3,4} \right| = \begin{vmatrix} a_{12} & a_{14} & a_{15} \\ a_{32} & a_{34} & a_{35} \\ a_{42} & a_{44} & a_{45} \end{vmatrix}$$

are a pair of complementary minors.

Let

$$(3.13) \qquad p = i_1 + i_2 + \cdots + i_m + j_1 + j_2 + \cdots + j_m$$

and

$$(3.14) \qquad q = i_{m+1} + i_{m+2} + \cdots + i_n + j_{m+1} + j_{m+2} + \cdots + j_n$$

The signed minor $(-1)^p \left| A^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m} \right|$ is called the **algebraic complement** of $\left| A^{j_{m+1},j_{m+2},\ldots,j_n}_{i_{m+1},i_{m+2},\ldots,i_n} \right|$

and $(-1)^q \left| A^{j_{m+1},j_{m+2},\ldots,j_n}_{i_{m+1},i_{m+2},\ldots,i_n} \right|$ is called the **algebraic complement** of $\left| A^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m} \right|$

**Example 4.** For the minors of Example 3, $(-1)^{2+5+1+3} \left| A^{1,3}_{2,5} \right| = -\left| A^{1,3}_{2,5} \right|$ is the algebraic complement of $\left| A^{2,4,5}_{1,3,4} \right|$ and $(-1)^{1+3+4+2+4+5} \left| A^{2,4,5}_{1,3,4} \right| = -\left| A^{2,4,5}_{1,3,4} \right|$ is the algebraic complement of $\left| A^{1,3}_{2,5} \right|$. Note that the sign given to the two complementary minors is the same. Is this always true?

When $m=1$, **(3.11)** becomes $A^{j_1}_{i_1} = \left[a_{i_1 j_1}\right]$ and $\left| A^{j_1}_{i_1} \right| = a_{i_1 j_1}$, an element of $A$. The complementary minor $\left| A^{j_2,j_3,\ldots,j_n}_{i_2,i_3,\ldots,i_n} \right|$ is $\left| M_{i_1,j_1} \right|$ in the notation of the section above, and the algebraic complement is the cofactor $\alpha_{i_1 j_1}$.

A minor of $A$, whose diagonal elements are also diagonal elements of $A$, is called a **principal minor** of $A$. The complement of a principal minor of $A$ is also a principal minor of $A$; the algebraic complement of a principal minor is its complement.

**Example 5.** For the 5-square matrix $A = [a_{ij}]$,

$$|A^{1,3}_{1,3}| = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \quad \text{and} \quad |A^{2,4,5}_{2,4,5}| = \begin{vmatrix} a_{22} & a_{24} & a_{25} \\ a_{42} & a_{44} & a_{45} \\ a_{52} & a_{54} & a_{55} \end{vmatrix}$$

are a pair of complementary principal minors of $A$. What is the algebraic complement of each?

The terms minor, complementary minor, algebraic complement, and principal minor as defined above for a square matrix $A$ will also be used without change in connection with $|A|$.

See Problems 12-13.

## SOLVED PROBLEMS

1. (a) $\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} = 2\cdot4 - 3(-1) = 11$

(b) $\begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix} = (1)\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} - 0\begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = (1)(4\cdot7 - 5\cdot6) - 0 + 2(3\cdot6 - 4\cdot5) = -2 - 4 = -6$

(c) $\begin{vmatrix} 1 & 0 & 6 \\ 3 & 4 & 15 \\ 5 & 6 & 21 \end{vmatrix} = 1(4\cdot21 - 15\cdot6) + 6(3\cdot6 - 4\cdot5) = -18$

(d) $\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \\ 4 & 1 & 3 \end{vmatrix} = 1(3\cdot3 - 5\cdot1) = 4$

2. Adding to the elements of the first column the corresponding elements of the other columns,

$$\begin{vmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & 1 & 1 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 1 & 1 & 1 & -4 \end{vmatrix} = 0$$

by Theorem I.

3. Adding the second column to the third, removing the common factor from this third column, and using Theorem VII

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = 0$$

4. Adding to the third row the first and second rows, then removing the common factor 2; subtracting the second row from the third; subtracting the third row from the first; subtracting the first row from the second; finally carrying the third row over the other rows

$$\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ c_1+a_1 & c_2+a_2 & c_3+a_3 \end{vmatrix} = 2\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+b_1+c_1 & a_2+b_2+c_2 & a_3+b_3+c_3 \end{vmatrix} = 2\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= 2\begin{vmatrix} b_1 & b_2 & b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 2\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 2\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

5. Without expanding, show that $|A| = \begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} = -(a_1-a_2)(a_2-a_3)(a_3-a_1)$.

Subtract the second row from the first; then

$$|A| = \begin{vmatrix} a_1^2-a_2^2 & a_1-a_2 & 0 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} = (a_1-a_2)\begin{vmatrix} a_1+a_2 & 1 & 0 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} \quad \text{by Theorem III}$$

and $a_1-a_2$ is a factor of $|A|$. Similarly, $a_2-a_3$ and $a_3-a_1$ are factors. Now $|A|$ is of order three in the letters; hence,

(*i*) $$|A| = k(a_1-a_2)(a_2-a_3)(a_3-a_1)$$

The product of the diagonal elements, $a_1^2a_2$, is a term of $|A|$ and, from (i), the term is $-ka_1^2a_2$. Thus, $k=-1$ and $|A| = -(a_1-a_2)(a_2-a_3)(a_3-a_1)$. Note that $|A|$ vanishes if and only if two of the $a_1, a_2, a_3$ are equal.

6. Prove: If $A$ is skew-symmetric and of odd order $2p-1$, then $|A| = 0$.

Since $A$ is skew-symmetric, $A' = -A$; then $|A'| = |-A| = (-1)^{2p-1}|A| = -|A|$. But, by Theorem II, $|A'| = |A|$; hence, $|A| = -|A|$ and $|A| = 0$.

7. Prove: If $A$ is Hermitian, then $|A|$ is a real number.

Since $A$ is Hermitian, $\bar{A} = A'$, and $|\bar{A}| = |A'| = |A|$ by Theorem II. But if

$$|A| = \sum_{\rho} \epsilon_{j_1j_2\ldots j_n} a_{1j_1}a_{2j_2}\cdots a_{nj_n} = a+bi$$

then

$$|\bar{A}| = \sum_{\rho} \epsilon_{j_1j_2\ldots j_n} \bar{a}_{1j_1}\bar{a}_{2j_2}\cdots\bar{a}_{nj_n} = a-bi$$

Now $|\bar{A}| = |A|$ requires $b=0$; hence, $|A|$ is a real number.

8. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$,

$$\alpha_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 2, \quad \alpha_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -2, \quad \alpha_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1$$

$$\alpha_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = 2, \quad \alpha_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1, \quad \alpha_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

$$\alpha_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5, \quad \alpha_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = 4, \quad \alpha_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$$

Note that the signs given to the minors of the elements in forming the cofactors follow the pattern

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

where each sign occupies the same position in the display as the element, whose cofactor is required, occupies in $A$. Write the display of signs for a 5-square matrix.

9. Prove: The value of the determinant $|A|$ of an $n$-square matrix $A$ is the sum of the products obtained by multiplying each element of a row (column) of $A$ by its cofactor.

We shall prove this for a row. The terms of (3.6) having $a_{11}$ as a factor are

(a) $$a_{11} \sum \epsilon_{1,\, j_2 j_3 \ldots j_n} a_{2j_2} a_{3j_3} \ldots a_{nj_n}$$

Now $\epsilon_{1,\, j_2 j_3 \ldots j_n} = \epsilon_{j_2 j_3 \ldots j_n}$ since in a permutation $1, j_1, j_2, \ldots j_n$, the 1 is in natural order. Then (a) may be written as

(b) $$a_{11} \sum_{\sigma} \epsilon_{j_2 j_3 \ldots j_n} a_{2j_2} a_{3j_3} \ldots a_{nj_n}$$

where the summation extends over the $\sigma = (n-1)!$ permutations of the integers $2, 3, \ldots, n$, and hence, as

(c) $$a_{11} \begin{vmatrix} a_{22} & a_{23} & \ldots & a_{2n} \\ a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix} = a_{11}|M_{11}|$$

Consider the matrix $B$ obtained from $A$ by moving its $s$th column over the first $s-1$ columns. By Theorem VI, $|B| = (-1)^{s-1}|A|$. Moreover, the element standing in the first row and first column of $B$ is $a_{1s}$ and the minor of $a_{1s}$ in $B$ is precisely the minor $|M_{1s}|$ of $a_{1s}$ in $A$. By the argument leading to (c), the terms of $a_{1s}|M_{1s}|$ are all the terms of $|B|$ having $a_{1s}$ as a factor and, thus, all the terms of $(-1)^{s-1}|A|$ having $a_{1s}$ as a factor. Then the terms of $a_{1s}\{(-1)^{s-1}|M_{1s}|\}$ are all the terms of $|A|$ having $a_{1s}$ as a factor. Thus,

(3.15) $$\begin{aligned} |A| &= a_{11}\{(-1)^{1+1}|M_{11}|\} + a_{12}\{(-1)^{1+2}|M_{12}|\} \\ &\quad + \cdots + a_{1s}\{(-1)^{1+s}|M_{1s}|\} + \cdots + a_{1n}\{(-1)^{1+n}|M_{1n}|\} \\ &= a_{11}\alpha_{11} + a_{12}\alpha_{12} + \cdots + a_{1n}\alpha_{1n} \end{aligned}$$

since $(-1)^{s-1} = (-1)^{s+1}$. We have (3.9) with $i = 1$. We shall call (3.15) the expansion of $|A|$ along its first row.

The expansion of $|A|$ along its $r$th row (that is, (3.9) for $i = r$) is obtained by repeating the above arguments. Let $B$ be the matrix obtained from $A$ by moving its $r$th row over the first $r-1$ rows and then its $s$th column over the first $s-1$ columns. Then

$$|B| = (-1)^{r-1} \cdot (-1)^{s-1}|A| = (-1)^{r+s}|A|$$

The element standing in the first row and the first column of $B$ is $a_{rs}$ and the minor of $a_{rs}$ in $B$ is precisely the minor of $a_{rs}$ in $A$. Thus, the terms of

$$a_{rs}\{(-1)^{r+s}|M_{rs}|\}$$

are all the terms of $|A|$ having $a_{rs}$ as a factor. Then

$$|A| = \sum_{k=1}^{n} a_{rk}\{(-1)^{r+k}|M_{rk}|\} = \sum_{k=1}^{n} a_{rk}\alpha_{rk}$$

and we have (3.9) for $i = r$.

**10.** When $\alpha_{ij}$ is the cofactor of $a_{ij}$ in the $n$-square matrix $A = [a_{ij}]$, show that

$$(i) \qquad k_1\alpha_{1j} + k_2\alpha_{2j} + \cdots + k_n\alpha_{nj} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & k_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & k_2 & a_{2,j+1} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & k_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

This relation follows from **(3.10)** by replacing $a_{1j}$ with $k_1$, $a_{2j}$ with $k_2, \ldots, a_{nj}$ with $k_n$. In making these replacements none of the cofactors $\alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{nj}$ appearing is affected since none contains an element from the $j$th column of $A$.

By Theorem **VII**, the determinant in $(i)$ is 0 when $k_r = a_{rs}$, $(r = 1, 2, \ldots, n$ and $s \neq j)$. By Theorems **VIII**, and **VII**, the determinant in $(i)$ is $|A|$ when $k_r = a_{rj} + ka_{rs}$, $(r = 1, 2, \ldots, n$ and $s \neq j)$.

Write the equality similar to $(i)$ obtained from **(3.9)** when the elements of the $i$th row of $A$ are replaced by $k_1, k_2, \ldots, k_n$.

**11.** Evaluate: $(a)\ |A| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 0 & 4 \\ 2 & -5 & 1 \end{vmatrix}$ $\quad (c)\ |A| = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ -2 & 5 & -4 \end{vmatrix}$ $\quad (e)\ |A| = \begin{vmatrix} 28 & 25 & 38 \\ 42 & 38 & 65 \\ 56 & 47 & 83 \end{vmatrix}$

$(b)\ |A| = \begin{vmatrix} 1 & 4 & 3 \\ -2 & 1 & 5 \\ -3 & 2 & 4 \end{vmatrix}$ $\quad (d)\ |A| = \begin{vmatrix} 2 & 3 & -4 \\ 5 & -6 & 3 \\ 4 & 2 & -3 \end{vmatrix}$

$(a)$ Expanding along the second column (see Theorem **X**)

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 0 & 4 \\ 2 & -5 & 1 \end{vmatrix} = a_{12}\alpha_{12} + a_{22}\alpha_{22} + a_{32}\alpha_{32} = 0\cdot\alpha_{12} + 0\cdot\alpha_{22} + (-5)\alpha_{32}$$

$$= -5(-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 5(4-6) = -10$$

$(b)$ Subtracting twice the second column from the third (see Theorem **IX**)

$$|A| = \begin{vmatrix} 1 & 4 & 8 \\ -2 & 1 & 5 \\ -3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 8-2\cdot4 \\ -2 & 1 & 5-2\cdot1 \\ -3 & 2 & 4-2\cdot2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 0 \\ -2 & 1 & 3 \\ -3 & 2 & 0 \end{vmatrix} = 3(-1)^{2+3}\begin{vmatrix} 1 & 4 \\ -3 & 2 \end{vmatrix}$$

$$= -3(14) = -42$$

$(c)$ Subtracting three times the second row from the first and adding twice the second row to the third

$$|A| = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ -2 & 5 & -4 \end{vmatrix} = \begin{vmatrix} 3-3(1) & 4-3(2) & 5-3(3) \\ 1 & 2 & 3 \\ -2+2(1) & 5+2(2) & -4+2(3) \end{vmatrix} = \begin{vmatrix} 0 & -2 & -4 \\ 1 & 2 & 3 \\ 0 & 9 & 2 \end{vmatrix} = -\begin{vmatrix} -2 & -4 \\ 9 & 2 \end{vmatrix}$$

$$= -(-4+36) = -32$$

$(d)$ Subtracting the first column from the second and then proceeding as in $(c)$

$$|A| = \begin{vmatrix} 2 & 3 & -4 \\ 5 & -6 & 3 \\ 4 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -4 \\ 5 & -11 & 3 \\ 4 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 2-2(1) & 1 & -4+4(1) \\ 5-2(-11) & -11 & 3+4(-11) \\ 4-2(-2) & -2 & -3+4(-2) \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 0 \\ 27 & -11 & -41 \\ 8 & -2 & -11 \end{vmatrix} = -\begin{vmatrix} 27 & -41 \\ 8 & -11 \end{vmatrix} = -31$$

(e) Factoring 14 from the first column, then using Theorem **IX** to reduce the elements in the remaining columns

$$|A| = \begin{vmatrix} 28 & 25 & 38 \\ 42 & 38 & 65 \\ 56 & 47 & 83 \end{vmatrix} = 14\begin{vmatrix} 2 & 25 & 38 \\ 3 & 38 & 65 \\ 4 & 47 & 83 \end{vmatrix} = 14\begin{vmatrix} 2 & 25-12(2) & 38-20(2) \\ 3 & 38-12(3) & 65-20(3) \\ 4 & 47-12(4) & 83-20(4) \end{vmatrix}$$

$$= 14\begin{vmatrix} 2 & 1 & -2 \\ 3 & 2 & 5 \\ 4 & -1 & 3 \end{vmatrix} = 14\begin{vmatrix} 0 & 1 & 0 \\ -1 & 2 & 9 \\ 6 & -1 & 1 \end{vmatrix} = -14\begin{vmatrix} -1 & 9 \\ 6 & 1 \end{vmatrix} = -14(-1-54) = 770$$

**12.** Show that $p$ and $q$, given by **(3.13)** and **(3.14)**, are either both even or both odd.

Since each row (column) index is found in either $p$ or $q$ but never in both,

$$p+q = (1+2+\cdots+n) + (1+2+\cdots+n) = 2\cdot\tfrac{1}{2}n(n+1) = n(n+1)$$

Now $p+q$ is even (either $n$ or $n+1$ is even); hence, $p$ and $q$ are either both even or both odd. Thus, $(-1)^p = (-1)^q$ and only one need be computed.

**13.** For the matrix $A = [a_{ij}] = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{bmatrix}$, the algebraic complement of $|A^{2,4}_{2,3}|$ is

$$(-1)^{2+3+2+4}|A^{1,3,5}_{1,4,5}| = -\begin{vmatrix} 1 & 3 & 5 \\ 16 & 18 & 20 \\ 21 & 23 & 25 \end{vmatrix} \qquad \text{(see Problem 12)}$$

and the algebraic complement of $|A^{1,3,5}_{1,4,5}|$ is $-|A^{2,4}_{2,3}| = -\begin{vmatrix} 7 & 9 \\ 12 & 14 \end{vmatrix}$.

## SUPPLEMENTARY PROBLEMS

**14.** Show that the permutation 12534 of the integers 1, 2, 3, 4, 5 is even, 24135 is odd, 41532 is even, 53142 is odd, and 52314 is even.

**15.** List the complete set of permutations of 1, 2, 3, 4, taken together; show that half are even and half are odd.

**16.** Let the elements of the diagonal of a 5-square matrix $A$ be $a, b, c, d, e$. Show, using **(3.6)**, that when $A$ is diagonal, upper triangular, or lower triangular then $|A| = abcde$.

**17.** Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, show that $AB \neq BA \neq A'B \neq AB' \neq A'B' \neq B'A'$ but that the determinant of each product is 4.

**18.** Evaluate, as in Problem 1.

(a) $\begin{vmatrix} 2 & -1 & 1 \\ 3 & 2 & 4 \\ -1 & 0 & 3 \end{vmatrix} = 27$ (b) $\begin{vmatrix} 2 & 2 & -2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 4$ (c) $\begin{vmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{vmatrix} = 0$

**19.** (*a*) Evaluate $|A| = \begin{vmatrix} 1 & 2 & 10 \\ 2 & 3 & 9 \\ 4 & 5 & 11 \end{vmatrix} = -4$.

(*b*) Denote by $|B|$ the determinant obtained from $|A|$ by multiplying the elements of its second column by 5. Evaluate $|B|$ to verify Theorem **III**.

(*c*) Denote by $|C|$ the determinant obtained from $|A|$ by interchanging its first and third columns. Evaluate $|C|$ to verify Theorem **V**.

(*d*) Show that $|A| = \begin{vmatrix} 1 & 2 & 7 \\ 2 & 3 & 5 \\ 4 & 5 & 8 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 3 \end{vmatrix}$, thus verifying Theorem **VIII**.

(*e*) Obtain from $|A|$ the determinant $|D| = \begin{vmatrix} 1 & 2 & 7 \\ 2 & 3 & 3 \\ 4 & 5 & -1 \end{vmatrix}$ by subtracting three times the elements of the first column from the corresponding elements of the third column. Evaluate $|D|$ to verify Theorem **IX**.

(*f*) In $|A|$ subtract twice the first row from the second and four times the first row from the third. Evaluate the resulting determinant.

(*g*) In $|A|$ multiply the first column by three and from it subtract the third column. Evaluate to show that $|A|$ has been tripled. Compare with (*e*). Do not confuse (*e*) and (*g*).

**20.** If $A$ is an $n$-square matrix and $k$ is a scalar, use (3.6) to show that $|kA| = k^n|A|$.

**21.** Prove: (*a*) If $|A| = k$, then $|\bar{A}| = \bar{k} = |\bar{A}'|$.

(*b*) If $A$ is skew-Hermitian, then $|A|$ is either real or is a pure imaginary number.

**22.** (*a*) Count the number of interchanges of adjacent rows (columns) necessary to obtain $B$ from $A$ in Theorem **V** and thus prove the theorem.

(*b*) Same, for Theorem **VI**.

**23.** Prove Theorem **VII**. Hint: Interchange the identical rows and use Theorem **V**.

**24.** Prove: If any two rows (columns) of a square matrix $A$ are proportional, then $|A| = 0$.

**25.** Use Theorems **VIII**, **III**, and **VII** to prove Theorem **IX**.

**26.** Evaluate the determinants of Problem 18 as in Problem 11.

**27.** Use (3.6) to evaluate $|A| = \begin{vmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{vmatrix}$; then check that $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & f \\ g & h \end{vmatrix}$. Thus, if $A = diag(A_1, A_2)$, where $A_1, A_2$ are 2-square matrices, $|A| = |A_1| \cdot |A_2|$.

**28.** Show that the cofactor of each element of $\begin{bmatrix} -1/3 & -2/3 & -2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$ is that element.

**29.** Show that the cofactor of an element of any row of $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ is the corresponding element of the same numbered column.

**30.** Prove: (*a*) If $A$ is symmetric then $\alpha_{ij} = \alpha_{ji}$ when $i \neq j$.

(*b*) If $A$ is $n$-square and skew-symmetric then $\alpha_{ij} = (-1)^{n-1}\alpha_{ji}$ when $i \neq j$.

31. For the matrix $A$ of Problem 8;
(a) show that $|A| = 1$
(b) form the matrix $C = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix}$ and show that $AC = I$.
(c) explain why the result in (b) is known as soon as (a) is known.

32. Multiply the columns of $|A| = \begin{vmatrix} bc & a^2 & a^2 \\ b^2 & ca & b^2 \\ c^2 & c^2 & ab \end{vmatrix}$ respectively by $a, b, c$; remove the common factor from each of the rows to show that $|A| = \begin{vmatrix} bc & ab & ca \\ ab & ca & bc \\ ca & bc & ab \end{vmatrix}$.

33. Without evaluating show that $\begin{vmatrix} a^2 & a & 1 & bcd \\ b^2 & b & 1 & acd \\ c^2 & c & 1 & abd \\ d^2 & d & 1 & abc \end{vmatrix} = \begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix} = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$.

34. Show that the $n$-square determinant $|A| = \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \cdots & & & & \\ \cdots & & & & \\ 1 & 1 & 1 & \ldots & 0 \end{vmatrix} = (n-1) \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 0 & \ldots & 1 & 1 \\ \cdots & & & & & \\ 1 & 1 & 1 & \ldots & 0 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 \end{vmatrix} = (-1)^{n-1}(n-1)$.

35. Prove: $\begin{vmatrix} a_1^{n-1} & a_1^{n-2} & \ldots & a_1 & 1 \\ a_2^{n-1} & a_2^{n-2} & \ldots & a_2 & 1 \\ \cdots & & & & \\ a_n^{n-1} & a_n^{n-2} & \ldots & a_n & 1 \end{vmatrix} = \{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)\}\{(a_2-a_3)(a_2-a_4)\ldots(a_2-a_n)\}\cdots\{a_{n-1}-a_n\}$.

36. Without expanding, show that $\begin{vmatrix} na_1+b_1 & na_2+b_2 & na_3+b_3 \\ nb_1+c_1 & nb_2+c_2 & nb_3+c_3 \\ nc_1+a_1 & nc_2+a_2 & nc_3+a_3 \end{vmatrix} = (n+1)(n^2-n+1) \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

37. Without expanding, show that the equation $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$ has 0 as a root.

38. Prove $\begin{vmatrix} a+b & a & a & \cdots & a \\ a & a+b & a & \cdots & a \\ \cdots & & & & \\ \cdots & & & & \\ a & a & a & \cdots & a+b \end{vmatrix} = b^{n-1}(na+b)$.

# Chapter 4

## Evaluation of Determinants

**PROCEDURES FOR EVALUATING** determinants of orders two and three are found in Chapter 3. In Problem 11 of that chapter, two uses of Theorem **IX** were illustrated: (*a*) to obtain an element 1 or $-1$ if the given determinant contains no such element, (*b*) to replace an element of a given determinant with 0.

For determinants of higher orders, the general procedure is to replace, by repeated use of Theorem **IX**, Chapter 3, the given determinant $|A|$ by another $|B| = |b_{ij}|$ having the property that all elements, except one, in some row (column) are zero. If $b_{pq}$ is this non-zero element and $\beta_{pq}$ is its cofactor,

$$|A| \quad = \quad |B| \quad = \quad b_{pq}\cdot\beta_{pq} \quad = \quad (-1)^{p+q}b_{pq}\cdot \text{minor of } b_{pq}$$

Then the minor of $b_{pq}$ is treated in similar fashion and the process is continued until a determinant of order two or three is obtained.

**Example 1.**

$$\begin{vmatrix} 2 & 3 & -2 & 4 \\ 3 & -2 & 1 & 2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 2+2(3) & 3+2(-2) & -2+2(1) & 4+2(2) \\ 3 & -2 & 1 & 2 \\ 3-3(3) & 2-3(-2) & 3-3(1) & 4-3(2) \\ -2 & 4 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 8 & -1 & 0 & 8 \\ 3 & -2 & 1 & 2 \\ -6 & 8 & 0 & -2 \\ -2 & 4 & 0 & 5 \end{vmatrix}$$

$$= (-1)^{2+3}\begin{vmatrix} 8 & -1 & 8 \\ -6 & 8 & -2 \\ -2 & 4 & 5 \end{vmatrix} = -\begin{vmatrix} 8+8(-1) & -1 & 8+8(-1) \\ -6+8(8) & 8 & -2+8(8) \\ -2+8(4) & 4 & 5+8(4) \end{vmatrix} = -\begin{vmatrix} 0 & -1 & 0 \\ 58 & 8 & 62 \\ 30 & 4 & 37 \end{vmatrix}$$

$$= -(-1)^{1+2}(-1)\begin{vmatrix} 58 & 62 \\ 30 & 37 \end{vmatrix} = -286$$

See Problems 1-3

For determinants having elements of the type in Example 2 below, the following variation may be used: divide the first row by one of its non-zero elements and proceed to obtain zero elements in a row or column.

**Example 2.**

$$\begin{vmatrix} 0.921 & 0.185 & 0.476 & 0.614 \\ 0.782 & 0.157 & 0.527 & 0.138 \\ 0.872 & 0.484 & 0.637 & 0.799 \\ 0.312 & 0.555 & 0.841 & 0.448 \end{vmatrix} = 0.921\begin{vmatrix} 1 & 0.201 & 0.517 & 0.667 \\ 0.782 & 0.157 & 0.527 & 0.138 \\ 0.872 & 0.484 & 0.637 & 0.799 \\ 0.312 & 0.555 & 0.841 & 0.448 \end{vmatrix} = 0.921\begin{vmatrix} 1 & 0.201 & 0.517 & 0.667 \\ 0 & 0 & 0.123 & -0.384 \\ 0 & 0.309 & 0.196 & 0.217 \\ 0 & 0.492 & 0.680 & 0.240 \end{vmatrix}$$

$$= 0.921\begin{vmatrix} 0 & 0.123 & -0.384 \\ 0.309 & 0.196 & 0.217 \\ 0.492 & 0.680 & 0.240 \end{vmatrix} = 0.921(-0.384)\begin{vmatrix} 0 & -0.320 & 1 \\ 0.309 & 0.196 & 0.217 \\ 0.492 & 0.680 & 0.240 \end{vmatrix}$$

$$= 0.921(-0.384)\begin{vmatrix} 0 & 0 & 1 \\ 0.309 & 0.265 & 0.217 \\ 0.492 & 0.757 & 0.240 \end{vmatrix} = 0.921(-0.384)\begin{vmatrix} 0.309 & 0.265 \\ 0.492 & 0.757 \end{vmatrix}$$

$$= 0.921(-0.384)(0.104) = -0.037$$

**THE LAPLACE EXPANSION**. The expansion of a determinant $|A|$ of order $n$ along a row (column) is a special case of the Laplace expansion. Instead of selecting one row of $|A|$, let $m$ rows numbered $i_1, i_2, \ldots, i_m$, when arranged in order of magnitude, be selected. From these $m$ rows

$$\rho \;=\; \frac{n(n-1)\ldots(n-m+1)}{1\cdot 2\;\ldots\; m} \text{ minors} \left| A^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_m} \right|$$

can be formed by making all possible selections of $m$ columns from the $n$ columns. Using these minors and their algebraic complements, we have the Laplace expansion

$$(4.1) \qquad |A| \;=\; \sum_{\rho} (-1)^s \left| A^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_m} \right| \cdot \left| A^{j_{m+1}, j_{m+2}, \ldots, j_n}_{i_{m+1}, i_{m+2}, \ldots, i_n} \right|$$

where $s = i_1 + i_2 + \cdots + i_m + j_1 + j_2 + \cdots + j_m$ and the summation extends over the $\rho$ selections of the column indices taken $m$ at a time.

**Example 3.**

Evaluate $|A| = \begin{vmatrix} 2 & 3 & -2 & 4 \\ 3 & -2 & 1 & 2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{vmatrix}$, using minors of the first two rows.

From (4.1),

$$|A| \;=\; (-1)^{1+2+1+2}\left|A^{1,2}_{1,2}\right|\cdot\left|A^{3,4}_{3,4}\right| \;+\; (-1)^{1+2+1+3}\left|A^{1,3}_{1,2}\right|\cdot\left|A^{2,4}_{3,4}\right|$$

$$+\; (-1)^{1+2+1+4}\left|A^{1,4}_{1,2}\right|\cdot\left|A^{2,3}_{3,4}\right| \;+\; (-1)^{1+2+2+3}\left|A^{2,3}_{1,2}\right|\cdot\left|A^{1,4}_{3,4}\right|$$

$$+\; (-1)^{1+2+2+4}\left|A^{2,4}_{1,2}\right|\cdot\left|A^{1,3}_{3,4}\right| \;+\; (-1)^{1+2+3+4}\left|A^{3,4}_{1,2}\right|\cdot\left|A^{1,2}_{3,4}\right|$$

$$= \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix}\cdot\begin{vmatrix} 3 & 4 \\ 0 & 5 \end{vmatrix} - \begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix}\cdot\begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}\cdot\begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} 3 & -2 \\ -2 & 1 \end{vmatrix}\cdot\begin{vmatrix} 3 & 4 \\ -2 & 5 \end{vmatrix} - \begin{vmatrix} 3 & 4 \\ -2 & 2 \end{vmatrix}\cdot\begin{vmatrix} 3 & 3 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 4 \\ 1 & 2 \end{vmatrix}\cdot\begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix}$$

$$= (-13)(15) - (8)(-6) + (-8)(-12) + (-1)(23) - (14)(6) + (-8)(16)$$

$$= -286$$

See Problems 4-6

**DETERMINANT OF A PRODUCT**. If $A$ and $B$ are $n$-square matrices, then

$$(4.2) \qquad |AB| \;=\; |A|\cdot|B|$$

See Problem 7

**EXPANSION ALONG THE FIRST ROW AND COLUMN**. If $A = [a_{ij}]$ is $n$-square, then

$$(4.3) \qquad |A| \;=\; a_{11}\alpha_{11} \;-\; \sum_{i=2}^{n}\sum_{j=2}^{n} a_{i1}a_{1j}\alpha^{i1}_{1j}$$

where $\alpha_{11}$ is the cofactor of $a_{11}$ and $\alpha^{i1}_{1j}$ is the algebraic complement of the minor $\begin{vmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{vmatrix}$ of $A$.

**DERIVATIVE OF A DETERMINANT**. Let the $n$-square matrix $A = [a_{ij}]$ have as elements differentiable functions of a variable $x$. Then

**I.** The derivative, $\frac{d}{dx}|A|$, of $|A|$ with respect to $x$ is the sum of $n$ determinants obtained by replacing in all possible ways the elements of one row (column) of $|A|$ by their derivatives with respect to $x$.

**Example 4.**

$$\frac{d}{dx}\begin{vmatrix} x^2 & x+1 & 3 \\ 1 & 2x-1 & x^3 \\ 0 & x & -2 \end{vmatrix} = \begin{vmatrix} 2x & 1 & 0 \\ 1 & 2x-1 & x^3 \\ 0 & x & -2 \end{vmatrix} + \begin{vmatrix} x^2 & x+1 & 3 \\ 0 & 2 & 3x^2 \\ 0 & x & -2 \end{vmatrix} + \begin{vmatrix} x^2 & x+1 & 3 \\ 1 & 2x-1 & x^3 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 5 + 4x - 12x^2 - 6x^5$$

See Problem 8

# SOLVED PROBLEMS

**1.**
$$\begin{vmatrix} 2 & 3 & -2 & 4 \\ 7 & 4 & -3 & 10 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -2 & 4 \\ 7-2(2) & 4-2(3) & -3-2(-2) & 10-2(4) \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -2 & 4 \\ 3 & -2 & 1 & 2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{vmatrix} = -286 \text{ (See Example 1)}$$

There are, of course, many other ways of obtaining an element +1 or −1; for example, subtract the first column from the second, the fourth column from the second, the first row from the second, etc.

**2.**
$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 2 & -2 \\ 2 & 4 & 2 & 1 \\ 3 & 1 & 5 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1+1 & 2-2(1) \\ 2 & 3 & 2+2 & -2-2(2) \\ 2 & 4 & 2+2 & 1-2(2) \\ 3 & 1 & 5+3 & -3-2(3) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & -6 \\ 2 & 4 & 4 & -3 \\ 3 & 1 & 8 & -9 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 & -6 \\ 4 & 4 & -3 \\ 1 & 8 & -9 \end{vmatrix} = \begin{vmatrix} 3-2(4) & 4-2(4) & -6-2(-3) \\ 4 & 4 & -3 \\ 1-3(4) & 8-3(4) & -9-3(-3) \end{vmatrix} = \begin{vmatrix} -5 & -4 & 0 \\ 4 & 4 & -3 \\ -11 & -4 & 0 \end{vmatrix}$$

$$= 3\begin{vmatrix} -5 & -4 \\ -11 & -4 \end{vmatrix} = -72$$

**3.** Evaluate $|A| = \begin{vmatrix} 0 & 1+i & 1+2i \\ 1-i & 0 & 2-3i \\ 1-2i & 2+3i & 0 \end{vmatrix}$.

Multiply the second row by $1+i$ and the third row by $1+2i$; then

$$(1+i)(1+2i)|A| = (-1+3i)|A| = \begin{vmatrix} 0 & 1+i & 1+2i \\ 2 & 0 & 5-i \\ 5 & -4+7i & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1+i & 1+2i \\ 2 & 0 & 5-i \\ 1 & -4+7i & -10+2i \end{vmatrix} = \begin{vmatrix} 0 & 1+i & 1+2i \\ 0 & 8-14i & 25-5i \\ 1 & -4+7i & -10+2i \end{vmatrix}$$

$$= \begin{vmatrix} 1+i & 1+2i \\ 8-14i & 25-5i \end{vmatrix} = -6 + 18i$$

and $|A| = 6$.

**4.** Derive the Laplace expansion of $|A| = |a_{ij}|$ of order $n$, using minors of order $m<n$.

Consider the $m$-square minor $\left|A^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m}\right|$ of $|A|$ in which the row and column indices are arranged in order of magnitude. Now by $i_1-1$ interchanges of adjacent rows of $|A|$, the row numbered $i_1$ can be brought into the first row, by $i_2-2$ interchanges of adjacent rows the row numbered $i_2$ can be brought into the second row, ..., by $i_m-m$ interchanges of adjacent rows the row numbered $i_m$ can be brought into the $m$th row. Thus, after $(i_1-1)+(i_2-2)+\cdots+(i_m-m) = i_1+i_2+\cdots+i_m-\frac{1}{2}m(m+1)$ interchanges of adjacent rows the rows numbered $i_1, i_2, \ldots, i_m$ occupy the position of the first $m$ rows. Similarly, after $j_1+j_2+\cdots+j_m-\frac{1}{2}m(m+1)$ interchanges of adjacent columns, the columns numbered $j_1, j_2, \ldots, j_m$ occupy the position of the first $m$ columns. As a result of the interchanges of adjacent rows and adjacent columns, the minor selected above occupies the upper left corner and its complement occupies the lower right corner of the determinant; moreover, $|A|$ has changed sign $\sigma = i_1+i_2+\cdots+i_m+j_1+j_2+\cdots+j_m-m(m+1)$ times which is equivalent to $s = i_1+i_2+\cdots+i_m+j_1+j_2+\cdots+j_m$ changes. Thus

$$\left|A^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m}\right| \cdot \left|A^{j_{m+1},j_{m+2},\ldots,j_n}_{i_{m+1},i_{m+2},\ldots,i_n}\right| \quad \text{yields} \quad m!(n-m)! \text{ terms of } (-1)^s|A| \quad \text{or}$$

(a) $$(-1)^s\left|A^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m}\right| \cdot \left|A^{j_{m+1},j_{m+2},\ldots,j_n}_{i_{m+1},i_{m+2},\ldots,i_n}\right| \quad \text{yields} \quad m!(n-m)! \text{ terms of } |A|.$$

Let $i_1, i_2, \ldots, i_m$ be held fixed. From these rows $\rho = \dfrac{n(n-1)\ldots(n-m+1)}{1\cdot 2\ldots m} = \dfrac{n!}{m!(n-m)!}$ different $m$-square minors may be selected. Each of these minors when multiplied by its algebraic complement yields $m!(n-m)!$ terms of $|A|$. Since, by their formation, there are no duplicate terms of $|A|$ among these products,

$$|A| = \sum_{\rho}(-1)^s\left|A^{j_1,j_2,\ldots,j_m}_{i_1,i_2,\ldots,i_m}\right| \cdot \left|A^{j_{m+1},j_{m+2},\ldots,j_n}_{i_{m+1},i_{m+2},\ldots,i_n}\right|$$

where $s = i_1+i_2+\cdots+i_m+j_1+j_2+\cdots+j_m$ and the summation extends over the $\rho$ different selections $j_1, j_2, \ldots, j_m$ of the column indices.

**5.** Evaluate $|A| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 3 & 4 & 1 & 2 \end{vmatrix}$, using minors of the first two columns.

$$\begin{aligned} |A| &= (-1)^{1+2+1+2}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + (-1)^{1+4+1+2}\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + (-1)^{2+4+1+2}\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} \\ &= (-3)(1) + (-2)(1) - (5)(-1) \\ &= 0 \end{aligned}$$

**6.** If $A$, $B$, and $C$ are $n$-square matrices, prove

$$|P| = \begin{vmatrix} A & O \\ C & B \end{vmatrix} = |A| \cdot |B|$$

From the first $n$ rows of $|P|$ only one non-zero $n$-square minor, $|A|$, can be formed. Its algebraic complement is $|B|$. Hence, by the Laplace expansion, $|P| = |A| \cdot |B|$.

**7.** Prove $|AB| = |A| \cdot |B|$.

Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ are $n$-square. Let $C = [c_{ij}] = AB$ so that $c_{ij} = \sum_1^n a_{ik}b_{kj}$. From Problem 6,

$$|P| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & -1 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix} = |A| \cdot |B|$$

To the $(n+1)$st column of $|P|$ add $b_{11}$ times the first column, $b_{21}$ times the second column, ..., $b_{n1}$ times the $n$th column; we have

$$|P| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & c_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & c_{21} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & c_{n1} & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 0 & b_{12} & \cdots & b_{1n} \\ 0 & -1 & \cdots & 0 & 0 & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 & 0 & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

Next, to the $(n+2)$nd column of $|P|$ add $b_{12}$ times the first column, $b_{22}$ times the second column, ..., $b_{n2}$ times the $n$th column. We have

$$|P| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & c_{11} & c_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & c_{21} & c_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & c_{n1} & c_{n2} & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 0 & 0 & b_{13} & \cdots & b_{1n} \\ 0 & -1 & \cdots & 0 & 0 & 0 & b_{23} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & b_{n3} & \cdots & b_{nn} \end{vmatrix}$$

Continuing this process, we obtain finally $|P| = \begin{vmatrix} A & C \\ -I_n & 0 \end{vmatrix}$. From the last $n$ rows of $|P|$ only one non-zero $n$-square minor, $|-I_n| = (-1)^n$ can be formed. Its algebraic complement is $(-1)^{1+2+\cdots+n+(n+1)+\cdots+2n}|C| = (-1)^{n(2n+1)}|C|$. Hence, $|P| = (-1)^n(-1)^{n(2n+1)}|C| = |C|$ and $|C| = |AB| = |A|\cdot|B|$.

8. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ where $a_{ij} = a_{ij}(x)$, $(i, j = 1, 2, 3)$, are differentiable functions of $x$. Then

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

and, denoting $\frac{d}{dx}a_{ij}$ by $a'_{ij}$,

$$\frac{d}{dx}|A| = a'_{11}a_{22}a_{33} + a'_{22}a_{11}a_{33} + a'_{33}a_{11}a_{22} + a'_{12}a_{23}a_{31} + a'_{23}a_{12}a_{31} + a'_{31}a_{12}a_{23}$$
$$+ a'_{13}a_{32}a_{21} + a'_{32}a_{13}a_{21} + a'_{21}a_{13}a_{32} - a'_{11}a_{23}a_{32} - a'_{23}a_{11}a_{32} - a'_{32}a_{11}a_{23}$$
$$- a'_{12}a_{21}a_{33} - a'_{21}a_{12}a_{33} - a'_{33}a_{12}a_{21} - a'_{13}a_{22}a_{31} - a'_{22}a_{13}a_{31} - a'_{31}a_{13}a_{22}$$

$$= a'_{11}\alpha_{11} + a'_{12}\alpha_{12} + a'_{13}\alpha_{13} + a'_{21}\alpha_{21} + a'_{22}\alpha_{22} + a'_{23}\alpha_{23} + a'_{31}\alpha_{31} + a'_{32}\alpha_{32} + a'_{33}\alpha_{33}$$

$$= \begin{vmatrix} a'_{11} & a'_{12} & a'_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{vmatrix}$$

by Problem 10, Chapter 3.

## SUPPLEMENTARY PROBLEMS

**9.** Evaluate:

(a) $\begin{vmatrix} 3 & 5 & 7 & 2 \\ 2 & 4 & 1 & 1 \\ -2 & 0 & 0 & 0 \\ 1 & 1 & 3 & 4 \end{vmatrix} = 156$ (c) $\begin{vmatrix} 1 & -2 & 3 & -4 \\ 2 & -1 & 4 & -3 \\ 2 & 3 & -4 & -5 \\ 3 & -4 & 5 & 6 \end{vmatrix} = -304$

(b) $\begin{vmatrix} 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & 6 \\ 4 & 1 & 2 & 9 \\ 2 & 4 & 2 & 7 \end{vmatrix} = 41$ (d) $\begin{vmatrix} 1 & -2 & 3 & -2 & -2 \\ 2 & -1 & 1 & 3 & 2 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & -4 & -3 & -2 & -5 \\ 3 & -2 & 2 & 2 & -2 \end{vmatrix} = 118$

**10.** If $A$ is $n$-square, show that $|\bar{A}'A|$ is real and non-negative.

**11.** Evaluate the determinant of Problem 9(*a*) using minors from the first two rows; also using minors from the first two columns.

**12.** (*a*) Let $A = \begin{bmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{bmatrix}$.

Use $|AB| = |A|\cdot|B|$ to show that $(a_1^2+a_2^2)(b_1^2+b_2^2) = (a_1b_1 - a_2b_2)^2 + (a_2b_1 + a_1b_2)^2$.

(*b*) Let $A = \begin{bmatrix} a_1+ia_3 & a_2+ia_4 \\ -a_2+ia_4 & a_1-ia_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1+ib_3 & b_2+ib_4 \\ -b_2+ib_4 & b_1-ib_3 \end{bmatrix}$.

Use $|AB| = |A|\cdot|B|$ to express $(a_1^2+a_2^2+a_3^2+a_4^2)(b_1^2+b_2^2+b_3^2+b_4^2)$ as a sum of four squares.

**13.** Evaluate $\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 4 & 3 & 2 & 1 \\ 0 & 5 & 4 & 3 & 2 & 1 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{vmatrix}$ using minors from the first three rows. *Ans.* $-720$

14. Evaluate $\begin{vmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 & 1 \end{vmatrix}$ using minors from the first two columns. *Ans.* 2

15. If $A_1, A_2, \ldots, A_s$ are square matrices, use the Laplace expansion to prove

$$|\operatorname{diag}(A_1, A_2, \ldots, A_s)| = |A_1|\cdot|A_2| \cdots |A_s|$$

16. Expand $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix}$ using minors of the first two rows and show that

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\cdot\begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\cdot\begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix}\cdot\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0$$

17. Use the Laplace expansion to show that the $n$-square determinant $\begin{vmatrix} 0 & A \\ B & C \end{vmatrix}$, where 0 is $k$-square, is zero when $k > \frac{1}{2}n$.

18. In $|A| = a_{11}\alpha_{11} + a_{12}\alpha_{12} + a_{13}\alpha_{13} + a_{14}\alpha_{14}$ expand each of the cofactors $\alpha_{12}, \alpha_{13}, \alpha_{14}$ along its first column to show

$$|A| = a_{11}\alpha_{11} - \sum_{i=2}^{4}\sum_{j=2}^{4} a_{i1}a_{1j}\alpha_{1j}^{i1}$$

where $\alpha_{1j}^{i1}$ is the algebraic complement of the minor $\begin{vmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{vmatrix}$ of $|A|$.

19. If $\alpha_{ij}$ denotes the cofactor of $a_{ij}$ in the $n$-square matrix $A = [a_{ij}]$, show that the bordered determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & p_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & p_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & p_n \\ q_1 & q_2 & \cdots & q_n & 0 \end{vmatrix} = \begin{vmatrix} 0 & q_1 & q_2 & \cdots & q_n \\ p_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_n & a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = -\sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j \alpha_{ij}$$

Hint. Use (4.3).

20. For each of the determinants $|A|$, find the derivative.

(a) $\begin{vmatrix} x^2 & x^3 \\ 2x & 3x+1 \end{vmatrix}$ (b) $\begin{vmatrix} x & 1 & 2 \\ x^2 & 2x+1 & x^3 \\ 0 & 3x-2 & x^2+1 \end{vmatrix}$ (c) $\begin{vmatrix} x^2-1 & x-1 & 1 \\ x^4 & x^3 & 2x+5 \\ x+1 & x^2 & x \end{vmatrix}$

*Ans.* (a) $2x + 9x^2 - 8x^3$, (b) $1 - 6x + 21x^2 + 12x^3 - 15x^4$, (c) $6x^5 - 5x^4 - 28x^3 + 9x^2 + 20x - 2$

21. Prove: If $A$ and $B$ are real $n$-square matrices with $A$ non-singular and if $H = A + iB$ is Hermitian, then

$$|H|^2 = |A|^2 \cdot |I + (A^{-1}B)^2|$$

# Chapter 5

## Equivalence

**THE RANK OF A MATRIX.** A non-zero matrix $A$ is said to have **rank** $r$ if at least one of its $r$-square minors is different from zero while every $(r+1)$-square minor, if any, is zero. A zero matrix is said to have rank 0.

**Example 1.** The rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ is $r=2$ since $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \neq 0$ while $|A| = 0$.

See Problem 1.

An $n$-square matrix $A$ is called **non-singular** if its rank $r=n$, that is, if $|A| \neq 0$. Otherwise, $A$ is called **singular**. The matrix of Example 1 is singular.

From $|AB| = |A|\cdot|B|$ follows

**I.** The product of two or more non-singular $n$-square matrices is non-singular; the product of two or more $n$-square matrices is singular if at least one of the matrices is singular.

**ELEMENTARY TRANSFORMATIONS.** The following operations, called **elementary transformations**, on a matrix do not change either its order or its rank:

(1) The interchange of the $i$th and $j$th rows, denoted by $H_{ij}$;
The interchange of the $i$th and $j$th columns, denoted by $K_{ij}$.

(2) The multiplication of every element of the $i$th row by a non-zero scalar $k$, denoted by $H_i(k)$;
The multiplication of every element of the $i$th column by a non-zero scalar $k$, denoted by $K_i(k)$.

(3) The addition to the elements of the $i$th row of $k$, a scalar, times the corresponding elements of the $j$th row, denoted by $H_{ij}(k)$;

The addition to the elements of the $i$th column of $k$, a scalar, times the corresponding elements of the $j$th column, denoted by $K_{ij}(k)$.

The transformations $H$ are called **elementary row transformations**; the transformations $K$ are called **elementary column transformations**.

The elementary transformations, being precisely those performed on the rows (columns) of a determinant, need no elaboration. It is clear that an elementary transformation cannot alter the order of a matrix. In Problem 2, it is shown that an elementary transformation does not alter its rank.

**THE INVERSE OF AN ELEMENTARY TRANSFORMATION.** The **inverse** of an elementary transformation is an operation which undoes the effect of the elementary transformation; that is, after $A$ has been subjected to one of the elementary transformations and then the resulting matrix has been subjected to the inverse of that elementary transformation, the final result is the matrix $A$.

**Example 2.** Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

The effect of the elementary row transformation $H_{21}(-2)$ is to produce $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{bmatrix}$.

The effect of the elementary row transformation $H_{21}(+2)$ on $B$ is to produce $A$ again. Thus, $H_{21}(-2)$ and $H_{21}(+2)$ are inverse elementary row transformations.

The inverse elementary transformations are:

(1') $\quad H_{ij}^{-1} = H_{ij} \qquad K_{ij}^{-1} = K_{ij}$

(2') $\quad H_i^{-1}(k) = H_i(1/k) \qquad K_i^{-1}(k) = K_i(1/k)$

(3') $\quad H_{ij}^{-1}(k) = H_{ij}(-k) \qquad K_{ij}^{-1}(k) = K_{ij}(-k)$

We have

**II.** The inverse of an elementary transformation is an elementary transformation of the same type.

**EQUIVALENT MATRICES.** Two matrices $A$ and $B$ are called **equivalent**, $A \sim B$, if one can be obtained from the other by a sequence of elementary transformations.

Equivalent matrices have the same order and the same rank.

**Example 3.** Applying in turn the elementary transformations $H_{21}(-2)$, $H_{31}(1)$, $H_{32}(-1)$,

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ -1 & -2 & 6 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Since all 3-square minors of $B$ are zero while $\begin{vmatrix} -1 & 4 \\ 5 & -3 \end{vmatrix} \neq 0$, the rank of $B$ is 2; hence, the rank of $A$ is 2. This procedure of obtaining from $A$ an equivalent matrix $B$ from which the rank is evident by inspection is to be compared with that of computing the various minors of $A$.

See Problem 3.

**ROW EQUIVALENCE.** If a matrix $A$ is reduced to $B$ by the use of elementary row transformations alone, $B$ is said to be **row equivalent** to $A$ and conversely. The matrices $A$ and $B$ of Example 3 are row equivalent.

Any non-zero matrix $A$ of rank $r$ is row equivalent to a **canonical matrix** $C$ in which

(*a*) one or more elements of each of the first $r$ rows are non-zero while all other rows have only zero elements.

(*b*) in the $i$th row, $(i = 1, 2, \ldots, r)$, the first non-zero element is 1; let the column in which this element stands be numbered $j_i$.

(*c*) $j_1 < j_2 < \cdots < j_r$.

(*d*) the only non-zero element in the column numbered $j_i$, $(i = 1, 2, \ldots, r)$, is the element 1 of the $i$th row.

To reduce $A$ to $C$, suppose $j_1$ is the number of the first non-zero column of $A$.

(i$_1$) If $a_{1j_1} \neq 0$, use $H_1(1/a_{1j_1})$ to reduce it to 1, when necessary.
(i$_2$) If $a_{ij_1} = 0$ but $a_{pj_1} \neq 0$, use $H_{1p}$ and proceed as in (i$_1$).

(ii) Use row transformations of type (3) with appropriate multiples of the first row to obtain zeros elsewhere in the $j_1$st column.

If non-zero elements of the resulting matrix $B$ occur only in the first row, $B = C$. Otherwise, suppose $j_2$ is the number of the first column in which this does not occur. If $b_{2j_2} \neq 0$, use $H_2(1/b_{2j_2})$ as in (i$_1$); if $b_{2j_2} = 0$ but $b_{qj_2} \neq 0$, use $H_{2q}$ and proceed as in (i$_1$). Then, as in (ii), clear the $j_2$nd column of all other non-zero elements.

If non-zero elements of the resulting matrix occur only in the first two rows, we have $C$. Otherwise, the procedure is repeated until $C$ is reached.

**Example 4.** The sequence of row transformations $H_{21}(-2), H_{31}(1)$; $H_2(1/5)$; $H_{12}(1), H_{32}(-5)$ applied to $A$ of Example 3 yields

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 17/5 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= C$$

having the properties $(a)$-$(d)$.

See Problem 4.

**THE NORMAL FORM OF A MATRIX.** By means of elementary transformations any matrix $A$ of rank $r > 0$ can be reduced to one of the forms

$$(5.1) \qquad I_r, \quad \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad [I_r \;\; 0], \quad \begin{bmatrix} I_r \\ 0 \end{bmatrix}$$

called its **normal form**. A zero matrix is its own normal form.

Since both row and column transformations may be used here, the element 1 of the first row obtained in the section above can be moved into the first column. Then both the first row **and** first column can be cleared of other non-zero elements. Similarly, the element 1 of the second row can be brought into the second column, and so on.

For example, the sequence $H_{21}(-2)$, $H_{31}(1)$, $K_{21}(-2)$, $K_{31}(1)$, $K_{41}(-4)$, $K_{23}$, $K_2(1/5)$, $H_{32}(-1)$, $K_{42}(3)$ applied to $A$ of Example 3 yields $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, the normal form.

See Problem 5.

**ELEMENTARY MATRICES.** The matrix which results when an elementary row (column) transformation is applied to the identity matrix $I_n$ is called an **elementary row (column) matrix**. Here, an elementary matrix will be denoted by the symbol introduced to denote the elementary transformation which produces the matrix.

**Example 5.** Examples of elementary matrices obtained from $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are:

$$H_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = K_{12}, \qquad H_3(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix} = K_3(k), \qquad H_{23}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} = K_{32}(k)$$

Every elementary matrix is non-singular. (Why?)

The effect of applying an elementary transformation to an $m \times n$ matrix $A$ can be produced by multiplying $A$ by an elementary matrix.

To effect a given elementary row transformation on $A$ of order $m \times n$, apply the transformation to $I_m$ to form the corresponding elementary matrix $H$ and multiply $A$ **on the left** by $H$.

To effect a given elementary column transformation on $A$, apply the transformation to $I_n$ to form the corresponding elementary matrix $K$ and multiply $A$ **on the right** by $K$.

**Example 6.** When $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $H_{13} \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$ interchanges the first and third rows of $A$; $AK_{13}(2) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 3 \\ 16 & 5 & 6 \\ 25 & 8 & 9 \end{bmatrix}$ adds to the first column of $A$ two times the third column.

**LET $A$ AND $B$ BE EQUIVALENT MATRICES.** Let the elementary row and column matrices corresponding to the elementary row and column transformations which reduce $A$ to $B$ be designated as $H_1, H_2, \ldots, H_s$; $K_1, K_2, \ldots, K_t$ where $H_1$ is the first row transformation, $H_2$ is the second, ...; $K_1$ is the first column transformation, $K_2$ is the second, .... Then

$$H_s \ldots H_2 \cdot H_1 \cdot A \cdot K_1 \cdot K_2 \ldots K_t \;=\; PAQ \;=\; B \tag{5.2}$$

where

$$P \;=\; H_s \ldots H_2 \cdot H_1 \quad \text{and} \quad Q \;=\; K_1 \cdot K_2 \ldots K_t \tag{5.3}$$

We have

**III.** Two matrices $A$ and $B$ are equivalent if and only if there exist non-singular matrices $P$ and $Q$ defined in (5.3) such that $PAQ = B$.

**Example 7.** When $A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix}$, $H_{31}(-1) \cdot H_{21}(-2) \cdot A \cdot K_{21}(-2) \cdot K_{31}(1) \cdot K_{41}(-2) \cdot K_{42}(1) \cdot K_3(\tfrac{1}{2})$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & -2 & \frac{1}{2} & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = PAQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = B$$

Since any matrix is equivalent to its normal form, we have

**IV.** If $A$ is an $n$-square non-singular matrix, there exist non-singular matrices $P$ and $Q$ as defined in (5.3) such that $PAQ = I_n$.

See Problem 6.

**INVERSE OF A PRODUCT OF ELEMENTARY MATRICES.** Let

$$P = H_s \dots H_2 \cdot H_1 \quad \text{and} \quad Q = K_1 \cdot K_2 \dots K_t$$

as in (5.3). Since each $H$ and $K$ has an inverse and since the inverse of a product is the product in reverse order of the inverses of the factors

(5.4) $$P^{-1} = H_1^{-1} \cdot H_2^{-1} \dots H_s^{-1} \quad \text{and} \quad Q^{-1} = K_t^{-1} \dots K_2^{-1} \cdot K_1^{-1}$$

Let $A$ be an $n$-square non-singular matrix and let $P$ and $Q$ defined above be such that $PAQ = I_n$. Then

(5.5) $$A = P^{-1}(PAQ)Q^{-1} = P^{-1} \cdot I_n \cdot Q^{-1} = P^{-1} \cdot Q^{-1}$$

We have proved

**V.** Every non-singular matrix can be expressed as a product of elementary matrices.

See Problem 7.

From this follow

**VI.** If $A$ is non-singular, the rank of $AB$ (also of $BA$) is that of $B$.

**VII.** If $P$ and $Q$ are non-singular, the rank of $PAQ$ is that of $A$.

**CANONICAL SETS UNDER EQUIVALENCE.** In Problem 8, we prove

**VIII.** Two $m \times n$ matrices $A$ and $B$ are equivalent if and only if they have the same rank.

A set of $m \times n$ matrices is called a **canonical set** under equivalence if every $m \times n$ matrix is equivalent to one and only one matrix of the set. Such a canonical set is given by (5.1) as $r$ ranges over the values $1, 2, \dots, m$ or $1, 2, \dots, n$ whichever is the smaller.

See Problem 9.

**RANK OF A PRODUCT.** Let $A$ be an $m \times p$ matrix of rank $r$. By Theorem **III** there exist non-singular matrices $P$ and $Q$ such that

$$PAQ = N = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Then $A = P^{-1}NQ^{-1}$. Let $B$ be a $p \times n$ matrix and consider the rank of

(5.6) $$AB = P^{-1}NQ^{-1}B$$

By Theorem **VI**, the rank of $AB$ is that of $NQ^{-1}B$. Now the rows of $NQ^{-1}B$ consist of the first $r$ rows of $Q^{-1}B$ and $m-r$ rows of zeros. Hence, the rank of $AB$ cannot exceed $r$, the rank of $A$. Similarly, the rank of $AB$ cannot exceed that of $B$. We have proved

**IX.** The rank of the product of two matrices cannot exceed the rank of either factor.

Suppose $AB = 0$; then from (5.6), $NQ^{-1}B = 0$. This requires that the first $r$ rows of $Q^{-1}B$ be zeros while the remaining rows may be arbitrary. Thus, the rank of $Q^{-1}B$ and, hence, the rank of $B$ cannot exceed $p-r$. We have proved

**X.** If the $m \times p$ matrix $A$ is of rank $r$ and if the $p \times n$ matrix $B$ is such that $AB = 0$, the rank of $B$ cannot exceed $p-r$.

## SOLVED PROBLEMS

1. (a) The rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 0 & 5 \end{bmatrix}$ is 2 since $\begin{vmatrix} 1 & 2 \\ -4 & 0 \end{vmatrix} \neq 0$ and there are no minors of order three.

(b) The rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 4 & 8 \end{bmatrix}$ is 2 since $|A| = 0$ and $\begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} \neq 0$.

(c) The rank of $A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$ is 1 since $|A| = 0$, each of the nine 2-square minors is 0, but not every element is 0.

2. Show that the elementary transformations do not alter the rank of a matrix.

We shall consider only row transformations here and leave consideration of the column transformations as an exercise. Let the rank of the $m \times n$ matrix $A$ be $r$ so that every $(r+1)$-square minor of $A$, if any, is zero. Let $B$ be the matrix obtained from $A$ by a row transformation. Denote by $|R|$ any $(r+1)$-square minor of $A$ and by $|S|$ the $(r+1)$-square minor of $B$ having the same position as $|R|$.

Let the row transformation be $H_{ij}$. Its effect on $|R|$ is either (i) to leave it unchanged, (ii) to interchange two of its rows, or (iii) to interchange one of its rows with a row not of $|R|$. In the case (i), $|S| = |R| = 0$; in the case (ii), $|S| = -|R| = 0$; in the case (iii), $|S|$ is, except possibly for sign, another $(r+1)$-square minor of $|A|$ and, hence, is 0.

Let the row transformation be $H_i(k)$. Its effect on $|R|$ is either (i) to leave it unchanged or (ii) to multiply one of its rows by $k$. Then, respectively, $|S| = |R| = 0$ or $|S| = k|R| = 0$.

Let the row transformation be $H_{ij}(k)$. Its effect on $|R|$ is either (i) to leave it unchanged, (ii) to increase one of its rows by $k$ times another of its rows, or (iii) to increase one of its rows by $k$ times a row not of $|R|$. In the cases (i) and (ii), $|S| = |R| = 0$; in the case (iii), $|S| = |R| \pm k$ (another $(r+1)$-square minor of $A$) $= 0 \pm k \cdot 0 = 0$.

Thus, an elementary row transformation cannot raise the rank of a matrix. On the other hand, it cannot lower the rank for, if it did, the inverse transformation would have to raise it. Hence, an elementary row transformation does not alter the rank of a matrix.

3. For each of the matrices $A$ obtain an equivalent matrix $B$ and from it, by inspection, determine the rank of $A$.

(a) $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & -4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = B$$

The transformations used were $H_{21}(-2)$, $H_{31}(-3)$; $H_2(-1/3)$, $H_3(-1/4)$; $H_{32}(-1)$. The rank is 3.

(b) $$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$ The rank is 3.

(c) $$A = \begin{bmatrix} 1 & 1+i & -i \\ 0 & i & 1+2i \\ 1 & 1+2i & 1+i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 1+2i \\ 1 & i & 1+2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 1+2i \\ 0 & 0 & 0 \end{bmatrix} = B.$$ The rank is 2.

Note. The equivalent matrices $B$ obtained here are not unique. In particular, since in (a) and (b) only row transformations were used, the reader may obtain others by using only column transformations. When the elements are rational numbers, there generally is no gain in mixing row and column transformations.

4. Obtain the canonical matrix $C$ row equivalent to each of the given matrices $A$.

$$(a)\quad A = \begin{bmatrix} 0&0&1&3&-2\\ 0&1&2&6&0\\ 0&2&3&9&2\\ 0&1&1&3&2 \end{bmatrix} \sim \begin{bmatrix} 0&1&1&3&2\\ 0&1&2&6&0\\ 0&2&3&9&2\\ 0&0&1&3&-2 \end{bmatrix} \sim \begin{bmatrix} 0&1&1&3&2\\ 0&0&1&3&-2\\ 0&0&1&3&-2\\ 0&0&1&3&-2 \end{bmatrix} \sim \begin{bmatrix} 0&1&0&0&4\\ 0&0&1&3&-2\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix} = C$$

$$(b)\quad A = \begin{bmatrix} 1&2&-2&3&1\\ 1&3&-2&3&0\\ 2&4&-3&6&4\\ 1&1&-1&4&6 \end{bmatrix} \sim \begin{bmatrix} 1&2&-2&3&1\\ 0&1&0&0&-1\\ 0&0&1&0&2\\ 0&-1&1&1&5 \end{bmatrix} \sim \begin{bmatrix} 1&0&-2&3&3\\ 0&1&0&0&-1\\ 0&0&1&0&2\\ 0&0&1&1&4 \end{bmatrix} \sim \begin{bmatrix} 1&0&0&3&7\\ 0&1&0&0&-1\\ 0&0&1&0&2\\ 0&0&0&1&2 \end{bmatrix} \sim \begin{bmatrix} 1&0&0&0&1\\ 0&1&0&0&-1\\ 0&0&1&0&2\\ 0&0&0&1&2 \end{bmatrix} = C$$

5. Reduce each of the following to normal form.

$$(a)\quad A = \begin{bmatrix} 1&2&0&-1\\ 3&4&1&2\\ -2&3&2&5 \end{bmatrix} \sim \begin{bmatrix} 1&2&0&-1\\ 0&-2&1&5\\ 0&7&2&3 \end{bmatrix} \sim \begin{bmatrix} 1&0&0&0\\ 0&-2&1&5\\ 0&7&2&3 \end{bmatrix} \sim \begin{bmatrix} 1&0&0&0\\ 0&1&-2&5\\ 0&2&7&3 \end{bmatrix} \sim \begin{bmatrix} 1&0&0&0\\ 0&1&-2&5\\ 0&0&11&-7 \end{bmatrix} \sim \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&11&-7 \end{bmatrix} \sim \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{bmatrix}$$

$$= [I_3\ 0]$$

The elementary transformations are:

$H_{21}(-3),\ H_{31}(2);\ K_{21}(-2),\ K_{41}(1);\ K_{23};\ H_{32}(-2);\ K_{32}(2),\ K_{42}(-5);\ K_3(1/11),\ K_{43}(7)$

$$(b)\quad A = \begin{bmatrix} 0&2&3&4\\ 2&3&5&4\\ 4&8&13&12 \end{bmatrix} \sim \begin{bmatrix} 2&3&5&4\\ 0&2&3&4\\ 4&8&13&12 \end{bmatrix} \sim \begin{bmatrix} 1&3&5&4\\ 0&2&3&4\\ 2&8&13&12 \end{bmatrix} \sim \begin{bmatrix} 1&3&5&4\\ 0&2&3&4\\ 0&2&3&4 \end{bmatrix} \sim \begin{bmatrix} 1&0&0&0\\ 0&2&3&4\\ 0&2&3&4 \end{bmatrix} \sim \begin{bmatrix} 1&0&0&0\\ 0&1&3&4\\ 0&1&3&4 \end{bmatrix} \sim \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&1&0&0 \end{bmatrix} \sim \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0 \end{bmatrix}$$

$$= \begin{bmatrix} I_2 & 0\\ 0 & 0 \end{bmatrix}$$

The elementary transformations are:

$H_{12};\ K_1(\tfrac{1}{2});\ H_{31}(-2);\ K_{21}(-3),\ K_{31}(-5),\ K_{41}(-4);\ K_2(\tfrac{1}{2});\ K_{32}(-3),\ K_{42}(-4);\ H_{32}(-1)$

6. Reduce $A = \begin{bmatrix} 1&2&3&-2\\ 2&-2&1&3\\ 3&0&4&1 \end{bmatrix}$ to normal form $N$ and compute the matrices $P_1$ and $Q_1$ such that $P_1 A Q_1 = N$.

Since $A$ is $3\times 4$, we shall work with the array $\begin{array}{cc} I_4 & \\ A & I_3 \end{array}$. Each row transformation is performed on a row of seven elements and each column transformation is performed on a column of seven elements.

$$\begin{array}{rrrrrrr} 1&0&0&0&&&\\ 0&1&0&0&&&\\ 0&0&1&0&&&\\ 0&0&0&1&&&\\ 1&2&3&-2&1&0&0\\ 2&-2&1&3&0&1&0\\ 3&0&4&1&0&0&1 \end{array} \rightarrow \begin{array}{rrrrrrr} 1&0&0&0&&&\\ 0&1&0&0&&&\\ 0&0&1&0&&&\\ 0&0&0&1&&&\\ 1&2&3&-2&1&0&0\\ 0&-6&-5&7&-2&1&0\\ 0&-6&-5&7&-3&0&1 \end{array} \rightarrow \begin{array}{rrrrrrr} 1&-2&-3&2&&&\\ 0&1&0&0&&&\\ 0&0&1&0&&&\\ 0&0&0&1&&&\\ 1&0&0&0&1&0&0\\ 0&-6&-5&7&-2&1&0\\ 0&-6&-5&7&-3&0&1 \end{array} \rightarrow \begin{array}{rrrrrrr} 1&-2&-3&2&&&\\ 0&1&0&0&&&\\ 0&0&1&0&&&\\ 0&0&0&1&&&\\ 1&0&0&0&1&0&0\\ 0&-6&-5&7&-2&1&0\\ 0&0&0&0&-1&-1&1 \end{array}$$

$$\rightarrow \begin{array}{rrrrrrr} 1&1/3&-3&2&&&\\ 0&-1/6&0&0&&&\\ 0&0&1&0&&&\\ 0&0&0&1&&&\\ 1&0&0&0&1&0&0\\ 0&1&-5&7&-2&1&0\\ 0&0&0&0&-1&-1&1 \end{array} \rightarrow \begin{array}{rrrrrrr} 1&1/3&-4/3&-1/3&&&\\ 0&-1/6&-5/6&7/6&&&\\ 0&0&1&0&&&\\ 0&0&0&1&&&\\ 1&0&0&0&1&0&0\\ 0&1&0&0&-2&1&0\\ 0&0&0&0&-1&-1&1 \end{array} \quad \text{or} \quad \begin{array}{cc} Q_1 & \\ N & P_1 \end{array}$$

Thus, $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$, $Q_1 = \begin{bmatrix} 1 & 1/3 & -4/3 & -1/3 \\ 0 & -1/6 & -5/6 & 7/6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $P_1AQ_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = N$.

7. Express $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ as a product of elementary matrices.

The elementary transformations $H_{21}(-1)$, $H_{31}(-1)$; $K_{21}(-3)$, $K_{31}(-3)$ reduce $A$ to $I_3$, that is, [see (5.2)]

$$I = H_2 \cdot H_1 \cdot A \cdot K_1 \cdot K_2 = H_{31}(-1) \cdot H_{21}(-1) \cdot A \cdot K_{21}(-3) \cdot K_{31}(-3)$$

From (5.5),
$$A = H_1^{-1} \cdot H_2^{-1} \cdot K_2^{-1} \cdot K_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. Prove: Two $m \times n$ matrices $A$ and $B$ are equivalent if and only if they have the same rank.

If $A$ and $B$ have the same rank, both are equivalent to the same matrix (5.1) and are equivalent to each other. Conversely, if $A$ and $B$ are equivalent, there exist non-singular matrices $P$ and $Q$ such that $B = PAQ$. By Theorem **VII**, $A$ and $B$ have the same rank.

9. A canonical set for non-zero matrices of order 3 is

$$I_3, \quad \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A canonical set for non-zero $3 \times 4$ matrices is

$$[I_3 \; 0] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

10. If from a square matrix $A$ of order $n$ and rank $r_A$, a submatrix $B$ consisting of $s$ rows (columns) of $A$ is selected, the rank $r_B$ of $B$ is equal to or greater than $r_A + s - n$.

The normal form of $A$ has $n - r_A$ rows whose elements are zeros and the normal form of $B$ has $s - r_B$ rows whose elements are zeros. Clearly

$$n - r_A \geq s - r_B$$

from which follows $r_B \geq r_A + s - n$ as required.

## SUPPLEMENTARY PROBLEMS

11. Find the rank of (a) $\begin{bmatrix}1&2&3&2\\2&3&5&1\\1&3&4&5\end{bmatrix}$, (b) $\begin{bmatrix}1&2&1&2\\1&3&2&2\\2&4&3&4\\3&7&4&6\end{bmatrix}$, (c) $\begin{bmatrix}1&2&-2&3\\2&5&-4&6\\-1&-3&2&-2\\2&4&-1&6\end{bmatrix}$, (d) $\begin{bmatrix}3&4&5&6&7\\4&5&6&7&8\\5&6&7&8&9\\10&11&12&13&14\\15&16&17&18&19\end{bmatrix}$.

*Ans.* (a) 2, (b) 3, (c) 4, (d) 2

12. Show by considering minors that $A, A', \bar{A}$, and $\bar{A}'$ have the same rank.

13. Show that the canonical matrix $C$, row equivalent to a given matrix $A$, is uniquely determined by $A$.

14. Find the canonical matrix row equivalent to each of the following:

(a) $\begin{bmatrix}1&2&-3\\2&5&-4\end{bmatrix} \sim \begin{bmatrix}1&0&-7\\0&1&2\end{bmatrix}$ (b) $\begin{bmatrix}1&2&3&4\\3&4&1&2\\4&3&1&2\end{bmatrix} \sim \begin{bmatrix}1&0&0&1/9\\0&1&0&1/9\\0&0&1&11/9\end{bmatrix}$ (c) $\begin{bmatrix}1&1&1&2\\2&1&-3&-6\\3&-3&1&2\end{bmatrix} \sim \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&2\end{bmatrix}$

(d) $\begin{bmatrix}1&2&1&0\\3&2&1&2\\2&-1&2&5\\5&6&3&2\\1&3&-1&-3\end{bmatrix} \sim \begin{bmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$ (e) $\begin{bmatrix}1&-1&1&1&1\\1&-1&2&3&1\\2&-2&1&0&2\\1&1&-1&-3&3\end{bmatrix} \sim \begin{bmatrix}1&0&0&-1&2\\0&1&0&0&1\\0&0&1&2&0\\0&0&0&0&0\end{bmatrix}$

15. Write the normal form of each of the matrices of Problem 14.

*Ans.* (a) $[I_2\ 0]$, (b),(c) $[I_3\ 0]$ (d) $\begin{bmatrix}I_3&0\\0&0\end{bmatrix}$ (e) $\begin{bmatrix}I_3&0\\0&0\end{bmatrix}$

16. Let $A = \begin{bmatrix}1&2&3&4\\2&3&4&1\\3&4&1&2\end{bmatrix}$.

(a) From $I_3$ form $H_{12}, H_2(3), H_{13}(-4)$ and check that each $HA$ effects the corresponding row transformation.
(b) From $I_4$ form $K_{24}, K_3(-1), K_{42}(3)$ and show that each $AK$ effects the corresponding column transformation.
(c) Write the inverses $H_{12}^{-1}, H_2^{-1}(3), H_{13}^{-1}(-4)$ of the elementary matrices of (a). Check that for each $H$, $H\cdot H^{-1} = I$.
(d) Write the inverses $K_{24}^{-1}, K_3^{-1}(-1), K_{42}^{-1}(3)$ of the elementary matrices of (b). Check that for each $K$, $KK^{-1} = I$.
(e) Compute $B = H_{12}\cdot H_2(3)\cdot H_{13}(-4) = \begin{bmatrix}0&3&0\\1&0&-4\\0&0&1\end{bmatrix}$ and $C = H_{13}^{-1}(-4)\cdot H_2^{-1}(3)\cdot H_{12}^{-1} = \begin{bmatrix}0&1&4\\1/3&0&0\\0&0&1\end{bmatrix}$.
(f) Show that $BC = CB = I$.

17. (a) Show that $K'_{ij} = H_{ij}$, $K'_i(k) = H_i(k)$, and $K'_{ij}(k) = H_{ij}(k)$.
(b) Show that if $R$ is a product of elementary column matrices, $R'$ is the product in reverse order of the same elementary row matrices.

18. Prove: (a) $AB$ and $BA$ are non-singular if $A$ and $B$ are non-singular $n$-square matrices.
(b) $AB$ and $BA$ are singular if at least one of the $n$-square matrices $A$ and $B$ is singular.

19. If $P$ and $Q$ are non-singular, show that $A, PA, AQ$, and $PAQ$ have the same rank.
*Hint.* Express $P$ and $Q$ as products of elementary matrices.

20. Reduce $B = \begin{bmatrix}1&3&6&-1\\1&4&5&1\\1&5&4&3\end{bmatrix}$ to normal form $N$ and compute the matrices $P_2$ and $Q_2$ such that $P_2BQ_2 = N$.

21. (*a*) Show that the number of matrices in a canonical set of $n$-square matrices under equivalence is $n+1$.
(*b*) Show that the number of matrices in a canonical set of $m \times n$ matrices under equivalence is the smaller of $m+1$ and $n+1$.

22. Given $A = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 1 & 3 & 2 & 6 \\ 2 & 5 & 6 & 10 \end{bmatrix}$ of rank 2. Find a 4-square matrix $B \neq 0$ such that $AB = 0$.

*Hint.* Follow the proof of Theorem **X** and take

$$Q^{-1}B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & b & c & d \\ e & f & g & h \end{bmatrix}$$

where $a, b, \ldots, h$ are arbitrary.

23. The matrix $A$ of Problem 6 and the matrix $B$ of Problem 20 are equivalent. Find $P$ and $Q$ such that $B = PAQ$.

24. If the $m \times n$ matrices $A$ and $B$ are of rank $r_A$ and $r_B$ respectively, show that the rank of $A+B$ cannot exceed $r_A + r_B$.

25. Let $A$ be an arbitrary $n$-square matrix and $B$ be an $n$-square elementary matrix. By considering each of the six different types of matrix $B$, show that $|AB| = |A| \cdot |B|$.

26. Let $A$ and $B$ be $n$-square matrices. (*a*) If at least one is singular show that $|AB| = |A| \cdot |B|$; (*b*) If both are non-singular, use **(5.5)** and Problem 25 to show that $|AB| = |A| \cdot |B|$.

27. Show that equivalence of matrices is an equivalence relation.

28. Prove: The row equivalent canonical form of a non-singular matrix $A$ is $I$ and conversely.

29. Prove: Not every matrix $A$ can be reduced to normal form by row transformations alone.
*Hint.* Exhibit a matrix which cannot be so reduced.

30. Show how to effect on any matrix $A$ the transformation $H_{ij}$ by using a succession of row transformations of types **(2)** and **(3)**.

31. Prove: If $A$ is an $m \times n$ matrix, $(m \leq n)$, of rank $m$ then $AA'$ is a non-singular symmetric matrix. State the theorem when the rank of $A$ is $< m$.

# Chapter 6

## The Adjoint of a Square Matrix

**THE ADJOINT**. Let $A = [a_{ij}]$ be an $n$-square matrix and $\alpha_{ij}$ be the cofactor of $a_{ij}$; then by definition,

$$\text{(6.1)} \qquad \text{adjoint } A \;=\; \text{adj } A \;=\; \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{1n} & \alpha_{2n} & & \alpha_{nn} \end{bmatrix}$$

Note carefully that the cofactors of the elements of the $i$th row (column) of $A$ are the elements of the $i$th column (row) of adj $A$.

**Example 1**. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$.

$$\alpha_{11} = 6,\; \alpha_{12} = -2,\; \alpha_{13} = -3,\; \alpha_{21} = 1,\; \alpha_{22} = -5,\; \alpha_{23} = 3,\; \alpha_{31} = -5,\; \alpha_{32} = 4,\; \alpha_{33} = -1$$

and

$$\text{adj } A = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

See Problems 1-2.

Using Theorems **X** and **XI** of Chapter 3, we find

$$\text{(6.2)} \qquad A(\text{adj } A) \;=\; \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{nn} \end{bmatrix}$$

$$= \text{diag}(|A|, |A|, \ldots, |A|) \;=\; |A| \cdot I_n \;=\; (\text{adj } A)\, A$$

**Example 2**. For the matrix $A$ of Example 1, $|A| = -7$ and

$$A(\text{adj } A) \;=\; \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix} \;=\; \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix} \;=\; -7I$$

By taking determinants in **(6.2)**, we have

$$\text{(6.3)} \qquad |A| \cdot |\text{adj } A| \;=\; |A|^n \;=\; |\text{adj } A| \cdot |A|$$

There follow

**I**. If $A$ is $n$-square and non-singular, then

$$\text{(6.4)} \qquad |\text{adj } A| \;=\; |A|^{n-1}$$

**II.** If $A$ is $n$-square and singular, then

$$A(\operatorname{adj} A) \quad = \quad (\operatorname{adj} A)A \quad = \quad 0$$

If $A$ is of rank $< n-1$, then $\operatorname{adj} A = 0$. If $A$ is of rank $n-1$, then $\operatorname{adj} A$ is of rank 1.

See Problem 3.

**THE ADJOINT OF A PRODUCT.** In Problem 4, we prove

**III.** If $A$ and $B$ are $n$-square matrices,

**(6.5)** $$\operatorname{adj} AB \quad = \quad \operatorname{adj} B \cdot \operatorname{adj} A$$

**MINOR OF AN ADJOINT.** In Problem 6, we prove

**IV.** Let $\left|A^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_m}\right|$ be an $m$-square minor of the $n$-square matrix $A = [a_{ij}]$,

let $\left|A^{j_{m+1}, j_{m+2}, \ldots, j_n}_{i_{m+1}, i_{m+2}, \ldots, i_n}\right|$ be its complement in $A$, and

let $\left|M^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_m}\right|$ denote the $m$-square minor of $\operatorname{adj} A$ whose elements occupy **the same position** in $\operatorname{adj} A$ as those of $\left|A^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_m}\right|$ occupy in $A$.

Then

**(6.6)** $$|A| \cdot \left|M^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_m}\right| \quad = \quad (-1)^s |A|^m \cdot \left|A^{j_{m+1}, j_{m+2}, \ldots, j_n}_{i_{m+1}, i_{m+2}, \ldots, i_n}\right|$$

where $s = i_1 + i_2 + \cdots + i_m + j_1 + j_2 + \cdots + j_m$.

If in **(6.6)**, $A$ is non-singular, then

**(6.7)** $$\left|M^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_m}\right| \quad = \quad (-1)^s |A|^{m-1} \cdot \left|A^{j_{m+1}, j_{m+2}, \ldots, j_n}_{i_{m+1}, i_{m+2}, \ldots, i_n}\right|$$

When $m = 2$, **(6.7)** becomes

**(6.8)** $$\begin{vmatrix} \alpha_{i_1,j_1} & \alpha_{i_2,j_1} \\ \alpha_{i_1,j_2} & \alpha_{i_2,j_2} \end{vmatrix} \quad = \quad (-1)^{i_1+i_2+j_1+j_2} |A| \cdot \left|A^{j_3, j_4, \ldots, j_n}_{i_3, i_4, \ldots, i_n}\right|$$

$$= \quad |A| \cdot \text{algebraic complement of } \left|A^{j_1, j_2}_{i_1, i_2}\right|$$

When $m = n-1$, **(6.7)** becomes

**(6.9)** $$\left|M^{j_1, j_2, \ldots, j_{n-1}}_{i_1, i_2, \ldots, i_{n-1}}\right| \quad = \quad (-1)^{i_n + j_n} |A|^{n-2} a_{i_n, j_n}$$

When $m = n$, **(6.7)** becomes **(6.4)**.

## SOLVED PROBLEMS

1. The adjoint of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

2. The adjoint of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$ is $\begin{bmatrix} \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$.

3. Prove: If $A$ is of order $n$ and rank $n-1$, then $\operatorname{adj} A$ is of rank 1.

First we note that, since $A$ is of rank $n-1$, there is at least one non-zero cofactor and the rank of $\operatorname{adj} A$ is at least one. By Theorem **X**, Chapter 5, the rank of $\operatorname{adj} A$ is at most $n-(n-1) = 1$. Hence, the rank is exactly one.

4. Prove: $\operatorname{adj} AB = \operatorname{adj} B \cdot \operatorname{adj} A$.

By **(6.2)**
$$AB \operatorname{adj} AB = |AB| \cdot I = (\operatorname{adj} AB) AB$$

Since
$$AB \cdot \operatorname{adj} B \cdot \operatorname{adj} A = A(B \cdot \operatorname{adj} B)\operatorname{adj} A = A(|B| \cdot I)\operatorname{adj} A = |B|(A \operatorname{adj} A) = |B| \cdot |A| \cdot I = |AB| \cdot I$$

and
$$(\operatorname{adj} B \cdot \operatorname{adj} A) AB = \operatorname{adj} B \{(\operatorname{adj} A) A\} B = \operatorname{adj} B \cdot |A| \cdot I \cdot B = |A| \{(\operatorname{adj} B) B\} = |AB| \cdot I$$

we conclude that
$$\operatorname{adj} AB = \operatorname{adj} B \cdot \operatorname{adj} A$$

5. Show that $\operatorname{adj}(\operatorname{adj} A) = |A|^{n-2} \cdot A$, if $|A| \neq 0$.

By **(6.2)** and **(6.4)**,
$$\begin{aligned} \operatorname{adj} A \cdot \operatorname{adj}(\operatorname{adj} A) &= \operatorname{diag}(|\operatorname{adj} A|, |\operatorname{adj} A|, \ldots, |\operatorname{adj} A|) \\ &= \operatorname{diag}(|A|^{n-1}, |A|^{n-1}, \ldots, |A|^{n-1}) \end{aligned}$$

Then
$$A \cdot \operatorname{adj} A \cdot \operatorname{adj}(\operatorname{adj} A) = |A|^{n-1} \cdot A$$
$$|A| \cdot \operatorname{adj}(\operatorname{adj} A) = |A|^{n-1} \cdot A$$

and
$$\operatorname{adj}(\operatorname{adj} A) = |A|^{n-2} \cdot A$$

6. Prove: Let $\left|A^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_m}\right|$ be an $m$-square minor of the $n$-square matrix $A = [a_{ij}]$,

let $\left|A^{j_{m+1}, j_{m+2}, \ldots, j_n}_{i_{m+1}, i_{m+2}, \ldots, i_n}\right|$ be its complement in $A$, and

let $\left|M^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_m}\right|$ denote the $m$-square minor of $\operatorname{adj} A$ whose elements occupy the same

position in $\operatorname{adj} A$ as those of $\left|A^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_m}\right|$ occupy in $A$. Then

$$|A| \cdot \left| M_{i_1, i_2, \ldots, i_m}^{j_1, j_2, \ldots, j_m} \right| \quad = \quad (-1)^s |A|^m \cdot \left| A_{i_{m+1}, i_{m+2}, \ldots, i_n}^{j_{m+1}, j_{m+2}, \ldots, j_n} \right|$$

where $s = i_1 + i_2 + \cdots + i_m + j_1 + j_2 + \cdots + j_m$.

From

$$\begin{bmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_m} & a_{i_1,j_{m+1}} & \cdots & a_{i_1,j_n} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_m} & a_{i_2,j_{m+1}} & \cdots & a_{i_2,j_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i_m,j_1} & a_{i_m,j_2} & \cdots & a_{i_m,j_m} & a_{i_m,j_{m+1}} & \cdots & a_{i_m,j_n} \\ \hline a_{i_{m+1},j_1} & a_{i_{m+1},j_2} & \cdots & a_{i_{m+1},j_m} & a_{i_{m+1},j_{m+1}} & \cdots & a_{i_{m+1},j_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i_n,j_1} & a_{i_n,j_2} & \cdots & a_{i_n,j_m} & a_{i_n,j_{m+1}} & \cdots & a_{i_n,j_n} \end{bmatrix} \begin{bmatrix} \alpha_{i_1,j_1} & \alpha_{i_2,j_1} & \cdots & \alpha_{i_m,j_1} & 0 & 0 & \cdots & 0 \\ \alpha_{i_1,j_2} & \alpha_{i_2,j_2} & \cdots & \alpha_{i_m,j_2} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{i_1,j_m} & \alpha_{i_2,j_m} & \cdots & \alpha_{i_m,j_m} & 0 & 0 & \cdots & 0 \\ \hline \alpha_{i_1,j_{m+1}} & \alpha_{i_2,j_{m+1}} & \cdots & \alpha_{i_m,j_{m+1}} & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{i_1,j_n} & \alpha_{i_2,j_n} & \cdots & \alpha_{i_m,j_n} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & \cdots & 0 & a_{i_1,j_{m+1}} & \cdots & a_{i_1,j_n} \\ 0 & |A| & \cdots & 0 & a_{i_2,j_{m+1}} & \cdots & a_{i_2,j_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & |A| & a_{i_m,j_{m+1}} & \cdots & a_{i_m,j_n} \\ \hline 0 & 0 & \cdots & 0 & a_{i_{m+1},j_{m+1}} & \cdots & a_{i_{m+1},j_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & a_{i_n,j_{m+1}} & \cdots & a_{i_n,j_n} \end{bmatrix}$$

by taking determinants of both sides, we have

$$(-1)^s |A| \cdot \left| M_{i_1, i_2, \ldots, i_m}^{j_1, j_2, \ldots, j_m} \right| \quad = \quad |A|^m \cdot \left| A_{i_{m+1}, i_{m+2}, \ldots, i_n}^{j_{m+1}, j_{m+2}, \ldots, j_n} \right|$$

where $s$ is as defined in the theorem. From this, the required form follows immediately.

7. Prove: If $A$ is a skew-symmetric of order $2n$, then $|A|$ is the square of a polynomial in the elements of $A$.

By its definition, $|A|$ is a polynomial in its elements; we are to show that under the conditions given above this polynomial is a perfect square.

The theorem is true for $n=1$ since, when $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, $|A| = a^2$.

Assume now that the theorem is true when $n = k$ and consider the skew-symmetric matrix $A = [a_{ij}]$ of order $2k+2$. By partitioning, write $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ where $E = \begin{bmatrix} 0 & a_{2k+1,\,2k+2} \\ a_{2k+2,\,2k+1} & 0 \end{bmatrix}$. Then $B$ is skew-symmetric of order $2k$ and, by assumption, $|B| = f^2$ where $f$ is a polynomial in the elements of $B$.

If $\alpha_{ij}$ denotes the cofactor of $a_{ij}$ in $A$, we have by Problem 6, Chapter 3, and (6.8)

$$\begin{vmatrix} \alpha_{2k+1,\,2k+1} & \alpha_{2k+2,\,2k+1} \\ \alpha_{2k+1,\,2k+2} & \alpha_{2k+2,\,2k+2} \end{vmatrix} = \begin{vmatrix} 0 & \alpha_{2k+2,\,2k+1} \\ \alpha_{2k+1,\,2k+2} & 0 \end{vmatrix} = |A|\cdot|B|$$

Moreover, $\alpha_{2k+2,\,2k+1} = -\alpha_{2k+1,\,2k+2}$; hence,

$$|A|\cdot f^2 = \alpha^2_{2k+1,\,2k+2} \qquad \text{and} \qquad |A| = \left\{\frac{\alpha_{2k+1,\,2k+2}}{f}\right\}^2$$

a perfect square.

## SUPPLEMENTARY PROBLEMS

8. Compute the adjoint of:

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, (d) $\begin{bmatrix} 5 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

*Ans.* (a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$, (d) $\begin{bmatrix} 2 & 0 & 0 & -4 \\ 2 & 6 & 0 & -16 \\ 1 & 0 & 3 & -5 \\ -2 & 0 & 0 & 10 \end{bmatrix}$

9. Verify:
   (a) The adjoint of a scalar matrix is a scalar matrix.
   (b) The adjoint of a diagonal matrix is a diagonal matrix.
   (c) The adjoint of a triangular matrix is a triangular matrix.

10. Write a matrix $A \neq 0$ of order 3 such that $\text{adj}\, A = 0$.

11. If $A$ is a 2-square matrix, show that $\text{adj}(\text{adj}\, A) = A$.

12. Show that the adjoint of $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ is $3A'$ and the adjoint of $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ is $A$ itself.

13. Prove: If an $n$-square matrix $A$ is of rank $< n-1$, then $\text{adj}\, A = 0$.

14. Prove: If $A$ is symmetric, so also is $\text{adj}\, A$.

15. Prove: If $A$ is Hermitian, so also is $\text{adj}\, A$.

16. Prove: If $A$ is skew-symmetric of order $n$, then $\text{adj}\, A$ is symmetric or skew-symmetric according as $n$ is odd or even.

17. Is there a theorem similar to that of Problem 16 for skew-Hermitian matrices?

18. For the elementary matrices, show that

(a) $\text{adj}\, H_{ij}^{-1} = -H_{ij}$

(b) $\text{adj}\, H_i^{-1}(k) = \text{diag}(1/k, 1/k, \ldots, 1/k, 1, 1/k, \ldots, 1/k)$, where the element 1 stands in the $i$th row

(c) $\text{adj}\, H_{ij}^{-1}(k) = H_{ij}(k)$, with similar results for the $K$'s.

19. Prove: If $A$ is an $n$-square matrix of rank $n$ or $n-1$ and if $H_s \ldots H_2 \cdot H_1 \cdot A \cdot K_1 \cdot K_2 \ldots K_t = \lambda$ where $\lambda$ is $I_n$ or $\begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix}$, then

$$\text{adj}\, A = \text{adj}\, K_1^{-1} \cdot \text{adj}\, K_2^{-1} \ldots \text{adj}\, K_t^{-1} \cdot \text{adj}\, \lambda \cdot \text{adj}\, H_s^{-1} \ldots \text{adj}\, H_2^{-1} \cdot \text{adj}\, H_1^{-1}$$

20. Use the method of Problem 19 to compute the adjoint of

(a) $A$ of Problem 7, Chapter 5 (b) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 3 & 2 \\ 1 & 2 & 3 & 2 \\ 4 & 6 & 7 & 4 \end{bmatrix}$

*Ans.* (a) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} -14 & 2 & -2 & 2 \\ 14 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ -7 & 1 & -1 & 1 \end{bmatrix}$

21. Let $A = [a_{ij}]$ and $B = [k - a_{ij}]$ be 3-square matrices. If $S(C)$ = sum of elements of matrix $C$, show that

$$S(\text{adj}\, A) = S(\text{adj}\, B) \quad \text{and} \quad |B| = k \cdot S(\text{adj}\, A) - |A|$$

22. Prove: If $A$ is $n$-square then $|\,\text{adj}(\text{adj}\, A)\,| = |A|^{(n-1)^2}$

23. Let $A_n = [a_{ij}]$ $(i, j = 1, 2, \ldots, n)$ be the lower triangular matrix whose triangle is the Pascal triangle; for example,

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

Define $b_{ij} = (-1)^{i+j} a_{ij}$ and verify for $n = 2, 3, 4$ that

(i) $$\text{adj}\, A_n = [b_{ij}] = A_n^{-1}$$

24. Let $B$ be obtained from $A$ by deleting its $i$th and $p$th rows and $j$th and $q$th columns. Show that

$$\begin{vmatrix} \alpha_{ij} & \alpha_{pj} \\ \alpha_{iq} & \alpha_{pq} \end{vmatrix} = (-1)^{i+p+j+q} |B| \cdot |A|$$

where $\alpha_{ij}$ is the cofactor of $a_{ij}$ in $|A|$.

# Chapter 7

## The Inverse of a Matrix

**IF** $A$ **AND** $B$ are $n$-square matrices such that $AB = BA = I$, $B$ is called the inverse of $A$, $(B = A^{-1})$ and $A$ is called the inverse of $B$, $(A = B^{-1})$.

In Problem 1, we prove

I. An $n$-square matrix $A$ has an inverse if and only if it is non-singular.

The inverse of a non-singular $n$-square matrix is unique. (See Problem 7, Chapter 2.)

II. If $A$ is non-singular, then $AB = AC$ implies $B = C$.

**THE INVERSE** of a non-singular diagonal matrix $\text{diag}(k_1, k_2, \ldots, k_n)$ is the diagonal matrix

$$\text{diag}(1/k_1, 1/k_2, \ldots, 1/k_n)$$

If $A_1, A_2, \ldots, A_s$ are non-singular matrices, then the inverse of the direct sum $\text{diag}(A_1, A_2, \ldots, A_s)$ is

$$\text{diag}(A_1^{-1}, A_2^{-1}, \ldots, A_s^{-1})$$

Procedures for computing the inverse of a general non-singular matrix are given below.

**INVERSE FROM THE ADJOINT.** From (6.2) $A \,\text{adj}\, A = |A| \cdot I$. If $A$ is non-singular

$$(7.1)\qquad A^{-1} = \frac{\text{adj}\, A}{|A|} = \begin{bmatrix} \alpha_{11}/|A| & \alpha_{21}/|A| & \ldots & \alpha_{n1}/|A| \\ \alpha_{12}/|A| & \alpha_{22}/|A| & \ldots & \alpha_{n2}/|A| \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{1n}/|A| & \alpha_{2n}/|A| & \ldots & \alpha_{nn}/|A| \end{bmatrix}$$

**Example 1.** From Problem 2, Chapter 6, the adjoint of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$ is $\begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$.

Since $|A| = -2$, $A^{-1} = \dfrac{\text{adj}\, A}{|A|} = \begin{bmatrix} 7/2 & -3 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}$.

See Problem 2.

**INVERSE FROM ELEMENTARY MATRICES.** Let the non-singular $n$-square matrix $A$ be reduced to $I$ by elementary transformations so that

$$H_s \ldots H_2 \cdot H_1 \cdot A \cdot K_1 \cdot K_2 \ldots K_t = PAQ = I$$

Then $A = P^{-1} \cdot Q^{-1}$ by (5.5) and, since $(B^{-1})^{-1} = B$,

$$A^{-1} = (P^{-1} \cdot Q^{-1})^{-1} = Q \cdot P = K_1 \cdot K_2 \ldots K_t \cdot H_s \ldots H_2 \cdot H_1 \qquad (7.2)$$

**Example 2.** From Problem 7, Chapter 5.

$$H_2 H_1 A K_1 K_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Then $A^{-1} = K_1 K_2 H_2 H_1 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

In Chapter 5 it was shown that a non-singular matrix can be reduced to normal form by row transformations alone. Then, from (7.2) with $Q = I$, we have

$$A^{-1} = P = H_s \ldots H_2 \cdot H_1 \qquad (7.3)$$

That is,

**III.** If $A$ is reduced to $I$ by a sequence of row transformations alone, then $A^{-1}$ is equal to the product in reverse order of the corresponding elementary matrices.

**Example 3.** Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ of Example 2 using only row transformations to reduce $A$ to $I$.

Write the matrix $[A\ I_3]$ and perform the sequence of row transformations which carry $A$ into $I_3$ on the rows of six elements. We have

$$[A\ I_3] = \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$= [I_3\ A^{-1}]$$

by (7.3). Thus, as $A$ is reduced to $I_3$, $I_3$ is carried into $A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

See Problem 3.

**INVERSE BY PARTITIONING.** Let the matrix $A = [a_{ij}]$ of order $n$ and its inverse $B = [b_{ij}]$ be partitioned into submatrices of indicated orders:

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ (p \times p) & (p \times q) \\ \hline A_{21} & A_{22} \\ (q \times p) & (q \times q) \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} B_{11} & B_{12} \\ (p \times p) & (p \times q) \\ \hline B_{21} & B_{22} \\ (q \times p) & (q \times q) \end{array}\right] \quad \text{where } p + q = n$$

Since $AB = BA = I_n$, we have

$$(7.4)\qquad \begin{cases} \text{(i)}\ A_{11}B_{11} + A_{12}B_{21} = I_p & \text{(iii)}\ B_{21}A_{11} + B_{22}A_{21} = 0 \\ \text{(ii)}\ A_{11}B_{12} + A_{12}B_{22} = 0 & \text{(iv)}\ B_{21}A_{12} + B_{22}A_{22} = I_q \end{cases}$$

Then, provided $A_{11}$ is non-singular,

$$(7.5)\qquad \begin{cases} B_{11} = A_{11}^{-1} + (A_{11}^{-1}\,A_{12})\xi^{-1}(A_{21}\,A_{11}^{-1}) & B_{21} = -\xi^{-1}(A_{21}\,A_{11}^{-1}) \\ B_{12} = -(A_{11}^{-1}\,A_{12})\xi^{-1} & B_{22} = \xi^{-1} \end{cases}$$

where $\xi = A_{22} - A_{21}(A_{11}^{-1}\,A_{12})$.

See Problem 4.

In practice, $A_{11}$ is usually taken of order $n-1$. To obtain $A_{11}^{-1}$, the following procedure is used. Let

$$G_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad G_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad G_4 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \qquad \ldots$$

After computing $G_2^{-1}$, partition $G_3$ so that $A_{22} = [a_{33}]$ and use (7.5) to obtain $G_3^{-1}$. Repeat the process on $G_4$ after partitioning it so that $A_{22} = [a_{44}]$, and so on.

**Example 4.** Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, using partitioning.

Take $A_{11} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $A_{21} = [1\ 3]$, and $A_{22} = [4]$. Now

$$A_{11}^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}, \quad A_{11}^{-1}A_{12} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad A_{21}A_{11}^{-1} = [1\ 3]\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} = [1\ 0],$$

$$\xi = A_{22} - A_{21}(A_{11}^{-1}A_{12}) = [4] - [1\ 3]\begin{bmatrix} 3 \\ 0 \end{bmatrix} = [1], \quad \text{and} \quad \xi^{-1} = [1]$$

Then

$$B_{11} = A_{11}^{-1} + (A_{11}^{-1}A_{12})\xi^{-1}(A_{21}A_{11}^{-1}) = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}[1]\cdot[1\ 0]$$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -1 & 1 \end{bmatrix}$$

$$B_{12} = -(A_{11}^{-1}A_{12})\xi^{-1} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$B_{21} = -\xi^{-1}(A_{21}A_{11}^{-1}) = [-1, 0]$$

$$B_{22} = \xi^{-1} = [1]$$

and

$$A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

See Problems 5-6.

**THE INVERSE OF A SYMMETRIC MATRIX.** When $A$ is symmetric, $\alpha_{ij} = \alpha_{ji}$ and only $\frac{1}{2}n(n+1)$ cofactors need be computed instead of the usual $n^2$ in obtaining $A^{-1}$ from adj $A$.

If there is to be any gain in computing $A^{-1}$ as the product of elementary matrices, the elementary transformations must be performed so that the property of being symmetric is preserved. This requires that the transformations occur in pairs, a row transformation followed immediately by the same column transformation. For example,

$$H_{12}\begin{bmatrix} 0 & b & c & \dots \\ b & a & \dots & \\ c & \dots & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{bmatrix}K_{12} = \begin{bmatrix} a & b & \dots & \\ b & 0 & c & \dots \\ \cdot & c & \dots & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{bmatrix}$$

$$H_{21}(-\tfrac{b}{a})\begin{bmatrix} a & b & c & \dots \\ b & & & \\ c & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{bmatrix}K_{21}(-\tfrac{b}{a}) = \begin{bmatrix} a & 0 & c & \dots \\ 0 & & & \\ c & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{bmatrix}$$

However, when the element $a$ in the diagonal is replaced by 1, the pair of transformations are $H_1(1/\sqrt{a})$ and $K_1(1/\sqrt{a})$. In general, $\sqrt{a}$ is either irrational or imaginary; hence, this procedure is not recommended.

The maximum gain occurs when the method of partitioning is used since then (7.5) reduces to

(7.6)
$$\begin{aligned} B_{11} &= A_{11}^{-1} + (A_{11}^{-1}A_{12})\xi^{-1}(A_{11}^{-1}A_{12})', \qquad & B_{21} &= B_{12}' \\ B_{12} &= -(A_{11}^{-1}A_{12})\xi^{-1}, & B_{22} &= \xi^{-1} \end{aligned}$$

where $\xi = A_{22} - A_{21}(A_{11}^{-1}A_{12})$.

See Problem 7.

When $A$ is not symmetric, the above procedure may be used to find the inverse of $A'A$, which is symmetric, and then the inverse of $A$ is found by

(7.7)
$$A^{-1} = (A'A)^{-1}A'$$

## SOLVED PROBLEMS

1. Prove: An $n$-square matrix $A$ has an inverse if and only if it is non-singular.

Suppose $A$ is non-singular. By Theorem IV, Chapter 5, there exist non-singular matrices $P$ and $Q$ such that $PAQ = I$. Then $A = P^{-1}\cdot Q^{-1}$ and $A^{-1} = Q\cdot P$ exists.

Suppose $A^{-1}$ exists. The $A\cdot A^{-1} = I$ is of rank $n$. If $A$ were singular, $AA^{-1}$ would be of rank $< n$; hence, $A$ is non-singular.

2. (*a*) When $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then $|A| = 5$, $\text{adj}\, A = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$, and $A^{-1} = \begin{bmatrix} 4/5 & -3/5 \\ -1/5 & 2/5 \end{bmatrix}$.

(*b*) When $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, then $|A| = 18$, $\text{adj}\, A = \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$, and $A^{-1} = \dfrac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$.

3. Find the inverse of $A = \begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix}$.

$$[A\ I_4] = \left[\begin{array}{cccc|cccc} 2 & 4 & 3 & 2 & 1 & 0 & 0 & 0 \\ 3 & 6 & 5 & 2 & 0 & 1 & 0 & 0 \\ 2 & 5 & 2 & -3 & 0 & 0 & 1 & 0 \\ 4 & 5 & 14 & 14 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 2 & 3/2 & 1 & 1/2 & 0 & 0 & 0 \\ 3 & 6 & 5 & 2 & 0 & 1 & 0 & 0 \\ 2 & 5 & 2 & -3 & 0 & 0 & 1 & 0 \\ 4 & 5 & 14 & 14 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 2 & 3/2 & 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & -1 & -3/2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -5 & -1 & 0 & 1 & 0 \\ 0 & -3 & 8 & 10 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 2 & 3/2 & 1 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1 & -5 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & -1 & -3/2 & 1 & 0 & 0 \\ 0 & -3 & 8 & 10 & -2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 7/2 & 11 & 5/2 & 0 & -2 & 0 \\ 0 & 1 & -1 & -5 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -3 & 2 & 0 & 0 \\ 0 & 0 & 5 & -5 & -5 & 0 & 3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 18 & 13 & -7 & -2 & 0 \\ 0 & 1 & 0 & -7 & -4 & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & -3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5 & 10 & -10 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 18 & 13 & -7 & -2 & 0 \\ 0 & 1 & 0 & -7 & -4 & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & -3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & 3/5 & 1/5 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -23 & 29 & -64/5 & -18/5 \\ 0 & 1 & 0 & 0 & 10 & -12 & 26/5 & 7/5 \\ 0 & 0 & 1 & 0 & 1 & -2 & 6/5 & 2/5 \\ 0 & 0 & 0 & 1 & 2 & -2 & 3/5 & 1/5 \end{array}\right]$$

$$= [I_4\ A^{-1}]$$

The inverse is $A^{-1} = \begin{bmatrix} -23 & 29 & -64/5 & -18/5 \\ 10 & -12 & 26/5 & 7/5 \\ 1 & -2 & 6/5 & 2/5 \\ 2 & -2 & 3/5 & 1/5 \end{bmatrix}$

4. Solve $\begin{cases} \text{(i)}\ A_{11}B_{11} + A_{12}B_{21} = I & \text{(iii)}\ B_{21}A_{11} + B_{22}A_{21} = 0 \\ \text{(ii)}\ A_{11}B_{12} + A_{12}B_{22} = 0 & \text{(iv)}\ B_{21}A_{12} + B_{22}A_{22} = I \end{cases}$ for $B_{11}, B_{12}, B_{21}$, and $B_{22}$.

Set $B_{22} = \xi^{-1}$. From **(ii)**, $B_{12} = -(A_{11}^{-1}A_{12})\xi^{-1}$; from **(iii)**, $B_{21} = -\xi^{-1}(A_{21}A_{11}^{-1})$; and, from **(i)**, $B_{11} = A_{11}^{-1} - A_{11}^{-1}A_{12}B_{21} = A_{11}^{-1} + (A_{11}^{-1}A_{12})\xi^{-1}(A_{21}A_{11}^{-1})$.

Finally, substituting in **(iv)**,

$$-\xi^{-1}(A_{21}A_{11}^{-1})A_{12} + \xi^{-1}A_{22} = I \quad \text{and} \quad \xi = A_{22} - (A_{21}A_{11}^{-1})A_{12}$$

5. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ by partitioning.

(*a*) Take $G_3 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}$ and partition so that

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad A_{21} = [2 \;\; 4], \quad \text{and} \quad A_{22} = [3]$$

Now $A_{11}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \quad A_{11}^{-1}A_{12} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad A_{21}A_{11}^{-1} = [2 \;\; 4]\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = [2 \;\; 0],$

$$\xi = A_{22} - A_{21}(A_{11}^{-1}A_{12}) = [3] - [2 \;\; 4]\begin{bmatrix} 3 \\ 0 \end{bmatrix} = [-3], \quad \text{and} \quad \xi^{-1} = [-1/3]$$

Then $B_{11} = A_{11}^{-1} + (A_{11}^{-1}A_{12})\xi^{-1}(A_{21}A_{11}^{-1}) = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}\left[-\frac{1}{3}\right][2 \;\; 0] = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} 3 & -6 \\ -3 & 3 \end{bmatrix}$$

$$B_{12} = -(A_{11}^{-1}A_{12})\xi^{-1} = \frac{1}{3}\begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad B_{21} = -\xi^{-1}(A_{21}A_{11}^{-1}) = \frac{1}{3}[2 \;\; 0], \quad B_{22} = \xi^{-1} = \left[-\frac{1}{3}\right]$$

and
$$G_3^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & -6 & 3 \\ -3 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

(*b*) Partition $A$ so that $A_{11} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}, \; A_{12} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; A_{21} = [1 \;\; 1 \;\; 1], \;$ and $\; A_{22} = [1]$.

Now $A_{11}^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -6 & 3 \\ -3 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix}, \quad A_{11}^{-1}A_{12} = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \quad A_{21}A_{11}^{-1} = \frac{1}{3}[2 \;\; -3 \;\; 2].$

$$\xi = [1] - [1 \;\; 1 \;\; 1](\tfrac{1}{3})\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} = \left[\frac{1}{3}\right], \quad \text{and} \quad \xi^{-1} = [3]$$

Then $B_{11} = \frac{1}{3}\begin{bmatrix} 3 & -6 & 3 \\ -3 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}[3]\frac{1}{3}[2 \;\; -3 \;\; 2] = \frac{1}{3}\begin{bmatrix} 3 & -6 & 3 \\ -3 & 3 & 0 \\ 2 & 0 & -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 6 & -9 & 6 \\ -2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix},$

$$B_{12} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \quad B_{21} = [-2 \;\; 3 \;\; -2], \quad B_{22} = [3]$$

and
$$A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$$

6. Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$ by partitioning.

We cannot take $A_{11} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$ since this is singular.

By Example 3, the inverse of $H_{23}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = B$ is $B^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Then

$$A^{-1} \;=\; B^{-1}H_{23} \;=\; \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \;=\; \begin{bmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

Thus, if the $(n-1)$-square minor $A_{11}$ of the $n$-square non-singular matrix $A$ is singular, we first bring a non-singular $(n-1)$-square matrix into the upper left corner to obtain $B$, find the inverse of $B$, and by the proper transformation on $B^{-1}$ obtain $A^{-1}$.

7. Compute the inverse of the symmetric matrix $A = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & -3 \\ -1 & 2 & 1 & -1 \\ 2 & -3 & -1 & 4 \end{bmatrix}$.

Consider first the submatrix $G_3 = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix}$ partitioned so that

$$A_{11} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad A_{21} = [-1 \;\; 2], \quad A_{22} = [1]$$

Now $$A_{11}^{-1} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}, \quad A_{11}^{-1}A_{12} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$\xi = A_{22} - A_{21}(A_{11}^{-1}A_{12}) = [1] - [-1 \;\; 2]\begin{bmatrix} -1 \\ 1 \end{bmatrix} = [-2] \quad \text{and} \quad \xi^{-1} = [-\tfrac{1}{2}]$$

Then $$B_{11} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}[-\tfrac{1}{2}][-1 \;\; 1] = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix},$$

$$B_{12} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad B_{21} = [-\tfrac{1}{2} \;\; \tfrac{1}{2}], \quad B_{22} = [-\tfrac{1}{2}]$$

and $$G_3^{-1} = \frac{1}{10}\begin{bmatrix} 1 & 3 & -5 \\ 3 & -1 & 5 \\ -5 & 5 & -5 \end{bmatrix}$$

Consider now the matrix $A$ partitioned so that

$$A_{11} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}, \quad A_{21} = [2 \;\; -3 \;\; -1], \quad A_{22} = [4]$$

Now $$A_{11}^{-1} = \frac{1}{10}\begin{bmatrix} 1 & 3 & -5 \\ 3 & -1 & 5 \\ -5 & 5 & -5 \end{bmatrix}, \quad A_{11}^{-1}A_{12} = \begin{bmatrix} -1/5 \\ 2/5 \\ -2 \end{bmatrix}, \quad \xi = [18/5], \quad \xi^{-1} = [5/18].$$

Then $B_{11} = \frac{1}{18}\begin{bmatrix} 2 & 5 & -7 \\ 5 & -1 & 5 \\ -7 & 5 & 11 \end{bmatrix}$, $B_{12} = \frac{1}{18}\begin{bmatrix} 1 \\ -2 \\ 10 \end{bmatrix}$, $B_{21} = \frac{1}{18}[1\ \ -2\ \ 10]$, $B_{22} = [5/18]$

and

$$A^{-1} = \frac{1}{18}\begin{bmatrix} 2 & 5 & -7 & 1 \\ 5 & -1 & 5 & -2 \\ -7 & 5 & 11 & 10 \\ 1 & -2 & 10 & 5 \end{bmatrix}$$

## SUPPLEMENTARY PROBLEMS

**8.** Find the adjoint and inverse of each of the following:

$(a)\ \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, $(b)\ \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$, $(c)\ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$, $(d)\ \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

*Ans.* Inverses $(a)\ \frac{1}{14}\begin{bmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{bmatrix}$, $(b)\ \frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$, $(c)\ \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$, $(d)\ \begin{bmatrix} 1 & -2/3 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & -1/6 \\ 0 & 0 & 0 & 1/3 \end{bmatrix}$

**9.** Find the inverse of the matrix of Problem 8(*d*) as a direct sum.

**10.** Obtain the inverses of the matrices of Problem 8 using the method of Problem 3.

**11.** Same, for the matrices $(a)\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$, $(b)\ \begin{bmatrix} 3 & 4 & 2 & 7 \\ 2 & 3 & 3 & 2 \\ 5 & 7 & 3 & 9 \\ 2 & 3 & 2 & 3 \end{bmatrix}$, $(c)\ \begin{bmatrix} 2 & 5 & 2 & 3 \\ 2 & 3 & 3 & 4 \\ 3 & 6 & 3 & 2 \\ 4 & 12 & 0 & 8 \end{bmatrix}$, $(d)\ \begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 1 & 4 & 3 & 3 & -1 \\ 1 & 3 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & -2 & -1 & 2 & 2 \end{bmatrix}$

*Ans.* $(a)\ \frac{1}{18}\begin{bmatrix} 2 & 16 & -6 & 4 \\ 22 & 41 & -30 & -1 \\ -10 & -44 & 30 & -2 \\ 4 & -13 & 6 & -1 \end{bmatrix}$ $\qquad (c)\ \frac{1}{48}\begin{bmatrix} -144 & 36 & 60 & 21 \\ 48 & -20 & -12 & -5 \\ 48 & -4 & -12 & -13 \\ 0 & 12 & -12 & 3 \end{bmatrix}$

$(b)\ \frac{1}{2}\begin{bmatrix} -1 & 11 & 7 & -26 \\ -1 & -7 & -3 & 16 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 2 \end{bmatrix}$ $\qquad (d)\ \frac{1}{15}\begin{bmatrix} 30 & -20 & -15 & 25 & -5 \\ 30 & -11 & -18 & 7 & -8 \\ -30 & 12 & 21 & -9 & 6 \\ -15 & 12 & 6 & -9 & 6 \\ 15 & -7 & -6 & -1 & -1 \end{bmatrix}$

**12.** Use the result of Example 4 to obtain the inverse of the matrix of Problem 11(*d*) by partitioning.

**13.** Obtain by partitioning the inverses of the matrices of Problems 8(*a*), 8(*b*), 11(*a*) – 11(*c*).

14. Obtain by partitioning the inverses of the symmetric matrices (a) $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 2 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 2 & 1 & -1 & 2 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$.

*Ans.* (a) $-\frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & -5 & -1 \\ -1 & 1 & -1 & -1 \end{bmatrix}$, (b) $\begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$

15. Prove: If $A$ is non-singular, then $AB = AC$ implies $B = C$.

16. Show that if the non-singular matrices $A$ and $B$ commute, so also do
(a) $A^{-1}$ and $B$, (b) $A$ and $B^{-1}$, (c) $A^{-1}$ and $B^{-1}$. *Hint.* (a) $A^{-1}(AB)A^{-1} = A^{-1}(BA)A^{-1}$.

17. Show that if the non-singular matrix $A$ is symmetric, so also is $A^{-1}$.
*Hint:* $A^{-1}A = I = (AA^{-1})' = (A^{-1})'A$.

18. Show that if the non-singular symmetric matrices $A$ and $B$ commute, then (a) $A^{-1}B$, (b) $AB^{-1}$, and (c) $A^{-1}B^{-1}$ are symmetric. *Hint:* (a) $(A^{-1}B)' = (BA^{-1})' = (A^{-1})'B' = A^{-1}B$.

19. An $m\times n$ matrix $A$ is said to have a *right inverse* $B$ if $AB = I$ and a *left inverse* $C$ if $CA = I$. Show that $A$ has a right inverse if and only if $A$ is of rank $m$ and has a left inverse if and only if the rank of $A$ is $n$.

20. Find a right inverse of $A = \begin{bmatrix} 1 & 3 & 2 & 3 \\ 1 & 4 & 1 & 3 \\ 1 & 3 & 5 & 4 \end{bmatrix}$ if one exists.

*Hint.* The rank of $A$ is 3 and the submatrix $S = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 1 \\ 1 & 3 & 5 \end{bmatrix}$ is non-singular with inverse $S^{-1}$. A right inverse of $A$ is the 4×3 matrix $B = \begin{bmatrix} S^{-1} \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 17 & -9 & -5 \\ -4 & 3 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

21. Show that the submatrix $T = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ of $A$ of Problem 20 is non-singular and obtain $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ as another right inverse of $A$.

22. Obtain $\begin{bmatrix} 7 & -1 & -1 & a \\ -3 & 1 & 0 & b \\ -3 & 0 & 1 & c \end{bmatrix}$ as a left inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, where $a$, $b$, and $c$ are arbitrary.

23. Show that $A = \begin{bmatrix} 1 & 3 & 4 & 7 \\ 1 & 4 & 5 & 9 \\ 2 & 3 & 5 & 8 \end{bmatrix}$ has neither a right nor a left inverse.

24. Prove: If $|A_{11}| \neq 0$, then $\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = |A_{11}|\cdot|A_{22} - A_{21}A_{11}^{-1}A_{12}|$.

25. If $|I+A| \neq 0$, then $(I+A)^{-1}$ and $(I-A)$ commute.

26. Prove:(i) of Problem 23, Chapter 6.

# Chapter 8

## Fields

**NUMBER FIELDS.** A collection or set $S$ of real or complex numbers, consisting of more than the element 0, is called a **number field** provided the operations of addition, subtraction, multiplication, and division (except by 0) on any two of the numbers yield a number of $S$.

Examples of number fields are:

(*a*) the set of all rational numbers,
(*b*) the set of all real numbers,
(*c*) the set of all numbers of the form $a+b\sqrt{3}$, where $a$ and $b$ are rational numbers,
(*d*) the set of all complex numbers $a+bi$, where $a$ and $b$ are real numbers.

The set of all integers and the set of all numbers of the form $b\sqrt{3}$, where $b$ is a rational number, are **not** number fields.

**GENERAL FIELDS.** A collection or set $S$ of two or more elements, together with two operations called addition (+) and multiplication (·), is called a field $F$ provided that ( $a, b, c, \ldots$ are elements of $F$, i.e. scalars),

$A_1$: $a+b$ is a unique element of $F$

$A_2$: $a+b = b+a$

$A_3$: $a+(b+c) = (a+b)+c$

$A_4$: For every element $a$ in $F$ there exists an element 0 in $F$ such that $a+0 = 0+a = a$.

$A_5$: For each element $a$ in $F$ there exists a unique element $-a$ in $F$ such that $a+(-a) = 0$.

$M_1$: $ab = a\cdot b$ is a unique element of $F$

$M_2$: $ab = ba$

$M_3$: $(ab)c = a(bc)$

$M_4$: For every element $a$ in $F$ there exists an element $1 \neq 0$ such that $1\cdot a = a\cdot 1 = a$.

$M_5$: For each element $a \neq 0$ in $F$ there exists a unique element $a^{-1}$ in $F$ such that $a\cdot a^{-1} = a^{-1}\cdot a = 1$.

$D_1$: $a(b+c) = ab+ac$

$D_2$: $(a+b)c = ac+bc$

In addition to the number fields listed above, other examples of fields are:

(*e*) the set of all quotients $\dfrac{P(x)}{Q(x)}$ of polynomials in $x$ with real coefficients,

(*f*) the set of all $2\times 2$ matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a$ and $b$ are real numbers.

(*g*) the set in which $a+a = 0$. This field, called **of characteristic 2**, will be **excluded** hereafter. In this field, for example, the customary proof that a determinant having two rows identical is 0 is not valid. By interchanging the two identical rows, we are led to $D = -D$ or $2D = 0$; but $D$ is not necessarily 0.

**SUBFIELDS.** If $S$ and $T$ are two sets and if every member of $S$ is also a member of $T$, then $S$ is called a **subset** of $T$.

If $S$ and $T$ are fields and if $S$ is a subset of $T$, then $S$ is called a **subfield** of $T$. For example, the field of all real numbers is a subfield of the field of all complex numbers; the field of all rational numbers is a subfield of the field of all real numbers and the field of all complex numbers.

**MATRICES OVER A FIELD.** When all of the elements of a matrix $A$ are in a field $F$, we say that "**$A$ is over $F$**". For example,

$$A = \begin{bmatrix} 1 & 1/2 \\ 1/4 & 2/3 \end{bmatrix} \text{ is over the rational field and } B = \begin{bmatrix} 1 & 1+i \\ 2 & 1-3i \end{bmatrix} \text{ is over the complex field}$$

Here, $A$ is also over the real field while $B$ is not; also $A$ is over the complex field.

Let $A, B, C, \ldots$ be matrices over the same field $F$ and let $F$ be the smallest field which contains all the elements; that is, if all the elements are rational numbers, the field $F$ is the rational field and not the real or complex field. An examination of the various operations defined on these matrices, individually or collectively, in the previous chapters shows that no elements other than those in $F$ are ever required. For example:

The sum, difference, and product are matrices over $F$.

If $A$ is non-singular, its inverse is over $F$.

If $A \sim I$ then there exist matrices $P$ and $Q$ over $F$ such that $PAQ = I$ and $I$ is over $F$.

If $A$ is over the rational field and is of rank $r$, its rank is unchanged when considered over the real or the complex field.

Hereafter when $A$ is said to be over $F$ it will be assumed that $F$ is the smallest field containing all of its elements.

In later chapters it will at times be necessary to restrict the field, say, to the real field. At other times, the field of the elements will be extended, say, from the rational field to the real field. Otherwise, the statement "$A$ over $F$" implies no restriction on the field, **except for the excluded field of characteristic two.**

## SOLVED PROBLEM

1. Verify that the set of all complex numbers constitutes a field.

To do this we simply check the properties $A_1 - A_5$, $M_1 - M_5$, and $D_1 - D_2$. The zero element ($A_4$) is 0 and the unit element ($M_4$) is 1. If $a+bi$ and $c+di$ are two elements, the negative ($A_5$) of $a+bi$ is $-a-bi$, the product ($M_1$) is $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$; the inverse ($M_5$) of $a+bi \neq 0$ is

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}$$

Verification of the remaining properties is left as an exercise for the reader.

# SUPPLEMENTARY PROBLEMS

2. Verify (*a*) the set of all real numbers of the form $a+b\sqrt{5}$ where $a$ and $b$ are rational numbers and

   (*b*) the set of all quotients $\frac{P(x)}{Q(x)}$ of polynomials in $x$ with real coefficients constitute fields.

3. Verify (*a*) the set of all rational numbers,

   (*b*) the set of all numbers $a+b\sqrt{3}$, where $a$ and $b$ are rational numbers, and

   (*c*) the set of all numbers $a+bi$, where $a$ and $b$ are rational numbers are subfields of the complex field.

4. Verify that the set of all 2×2 matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a$ and $b$ are rational numbers, forms a field.

   Show that this is a subfield of the field of all 2×2 matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a$ and $b$ are real numbers.

5. Why does not the set of all $2\times 2$ matrices with real elements form a field?

6. A set $R$ of elements $a, b, c, \ldots$ satisfying the conditions ($A_1, A_2, A_3, A_4, A_5$; $M_1, M_3$; $D_1, D_2$) of Page 64 is called a **ring**. To emphasize the fact that multiplication is not commutative, $R$ may be called a **non-commutative ring**. When a ring $R$ satisfies $M_2$, it is called **commutative**. When a ring $R$ satisfies $M_4$, it is spoken of as a ring **with unit element**.

   Verify:

   (*a*) the set of even integers $0, \pm 2, \pm 4, \ldots$ is an example of a commutative ring without unit element.

   (*b*) the set of all integers $0, \pm 1, \pm 2, \pm 3, \ldots$ is an example of a commutative ring with unit element.

   (*c*) the set of all $n$-square matrices over $F$ is an example of a non-commutative ring with unit element.

   (*d*) the set of all $2\times 2$ matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a$ and $b$ are real numbers, is an example of a commutative ring with unit element.

7. Can the set (*a*) of Problem 6 be turned into a commutative ring with unit element by simply adjoining the elements $\pm 1$ to the set?

8. By Problem 4, the set (*d*) of Problem 6 is a field. Is every field a ring? Is every commutative ring with unit element a field?

9. Describe the ring of all $2\times 2$ matrices $\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$, where $a$ and $b$ are in $F$. If $A$ is any matrix of the ring and $L = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, show that $LA = A$. Call $L$ a **left unit element**. Is there a right unit element?

10. Let $C$ be the field of all complex numbers $p+qi$ and $K$ be the field of all $2\times 2$ matrices $\begin{bmatrix} u & -v \\ v & u \end{bmatrix}$ where $p, q, u, v$ are real numbers. Take the complex number $a+bi$ and the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ as corresponding elements of the two sets and call each the **image** of the other.

    (*a*) Write the image of $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$; $3+2i$; 5.

    (*b*) Show that the image of the sum (product) of two elements of $K$ is the sum (product) of their images in $C$.

    (*c*) Show that the image of the identity element of $K$ is the identity element of $C$.

    (*d*) What is the image of the conjugate of $a+bi$?

    (*e*) What is the image of the inverse of $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$?

    This is an example of an **isomorphism** between two sets.

# Chapter 9

## Linear Dependence of Vectors and Forms

**THE ORDERED PAIR** of real numbers $(x_1, x_2)$ is used to denote a point $X$ in a plane. The same pair of numbers, written as $[x_1, x_2]$, will be used here to denote the **two-dimensional vector** or **2-vector** $OX$ (see Fig. 9-1).

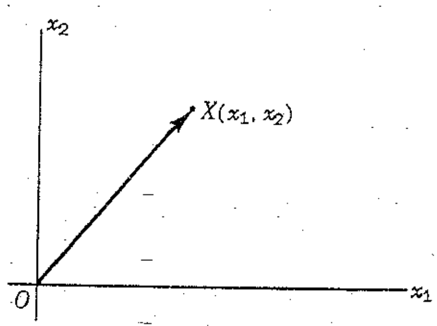


**Fig. 9-1**

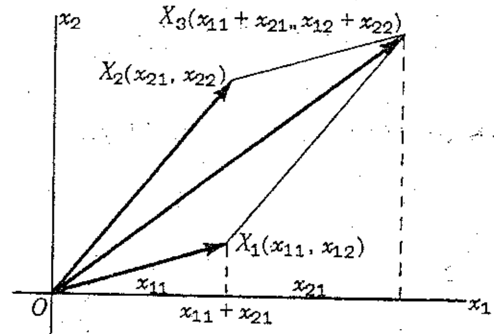


**Fig. 9-2**

If $X_1 = [x_{11}, x_{12}]$ and $X_2 = [x_{21}, x_{22}]$ are distinct 2-vectors, the parallelogram law for their sum (see Fig. 9-2) yields

$$X_3 \quad = \quad X_1 + X_2 \quad = \quad [x_{11}+x_{21},\ x_{12}+x_{22}]$$

Treating $X_1$ and $X_2$ as 1×2 matrices, we see that this is merely the rule for adding matrices given in Chapter 1. Moreover, if $k$ is any scalar,

$$kX_1 \quad = \quad [kx_{11},\ kx_{12}]$$

is the familiar multiplication of a vector by a real number of physics.

**VECTORS.** By an **$n$-dimensional vector** or **$n$-vector** $X$ over $F$ is meant an ordered set of $n$ elements $x_i$ of $F$, as

**(9.1)** $$X \quad = \quad [x_1, x_2, \ldots, x_n]$$

The elements $x_1, x_2, \ldots, x_n$ are called respectively the first, second, ..., $n$th **components** of $X$.

Later we shall find it more convenient to write the components of a vector in a column, as

**(9.1′)** $$X \quad = \quad [x_1, x_2, \ldots, x_n]' \quad = \quad \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

Now (9.1) and (9.1′) denote the same vector; however, we shall speak of (9.1) as a **row vector** and (9.1′) as a **column vector**. We may, then, consider the $p\times q$ matrix $A$ as defining $p$ row vectors (the elements of a row being the components of a $q$-vector) or as defining $q$ column vectors.

The vector, all of whose components are zero, is called the **zero vector** and is denoted by 0.

The sum and difference of two row (column) vectors and the product of a scalar and a vector are formed by the rules governing matrices.

**Example 1.** Consider the 3-vectors

$$X_1 = [3,1,-4],\quad X_2 = [2,2,-3],\quad X_3 = [0,-4,1],\quad \text{and}\quad X_4 = [-4,-4,6]$$

(a) $2X_1 - 5X_2 = 2[3,1,-4] - 5[2,2,-3] = [6,2,-8] - [10,10,-15] = [-4,-8,7]$

(b) $2X_2 + X_4 = 2[2,2,-3] + [-4,-4,6] = [0,0,0] = 0$

(c) $2X_1 - 3X_2 - X_3 = 0$

(d) $2X_1 - X_2 - X_3 + X_4 = 0$

The vectors used here are row vectors. Note that if each bracket is primed to denote column vectors, the results remain correct.

**LINEAR DEPENDENCE OF VECTORS.** The $m$ $n$-vectors over $F$

$$(9.2)\qquad \begin{aligned} X_1 &= [x_{11}, x_{12}, \ldots, x_{1n}] \\ X_2 &= [x_{21}, x_{22}, \ldots, x_{2n}] \\ &\cdots\cdots\cdots\cdots\cdots \\ X_m &= [x_{m1}, x_{m2}, \ldots, x_{mn}] \end{aligned}$$

are said to be **linearly dependent** over $F$ provided there exist $m$ elements $k_1, k_2, \ldots, k_m$ of $F$, not all zero, such that

$$(9.3)\qquad k_1X_1 + k_2X_2 + \cdots + k_mX_m = 0$$

Otherwise, the $m$ vectors are said to be **linearly independent**.

**Example 2.** Consider the four vectors of Example 1. By (b) the vectors $X_2$ and $X_4$ are linearly dependent; so also are $X_1, X_2$, and $X_3$ by (c) and the entire set by (d).

The vectors $X_1$ and $X_2$, however, are linearly independent. For, assume the contrary so that

$$k_1X_1 + k_2X_2 = [3k_1 + 2k_2,\ k_1 + 2k_2,\ -4k_1 - 3k_2] = [0,0,0]$$

Then $3k_1 + 2k_2 = 0$, $k_1 + 2k_2 = 0$, and $-4k_1 - 3k_2 = 0$. From the first two relations $k_1 = 0$ and then $k_2 = 0$.

Any $n$-vector $X$ and the $n$-zero vector 0 are linearly dependent.

A vector $X_{m+1}$ is said to be expressible as a **linear combination** of the vectors $X_1, X_2, \ldots, X_m$ if there exist elements $k_1, k_2, \ldots, k_m$ of $F$ such that

$$X_{m+1} = k_1X_1 + k_2X_2 + \cdots + k_mX_m$$

**BASIC THEOREMS.** If in (9.3), $k_i \neq 0$, we may solve for

$$X_i = -\frac{1}{k_i}\{k_1X_1 + \cdots + k_{i-1}X_{i-1} + k_{i+1}X_{i+1} + \cdots + k_mX_m\} \qquad \text{or}$$

$$(9.4)\qquad X_i = s_1X_1 + \cdots + s_{i-1}X_{i-1} + s_{i+1}X_{i+1} + \cdots + s_mX_m$$

Thus,

I. If $m$ vectors are linearly dependent, some one of them may always be expressed as a linear combination of the others.

**II.** If $m$ vectors $X_1, X_2, \ldots, X_m$ are linearly independent while the set obtained by adding another vector $X_{m+1}$ is linearly dependent, then $X_{m+1}$ can be expressed as a linear combination of $X_1, X_2, \ldots, X_m$.

**Example 3.** From Example 2, the vectors $X_1$, and $X_2$ are linearly independent while $X_1, X_2$, and $X_3$ are linearly dependent, satisfying the relations $2X_1 - 3X_2 - X_3 = 0$. Clearly, $X_3 = 2X_1 - 3X_2$.

**III.** If among the $m$ vectors $X_1, X_2, \ldots, X_m$ there is a subset of $r<m$ vectors which are linearly dependent, the vectors of the entire set are linearly dependent.

**Example 4.** By (*b*) of Example 1, the vectors $X_2$ and $X_4$ are linearly dependent; by (*d*), the set of four vectors is linearly dependent. See Problem 1.

**IV.** If the rank of the matrix

**(9.5)**
$$A = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}, \qquad m \leq n,$$

associated with the $m$ vectors **(9.2)** is $r<m$, there are exactly $r$ vectors of the set which are linearly independent while each of the remaining $m-r$ vectors can be expressed as a linear combination of these $r$ vectors. See Problems 2-3.

**V.** A necessary and sufficient condition that the vectors **(9.2)** be linearly dependent is that the matrix **(9.5)** of the vectors be of rank $r<m$. If the rank is $m$, the vectors are linearly independent.

The set of vectors **(9.2)** is necessarily linearly dependent if $m>n$.

If the set of vectors **(9.2)** is linearly independent so also is every subset of them.

**A LINEAR FORM** over $F$ in $n$ variables $x_1, x_2, \ldots, x_n$ is a polynomial of the type

**(9.6)**
$$\sum_{i=1}^{n} a_i x_i = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

where the coefficients are in $F$.

Consider a system of $m$ linear forms in $n$ variables

**(9.7)**
$$\begin{cases} f_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ f_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{cases}$$

and the associated matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

If there exist elements $k_1, k_2, \ldots, k_m$, not all zero, in $F$ such that

$$k_1f_1 + k_2f_2 + \ldots + k_m f_m = 0$$

the forms (**9.7**) are said to be **linearly dependent**; otherwise the forms are said to be **linearly independent**. Thus, the linear dependence or independence of the forms of (**9.7**) is equivalent to the linear dependence or independence of the row vectors of $A$.

**Example 5.** The forms $f_1 = 2x_1 - x_2 + 3x_3$, $f_2 = x_1 + 2x_2 + 4x_3$, $f_3 = 4x_1 - 7x_2 + x_3$ are linearly dependent since $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 4 & -7 & 1 \end{bmatrix}$ is of rank 2. Here, $3f_1 - 2f_2 - f_3 = 0$.

The system (**9.7**) is necessarily dependent if $m > n$. Why?

## SOLVED PROBLEMS

1. Prove: If among the $m$ vectors $X_1, X_2, \ldots, X_m$ there is a subset, say, $X_1, X_2, \ldots, X_r$, $r<m$, which is linearly dependent, so also are the $m$ vectors.

Since, by hypothesis, $k_1X_1 + k_2X_2 + \cdots + k_rX_r = 0$ with not all of the $k$'s equal to zero, then

$$k_1X_1 + k_2X_2 + \cdots + k_rX_r + 0\cdot X_{r+1} + \cdots + 0\cdot X_m = 0$$

with not all of the $k$'s equal to zero and the entire set of vectors is linearly dependent.

2. Prove: If the rank of the matrix associated with a set of $m$ $n$-vectors is $r<m$, there are exactly $r$ vectors which are linearly independent while each of the remaining $m$-$r$ vectors can be expressed as a linear combination of these $r$ vectors.

Let (**9.5**) be the matrix and suppose first that $m \leq n$. If the $r$-rowed minor in the upper left hand corner is equal to zero, we interchange rows and columns as are necessary to bring a non-vanishing $r$-rowed minor into this position and then renumber all rows and columns in natural order. Thus, we have

$$\Delta = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1r} \\ x_{21} & x_{22} & \cdots & x_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ x_{r1} & x_{r2} & \cdots & x_{rr} \end{vmatrix} \neq 0$$

Consider now an $(r+1)$-rowed minor

$$\nabla = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1r} & x_{1q} \\ x_{21} & x_{22} & \cdots & x_{2r} & x_{2q} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{r1} & x_{r2} & \cdots & x_{rr} & x_{rq} \\ x_{p1} & x_{p2} & \cdots & x_{pr} & x_{pq} \end{vmatrix} = 0$$

where the elements $x_{pj}$ and $x_{iq}$ are respectively from any row and any column not included in $\Delta$. Let $k_1, k_2, \ldots, k_{r+1} = \Delta$ be the respective cofactors of the elements $x_{1q}, x_{2q}, \ldots, x_{rq}, x_{pq}$ of the last column of $\nabla$. Then, by (**3.10**)

$$k_1x_{1i} + k_2x_{2i} + \cdots + k_rx_{ri} + k_{r+1}x_{pi} = 0 \qquad (i = 1, 2, \ldots, r)$$

and by hypothesis $\quad k_1x_{1q} + k_2x_{2q} + \cdots + k_rx_{rq} + k_{r+1}x_{pq} = \nabla = 0$

Now let the last column of $\nabla$ be replaced by another of the remaining columns, say the column numbered $u$, not appearing in $\Delta$. The cofactors of the elements of this column are precisely the $k$'s obtained above so that

$$k_1x_{1u} + k_2x_{2u} + \cdots + k_rx_{ru} + k_{r+1}x_{pu} = 0$$

Thus,

$$k_1x_{1t} + k_2x_{2t} + \cdots + k_rx_{rt} + k_{r+1}x_{pt} = 0 \qquad (t = 1, 2, \ldots, n)$$

and, summing over all values of $t$,

$$k_1X_1 + k_2X_2 + \cdots + k_rX_r + k_{r+1}X_p = 0$$

Since $k_{r+1} = \Delta \neq 0$, $X_p$ is a linear combination of the $r$ linearly independent vectors $X_1, X_2, \ldots, X_r$. But $X_p$ was any one of the $m$-$r$ vectors $X_{r+1}, X_{r+2}, \ldots, X_m$; hence, each of these may be expressed as a linear combination of $X_1, X_2, \ldots, X_r$.

For the case $m > n$, consider the matrix when to each of the given $m$ vectors $m$-$n$ additional zero components are added. This matrix is $[A \,\vdots\, 0]$. Clearly the linear dependence or independence of the vectors and also the rank of $A$ have not been changed.

Thus, in either case, the vectors $X_{r+1}, \ldots, X_m$ are linear combinations of the linearly independent vectors $X_1, X_2, \ldots, X_r$ as was to be proved.

3. Show, using a matrix, that each triple of vectors

$$(a) \quad \begin{matrix} X_1 = [1, 2, -3, 4] \\ X_2 = [3, -1, 2, 1] \\ X_3 = [1, -5, 8, -7] \end{matrix} \qquad \text{and} \qquad (b) \quad \begin{matrix} X_1 = [2, 3, 1, -1] \\ X_2 = [2, 3, 1, -2] \\ X_3 = [4, 6, 2, -3] \end{matrix}$$

is linearly dependent. In each determine a maximum subset of linearly independent vectors and express the others as linear combinations of these.

(a) Here, $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 2 & 1 \\ 1 & -5 & 8 & -7 \end{bmatrix}$ is of rank 2; there are two linearly independent vectors, say $X_1$ and $X_2$. The minor $\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \neq 0$. Consider then the minor $\begin{vmatrix} 1 & 2 & -3 \\ 3 & -1 & 2 \\ 1 & -5 & 8 \end{vmatrix}$. The cofactors of the elements of the third column are respectively $-14$, $7$, and $-7$. Then $-14X_1 + 7X_2 - 7X_3 = 0$ and $X_3 = -2X_1 + X_2$.

(b) Here $\begin{bmatrix} 2 & 3 & 1 & -1 \\ 2 & 3 & 1 & -2 \\ 4 & 6 & 2 & -3 \end{bmatrix}$ is of rank 2; there are two linearly independent vectors, say $X_1$ and $X_2$. Now the minor $\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 0$; we interchange the 2nd and 4th columns to obtain $\begin{bmatrix} 2 & -1 & 1 & 3 \\ 2 & -2 & 1 & 3 \\ 4 & -3 & 2 & 6 \end{bmatrix}$ for which $\begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} \neq 0$.

The cofactors of the elements of the last column of $\begin{vmatrix} 2 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & -3 & 2 \end{vmatrix}$ are $2, 2, -2$ respectively. Then

$$2X_1 + 2X_2 - 2X_3 = 0 \qquad \text{and} \qquad X_3 = X_1 + X_2$$

4. Let $P_1(1,1,1)$, $P_2(1,2,3)$, $P_3(3,1,2)$, and $P_4(2,3,4)$ be points in ordinary space. The points $P_1$, $P_2$ and the origin of coordinates determine a plane $\pi$ of equation

(i) $$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = x - 2y + z = 0$$

Substituting the coordinates of $P_4$ into the left member of (i), we have

$$\begin{vmatrix} 2 & 3 & 4 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

Thus, $P_4$ lies in $\pi$. The significant fact here is that $[P_4, P_1, P_2]' = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ is of rank 2.

We have verified: Any three points of ordinary space lie in a plane through the origin provided the matrix of their coordinates is of rank 2.

Show that $P_3$ does not lie in $\pi$.

## SUPPLEMENTARY PROBLEMS

5. Prove: If $m$ vectors $X_1, X_2, \ldots, X_m$ are linearly independent while the set obtained by adding another vector $X_{m+1}$ is linearly dependent, then $X_{m+1}$ can be expressed as a linear combination of $X_1, X_2, \ldots, X_m$.

6. Show that the representation of $X_{m+1}$ in Problem 5 is unique.

*Hint*: Suppose $X_{m+1} = \sum_{i=1}^{m} k_i X_i = \sum_{i=1}^{n} s_i X_i$ and consider $\sum_{i=1}^{m} (k_i - s_i) X_i$.

7. Prove: A necessary and sufficient condition that the vectors (9.2) be linearly dependent is that the matrix (9.5) of the vectors be of rank $r < m$.

*Hint*: Suppose the $m$ vectors are linearly dependent so that (9.4) holds. In (9.5) subtract from the $i$th row the product of the first row by $s_1$, the product of the second row by $s_2$, ... as indicated in (9.4). For the converse, see Problem 2.

8. Examine each of the following sets of vectors over the real field for linear dependence or independence. In each dependent set select a maximum linearly independent subset and express each of the remaining vectors as a linear combination of these.

(a) $X_1 = [2,-1,3,2]$, $X_2 = [1,3,4,2]$, $X_3 = [3,-5,2,2]$

(b) $X_1 = [1,2,1]$, $X_2 = [2,1,4]$, $X_3 = [4,5,6]$, $X_4 = [1,8,-3]$

(c) $X_1 = [2,1,3,2,-1]$, $X_2 = [4,2,1,-2,3]$, $X_3 = [0,0,5,6,-5]$, $X_4 = [6,3,-1,-6,7]$

*Ans.* (a) $X_3 = 2X_1 - X_2$ (b) $X_3 = 2X_1 + X_2$, $X_4 = 5X_1 - 2X_2$ (c) $X_3 = 2X_1 - X_2$, $X_4 = 2X_2 - X_1$

9. Why can there be no more than $n$ linearly independent $n$-vectors over $F$?

10. Show that if in (9.2) either $X_i = X_j$ or $X_i = aX_j$, $a$ in $F$, the set of vectors is linearly dependent. Is the converse true?

11. Show that any $n$-vector $X$ and the $n$-zero vector are linearly dependent; hence, $X$ and 0 are considered proportional.
*Hint*: Consider $k_1X + k_2 \cdot 0 = 0$ where $k_1 = 0$ and $k_2 \neq 0$.

12. (*a*) Show that $X_1 = [1, 1+i, i]$, $X_2 = [i, -i, 1-i]$ and $X_3 = [1+2i, 1-i, 2-i]$ are linearly dependent over the rational field and, hence, over the complex field.
(*b*) Show that $X_1 = [1, 1+i, i]$, $X_2 = [i, -i, 1-i]$, and $X_3 = [0, 1-2i, 2-i]$ are linearly independent over the real field but are linearly dependent over the complex field.

13. Investigate the linear dependence or independence of the linear forms:

$$(a)\quad \begin{aligned} f_1 &= 3x_1 - x_2 + 2x_3 + x_4 \\ f_2 &= 2x_1 + 3x_2 - x_3 + 2x_4 \\ f_3 &= 5x_1 - 9x_2 + 8x_3 - x_4 \end{aligned} \qquad (b)\quad \begin{aligned} f_1 &= 2x_1 - 3x_2 + 4x_3 - 2x_4 \\ f_2 &= 3x_1 + 2x_2 - 2x_3 + 5x_4 \\ f_3 &= 5x_1 - x_2 + 2x_3 + x_4 \end{aligned}$$

*Ans.* (*a*) $3f_1 - 2f_2 - f_3 = 0$

14. Consider the linear dependence or independence of a system of polynomials

$$P_i = a_{i0}x^n + a_{i1}x^{n-1} + \cdots + a_{in-1}x + a_{in} \qquad (i = 1, 2, \ldots, m)$$

and show that the system is linearly dependent or independent according as the row vectors of the coefficient matrix

$$A = \begin{bmatrix} a_{10} & a_{11} & \ldots & a_{1n} \\ a_{20} & a_{21} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m0} & a_{m1} & \ldots & a_{mn} \end{bmatrix}$$

are linearly dependent or independent, that is, according as the rank $r$ of $A$ is less than or equal to $m$.

15. If the polynomials of either system are linearly dependent, find a linear combination which is identically zero.

$$(a)\quad \begin{aligned} P_1 &= x^3 - 3x^2 + 4x - 2 \\ P_2 &= 2x^2 - 6x + 4 \\ P_3 &= x^3 - 2x^2 + x \end{aligned} \qquad (b)\quad \begin{aligned} P_1 &= 2x^4 + 3x^3 - 4x^2 + 5x + 3 \\ P_2 &= x^3 + 2x^2 - 3x + 1 \\ P_3 &= x^4 + 2x^3 - x^2 + x + 2 \end{aligned}$$

*Ans.* (*a*) $2P_1 + P_2 - 2P_3 = 0$ (*b*) $P_1 + P_2 - 2P_3 = 0$

16. Consider the linear dependence or independence of a set of 2×2 matrices $M_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $M_2 = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, $M_3 = \begin{bmatrix} p & q \\ s & t \end{bmatrix}$ over $F$.

Show that $k_1M_1 + k_2M_2 + k_3M_3 = 0$, when not all the $k$'s (in $F$) are zero, requires that the rank of the matrix $\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ p & q & s & t \end{bmatrix}$ be $< 3$. (Note that the matrices $M_1, M_2, M_3$ are considered as defining vectors of four components.)

Extend the result to a set of $m \times n$ matrices.

17. Show that $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 3 & 3 \\ 3 & 0 & 6 \\ 0 & 4 & 3 \end{bmatrix}$ are linearly dependent.

18. Show that any 2×2 matrix can be written as a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Generalize to $n \times n$ matrices.

19. If the $n$-vectors $X_1, X_2, \ldots, X_n$ are linearly independent, show that the vectors $Y_1, Y_2, \ldots, Y_n$, where $Y_i = \sum_{j=1}^{n} a_{ij}X_j$, are linearly independent if and only if $A = [a_{ij}]$ is non-singular.

20. If $A$ is of rank $r$, show how to construct a non-singular matrix $B$ such that $AB = [C_1, C_2, \ldots, C_r, 0, \ldots, 0]$ where $C_1, C_2, \ldots, C_r$ are a given set of linearly independent columns of $A$.

21. Given the points $P_1(1, 1, 1, 1)$, $P_2(1, 2, 3, 4)$, $P_3(2, 2, 2, 2)$, and $P_4(3, 4, 5, 6)$ of four-dimensional space.
(a) Show that the rank of $[P_1, P_3]'$ is 1 so that the points lie on a line through the origin.
(b) Show that $[P_1, P_2, P_3, P_4]'$ is of rank 2 so that these points lie in a plane through the origin.
(c) Does $P_5(2, 3, 2, 5)$ lie in the plane of (b)?

22. Show that every $n$-square matrix $A$ over $F$ satisfies an equation of the form

$$A^p + k_1A^{p-1} + k_2A^{p-2} + \ldots + k_{p-1}A + k_pI = 0$$

where the $k_i$ are scalars of $F$.
*Hint*: Consider $I, A, A^2, A^3, \ldots, A^{n^2}$ in the light of Problem 16.

23. Find the equation of minimum degree (see Problem 22) which is satisfied by

$$(a)\quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad (b)\quad A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad (c)\quad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

*Ans.* (a) $A^2 - 2A = 0$, (b) $A^2 - 2A + 2I = 0$, (c) $A^2 - 2A + I = 0$

24. In Problem 23(b) and (c), multiply each equation by $A^{-1}$ to obtain (b) $A^{-1} = I - \frac{1}{2}A$, (c) $A^{-1} = 2I - A$, and thus verify: If $A$ over $F$ is non-singular, then $A^{-1}$ can be expressed as a polynomial in $A$ whose coefficients are scalars of $F$.

# Chapter 10

## Linear Equations

**DEFINITIONS.** Consider a system of $m$ linear equations in the $n$ unknowns $x_1, x_2, \dots, x_n$

$$(10.1) \qquad \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = h_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = h_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = h_m \end{cases}$$

in which the coefficients ($a$'s) and the constant terms ($h$'s) are in $F$.

By a solution in $F$ of the system is meant any set of values of $x_1, x_2, \dots, x_n$ in $F$ which satisfy simultaneously the $m$ equations. When the system has a solution, it is said to be **consistent**; otherwise, the system is said to be **inconsistent**. A consistent system has either just one solution or infinitely many solutions.

Two systems of linear equations over $F$ in the same number of unknowns are called **equivalent** if every solution of either system is a solution of the other. A system of equations equivalent to **(10.1)** may be obtained from it by applying one or more of the transformations: ($a$) interchanging any two of the equations, ($b$) multiplying any equation by any non-zero constant in $F$, or ($c$) adding to any equation a constant multiple of another equation. Solving a system of consistent equations consists in replacing the given system by an equivalent system of prescribed form.

**SOLUTION USING A MATRIX.** In matrix notation the system of linear equations **(10.1)** may be written as

$$(10.2) \qquad \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \cdot \\ h_m \end{bmatrix}$$

or, more compactly, as

$$(10.3) \qquad AX = H$$

where $A = [a_{ij}]$ is the coefficient matrix, $X = [x_1, x_2, \dots, x_n]'$, and $H = [h_1, h_2, \dots, h_m]'$.

Consider now for the system **(10.1)** the **augmented matrix**

$$(10.4) \qquad \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & h_1 \\ a_{21} & a_{22} & \dots & a_{2n} & h_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & h_m \end{bmatrix} = [A \quad H]$$

(Each row of **(10.4)** is simply an abbreviation of a corresponding equation of **(10.1)**; to read the equation from the row, we simply supply the unknowns and the + and = signs properly.)

To solve the system **(10.1)** by means of **(10.4)**, we proceed by elementary row transformations to replace $A$ by the row equivalent canonical matrix of Chapter 5. In doing this, we operate on the entire rows of **(10.4)**.

**Example 1.** Solve the system $\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ 3x_1 + x_2 - 2x_3 = 1 \\ 4x_1 - 3x_2 - x_3 = 3 \\ 2x_1 + 4x_2 + 2x_3 = 4 \end{cases}$

The augmented matrix $[A\ \ H] = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the solution is the equivalent system of equations: $x_1 = 1,\ x_2 = 0,\ x_3 = 1$. Expressed in vector form, we have $X = [1, 0, 1]'$.

**FUNDAMENTAL THEOREMS.** When the coefficient matrix $A$ of the system **(10.1)** is reduced to the row equivalent canonical form $C$, suppose $[A\ \ H]$ is reduced to $[C\ \ K]$, where $K = [k_1, k_2, \ldots, k_m]'$. If $A$ is of rank $r$, the first $r$ rows of $C$ contain one or more non-zero elements. The first non-zero element in each of these rows is 1 and the column in which that 1 stands has zeros elsewhere. The remaining rows consist of zeros. From the first $r$ rows of $[C\ \ K]$, we may obtain each of the variables $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$ (the notation is that of Chapter 5) in terms of the remaining variables $x_{j_{r+1}}, x_{j_{r+2}}, \ldots, x_{j_n}$ and one of the $k_1, k_2, \ldots, k_r$.

If $k_{r+1} = k_{r+2} = \ldots = k_m = 0$, then **(10.1)** is consistent and an arbitrarily selected set of values for $x_{j_{r+1}}, x_{j_{r+2}}, \ldots, x_{j_n}$ together with the resulting values of $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$ constitute a solution. On the other hand, if at least one of $k_{r+1}, k_{r+2}, \ldots, k_m$ is different from zero, say $k_t \neq 0$, the corresponding equation reads

$$0x_1 + 0x_2 + \ldots + 0x_n = k_t \neq 0$$

and **(10.1)** is inconsistent.

In the consistent case, $A$ and $[A\ \ H]$ have the same rank; in the inconsistent case, they have different ranks. Thus

**I.** A system $AX = H$ of $m$ linear equations in $n$ unknowns is consistent if and only if the coefficient matrix and the augmented matrix of the system have the same rank.

**II.** In a consistent system **(10.1)** of rank $r<n$, $n-r$ of the unknowns may be chosen so that the coefficient matrix of the remaining $r$ unknowns is of rank $r$. When these $n-r$ unknowns are assigned any values whatever, the other $r$ unknowns are uniquely determined.

**Example 2.** For the system $\begin{cases} x_1 + 2x_2 - 3x_3 - 4x_4 = 6 \\ x_1 + 3x_2 + x_3 - 2x_4 = 4 \\ 2x_1 + 5x_2 - 2x_3 - 5x_4 = 10 \end{cases}$

$$[A \quad H] = \left[\begin{array}{cccc|c} 1 & 2 & -3 & -4 & 6 \\ 1 & 3 & 1 & -2 & 4 \\ 2 & 5 & -2 & -5 & 10 \end{array}\right] \sim \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 1 & 4 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -11 & -8 & 10 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -11 & 0 & 10 \\ 0 & 1 & 4 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] = [C \quad K]$$

Since $A$ and $[A \quad H]$ are each of rank $r = 3$, the given system is consistent; moreover, the general solution contains $n-r = 4-3 = 1$ arbitrary constants. From the last row of $[C \quad K]$, $x_4 = 0$. Let $x_3 = a$, where $a$ is arbitrary; then $x_1 = 10+11a$ and $x_2 = -2-4a$. The solution of the system is given by $x_1 = 10+11a$, $x_2 = -2-4a$, $x_3 = a$, $x_4 = 0$ or $X = [10+11a, \ -2-4a, \ a, \ 0]'$.

If a consistent system of equations over $F$ has a unique solution (Example 1) that solution is over $F$. If the system has infinitely many solutions (Example 2) it has infinitely many solutions over $F$ when the arbitrary values to be assigned are over $F$. However, the system has infinitely many solutions over any field $\mathfrak{F}$ of which $F$ is a subfield. For example, the system of Example 2 has infinitely many solutions over $F$ (the rational field) if $a$ is restricted to rational numbers, it has infinitely many real solutions if $a$ 's restricted to real numbers, it has infinitely many complex solutions if $a$ is any complex number whatever.

See Problems 1-2.

**NON-HOMOGENEOUS EQUATIONS.** A linear equation

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = h$$

is called **non-homogeneous** if $h \neq 0$. A system $AX = H$ is called a system of non-homogeneous equations provided $H$ is not a zero vector. The systems of Examples 1 and 2 are non-homogeneous systems.

In Problem 3 we prove

**III.** A system of $n$ non-homogeneous equations in $n$ unknowns has a unique solution provided the rank of its coefficient matrix $A$ is $n$, that is, provided $|A| \neq 0$.

In addition to the method above, two additional procedures for solving a consistent system of $n$ non-homogeneous equations in as many unknowns $AX = H$ are given below. The first of these is the familiar solution by determinants.

(*a*) Solution by Cramer's Rule. Denote by $A_i$, $(i = 1, 2, \ldots, n)$ the matrix obtained from $A$ by replacing its $i$th column with the column of constants (the $h$'s). Then, if $|A| \neq 0$, the system $AX = H$ has the unique solution

**(10.5)** $$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad \ldots, \quad x_n = \frac{|A_n|}{|A|}$$

See Problem 4.

**Example 3.** Solve the system $\begin{cases} 2x_1 + x_2 + 5x_3 + x_4 = 5 \\ x_1 + x_2 - 3x_3 - 4x_4 = -1 \\ 3x_1 + 6x_2 - 2x_3 + x_4 = 8 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 = 2 \end{cases}$ using Cramer's Rule.

We find

$$|A| = \begin{bmatrix} 2 & 1 & 5 & 1 \\ 1 & 1 & -3 & -4 \\ 3 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{bmatrix} = -120, \qquad |A_1| = \begin{bmatrix} 5 & 1 & 5 & 1 \\ -1 & 1 & -3 & -4 \\ 8 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{bmatrix} = -240$$

$$|A_2| = \begin{vmatrix} 2 & 5 & 5 & 1 \\ 1 & -1 & -3 & -4 \\ 3 & 8 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = -24, \qquad |A_3| = \begin{vmatrix} 2 & 1 & 5 & 1 \\ 1 & 1 & -1 & -4 \\ 3 & 6 & 8 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = 0$$

and

$$|A_4| = \begin{vmatrix} 2 & 1 & 5 & 5 \\ 1 & 1 & -3 & -1 \\ 3 & 6 & -2 & 8 \\ 2 & 2 & 2 & 2 \end{vmatrix} = -96$$

Then $x_1 = \dfrac{|A_1|}{|A|} = \dfrac{-240}{-120} = 2$, $x_2 = \dfrac{|A_2|}{|A|} = \dfrac{-24}{-120} = \dfrac{1}{5}$, $x_3 = \dfrac{|A_3|}{|A|} = \dfrac{0}{-120} = 0$, and

$$x_4 = \frac{|A_4|}{|A|} = \frac{-96}{-120} = \frac{4}{5}.$$

(*b*) Solution using $A^{-1}$. If $|A| \neq 0$, $A^{-1}$ exists and the solution of the system $AX = H$ is given by

**(10.6)** $$A^{-1}\cdot AX = A^{-1}H \quad \text{or} \quad X = A^{-1}H$$

**Example 4.** The coefficient matrix of the system $\begin{cases} 2x_1 + 3x_2 + x_3 = 9 \\ x_1 + 2x_2 + 3x_3 = 6 \\ 3x_1 + x_2 + 2x_3 = 8 \end{cases}$ is $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$.

From Problem 2(*b*), Chapter 7, $A^{-1} = \dfrac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$. Then

$$A^{-1}\cdot AX = X = A^{-1}H = \frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 35 \\ 29 \\ 5 \end{bmatrix}$$

The solution of the system is $x_1 = 35/18$, $x_2 = 29/18$, $x_3 = 5/18$.

See Problem 5.

**HOMOGENEOUS EQUATIONS.** A linear equation

**(10.7)** $$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$$

is called **homogeneous.** A system of linear equations

**(10.8)** $$AX = 0$$

in $n$ unknowns is called a **system of homogeneous equations**. For the system **(10.8)** the rank of the coefficient matrix $A$ and the augmented matrix $[A\ 0]$ are the same; thus, the system is always consistent. Note that $X = 0$, that is, $x_1 = x_2 = \ldots = x_n = 0$ is always a solution; it is called **the trivial solution.**

If the rank of $A$ is $n$, then $n$ of the equations of **(10.8)** can be solved by Cramer's rule for the unique solution $x_1 = x_2 = \ldots = x_n = 0$ and the system has only the trivial solution. If the rank of $A$ is $r < n$, Theorem **II** assures the existence of non-trivial solutions. Thus,

**IV.** A necessary and sufficient condition for **(10.8)** to have a solution other than the trivial solution is that the rank of $A$ be $r < n$.

**V.** A necessary and sufficient condition that a system of $n$ homogeneous equations in $n$ unknowns has a solution other than the trivial solution is $|A| = 0$.

**VI.** If the rank of **(10.8)** is $r < n$, the system has exactly $n$-$r$ linearly independent solutions such that every solution is a linear combination of these $n$-$r$ and every such linear combination is a solution.

See Problem 6.

LET $X_1$ and $X_2$ be two distinct solutions of $AX = H$. Then $AX_1 = H$, $AX_2 = H$, and $A(X_1 - X_2) = AY = 0$. Thus, $Y = X_1 - X_2$ is a non-trivial solution of $AX = 0$.

Conversely, if $Z$ is any non-trivial solution of $AX = 0$ and if $X_p$ is any solution of $AX = H$, then $X = X_p + Z$ is also a solution of $AX = H$. As $Z$ ranges over the complete solution of $AX = 0$, $X_p + Z$ ranges over the complete solution of $AX = H$. Thus,

**VII.** If the system of non-homogeneous equations $AX = H$ is consistent, a complete solution of the system is given by the complete solution of $AX = 0$ plus any particular solution of $AX = H$.

**Example 5.** In the system $\begin{cases} x_1 - 2x_2 + 3x_3 = 4 \\ x_1 + x_2 + 2x_3 = 5 \end{cases}$ set $x_1 = 0$; then $x_3 = 2$ and $x_2 = 1$. A particular solution is $X_p = [0, 1, 2]'$. The complete solution of $\begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ x_1 + x_2 + 2x_3 = 0 \end{cases}$ is $[-7a, a, 3a]'$, where $a$ is arbitrary. Then the complete solution of the given system is

$$X = [-7a, a, 3a]' + [0, 1, 2]' = [-7a,\ 1+a,\ 2+3a]'$$

Note. The above procedure may be extended to larger systems. However, it is first necessary to show that the system is consistent. This is a long step in solving the system by the augmented matrix method given earlier.

## SOLVED PROBLEMS

1. Solve $\begin{cases} x_1 + x_2 - 2x_3 + x_4 + 3x_5 = 1 \\ 2x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 = 2 \\ 3x_1 + 2x_2 - 4x_3 - 3x_4 - 9x_5 = 3 \end{cases}$.

**Solution:**

The augmented matrix

$$[A\ H] = \left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 1 & 3 & 1 \\ 2 & -1 & 2 & 2 & 6 & 2 \\ 3 & 2 & -4 & -3 & -9 & 3 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 1 & 3 & 1 \\ 0 & -3 & 6 & 0 & 0 & 0 \\ 0 & -1 & 2 & -6 & -18 & 0 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 1 & 3 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & -1 & 2 & -6 & -18 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrrr|r} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & -18 & 0 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrrr|r} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right]$$

Then $x_1 = 1$, $x_2 - 2x_3 = 0$, and $x_4 + 3x_5 = 0$. Take $x_3 = a$ and $x_5 = b$, where $a$ and $b$ are arbitrary; the complete solution may be given as $x_1 = 1$, $x_2 = 2a$, $x_3 = a$, $x_4 = -3b$, $x_5 = b$ or as $X = [1, 2a, a, -3b, b]'$.

2. Solve $\begin{cases} x_1 + x_2 + 2x_3 + x_4 = 5 \\ 2x_1 + 3x_2 - x_3 - 2x_4 = 2 \\ 4x_1 + 5x_2 + 3x_3 = 7 \end{cases}$.

**Solution:**

$$[A\ H] = \left[\begin{array}{rrrrr} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{array}\right] \sim \left[\begin{array}{rrrrr} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 1 & -5 & -4 & -13 \end{array}\right] \sim \left[\begin{array}{rrrrr} 1 & 0 & 7 & 5 & 13 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right].$$

The last row reads $0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + 0\cdot x_4 = -5$; thus the given system is inconsistent and has no solution.

3. Prove: A system $AX = H$ of $n$ non-homogeneous equations in $n$ unknowns has a unique solution provided $|A| \neq 0$.

If $A$ is non-singular, it is equivalent to $I$. When $A$ is reduced by row transformations only to $I$, suppose $[A\ H]$ is reduced to $[I\ K]$. Then $X = K$ is a solution of the system.

Suppose next that $X = L$ is a second solution of the system; then $AK = H$ and $AL = H$, and $AK = AL$. Since $A$ is non-singular, $K = L$, and the solution is unique.

4. Derive Cramer's Rule.

Let the system of non-homogeneous equations be

$$(1)\qquad \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = h_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = h_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = h_n \end{cases}$$

Denote by $A$ the coefficient matrix $[a_{ij}]$ and let $\alpha_{ij}$ be the cofactor of $a_{ij}$ in $A$. Multiply the first equation of (1) by $\alpha_{11}$, the second equation by $\alpha_{21}$, ...., the last equation by $\alpha_{n1}$, and add. We have

$$\sum_{i=1}^{n} a_{i1}\alpha_{i1}x_1 + \sum_{i=1}^{n} a_{i2}\alpha_{i1}x_2 + \dots + \sum_{i=1}^{n} a_{in}\alpha_{i1}x_n = \sum_{i=1}^{n} h_i\alpha_i$$

which by Theorems **X** and **XI**, and Problem 10, Chapter-3, reduces to

$$|A|\cdot x_1 = \begin{vmatrix} h_1 & a_{12} & \dots & a_{1n} \\ h_2 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ h_n & a_{n2} & \dots & a_{nn} \end{vmatrix} = |A_1| \quad \text{so that} \quad x_1 = \frac{|A_1|}{|A|}$$

Next, multiply the equations of (1) respectively by $\alpha_{12}, \alpha_{22}, \dots, \alpha_{n2}$ and sum to obtain

$$|A|\cdot x_2 = \begin{vmatrix} a_{11} & h_1 & a_{13} & \dots & a_{1n} \\ a_{21} & h_2 & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & h_n & a_{n3} & \dots & a_{nn} \end{vmatrix} = |A_2| \quad \text{so that} \quad x_2 = \frac{|A_2|}{|A|}$$

Continuing in this manner, we finally multiply the equations of (1) respectively by $\alpha_{1n}, \alpha_{2n}, \dots, \alpha_{nn}$ and sum to obtain

$$|A|\cdot x_n = \begin{vmatrix} a_{11} & \dots & a_{1,n-1} & h_1 \\ a_{21} & \dots & a_{2,n-1} & h_2 \\ \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{n,n-1} & h_n \end{vmatrix} = |A_n| \quad \text{so that} \quad x_n = \frac{|A_n|}{|A|}$$

5. Solve the system $\begin{cases} 2x_1 + x_2 + 5x_3 + x_4 = 5 \\ x_1 + x_2 - 3x_3 - 4x_4 = -1 \\ 3x_1 + 6x_2 - 2x_3 + x_4 = 8 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 = 2 \end{cases}$ using the inverse of the coefficient matrix.

**Solution:**

The inverse of $A = \begin{bmatrix} 2 & 1 & 5 & 1 \\ 1 & 1 & -3 & -4 \\ 3 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{bmatrix}$ is $\frac{1}{120}\begin{bmatrix} 120 & 120 & 0 & -120 \\ -69 & -73 & 17 & 80 \\ -15 & -35 & -5 & 40 \\ 24 & 8 & 8 & -40 \end{bmatrix}$. Then

$$X = \frac{1}{120}\begin{bmatrix} 120 & 120 & 0 & -120 \\ -69 & -73 & 17 & 80 \\ -15 & -35 & -5 & 40 \\ 24 & 8 & 8 & -40 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1/5 \\ 0 \\ 4/5 \end{bmatrix}$$

(See Example 3.)

6. Solve $\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + 3x_2 + 2x_3 + 4x_4 = 0 \\ 2x_1 + x_3 - x_4 = 0 \end{cases}$

**Solution:**

$$[A\ \ H] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 3 & 2 & 4 & 0 \\ 2 & 0 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & -2 & -1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The complete solution of the system is $x_1 = -\frac{1}{2}a + \frac{1}{2}b$, $x_2 = -\frac{1}{2}a - \frac{3}{2}b$, $x_3 = a$, $x_4 = b$. Since the rank of $A$ is 2, we may obtain exactly $n - r = 4 - 2 = 2$ linearly independent solutions. One such pair, obtained by first taking $a = 1$, $b = 1$ and then $a = 3$, $b = 1$ is

$$x_1 = 0,\ x_2 = -2,\ x_3 = 1,\ x_4 = 1 \quad \text{and} \quad x_1 = -1,\ x_2 = -3,\ x_3 = 3,\ x_4 = 1$$

What can be said of the pair of solutions obtained by taking $a = b = 1$ and $a = b = 3$?

7. Prove: In a square matrix $A$ of order $n$ and rank $n$-1, the cofactors of the elements of any two rows (columns) are proportional.

Since $|A| = 0$, the cofactors of the elements of any row (column) of $A$ are a solution $X_1$ of the system $AX = 0$ $(A'X = 0)$.

Now the system has but one linearly independent solution since $A$ $(A')$ is of rank $n$-1. Hence, for the cofactors of another row (column) of $A$ (another solution $X_2$ of the system), we have $X_2 = kX_1$.

8. Prove: If $f_1, f_2, \ldots, f_m$ are $m < n$ linearly independent linear forms over $F$ in $n$ variables, then the $p$ linear forms

$$g_j = \sum_{i=1}^{m} s_{ij} f_i, \qquad (j = 1, 2, \ldots, p)$$

are linearly dependent if and only if the $m \times p$ matrix $[s_{ij}]$ is of rank $r < p$.

The $g$'s are linearly dependent if and only if there exist scalars $a_1, a_2, \ldots, a_p$ in $F$, not all zero, such that

$$\begin{aligned} a_1 g_1 + a_2 g_2 + \ldots + a_p g_p &= a_1 \sum_{i=1}^{m} s_{i1} f_i + a_2 \sum_{i=1}^{m} s_{i2} f_i + \ldots + a_p \sum_{i=1}^{m} s_{ip} f_i \\ &= (\sum_{j=1}^{p} a_j s_{1j}) f_1 + (\sum_{j=1}^{p} a_j s_{2j}) f_2 + \ldots + (\sum_{j=1}^{p} a_j s_{mj}) f_m \\ &= \sum_{i=1}^{m} (\sum_{j=1}^{p} a_j s_{ij}) f_i = 0 \end{aligned}$$

Since the $f$'s are linearly independent, this requires

$$\sum_{j=1}^{p} a_j s_{ij} = a_1 s_{i1} + a_2 s_{i2} + \dots + a_p s_{ip} = 0 \qquad (i = 1, 2, \dots, m)$$

Now, by Theorem **IV**, the system of $m$ homogeneous equations in $p$ unknowns $\sum_{j=1}^{p} s_{ij} x_j = 0$ has a non-trivial solution $X = [a_1, a_2, \dots, a_p]'$ if and only if $[s_{ij}]$ is of rank $r < p$.

**9.** Suppose $A = [a_{ij}]$ of order $n$ is singular. Show that there always exists a matrix $B = [b_{ij}] \neq 0$ of order $n$ such that $AB = 0$.

Let $B_1, B_2, \dots, B_n$ be the column vectors of $B$. Then, by hypothesis, $AB_1 = AB_2 = \dots = AB_n = 0$. Consider any one of these, say $AB_t = 0$, or

$$\begin{cases} a_{11}b_{1t} + a_{12}b_{2t} + \dots + a_{1n}b_{nt} = 0 \\ a_{21}b_{1t} + a_{22}b_{2t} + \dots + a_{2n}b_{nt} = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}b_{1t} + a_{n2}b_{2t} + \dots + a_{nn}b_{nt} = 0 \end{cases}$$

Since the coefficient matrix $A$ is singular, the system in the unknowns $b_{1t}, b_{2t}, \dots, b_{nt}$ has solutions other than the trivial solution. Similarly, $AB_1 = 0$, $AB_2 = 0$, ... have solutions, each being a column of $B$.

## SUPPLEMENTARY PROBLEMS

**10.** Find all solutions of:

(a) $x_1 - 2x_2 + x_3 - 3x_4 = 1$

(b) $\begin{cases} x_1 + x_2 + x_3 = 4 \\ 2x_1 + 5x_2 - 2x_3 = 3 \end{cases}$

(c) $\begin{cases} x_1 + x_2 + x_3 = 4 \\ 2x_1 + 5x_2 - 2x_3 = 3 \\ x_1 + 7x_2 - 7x_3 = 5 \end{cases}$

(d) $\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 - x_4 = 4 \\ x_1 + x_2 - x_3 + x_4 = -4 \\ x_1 - x_2 + x_3 + x_4 = 2 \end{cases}$

*Ans.* (a) $x_1 = 1 + 2a - b + 3c$, $x_2 = a$, $x_3 = b$, $x_4 = c$

(b) $x_1 = -7a/3 + 17/3$, $x_2 = 4a/3 - 5/3$, $x_3 = a$

(d) $x_1 = -x_2 = 1$, $x_3 = -x_4 = 2$

**11.** Find all non-trivial solutions of:

(a) $\begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ 2x_1 + 5x_2 + 6x_3 = 0 \end{cases}$

(b) $\begin{cases} 2x_1 - x_2 + 3x_3 = 0 \\ 3x_1 + 2x_2 + x_3 = 0 \\ x_1 - 4x_2 + 5x_3 = 0 \end{cases}$

(c) $\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + x_2 + 3x_3 = 0 \\ 3x_1 + 2x_2 + x_3 = 0 \end{cases}$

(d) $\begin{cases} 4x_1 - x_2 + 2x_3 + x_4 = 0 \\ 2x_1 + 3x_2 - x_3 - 2x_4 = 0 \\ 7x_2 - 4x_3 - 5x_4 = 0 \\ 2x_1 - 11x_2 + 7x_3 + 8x_4 = 0 \end{cases}$

*Ans.* (a) $x_1 = -3a$, $x_2 = 0$, $x_3 = a$

(b) $x_1 = -x_2 = -x_3 = a$

(d) $x_1 = -\frac{5}{8}a + \frac{3}{8}b$, $x_2 = a$, $x_3 = \frac{7}{4}a - \frac{5}{4}b$, $x_4 = b$

12. Reconcile the solution of 10(*d*) with another $x_1 = c,\ x_2 = d,\ x_3 = -\frac{10}{3}c - \frac{d}{3},\ x_4 = \frac{8}{3}c + \frac{5}{3}d.$

13. Given $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{bmatrix}$, find a matrix $B$ of rank 2 such that $AB = 0$. Hint. Select the columns of $B$ from the solutions of $AX = 0$.

14. Show that a square matrix is singular if and only if its rows (columns) are linearly dependent.

15. Let $AX = 0$ be a system of $n$ homogeneous equations in $n$ unknowns and suppose $A$ of rank $r = n-1$. Show that any non-zero vector of cofactors $[\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}]'$ of a row of $A$ is a solution of $AX = 0$.

16. Use Problem 15 to solve:

$$(a)\ \begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ 2x_1 + 5x_2 + 6x_3 = 0 \end{cases}, \qquad (b)\ \begin{cases} 2x_1 + 3x_2 - x_3 = 0 \\ 3x_1 - 4x_2 + 2x_3 = 0 \end{cases} \qquad (c)\ \begin{cases} 2x_1 + 3x_2 + 4x_3 = 0 \\ 2x_1 - x_2 + 6x_3 = 0 \end{cases}$$

Hint. To the equations of (*a*) adjoin $0x_1 + 0x_2 + 0x_3 = 0$ and find the cofactors of the elements of the third row of $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$.

*Ans.* (*a*) $x_1 = -27a,\ x_2 = 0,\ x_3 = 9a$ or $[3a, 0, -a]'$, (*b*) $[2a, -7a, -17a]'$, (*c*) $[11a, -2a, -4a]'$

17. Let the coefficient and the augmented matrix of the system of 3 non-homogeneous equations in 5 unknowns $AX = H$ be of rank 2 and assume the canonical form of the augmented matrix to be

$$\begin{bmatrix} 1 & 0 & b_{13} & b_{14} & b_{15} & c_1 \\ 0 & 1 & b_{23} & b_{24} & b_{25} & c_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with not both of $c_1, c_2$ equal to 0. First choose $x_3 = x_4 = x_5 = 0$ and obtain $X_1 = [c_1, c_2, 0, 0, 0]'$ as a solution of $AX = H$. Then choose $x_3 = 1,\ x_4 = x_5 = 0$, also $x_3 = x_5 = 0,\ x_4 = 1$ and $x_3 = x_4 = 0,\ x_5 = 1$ to obtain other solutions $X_2, X_3$, and $X_4$. Show that these $5-2+1 = 4$ solutions are linearly independent.

18. Consider the linear combination $Y = s_1X_1 + s_2X_2 + s_3X_3 + s_4X_4$ of the solutions of Problem 17. Show that $Y$ is a solution of $AX = H$ if and only if (*i*) $s_1 + s_2 + s_3 + s_4 = 1$. Thus, with $s_1, s_2, s_3, s_4$ arbitrary except for (*i*), $Y$ is a complete solution of $AX = H$.

19. Prove: Theorem **VI**. Hint. Follow Problem 17 with $c_1 = c_2 = 0$.

20. Prove: If $A$ is an $m \times p$ matrix of rank $r_1$ and $B$ is a $p \times n$ matrix of rank $r_2$ such that $AB = 0$, then $r_1 + r_2 \leq p$. Hint. Use Theorem **VI**.

21. Using the $4 \times 5$ matrix $A = [a_{ij}]$ of rank 2, verify: In an $m \times n$ matrix $A$ of rank $r$, the $r$-square determinants formed from the columns of a submatrix consisting of any $r$ rows of $A$ are proportional to the $r$-square determinants formed from any other submatrix consisting of $r$ rows of $A$.
Hint. Suppose the first two rows are linearly independent so that $a_{3j} = p_{31}a_{1j} + p_{32}a_{2j},\ a_{4j} = p_{41}a_{1j} + p_{42}a_{2j}$, $(j = 1, 2, \ldots, 5)$. Evaluate the 2-square determinants

$$\begin{vmatrix} a_{1q} & a_{1s} \\ a_{2q} & a_{2s} \end{vmatrix}, \qquad \begin{vmatrix} a_{1q} & a_{1s} \\ a_{3q} & a_{3s} \end{vmatrix}, \qquad \text{and} \qquad \begin{vmatrix} a_{3q} & a_{3s} \\ a_{4q} & a_{4s} \end{vmatrix}$$

22. Write a proof of the theorem of Problem 21.

23. From Problem 7, obtain: If the $n$-square matrix $A$ is of rank $n$-1, then the following relations among its cofactors hold

$$(a)\ \alpha_{ij}\alpha_{hk} = \alpha_{ik}\alpha_{hj}, \qquad (b)\ \alpha_{ii}\alpha_{jj} = \alpha_{ij}\alpha_{ji}$$

where $(h, i, j, k = 1, 2, \ldots, n)$.

**24.** Show that $B = \begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & -4 & 2 \\ 2 & 1 & 1 & 2 & 6 \\ 3 & 2 & -1 & -1 & 3 \\ 1 & 2 & 2 & -2 & 4 \\ 2 & 3 & -3 & 1 & 1 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. From $B = [A \quad H]$ infer that the system of 6 linear equations in 4 unknowns has 5 linearly independent equations. Show that a system of $m > n$ linear equations in $n$ unknowns can have at most $n+1$ linearly independent equations. Show that when there are $n+1$, the system is inconsistent.

**25.** If $AX = H$ is consistent and of rank $r$, for what set of $r$ variables can one solve?

**26.** Generalize the results of Problems 17 and 18 to $m$ non-homogeneous equations in $n$ unknowns with coefficient and augmented matrix of the same rank $r$ to prove: If the coefficient and the augmented matrix of the system $AX = H$ of $m$ non-homogeneous equations in $n$ unknowns have rank $r$ and if $X_1, X_2, \ldots, X_{n-r+1}$ are linearly independent solutions of the system, then

$$X = s_1 X_1 + s_1 X_2 + \ldots + s_{n-r+1} X_{n-r+1}$$

where $\sum_{i=1}^{n-r+1} s_i = 1$, is a complete solution.

**27.** In a four-pole electrical network, the input quantities $E_1$ and $I_1$ are given in terms of the output quantities $E_2$ and $I_2$ by

$$\begin{aligned} E_1 &= aE_2 + bI_2 \\ I_1 &= cE_2 + dI_2 \end{aligned} \quad \text{or} \quad \begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} E_2 \\ I_2 \end{bmatrix} = A \begin{bmatrix} E_2 \\ I_2 \end{bmatrix}$$

Show that $\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \frac{1}{c} \begin{bmatrix} a & -|A| \\ 1 & -d \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$ and $\begin{bmatrix} E_1 \\ I_2 \end{bmatrix} = \frac{1}{d} \begin{bmatrix} b & |A| \\ 1 & -c \end{bmatrix} \begin{bmatrix} I_1 \\ E_2 \end{bmatrix}$.

Solve also for $E_2$ and $I_2$, $I_1$ and $I_2$, $I_1$ and $E_2$.

**28.** Let the system of $n$ linear equations in $n$ unknowns $AX = H$, $H \neq 0$, have a unique solution. Show that the system $AX = K$ has a unique solution for any $n$-vector $K \neq 0$.

**29.** Solve the set of linear forms $AX = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ for the $x_i$ as linear forms in the $y$'s.

Now write down the solution of $A'X = Y$.

**30.** Let $A$ be $n$-square and non-singular, and let $S_i$ be the solution of $AX = E_i$, $(i = 1, 2, \ldots, n)$, where $E_i$ is the $n$-vector whose $i$th component is 1 and whose other components are 0. Identify the matrix $[S_1, S_2, \ldots, S_n]$.

**31.** Let $A$ be an $m \times n$ matrix with $m < n$ and let $S_i$ be a solution of $AX = E_i$, $(i = 1, 2, \ldots, m)$, where $E_i$ is the $m$-vector whose $i$th component is 1 and whose other components are 0. If $K = [k_1, k_2, \ldots, k_m]'$, show that

$$k_1 S_1 + k_2 S_2 + \ldots + k_m S_m$$

is a solution of $AX = K$.

# Chapter 11

## Vector Spaces

**UNLESS STATED OTHERWISE**, all vectors will now be column vectors. When components are displayed, we shall write $[x_1, x_2, \ldots, x_n]'$. The transpose mark ($'$) indicates that the elements are to be written in a column.

A set of such $n$-vectors over $F$ is said to be **closed under addition** if the sum of any two of them is a vector of the set. Similarly, the set is said to be **closed under scalar multiplication** if every scalar multiple of a vector of the set is a vector of the set.

**Example 1.** (*a*) The set of all vectors $[x_1, x_2, x_3]'$ of ordinary space having equal components ($x_1 = x_2 = x_3$) is closed under both addition and scalar multiplication. For, the sum of any two of the vectors and $k$ times any vector ($k$ real) are again vectors having equal components.

(*b*) The set of all vectors $[x_1, x_2, x_3]'$ of ordinary space is closed under addition and scalar multiplication.

**VECTOR SPACES.** Any set of $n$-vectors over $F$ which is closed under both addition and scalar multiplication is called a **vector space**. Thus, if $X_1, X_2, \ldots, X_m$ are $n$-vectors over $F$, the set of all linear combinations

**(11.1)** $$k_1X_1 + k_2X_2 + \cdots + k_mX_m \qquad (k_i \text{ in } F)$$

is a vector space over $F$. For example, both of the sets of vectors (*a*) and (*b*) of Example 1 are vector spaces. Clearly, every vector space (**11.1**) contains the zero $n$-vector while the zero $n$-vector alone is a vector space. (The space (**11.1**) is also called a **linear vector space.**)

The totality $V_n(F)$ of **all** $n$-vectors over $F$ is called the **$n$-dimensional vector space** over $F$.

**SUBSPACES.** A set $V$ of the vectors of $V_n(F)$ is called a **subspace** of $V_n(F)$ provided $V$ is closed under addition and scalar multiplication. Thus, the zero $n$-vector is a subspace of $V_n(F)$; so also is $V_n(F)$ itself. The set (*a*) of Example 1 is a subspace (a line) of ordinary space. In general, if $X_1, X_2, \ldots, X_m$ belong to $V_n(F)$, the space of all linear combinations (**11.1**) is a subspace of $V_n(F)$.

A vector space $V$ is said to be **spanned** or generated by the $n$-vectors $X_1, X_2, \ldots, X_m$ provided (*a*) the $X_i$ lie in $V$ and (*b*) every vector of $V$ is a linear combination (**11.1**). Note that the vectors $X_1, X_2, \ldots, X_m$ are not restricted to be linearly independent.

**Example 2.** Let $F$ be the field $R$ of real numbers so that the 3-vectors $X_1 = [1, 1, 1]'$, $X_2 = [1, 2, 3]'$, $X_3 = [1, 3, 2]'$, and $X_4 = [3, 2, 1]'$ lie in ordinary space $S = V_3(R)$. Any vector $[a, b, c]'$ of $S$ can be expressed as

$$y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4 = \begin{bmatrix} y_1 + y_2 + y_3 + 3y_4 \\ y_1 + 2y_2 + 3y_3 + 2y_4 \\ y_1 + 3y_2 + 2y_3 + y_4 \end{bmatrix}$$

since the resulting system of equations

$$\text{(i)} \qquad \begin{aligned} y_1 + y_2 + y_3 + 3y_4 &= a \\ y_1 + 2y_2 + 3y_3 + 2y_4 &= b \\ y_1 + 3y_2 + 2y_3 + y_4 &= c \end{aligned}$$

is consistent. Thus, the vectors $X_1, X_2, X_3, X_4$ span $S$.

The vectors $X_1$ and $X_2$ are linearly independent. They span a subspace (the plane $\pi$) of $S$ which contains every vector $hX_1 + kX_2$, where $h$ and $k$ are real numbers.

The vector $X_4$ spans a subspace (the line $L$) of $S$ which contains every vector $hX_4$, where $h$ is a real number.

See Problem 1.

**BASIS AND DIMENSION.** By the dimension of a vector space $V$ is meant the maximum number of linearly independent vectors in $V$ or, what is the same thing, the minimum number of linearly independent vectors required to span $V$. In elementary geometry, ordinary space is considered as a 3-space (space of dimension three) of points $(a, b, c)$. Here we have been considering it as a 3-space of vectors $[a, b, c]'$. The plane $\pi$ of Example 2 is of dimension 2 and the line $L$ is of dimension 1.

A vector space of dimension $r$ consisting of $n$-vectors will be denoted by $V_n^r(F)$. When $r = n$, we shall agree to write $V_n(F)$ for $V_n^n(F)$.

A set of $r$ linearly independent vectors of $V_n^r(F)$ is called a **basis** of the space. Each vector of the space is then a **unique** linear combination of the vectors of this basis. All bases of $V_n^r(F)$ have exactly the same number of vectors but any $r$ linearly independent vectors of the space will serve as a basis.

**Example 3.** The vectors $X_1, X_2, X_3$ of Example 2 span $S$ since any vector $[a, b, c]'$ of $S$ can be expressed as

$$y_1X_1 + y_2X_2 + y_3X_3 = \begin{bmatrix} y_1 + y_2 + y_3 \\ y_1 + 2y_2 + 3y_3 \\ y_1 + 3y_2 + 2y_3 \end{bmatrix}$$

The resulting system of equations $\begin{cases} y_1 + y_2 + y_3 = a \\ y_1 + 2y_2 + 3y_3 = b \\ y_1 + 3y_2 + 2y_3 = c \end{cases}$, unlike the system (i), has a unique solution. The vectors $X_1, X_2, X_3$ are a basis of $S$. The vectors $X_1, X_2, X_4$ are not a basis of $S$. (Show this.) They span the subspace $\pi$ of Example 2, whose basis is the set $X_1, X_2$.

Theorems **I-V** of Chapter 9 apply here, of course. In particular, Theorem **IV** may be restated as:

**I.** If $X_1, X_2, \ldots, X_m$ are a set of $n$-vectors over $F$ and if $r$ is the rank of the $n \times m$ matrix of their components, then from the set $r$ linearly independent vectors may be selected. These $r$ vectors span a $V_n^r(F)$ in which the remaining $m-r$ vectors lie.

See Problems 2-3.

Of considerable importance are:

**II.** If $X_1, X_2, \ldots, X_m$ are $m<n$ linearly independent $n$-vectors of $V_n(F)$ and if $X_{m+1}, X_{m+2}, \ldots, X_n$ are any $n-m$ vectors of $V_n(F)$ which together with $X_1, X_2, \ldots, X_m$ form a linearly independent set, then the set $X_1, X_2, \ldots, X_n$ is a basis of $V_n(F)$.

See Problem 4.

**III.** If $X_1, X_2, \ldots, X_m$ are $m<n$ linearly independent $n$-vectors over $F$, then the $p$ vectors

$$Y_j = \sum_{i=1}^{m} s_{ij} X_i \qquad (j = 1, 2, \ldots, p)$$

are linearly dependent if $p>m$ or, when $p \leq m$, if $[s_{ij}]$ is of rank $r<p$.

**IV.** If $X_1, X_2, \ldots, X_n$ are linearly independent $n$-vectors over $F$, then the vectors

$$Y_i = \sum_{j=1}^{n} a_{ij} X_j \qquad (i = 1, 2, \ldots, n)$$

are linearly independent if and only if $[a_{ij}]$ is nonsingular.

**IDENTICAL SUBSPACES.** If ${}_1V_n^r(F)$ and ${}_2V_n^r(F)$ are two subspaces of $V_n(F)$, they are identical if and only if each vector of ${}_1V_n^r(F)$ is a vector of ${}_2V_n^r(F)$ and conversely, that is, if and only if each is a subspace of the other.

See Problem 5.

**SUM AND INTERSECTION OF TWO SPACES.** Let $V_n^h(F)$ and $V_n^k(F)$ be two vector spaces. By their **sum** is meant the totality of vectors $X+Y$ where $X$ is in $V_n^h(F)$ and $Y$ is in $V_n^k(F)$. Clearly, this is a vector space; we call it the **sum space** $V_n^s(F)$. The dimension $s$ of the sum space of two vector spaces does not exceed the sum of their dimensions.

By the **intersection** of the two vector spaces is meant the totality of vectors common to the two spaces. Now if $X$ is a vector common to the two spaces, so also is $aX$; likewise, if $X$ and $Y$ are common to the two spaces so also is $aX+bY$. Thus, the intersection of two spaces is a vector space; we call it the **intersection space** $V_n^t(F)$. The dimension of the intersection space of two vector spaces cannot exceed the smaller of the dimensions of the two spaces.

**V.** If two vector spaces $V_n^h(F)$ and $V_n^k(F)$ have $V_n^s(F)$ as sum space and $V_n^t(F)$ as intersection space, then $h+k = s+t$.

**Example 4.** Consider the subspace $\pi_1$ spanned by $X_1$ and $X_2$ of Example 2 and the subspace $\pi_2$ spanned by $X_3$ and $X_4$. Since $\pi_1$ and $\pi_2$ are not identical (prove this) and since the four vectors span $S$, the sum space of $\pi_1$ and $\pi_2$ is $S$.

Now $4X_1 - X_2 = X_4$; thus, $X_4$ lies in both $\pi_1$ and $\pi_2$. The subspace (line $L$) spanned by $X_4$ is then the intersection space of $\pi_1$ and $\pi_2$. Note that $\pi_1$ and $\pi_2$ are each of dimension 2, $S$ is of dimension 3, and $L$ is of dimension 1. This agrees with Theorem **V**.

See Problems 6-8.

**NULLITY OF A MATRIX.** For a system of homogeneous equations $AX = 0$, the solution vectors $X$ constitute a vector space called the **null space** of $A$. The dimension of this space, denoted by $N_A$, is called the **nullity** of $A$.

Restating Theorem **VI**, Chapter 10, we have

**VI.** If $A$ has nullity $N_A$, then $AX = 0$ has $N_A$ linearly independent solutions $X_1, X_2, \ldots,$

$X_{N_A}$ such that every solution of $AX = 0$ is a linear combination of them and every such linear combination is a solution.

A basis for the null space of $A$ is any set of $N_A$ linearly independent solutions of $AX = 0$.

See Problem 9.

**VII.** For an $m \times n$ matrix $A$ of rank $r_A$ and nullity $N_A$,

$$r_A + N_A = n \tag{11.2}$$

**SYLVESTER'S LAWS OF NULLITY.** If $A$ and $B$ are of order $n$ and respective ranks $r_A$ and $r_B$, the rank and nullity of their product $AB$ satisfy the inequalities

$$\begin{aligned} r_{AB} &\geq r_A + r_B - n \\ N_{AB} > N_A, &\quad N_{AB} > N_B \\ N_{AB} &\leq N_A + N_B \end{aligned} \tag{11.3}$$

See Problem 10.

**BASES AND COORDINATES.** The $n$-vectors

$$E_1 = [1, 0, 0, \ldots, 0]', \quad E_2 = [0, 1, 0, \ldots, 0]', \quad \ldots, \quad E_n = [0, 0, 0, \ldots, 1]'$$

are called **elementary** or **unit** vectors over $F$. The elementary vector $E_j$, whose $j$th component is 1, is called the $j$th elementary vector. The elementary vectors $E_1, E_2, \ldots, E_n$ constitute an important basis for $V_n(F)$.

Every vector $X = [x_1, x_2, \ldots, x_n]'$ of $V_n(F)$ can be expressed uniquely as the sum

$$X = \sum_{i=1}^{n} x_i E_i = x_1 E_1 + x_2 E_2 + \cdots + x_n E_n$$

of the elementary vectors. The components $x_1, x_2, \ldots, x_n$ of $X$ are now called the **coordinates** of $X$ relative to the $E$-basis. Hereafter, unless otherwise specified, we shall assume that a vector $X$ is given relative to this basis.

Let $Z_1, Z_2, \ldots, Z_n$ be another basis of $V_n(F)$. Then there exist unique scalars $a_1, a_2, \ldots, a_n$ in $F$ such that

$$X = \sum_{i=1}^{n} a_i Z_i = a_1 Z_1 + a_2 Z_2 + \cdots + a_n Z_n$$

These scalars $a_1, a_2, \ldots, a_n$ are called the coordinates of $X$ relative to the $Z$-basis. Writing $X_Z = [a_1, a_2, \ldots, a_n]'$, we have

$$X = [Z_1, Z_2, \ldots, Z_n]X_Z = Z \cdot X_Z \tag{11.4}$$

where $Z$ is the matrix whose columns are the basis vectors $Z_1, Z_2, \ldots, Z_n$.

**Example 5.** If $Z_1 = [2, -1, 3]'$, $Z_2 = [1, 2, -1]'$, $Z_3 = [1, -1, -1]'$ is a basis of $V_3(F)$ and $X_Z = [1, 2, 3]'$ is a vector of $V_3(F)$ relative to that basis, then

$$X = [Z_1, Z_2, Z_3]X_Z = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ -2 \end{bmatrix} = [7, 0, -2]'$$

relative to the $E$-basis.

See Problem 11.

$X_E = (7, 0, 2)'$
↑
coordinate of x in E-basis

Let $W_1, W_2, \ldots, W_n$ be yet another basis of $V_n(F)$. Suppose $X_W = [b_1, b_2, \ldots, b_n]'$ so that

(11.5) $$X = [W_1, W_2, \ldots, W_n]X_W = W \cdot X_W$$

From (11.4) and (11.5), $X = Z \cdot X_Z = W \cdot X_W$ and

(11.6) $$X_W = W^{-1} \cdot Z \cdot X_Z = PX_Z$$

where $P = W^{-1}Z$.

Thus,

**VIII.** If a vector of $V_n(F)$ has coordinates $X_Z$ and $X_W$ respectively relative to two bases of $V_n(F)$, then there exists a non-singular matrix $P$, determined solely by the two bases and given by (11.6) such that $X_W = PX_Z$.

See Problem 12.

## SOLVED PROBLEMS

1. The set of all vectors $X = [x_1, x_2, x_3, x_4]'$, where $x_1 + x_2 + x_3 + x_4 = 0$ is a subspace $V$ of $V_4(F)$ since the sum of any two vectors of the set and any scalar multiple of a vector of the set have components whose sum is zero, that is, are vectors of the set.

2. Since $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 2 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ is of rank 2, the vectors $X_1 = [1, 2, 2, 1]'$, $X_2 = [3, 4, 4, 3]'$, and $X_3 = [1, 0, 0, 1]'$ are linearly dependent and span a vector space $V_4^2(F)$.

Now any two of these vectors are linearly independent; hence, we may take $X_1$ and $X_2$, $X_1$ and $X_3$, or $X_2$ and $X_3$ as a basis of the $V_4^2(F)$.

3. Since $\begin{bmatrix} 1 & 4 & 2 & 4 \\ 1 & 3 & 1 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & -1 & -1 & -2 \end{bmatrix}$ is of rank 2, the vectors $X_1 = [1,1,1,0]'$, $X_2 = [4,3,2,-1]'$, $X_3 = [2,1,0,-1]'$, and $X_4 = [4,2,0,-2]'$ are linearly dependent and span a $V_4^2(F)$.

For a basis, we may take any two of the vectors except the pair $X_3, X_4$.

4. The vectors $X_1, X_2, X_3$ of Problem 2 lie in $V_4(F)$. Find a basis.

For a basis of this space we may take $X_1, X_2, X_4 = [1, 0, 0, 0]'$, and $X_5 = [0, 1, 0, 0]'$ or $X_1, X_2, X_6 = [1, 2, 3, 4]'$, and $X_7 = [1, 3, 6, 8]'$, .... since the matrices $[X_1, X_2, X_4, X_5]$ and $[X_1, X_2, X_6, X_7]$ are of rank 4.

5. Let $X_1 = [1,2,1]'$, $X_2 = [1,2,3]'$, $X_3 = [3,6,5]'$, $Y_1 = [0,0,1]'$, $Y_2 = [1,2,5]'$ be vectors of $V_3(F)$. Show that the space spanned by $X_1, X_2, X_3$ and the space spanned by $Y_1, Y_2$ are identical.

First, we note that $X_1$ and $X_2$ are linearly independent while $X_3 = 2X_1 + X_2$. Thus, the $X_i$ span a space of dimension two, say ${}_1V_3^2(F)$. Also, the $Y_i$ being linearly independent span a space of dimension two, say ${}_2V_3^2(F)$.

Next, $Y_1 = \frac{1}{2}X_2 - \frac{1}{2}X_1$, $Y_2 = 2X_2 - X_1$; $X_1 = Y_2 - 4Y_1$, $X_2 = Y_2 - 2Y_1$. Thus, any vector $aY_1 + bY_2$ of ${}_2V_3^2(F)$ is a vector $(\frac{1}{2}a+2b)X_2 - (\frac{1}{2}a+b)X_1$ of ${}_1V_3^2(F)$ and any vector $cX_1 + dX_2$ of ${}_1V_3^2(F)$ is a vector $(c+d)Y_2 - (4c+2d)Y_1$ of ${}_2V_3^2(F)$. Hence, the two spaces are identical.

6. (*a*) If $X = [x_1, x_2, x_3]'$ lies in the $V_3^2(F)$ spanned by $X_1 = [1,-1,1]'$ and $X_2 = [3,4,-2]'$, then

$$\begin{vmatrix} x_1 & 1 & 3 \\ x_2 & -1 & 4 \\ x_3 & 1 & -2 \end{vmatrix} = -2x_1 + 5x_2 + 7x_3 = 0.$$

(*b*) If $X = [x_1, x_2, x_3, x_4]'$ lies in the $V_4^2(F)$ spanned by $X_1 = [1,1,2,3]'$ and $X_2 = [1,0,-2,1]'$, then

$$\begin{bmatrix} x_1 & 1 & 1 \\ x_2 & 1 & 0 \\ x_3 & 2 & -2 \\ x_4 & 3 & 1 \end{bmatrix}$$ is of rank 2. Since $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$, this requires $\begin{vmatrix} x_1 & 1 & 1 \\ x_2 & 1 & 0 \\ x_3 & 2 & -2 \end{vmatrix} = -2x_1 + 4x_2 - x_3 = 0$ and

$$\begin{vmatrix} x_1 & 1 & 1 \\ x_2 & 1 & 0 \\ x_4 & 3 & 1 \end{vmatrix} = x_1 + 2x_2 - x_4 = 0.$$

These problems illustrate that every $V_n^k(F)$ may be defined as the totality of solutions over $F$ of a system of $n$-$k$ linearly independent homogeneous linear equations over $F$ in $n$ unknowns.

7. Prove: If two vector spaces $V_n^h(F)$ and $V_n^k(F)$ have $V_n^s(F)$ as sum space and $V_n^t(F)$ as intersection space, then $h+k = s+t$.

Suppose $t = h$; then $V_n^h(F)$ is a subspace of $V_n^k(F)$ and their sum space is $V_n^k$ itself. Thus, $s = k$, $t = h$ and $s+t = h+k$. The reader will show that the same is true if $t = k$.

Suppose next that $t<h$, $t<k$ and let $X_1, X_2, \ldots, X_t$ span $V_n^t(F)$. Then by Theorem **II** there exist vectors $Y_{t+1}, Y_{t+2}, \ldots, Y_h$ so that $X_1, X_2, \ldots, X_t, Y_{t+1}, \ldots, Y_h$ span $V_n^h(F)$ and vectors $Z_{t+1}, Z_{t+2}, \ldots, Z_k$ so that $X_1, X_2, \ldots, X_t, Z_{t+1}, \ldots, Z_k$ span $V_n^k(F)$.

Now suppose there exist scalars $a$'s and $b$'s such that

$$(11.4) \qquad \sum_{i=1}^{t} a_i X_i + \sum_{i=t+1}^{h} a_i Y_i + \sum_{i=t+1}^{k} b_i Z_i = 0 \qquad \text{or}$$

$$\sum_{i=1}^{t} a_i X_i + \sum_{i=t+1}^{h} a_i Y_i = \sum_{i=t+1}^{k} -b_i Z_i$$

The vector on the left belongs to $V_n^h(F)$, and from the right member, belongs also to $V_n^k(F)$; thus it belongs to $V_n^t(F)$. But $X_1, X_2, \ldots, X_t$ span $V_n^t(F)$; hence, $a_{t+1} = a_{t+2} = \ldots = a_h = 0$.

Now from (11.4), $$\sum_{i=1}^{t} a_i X_i + \sum_{i=t+1}^{k} b_i Z_i = 0$$

But the $X$'s and $Z$'s are linearly independent so that $a_1 = a_2 = \ldots = a_t = b_{t+1} = b_{t+2} = \ldots = b_k = 0$; thus, the $X$'s, $Y$'s, and $Z$'s are a linearly independent set and span $V_n^s(F)$. Then $s = h+k-t$ as was to be proved.

8. Consider ${}_1V_3^2(F)$ having $X_1 = [1,2,3]'$ and $X_2 = [1,1,1]'$ as basis and ${}_2V_3^2(F)$ having $Y_1 = [3,1,2]'$ and $Y_2 = [1,0,1]'$ as basis. Since the matrix of the components $\begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 2 & 1 \end{bmatrix}$ is of rank 3, the sum space is $V_3(F)$. As a basis, we may take $X_1$, $X_2$, and $Y_1$.

From $h + k = s + t$, the intersection space is a $V_3^1(F)$. To find a basis, we equate linear combinations of the vectors of the bases of ${}_1V_3^2(F)$ and ${}_2V_3^2(F)$ as

$$aX_1 + bX_2 = cY_1 + dY_2$$

take $d = 1$ for convenience, and solve $\begin{cases} a + b - 3c = 1 \\ 2a + b - c = 0 \\ 3a + b - 2c = 1 \end{cases}$ obtaining $a = 1/3$, $b = -4/3$, $c = -2/3$. Then $aX_1 + bX_2 = [-1,-2/3,-1/3]'$ is a basis of the intersection space. The vector $[3,2,1]'$ is also a basis.

9. Determine a basis for the null space of $A = \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 2 & 2 & 4 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 3 \end{bmatrix}$.

Consider the system of equations $AX = 0$ which reduces to $\begin{cases} x_1 + 2x_3 + x_4 = 0 \\ x_2 + x_3 + 2x_4 = 0 \end{cases}$

A basis for the null space of $A$ is the pair of linearly independent solutions $[1,2,0,-1]'$ and $[2,1,-1,0]'$ of these equations.

10. Prove: $r_{AB} \geq r_A + r_B - n$.

Suppose first that $A$ has the form $\begin{bmatrix} I_{r_A} & 0 \\ 0 & 0 \end{bmatrix}$. Then the first $r_A$ rows of $AB$ are the first $r_A$ rows of $B$ while the remaining rows are zeros. By Problem 10, Chapter 5, the rank of $AB$ is $r_{AB} \geq r_A + r_B - n$.

Suppose next that $A$ is not of the above form. Then there exist nonsingular matrices $P$ and $Q$ such that $PAQ$ has that form while the rank of $PAQB$ is exactly that of $AB$ (why?).

The reader may consider the special case when $B = \begin{bmatrix} I_{r_B} & 0 \\ 0 & 0 \end{bmatrix}$.

11. Let $X = [1,2,1]'$ relative to the $E$-basis. Find its coordinates relative to a new basis $Z_1 = [1,1,0]'$, $Z_2 = [1,0,1]'$, and $Z_3 = [1,1,1]'$.

**Solution** $(a)$. Write

(i) $X = aZ_1 + bZ_2 + cZ_3$, that is, $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then $\begin{cases} a + b + c = 1 \\ a \quad\;\; + c = 2 \\ \quad\; b + c = 1 \end{cases}$ and $a = 0$, $b = -1$, $c = 2$. Thus relative to the $Z$-basis, we have $X_Z = [0,-1,2]'$.

**Solution** (*b*). Rewriting (i) as $X = [Z_1, Z_2, Z_3]X_Z = ZX_Z$, we have

$$X_Z = Z^{-1}X = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = [0,-1,2]'$$

**12.** Let $X_Z$ and $X_W$ be the coordinates of a vector $X$ with respect to the two bases $Z_1 = [1,1,0]'$, $Z_2 = [1,0,1]'$, $Z_3 = [1,1,1]'$ and $W_1 = [1,1,2]'$, $W_2 = [2,2,1]'$, $W_3 = [1,2,2]'$. Determine the matrix $P$ such that $X_W = PX_Z$.

Here $Z = [Z_1, Z_2, Z_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $W = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix}$, and $W^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -3 & 2 \\ 2 & 0 & -1 \\ -3 & 3 & 0 \end{bmatrix}$.

Then $P = W^{-1}Z = \frac{1}{3}\begin{bmatrix} -1 & 4 & 1 \\ 2 & 1 & 1 \\ 0 & -3 & 0 \end{bmatrix}$ by (**11.6**).

## SUPPLEMENTARY PROBLEMS

**13.** Let $[x_1, x_2, x_3, x_4]'$ be an arbitrary vector of $V_4(R)$, where $R$ denotes the field of real numbers. Which of the following sets are subspaces of $V_4(R)$?

(*a*) All vectors with $x_1 = x_2 = x_3 = x_4$.
(*b*) All vectors with $x_1 = x_2$, $x_3 = 2x_4$.
(*c*) All vectors with $x_4 = 0$.
(*d*) All vectors with $x_1 = 1$.
(*e*) All vectors with $x_1, x_2, x_3, x_4$ integral.

*Ans.* All except (*d*) and (*e*).

**14.** Show that $[1,1,1,1]'$ and $[2,3,3,2]'$ are a basis of the $V_4^2(F)$ of Problem 2.

**15.** Determine the dimension of the vector space spanned by each set of vectors. Select a basis for each.

(*a*) $[1,2,3,4,5]'$, $[5,4,3,2,1]'$, $[1,1,1,1,1]'$

(*b*) $[1,1,0,-1]'$, $[1,2,3,4]'$, $[2,3,3,3]'$

(*c*) $[1,1,1,1]'$, $[3,4,5,6]'$, $[1,2,3,4]'$, $[1,0,-1,-2]'$

*Ans.* (*a*), (*b*), (*c*), $r = 2$

**16.** (*a*) Show that the vectors $X_1 = [1,-1,1]'$ and $X_2 = [3,4,-2]'$ span the same space as $Y_1 = [9,5,-1]'$ and $Y_2 = [-17,-11,3]'$.

(*b*) Show that the vectors $X_1 = [1,-1,1]'$ and $X_2 = [3,4,-2]'$ do not span the same space as $Y_1 = [-2,2,-2]'$ and $Y_2 = [4,3,1]'$.

**17.** Show that if the set $X_1, X_2, \ldots, X_k$ is a basis for $V_n^k(F)$, then any other vector $Y$ of the space can be represented *uniquely* as a linear combination of $X_1, X_2, \ldots, X_k$.

Hint. Assume $Y = \sum_{i=1}^{k} a_i X_i = \sum_{i=1}^{k} b_i X_i$.

18. Consider the $4\times 4$ matrix whose columns are the vectors of a basis of the $V_4^2(R)$ of Problem 2 and a basis of the $V_4^2(R)$ of Problem 3. Show that the rank of this matrix is 4; hence, $V_4(R)$ is the sum space and $V_4^0(R)$, the zero space, is the intersection space of the two given spaces.

19. Follow the proof given in Problem 8, Chapter 10, to prove Theorem **III**.

20. Show that the space spanned by $[1,0,0,0,0]'$, $[0,0,0,0,1]'$, $[1,0,1,0,0]'$, $[0,0,1,0,0]'$, $[1,0,0,1,1]'$ and the space spanned by $[1,0,0,0,1]'$, $[0,1,0,1,0]'$, $[0,1,-2,1,0]'$, $[1,0,-1,0,1]'$, $[0,1,1,1,0]'$ are of dimensions 4 and 3, respectively. Show that $[1,0,1,0,1]'$ and $[1,0,2,0,1]'$ are a basis for the intersection space.

21. Find, relative to the basis $Z_1 = [1,1,2]'$, $Z_2 = [2,2,1]'$, $Z_3 = [1,2,2]'$ the coordinates of the vectors (*a*) $[1,1,0]'$, (*b*) $[1,0,1]'$, (*c*) $[1,1,1]'$.
*Ans.* (*a*) $[-1/3, 2/3, 0]'$, (*b*) $[4/3, 1/3, -1]'$, (*c*) $[1/3, 1/3, 0]'$

22. Find, relative to the basis $Z_1 = [0,1,0]'$, $Z_2 = [1,1,1]'$, $Z_3 = [3,2,1]'$ the coordinates of the vectors (*a*) $[2,-1,0]'$, (*b*) $[1,-3,5]'$, (*c*) $[0,0,1]'$.
*Ans.* (*a*) $[-2,-1,1]'$, (*b*) $[-6,7,-2]'$, (*c*) $[-1/2, 3/2, -1/2]'$

23. Let $X_Z$ and $X_W$ be the coordinates of a vector $X$ with respect to the given pair of bases. Determine the matrix $P$ such that $X_W = PX_Z$.

(*a*) $Z_1 = [1,0,0]'$, $Z_2 = [1,0,1]'$, $Z_3 = [1,1,1]'$; $W_1 = [0,1,0]'$, $W_2 = [1,2,3]'$, $W_3 = [1,-1,1]'$

(*b*) $Z_1 = [0,1,0]'$, $Z_2 = [1,1,0]'$, $Z_3 = [1,2,3]'$; $W_1 = [1,1,0]'$, $W_2 = [1,1,1]'$, $W_3 = [1,2,1]'$

*Ans.* (*a*) $P = \frac{1}{2}\begin{bmatrix} 5 & 2 & 4 \\ -1 & 0 & 0 \\ 3 & 2 & 2 \end{bmatrix}$, (*b*) $P = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

24. Prove: If $P_j$ is a solution of $AX = E_j$, $(j = 1,2,\ldots,n)$, then $\sum_{i=1}^{n} h_j P_j$ is a solution of $AX = H$, where $H = [h_1, h_2, \ldots, h_n]'$.
*Hint.* $H = h_1E_1 + h_2E_2 + \cdots + h_nE_n$.

25. The vector space defined by all linear combinations of the columns of a matrix $A$ is called the **column space** of $A$. The vector space defined by all linear combinations of the rows of $A$ is called the **row space** of $A$. Show that the columns of $AB$ are in the column space of $A$ and the rows of $AB$ are in the row space of $B$.

26. Show that $AX = H$, a system of $m$ non-homogeneous equations in $n$ unknowns, is consistent if and only if the the vector $H$ belongs to the column space of $A$.

27. Determine a basis for the null space of (*a*) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$, (*b*) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \end{bmatrix}$.
*Ans.* (*a*) $[1,-1,-1]'$, (*b*) $[1,1,-1,-1]'$, $[1,2,-1,-2]'$

28. Prove: (*a*) $N_{AB} \geq N_A$, $N_{AB} \geq N_B$ (*b*) $N_{AB} \leq N_A + N_B$

Hint: (*a*) $N_{AB} = n - r_{AB}$; $r_{AB} \leq r_A$ and $r_B$.

(*b*) Consider $n - r_{AB}$, using the theorem of Problem 10.

29. Derive a procedure for Problem 16 using only column transformations on $A = [X_1, X_2, Y_1, Y_2]$. Then resolve Problem 5.

# Chapter 12

## Linear Transformations

**DEFINITION**. Let $X = [x_1, x_2, \ldots, x_n]'$ and $Y = [y_1, y_2, \ldots, y_n]'$ be two vectors of $V_n(F)$, their coordinates being relative to the **same** basis of the space. Suppose that the coordinates of $X$, $Y$ are related by

$$\textbf{(12.1)} \qquad \begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{cases}$$

or, briefly,

$$Y = AX$$

where $A = [a_{ij}]$ is over $F$. Then **(12.1)** is a transformation $T$ which carries any vector $X$ of $V_n(F)$ into (usually) another vector $Y$ of the same space, called its **image**.

If **(12.1)** carries $X_1$ into $Y_1$ and $X_2$ into $Y_2$, then

($a$) it carries $kX_1$ into $kY_1$, for every scalar $k$, and

($b$) it carries $aX_1 + bX_2$ into $aY_1 + bY_2$, for every pair of scalars $a$ and $b$. For this reason, the transformation is called **linear**.

**Example 1.** Consider the linear transformation $Y = AX = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 3 & 3 \end{bmatrix} X$ in ordinary space $V_3(R)$.

($a$) The image of $X = [2,0,5]'$ is $Y = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 3 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 \\ 27 \\ 17 \end{bmatrix} = [12,27,17]'$.

($b$) The vector $X$ whose image is $Y = [2,0,5]'$ is obtained by solving $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$.

Since $\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 5 & 0 \\ 1 & 3 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 13/5 \\ 0 & 1 & 0 & 11/5 \\ 0 & 0 & 1 & -7/5 \end{bmatrix}$, $\quad X = [13/5, 11/5, -7/5]'$.

**BASIC THEOREMS**. If in **(12.1)**, $X = [1,0,\ldots,0]' = E_1$ then $Y = [a_{11}, a_{21}, \ldots, a_{n1}]'$ and, in general, if $X = E_j$ then $Y = [a_{1j}, a_{2j}, \ldots, a_{nj}]'$. Hence,

**I.** A linear transformation **(12.1)** is uniquely determined when the images ($Y$'s) of the basis vectors are known, the respective columns of $A$ being the coordinates of the images of these vectors. See Problem 1.

A linear transformation (**12.1**) is called **non-singular** if the images of distinct vectors $X_i$ are distinct vectors $Y_i$. Otherwise the transformation is called **singular**.

**II**. A linear transformation (**12.1**) is non-singular if and only if $A$, the matrix of the transformation, is non-singular. See Problem 2.

**III**. A non-singular linear transformation carries linearly independent (dependent) vectors into linearly independent (dependent) vectors. See Problem 3.

From Theorem **III** follows

**IV**. Under a non-singular transformation (**12.1**) the image of a vector space $V_n^k(F)$ is a vector space $V_n^k(F)$, that is, the dimension of the vector space is preserved. In particular, the transformation is a mapping of $V_n(F)$ onto itself.

When $A$ is non-singular, the inverse of (**12.1**)

$$X = A^{-1}Y$$

carries the set of vectors $Y_1, Y_2, \ldots, Y_n$ whose components are the columns of $A$ into the basis vectors of the space. It is also a linear transformation.

**V**. The elementary vectors $E_i$ of $V_n(F)$ may be transformed into any set of $n$ linearly independent $n$-vectors by a non-singular linear transformation and conversely.

**VI**. If $Y = AX$ carries a vector $X$ into a vector $Y$, if $Z = BY$ carries $Y$ into $Z$, and if $W = CZ$ carries $Z$ into $W$, then $Z = BY = (BA)X$ carries $X$ into $Z$ and $W = (CBA)X$ carries $X$ into $W$.

**VII**. When any two sets of $n$ linearly independent $n$-vectors are given, there exists a non-singular linear transformation which carries the vectors of one set into the vectors of the other.

**CHANGE OF BASIS**. Relative to a $Z$-basis, let $Y_Z = AX_Z$ be a linear transformation of $V_n(F)$. Suppose that the basis is changed and let $X_W$ and $Y_W$ be the coordinates of $X_Z$ and $Y_Z$ respectively relative to the new basis. By Theorem **VIII**, Chapter 11, there exists a non-singular matrix $P$ such that $X_W = PX_Z$ and $Y_W = PY_Z$ or, setting $P^{-1} = Q$, such that

$$X_Z = QX_W \quad \text{and} \quad Y_Z = QY_W$$

Then

$$Y_W = Q^{-1}Y_Z = Q^{-1}AX_Z = Q^{-1}AQX_W = BX_W$$

where

$$B = Q^{-1}AQ \tag{12.2}$$

Two matrices $A$ and $B$ such that there exists a non-singular matrix $Q$ for which $B = Q^{-1}AQ$ are called **similar**. We have proved

**VIII**. If $Y_Z = AX_Z$ is a linear transformation of $V_n(F)$ relative to a given basis ($Z$-basis) and $Y_W = BX_W$ is the same linear transformation relative to another basis ($W$-basis), then $A$ and $B$ are similar.

*Note.* Since $Q = P^{-1}$, (**12.2**) might have been written as $B = PAP^{-1}$. A study of similar matrices will be made later. There we shall agree to write $B = R^{-1}AR$ instead of $B = SAS^{-1}$ but for no compelling reason.

**Example 2.** Let $Y = AX = \begin{bmatrix}1&1&3\\1&2&1\\1&3&2\end{bmatrix}X$ be a linear transformation relative to the $E$-basis and let $W_1 = [1,2,1]'$, $W_2 = [1,-1,2]'$, $W_3 = [1,-1,-1]'$ be a new basis. (*a*) Given the vector $X = [3,0,2]'$, find the coordinates of its image relative to the $W$-basis. (*b*) Find the linear transformation $Y_W = BX_W$ corresponding to $Y = AX$. (*c*) Use the result of (*b*) to find the image $Y_W$ of $X_W = [1,3,3]'$.

Write $W = [W_1, W_2, W_3] = \begin{bmatrix}1&1&1\\2&-1&-1\\1&2&-1\end{bmatrix}$; then $W^{-1} = \frac{1}{9}\begin{bmatrix}3&3&0\\1&-2&3\\5&-1&-3\end{bmatrix}$.

(*a*) Relative to the $W$-basis, the vector $X = [3,0,2]'$ has coordinates $X_W = W^{-1}X = [1,1,1]'$. The image of $X$ is $Y = AX = [9,5,7]'$ which, relative to the $W$-basis is $Y_W = W^{-1}Y = [14/3, 20/9, 19/9]'$.

(*b*) $Y_W = W^{-1}Y = W^{-1}AX = (W^{-1}AW)X_W = BX_W = \frac{1}{9}\begin{bmatrix}36&21&-15\\21&10&-11\\-3&23&-1\end{bmatrix}X_W$

(*c*) $Y_W = \frac{1}{9}\begin{bmatrix}36&21&-15\\21&10&-11\\-3&23&-1\end{bmatrix}\begin{bmatrix}1\\3\\3\end{bmatrix} = \begin{bmatrix}6\\2\\7\end{bmatrix} = [6,2,7]'$.

See Problem 5.

## SOLVED PROBLEMS

1. (*a*) Set up the linear transformation $Y = AX$ which carries $E_1$ into $Y_1 = [1,2,3]'$, $E_2$ into $[3,1,2]'$, and $E_3$ into $Y_3 = [2,1,3]'$.
   (*b*) Find the images of $X_1 = [1,1,1]'$, $X_2 = [3,-1,4]'$, and $X_3 = [4,0,5]'$.
   (*c*) Show that $X_1$ and $X_2$ are linearly independent as also are their images.
   (*d*) Show that $X_1$, $X_2$, and $X_3$ are linearly dependent as also are their images.

(*a*) By Theorem I, $A = [Y_1, Y_2, Y_3]$; the equation of the linear transformation is $Y = AX = \begin{bmatrix}1&3&2\\2&1&1\\3&2&3\end{bmatrix}X$.

(*b*) The image of $X_1 = [1,1,1]'$ is $Y_1 = \begin{bmatrix}1&3&2\\2&1&1\\3&2&3\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = [6,4,8]'$. The image of $X_2$ is $Y_2 = [8,9,19]'$ and the image of $X_3$ is $Y_3 = [14,13,27]'$.

(*c*) The rank of $[X_1, X_2] = \begin{bmatrix}1&3\\1&-1\\1&4\end{bmatrix}$ is 2 as also is that of $[Y_1, Y_2] = \begin{bmatrix}6&8\\4&9\\8&19\end{bmatrix}$. Thus, $X_1$ and $X_2$ are linearly independent as also are their images.

(*d*) We may compare the ranks of $[X_1, X_2, X_3]$ and $[Y_1, Y_2, Y_3]$; however, $X_3 = X_1 + X_2$ and $Y_3 = Y_1 + Y_2$ so that both sets are linearly dependent.

2. Prove: A linear transformation (12.1) is non-singular if and only if $A$ is non-singular.

Suppose $A$ is non-singular and the transforms of $X_1 \neq X_2$ are $Y = AX_1 = AX_2$. Then $A(X_1 - X_2) = 0$ and the system of homogeneous linear equations $AX = 0$ has the non-trivial solution $X = X_1 - X_2$. This is possible if and only if $|A| = 0$, a contradiction of the hypothesis that $A$ is non-singular.

3. Prove: A non-singular linear transformation carries linearly independent vectors into linearly independent vectors.

Assume the contrary, that is, suppose that the images $Y_i = AX_i$, $(i = 1,2,\ldots,p)$ of the linearly independent vectors $X_1, X_2, \ldots, X_p$ are linearly dependent. Then there exist scalars $s_1, s_2, \ldots, s_p$, not all zero, such that

$$\sum_{i=1}^{p} s_i Y_i \quad = \quad s_1Y_1 + s_2Y_2 + \cdots + s_pY_p \quad = \quad 0$$

or

$$\sum_{i=1}^{p} s_i(AX_i) \quad = \quad A(s_1X_1 + s_2X_2 + \cdots + s_pX_p) \quad = \quad 0$$

Since $A$ is non-singular, $s_1X_1 + s_2X_2 + \cdots + s_pX_p = 0$. But this is contrary to the hypothesis that the $X_i$ are linearly independent. Hence, the $Y_i$ are linearly independent.

4. A certain linear transformation $Y = AX$ carries $X_1 = [1,0,1]'$ into $[2,3,-1]'$, $X_2 = [1,-1,1]'$ into $[3,0,-2]'$, and $X_3 = [1,2,-1]'$ into $[-2,7,-1]'$. Find the images of $E_1, E_2, E_3$ and write the equation of the transformation.

Let $aX_1 + bX_2 + cX_3 = E_1$; then $\begin{cases} a + b + c = 1 \\ -b + 2c = 0 \\ a + b - c = 0 \end{cases}$ and $a = -\frac{1}{2}$, $b = 1$, $c = \frac{1}{2}$. Thus, $E_1 = -\frac{1}{2}X_1 + X_2 + \frac{1}{2}X_3$

and its image is $Y_1 = -\frac{1}{2}[2,3,-1]' + [3,0,-2]' + \frac{1}{2}[-2,7,-1]' = [1,2,-2]'$. Similarly, the image of $E_2$ is $Y_2 = [-1,3,1]'$ and the image of $E_3$ is $Y_3 = [1,1,1]'$. The equation of the transformation is

$$Y \quad = \quad [Y_1, Y_2, Y_3]X \quad = \quad \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix} X$$

5. If $Y_Z = AX_Z = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} X_Z$ is a linear transformation relative to the $Z$-basis of Problem 12, Chapter 11, find the same transformation $Y_W = BX_W$ relative to the $W$-basis of that problem.

From Problem 12, Chapter 11, $X_W = PX_Z = \frac{1}{3}\begin{bmatrix} -1 & 4 & 1 \\ 2 & 1 & 1 \\ 0 & -3 & 0 \end{bmatrix} X_Z$. Then

$$X_Z \quad = \quad P^{-1}X_W \quad = \quad \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix} X_W \quad = \quad QX_W$$

and

$$Y_W \quad = \quad PY_Z \quad = \quad Q^{-1}AX_Z \quad = \quad Q^{-1}AQX_W \quad = \quad \frac{1}{3}\begin{bmatrix} -2 & 14 & -6 \\ 7 & 14 & 9 \\ 0 & -9 & 3 \end{bmatrix} X_W$$

## SUPPLEMENTARY PROBLEMS

6. In Problem 1 show: (*a*) the transformation is non-singular, (*b*) $X = A^{-1}Y$ carries the column vectors of $A$ into the elementary vectors.

7. Using the transformation of Problem 1, find (*a*) the image of $X = [1,1,2]'$, (*b*) the vector $X$ whose image is $[-2,-5,-5]'$. *Ans.* (*a*) $[8,5,11]'$, (*b*) $[-3,-1,2]'$

8. Study the effect of the transformation $Y = IX$, also $Y = kIX$.

9. Set up the linear transformation which carries $E_1$ into $[1,2,3]'$, $E_2$ into $[3,1,2]'$, and $E_3$ into $[2,-1,-1]'$. Show that the transformation is singular and carries the linearly independent vectors $[1,1,1]'$ and $[2,0,2]'$ into the same image vector.

10. Suppose (12.1) is non-singular and show that if $X_1, X_2, \ldots, X_n$ are linearly dependent so also are their images $Y_1, Y_2, \ldots, Y_n$.

11. Use Theorem III to show that under a non-singular transformation the dimension of a vector space is unchanged. *Hint.* Consider the images of a basis of $V_n^k(F)$.

12. Given the linear transformation $Y = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ -2 & 3 & 5 \end{bmatrix} X$, show (*a*) it is singular, (*b*) the images of the linearly independent vectors $X_1 = [1,1,1]'$, $X_2 = [2,1,2]'$, and $X_3 = [1,2,3]'$ are linearly dependent, (*c*) the image of $V_3(R)$ is a $V_3^2(R)$.

13. Given the linear transformation $Y = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix} X$, show (*a*) it is singular, (*b*) the image of every vector of the $V_3^2(R)$ spanned by $[1,1,1]'$ and $[3,2,0]'$ lies in the $V_3^1(R)$ spanned by $[5,7,5]'$.

14. Prove Theorem VII.
*Hint.* Let $X_i$ and $Y_i$, $(i = 1,2,\ldots,n)$ be the given sets of vectors. Let $Z = AX$ carry the set $X_i$ into $E_i$ and $Y = BZ$ carry the $E_i$ into $Y_i$.

15. Prove: Similar matrices have equal determinants.

16. Let $Y = AX = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} X$ be a linear transformation relative to the $E$-basis and let a new basis, say $Z_1 = [1,1,0]'$, $Z_2 = [1,0,1]'$, $Z_3 = [1,1,1]'$ be chosen. Let $X = [1,2,3]'$ relative to the $E$-basis. Show that
(*a*) $Y = [14,10,6]'$ is the image of $X$ under the transformation,
(*b*) $X$, when referred to the new basis, has coordinates $X_Z = [-2,-1,4]'$ and $Y$ has coordinates $Y_Z = [8,4,2]'$.
(*c*) $X_Z = PX$ and $Y_Z = PY$, where $P = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = [Z_1, Z_2, Z_3]^{-1}$
(*d*) $Y_Z = Q^{-1}AQX_Z$, where $Q = P^{-1}$.

17. Given the linear transformation $Y_W = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} X_W$, relative to the $W$-basis: $W_1 = [0,-1,2]'$, $W_2 = [4,1,0]'$,

$W_3 = [-2,0,-4]'$. Find the representation relative to the $Z$-basis: $Z_1 = [1,-1,1]'$, $Z_2 = [1,0,-1]'$, $Z_3 = [1,2,1]'$.

*Ans.* $Y_Z = \begin{bmatrix} -1 & 0 & 3 \\ 2 & 2 & -5 \\ -1 & 0 & 2 \end{bmatrix} X_Z$

**18.** If, in the linear transformation $Y = AX$, $A$ is singular, then the null space of $A$ is the vector space each of whose vectors transforms into the zero vector. Determine the null space of the transformation of

(a) Problem 12, (b) Problem 13, (c) $Y = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} X$.

*Ans.* (a) $V_3^1(R)$ spanned by $[1,-1,1]'$
(b) $V_3^1(R)$ spanned by $[2,1,-1]'$
(c) $V_3^2(R)$ spanned by $[2,-1,0]'$ and $[3,0,-1]'$

**19.** If $Y = AX$ carries every vector of a vector space $V_n^h$ into a vector of that same space, $V_n^h$ is called an **invariant space of the transformation**. Show that in the real space $V_3(R)$ under the linear transformation

(a) $Y = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} X$, the $V_3^1$ spanned by $[1,-1,0]'$, the $V_3^1$ spanned by $[2,-1,-2]'$, and the $V_3^1$ spanned by $[1,-1,-2]'$ are invariant vector spaces.

(b) $Y = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} X$, the $V_3^1$ spanned by $[1,1,1]'$ and the $V_3^2$ spanned by $[1,0,-1]'$ and $[2,-1,0]'$ are invariant spaces. (Note that every vector of the $V_3^2$ is carried into itself.)

(c) $Y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & -6 & 4 \end{bmatrix} X$, the $V_4^1$ spanned by $[1,1,1,1]'$ is an invariant vector space.

**20.** Consider the linear transformation $Y = PX$: $y_i = x_{j_i}$, $(i = 1, 2, \ldots, n)$ in which $j_1, j_2, \ldots, j_n$ is a permutation of $1, 2, \ldots, n$.

(a) Describe the **permutation matrix** $P$.

(b) Prove: There are $n!$ permutation matrices of order $n$.

(c) Prove: If $P_1$ and $P_2$ are permutation matrices so also are $P_3 = P_1P_2$ and $P_4 = P_2P_1$.

(d) Prove: If $P$ is a permutation matrix so also are $P'$ and $PP' = I$.

(e) Show that each permutation matrix $P$ can be expressed as a product of a number of the elementary column matrices $K_{12}, K_{23}, \ldots, K_{n-1,n}$.

(f) Write $P = [E_{i_1}, E_{i_2}, \ldots, E_{i_n}]$ where $i_1, i_2, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$ and $E_{i_j}$ are the elementary $n$-vectors. Find a rule (other than $P^{-1} = P'$) for writing $P^{-1}$. For example, when $n = 4$ and $P = [E_3, E_1, E_4, E_2]$, then $P^{-1} = [E_2, E_4, E_1, E_3]$; when $P = [E_4, E_2, E_1, E_3]$, then $P^{-1} = [E_3, E_2, E_4, E_1]$.

# INDEX

# Index of Symbols