ZCA 110 (Group C)

Linear Algebra and Calculus course materials

By Yoon Tiem Leong
School of Physics
USM
3 Sept 2014

PUSAT PENGAJIAN SAINS FIZIK
UNIVERSITI SAINS MALAYSIA
PULAU PINANG
Sidang Akademik 2014/2015

Kursus:	KALKULUS DAN ALGEBRA LINEAR
Kode:	ZCA 110/4/4 (C)
Jam Kredit:	4
Semester:	1
Prerequisite:	Tiada
Status:	Asas
Jam Kontak:	4 kuliah/(tutorial - dua minggu sekali) seminggu untuk 14 minggu [56 kelas se semester]

Waktu Pengajaran:

Isnin	11.00-11.50 (DK L)
Rabu	10.00-10.50 (BT 141)
Khamis	9.00-9.50 (BT 145)
Jumaat	11.00-11.50 (BT 145)

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OBJEKTIF: Untuk mempelajari beberapa teorem dan kaedah dalam kalkulus dan algebra linear bagi digunakan dalam matapelajaran-matapelajaran fizik.

PENGHARAPAN KURSUS / COURSE EXPECTATIONS:
Selepas habis kursus ini, pelajar akan jadi lebih cerdik dalam bahasa matematik yang diperlukan untuk beberapa applikasi dalam fizik dan juga lebih bersedia untuk kursus-kursus matematik maju.

BUKU-BUKU TEKS:
1. Schaum's Outline of Theory and Problems of Matrices SI (Metric) Edition by Frank Ayres, McGraw-Hill (1974) bagi Algebra Linear
2. Thomas Calculus, Twelfth Edition, by George B. Thomas, Maurice D. Weir, and Joel Hass, Addison-Wesley (2010) bagi Kalkulus.

RUJUKAN:

1. S.L. Salas, E. Hille, and G.J. Etgen, Calculus, John Wiley & Sons, New York, 9th Edition, 2003, John Wiley & Sons.
2. Edwards and Penny, Calculus, 6th Edition, 2002, Prentice Hall.
3. Gerald L. Bradley and Karl J. Smith, Calculus, 2nd Edition, 1999, Prentice Hall.
4. Seymour Lipschutz and Marc Lipson, Schaum’s Outlines, Linear Algebra, 3rd Edition, 2001, McGraw-Hill.
5. Introductory Linear Algebra with Application by Bernard Kolman and David R. Hill, 7th Edition, 2001, Prentice Hall.

TUTORIAL:
Kelas tutorial diadakan sekali setiap minggu. Waktu kelas tutorial adalah 1 – 2 jam, bergantung kepada keadaan.

JAM NASIHAT (CONSULTATION HOURS):
Bila-bila and mana-mana sahaja. Disarankan sms pensyarah dulu sebelum dating jumpa dibilik saya.

PENILAIAN (ASSESMENT):

TUTORIAL	30%	Kerja-kerja tutorial yang diserahkan oleh pelajar-pelajar akan digradkan oleh tutor-tutor.
4 Kuiz	10%	KALKULUS (20 soalan objectif per kuiz)
PERPERIKSAAN AKHIR (3 jam)	60%	Semua TOPIK (Pilih Enam daripada Lapan soalan subjektif yang berkeberatan sama)

Jadual di atas adalah tertakluk kepada perubahan, sebab penyesuaian memang diperlukan disepanjang semester itu.

Yoon Tiem Leong
3 Sept 2014

SEPTEMBER 2014

ACADEMIC YEAR 2014/2015

ZCA 110 (C)

	MONDAY (DK L)	**TUE**	**WEDNESDAY (BT 141)**	**THURSDAY (BT 145)**	**FRIDAY (BT 145)**
1	8 11.00-11.50 (1) **PART 1** 1 Matrix 1.1 Matrix Algebra	9	10 10.00-10.50 (2) 1.2 Type of Matrices: Identity Matrix, Special Square Matrices, Inverse of a Matrix, Transpose of a Matrix	11 9.00-9.50 (3) 1.2 Type of Matrices: Symmetric Matrices, Conjugate of a Matrix, Hermitian Matrices, Direct Sums.	12 11.00-11.50 (4) 1.3 Determinant of a Square Matrix: Determinants of orders 2 and 3, Properties of Determinants, Minors and Cofactors, Adjoint of a Square Matrix
2	15 11.00-11.50 (5)	**16** **H.D.**	17 10.00-10.50 (6) 1.3 Determinant of a Square Matrix: Evaluation of Determinant, The Inverse of a Matrix, Elementary Transformation.	18 9.00-9.50 (7) 1.4 System of Linear Equations: Solution using a Matrix, Fundamental Theorems, Homogeneous Equations.	19 11.00-11.50 (8) 2 Vector Spaces 2.1 Vector Spaces: Subspace
3	22 11.00-11.50 (9) 2.2 Basis and Dimension: Basis and Coordinate	23	24 10.00-10.50 (10) 2.3 Linear Transformation: Definition, Basic Theorems, Change of Basis	25 9.00-9.50 (11) **Tutorial 1 (Linear Algebra)**	26 11.00-11.50 DK T (12) **Tutorial 2 (Linear Algebra)**
4	29 11.00-11.50 (13) **PART 2** Preliminaries 0.1 Functions and Graphs 0.2 Exponential Functions 0.3 Inverse Functions and Logarithms:	30			

OCTOBER 2014

	MONDAY(DK L)	TUE	WEDNESDAY (BT 141)	THURSDAY (BT 145)	FRIDAY (BT 145)
4			1 10.00-10.50 (14) One-to-One Functions, Inverses, Logarithm Functions 0.4 Trigonometric Functions and Their Inverses	2 9.00-9.50 (15) 1. Limits and Continuity 1.1 Rates of Change and Limits	3 11.00-11.50 (16) 1.2 Limits and One-Sided Limits: Properties of Limits, One-Sided Limits
5	**6** 11.00-11.50 (17) 1.3 Limits Involving Infinity 1.4 Continuity **Hoilday**	7	8 10.00-10.50 (18) 2 Derivatives 2.1 The Derivative as a Function 2.2 The Derivative as a Rate of Change	9 9.00-9.50 (19) 2.3 Derivatives of Products, Quotients, and Negative Powers	10 11.00-11.50 (20) 2.4 Derivatives of Trigonometric Functions 2.5 The Chain Rule
6	13 11.00-11.50 (21) 2.6 Implicit Differentiation	14	15 10.00-10.50 (22) 3 Applications of the Derivatives 3.1 Extreme Values of Functions	16 9.00-9.50 (23) **Tutorial 3 Calculus Chap. 0, 1, 2.**	17 11.00-11.50 (24) 3.2 The Mean Value Theorem 3.3 The Shape of a Graph
7	**BREAK**				
8	27 11.00-11.50 (25) 3.4 Optimization	28	29 10.00-10.50 (26) 4. Integration 4.1 Indefinite Integrals	30 9.00-9.50 (27) 4.2 Integration by Substitution	31 11.00-11.50 (28)

NOVEMBER 2014

	MONDAY (DK L)	TUE	WEDNESDAY (BT 141)	THURSDAY (BT 145)	FRIDAY (BT 145)
9	3 11.00-11.50 (29) 4.3 Definite Integrals	4	5 10.00-10.50 (30) 4.4 The Mean Value and Fundamental Theorems	6 9.00-9.50 (31) **Tutorial 4 Calculus Chap. 3, 4, 5.**	7 11.00-11.50 (32) 4.5 Substitution in Indefinite Integrals
10	10 11.00-11.50 (33) 5 Applications of Integrals 5.1 Length of Plane Curves	11	12 10.00-10.50 (34) 6 Transcendental Functions 6.1 Logarithms	13 9.00-9.50 (35) 6.2 Exponential Functions	14 11.00-11.50 (36) 6.3 Derivatives 6.4 Hyperbolic Functions
11	**17** 11.00-11.50 (37) Revision	18	19 10.00-10.50 (38) Revision	**20** 9.00-9.50 (39) Revision	21 11.00-11.50 (40) Revision
12	24 11.00-11.50 (41) 7 Integration Techniques 7.1 Basic Integration Formulas	25	26 10.00-10.50 (42) 7.2 Integration by Parts 7.3 Partial Fractions	27 9.00-9.50 (43) **Tutorial 5 Calculus Chap. 6, 7.**	28 11.00-11.50 (44) 7.4 Trigonometric Substitutions

DECEMBER 2014

	MONDAY (DK L)	TUE	WEDNESDAY (BT 141)	THURSDAY (BT 145)	FRIDAY (BT 145)
13	1 11.00-11.50 (45) 7.5 Integral Tables 7.6 L' Hôpital's Rule 7.7 Improper Integrals	2	3 10.00-10.50 (46) 8 Infinite Series 8.1 Limits of Sequences of Numbers	4 9.00-9.50 (47) 8.2 Bounded Sequences	5 11.00-11.50 (48) 8.3 Infinite Series – Geometric Series
14	8 11.00-11.50 (49) 8.4 Series of Nonnegative Terms	9	10 10.00-10.50 (50) 8.5 Alternating Series	11 14.00-14.50 (51) **Tutorial 6 Calculus Chap. 8.**	12 10.00-10.50 (52) 8.6 Power Series
15	15 11.00-11.50 (53) 8.7 Taylor and Maclaurin Series	16	17 10.00-10.50 (54) 8.8 Binomial Series	18 9.00-9.50 (55) 8.9 Fourier Series	19 10.00-10.50 (56) 8.10 Fourier Cosine and Sine Series
16	STUDY WEEK				

Examination 29 December 2014 - 16 January 2015

Chapter 1

Preliminaries

1.3

Functions and Their Graphs

Function

- $y = f(x)$
- f represents function (a rule that tell us how to calculate the value of y from the variable x
- x : independent variable (input of f)
- y : dependent variable (the correspoinding output value of f at x)

DEFINITION Function

A **function** from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

Definition Domain of the function

The set of D of all possible input values

Definition Range of the function

The set of all values of $f(x)$ as x varies throughout D

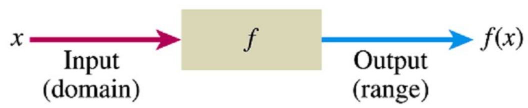


FIGURE 1.22 A diagram showing a function as a kind of machine.

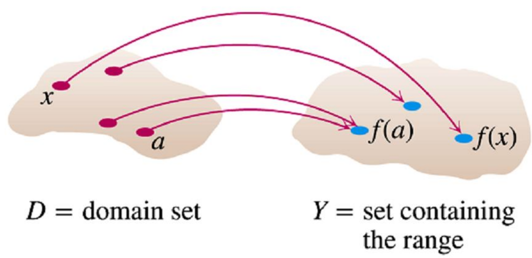


FIGURE 1.23 A function from a set D to a set Y assigns a unique element of Y to each element in D.

Natural Domain

- When a function $y = f(x)$ is defined and the domain is not stated explicitly, the domain is assumed to be the largest set of real x-values for the formula gives real y-values.
- e.g. compare "$y = x^2$" c.f. "$y = x^2, x \geqslant 0$"
- Domain may be open, closed, half open, finite, infinite.

Verify the domains and ranges of these functions

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Graphs of functions

- Graphs provide another way to visualise a function
- In set notation, a graph is

$$\{(x,f(x)) \mid x \in D\}$$

- The graph of a function is a useful picture of its behaviour.

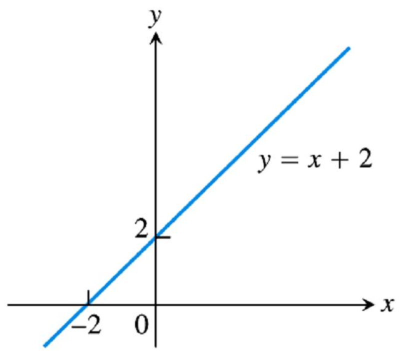


FIGURE 1.24 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

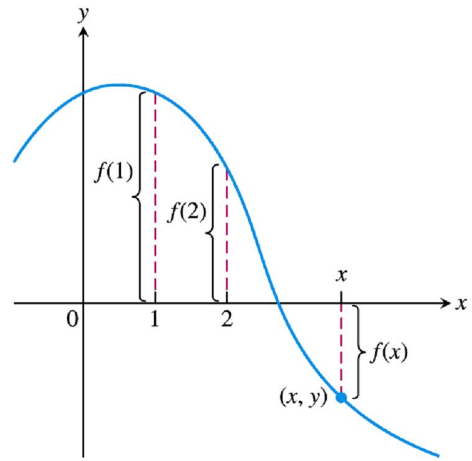


FIGURE 1.25 If (x, y) lies on the graph of f, then the value $y = f(x)$ is the height of the graph above the point x (or below x if $f(x)$ is negative).

Example 2 Sketching a graph

- Graph the function $y = x^2$ over the interval [-2,2]

The vertical line test

- Since a function must be single valued over its domain, no vertical line can intersect the graph of a function more than once.
- If *a* is a point in the domain of a function *f*, the vertical line *x*=*a* can intersect the graph of *f* in a single point *(a*, *f*(*a*)).

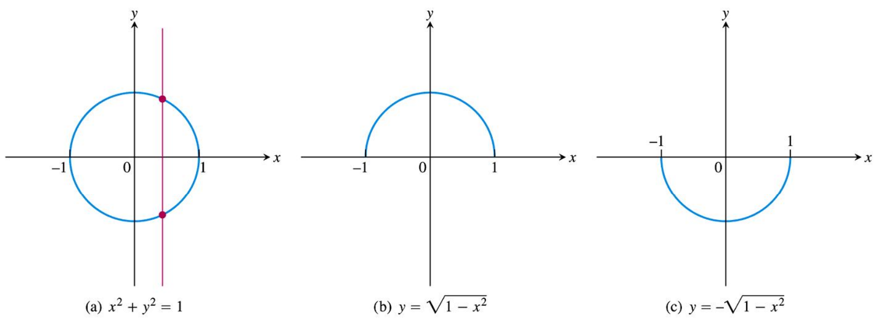


FIGURE 1.28 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of a function $f(x) = \sqrt{1-x^2}$. (c) The lower semicircle is the graph of a function $g(x) = -\sqrt{1-x^2}$.

Piecewise-defined functions

- The absolute value function

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

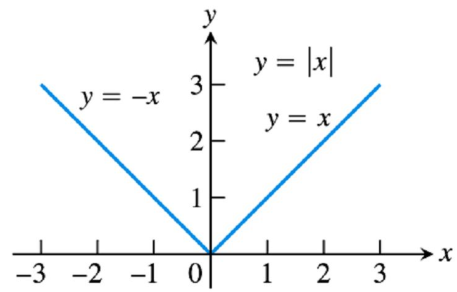


FIGURE 1.29 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

Graphing piecewise-defined functions

- Note: this is *just one function* with a domain covering all real number

$$f(x)=\begin{cases}-x & x<0\\ x^2 & 0\le x\le 1\\ 1 & x>1\end{cases}$$

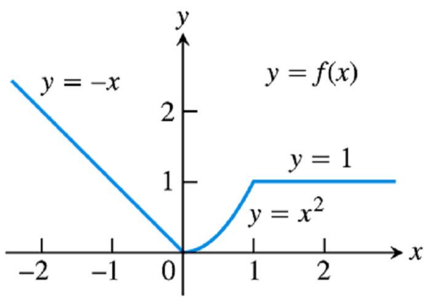


FIGURE 1.30 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 5).

The greatest integer function

- Also called integer floor function
- *f* = [*x*], defined as greatest integer less than or equal to *x*.
- e.g.
- [2.4] = 2
- [2]=2
- [-2] = -2, etc.

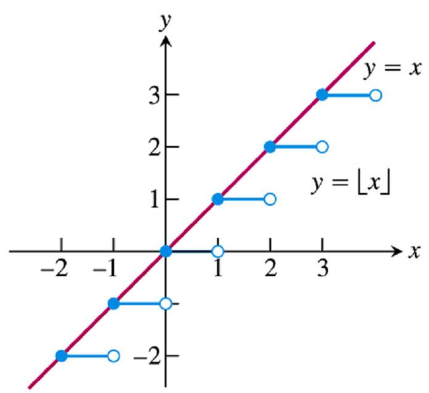


FIGURE 1.31 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x (Example 6).

Note: the graph is the blue colour lines, not the one in red

Writing formulas for piecewise-defined functions

- Write a formula for the function $y=f(x)$ in Figure 1.33

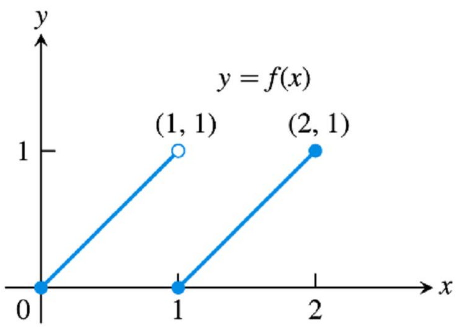


FIGURE 1.33 The segment on the left contains (0, 0) but not (1, 1). The segment on the right contains both of its endpoints (Example 8).

1.4

Identifying Functions;
Mathematical Models

Linear functions

- Linear function takes the form of
- $y=mx + b$
- m, b constants
- m slope of the graph
- b intersection with the y-axis
- The linear function reduces to a constant function $f = c$ when $m = 0$,

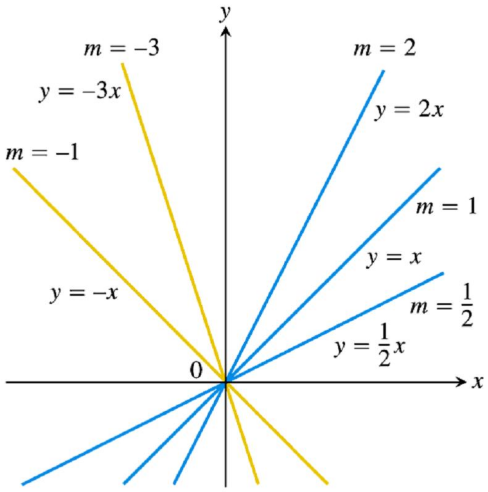


FIGURE 1.34 The collection of lines $y = mx$ has slope m and all lines pass through the origin.

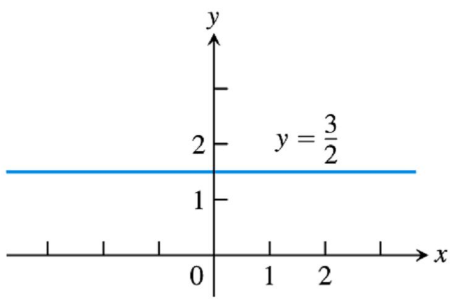


FIGURE 1.35 A constant function has slope $m = 0$.

Power functions

- $f(x) = x^a$
- a constant
- Case (a): $a = n$, a positive integer

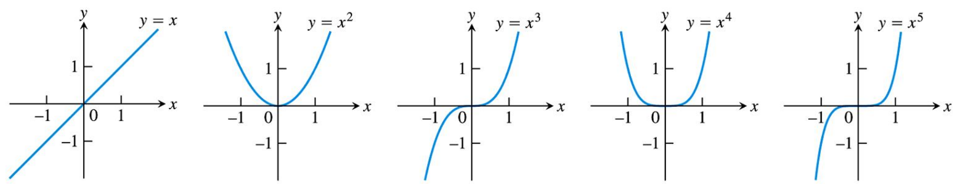


FIGURE 1.36 Graphs of $f(x) = x^n, n = 1, 2, 3, 4, 5$ defined for $-\infty < x < \infty$.

go back

Power functions

- Case (b):
- a = -1 (hyperbola)
- or a=-2

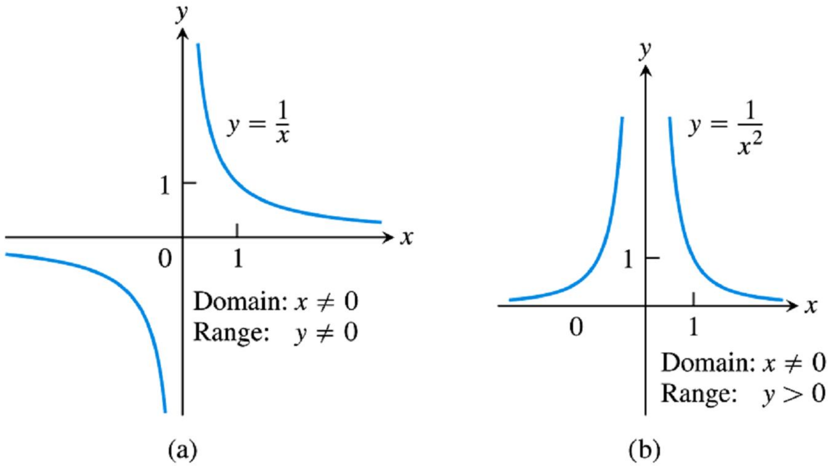


FIGURE 1.37 Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

go back

Power functions

- Case (c):
- a = ½, 1/3, 3/2, and 2/3
- $f(x) = x^{1/2} = \sqrt{x}$ (square root) , domain = $[0 \leqslant x < \infty)$
- $g(x) = x^{1/3} = \sqrt[3]{x}$(cube root), domain = $(-\infty < x < \infty)$

- $p(x) = x^{2/3} = (x^{1/3})^2$, domain = ?
- $q(x) = x^{3/2} = (x^3)^{1/2}$ domain = ?

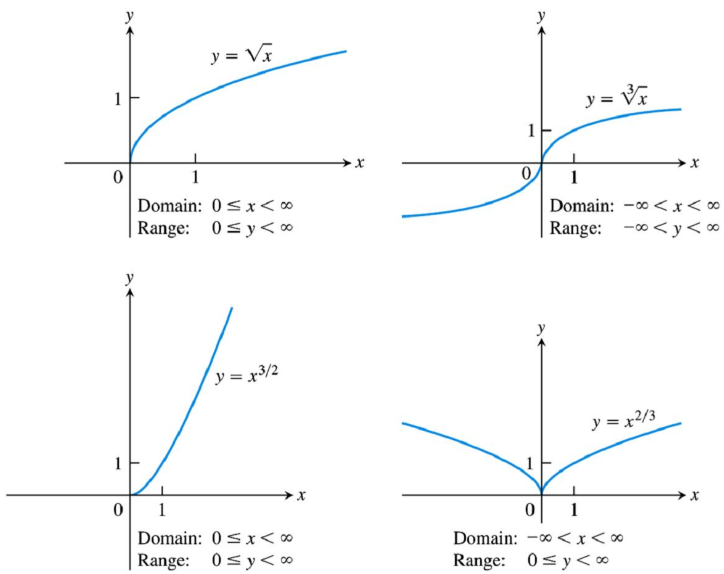


FIGURE 1.38 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

go back

Polynomials

- $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots a_1 x + a_0$
- n nonnegative integer (1,2,3…)
- a's coefficients (real constants)
- If $a_n \neq 0$, n is called the degree of the polynomial

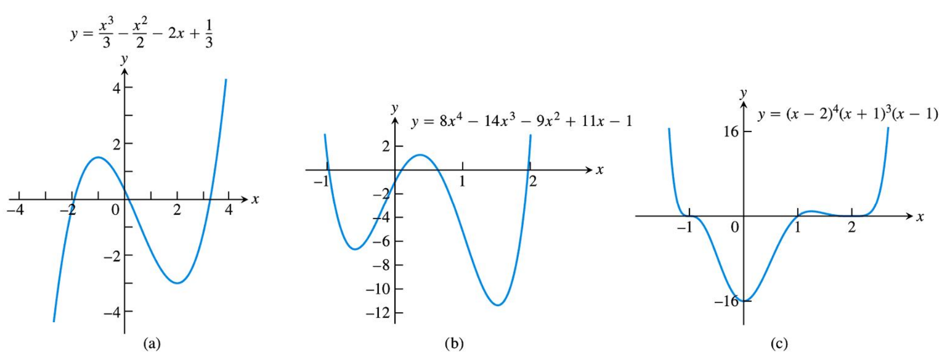


FIGURE 1.39 Graphs of three polynomial functions.

Rational functions

- A rational function is a quotient of two polynomials:
- $f(x) = p(x) / q(x)$
- p,q are polynomials.
- Domain of $f(x)$ is the set of all real number x for which $q(x) \neq 0$.

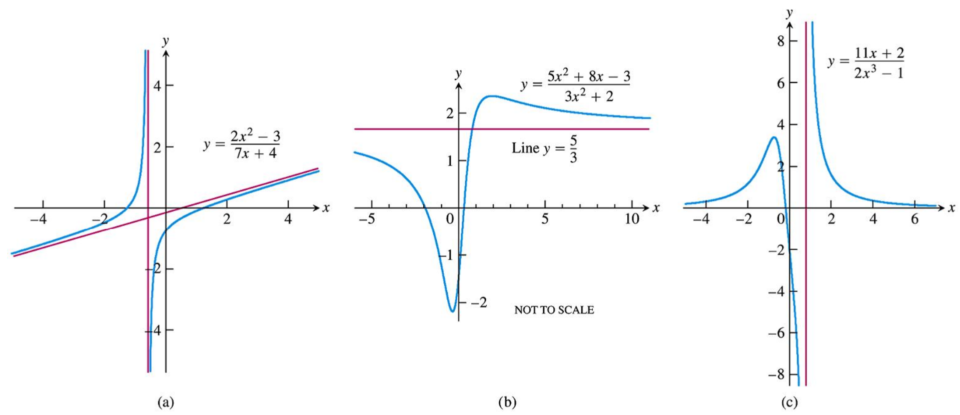


FIGURE 1.40 Graphs of three rational functions.

Algebraic functions

- Functions constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots)

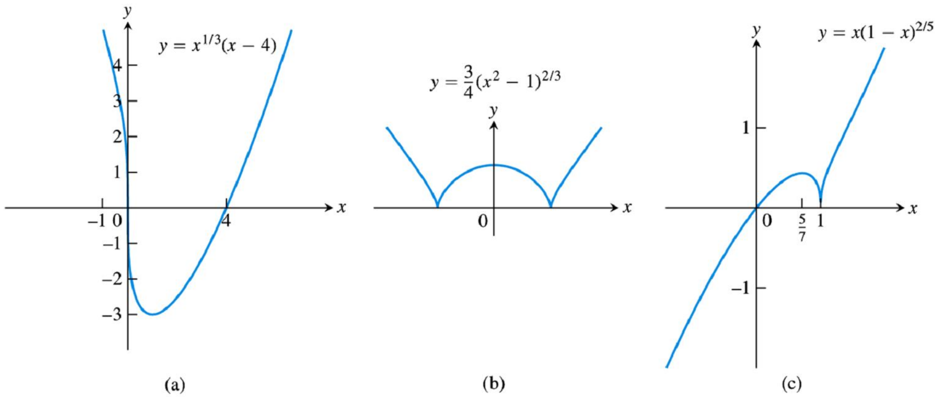


FIGURE 1.41 Graphs of three algebraic functions.

Trigonometric functions

- More details in later chapter

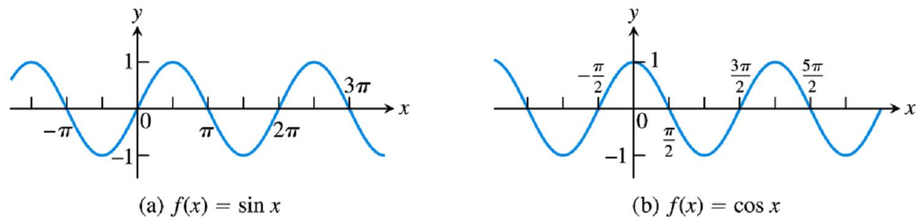


FIGURE 1.42 Graphs of the sine and cosine functions.

Exponential functions

- $f(x) = a^x$
- Where $a > 0$ and $a \neq 0$. a is called the 'base'.
- Domain $(-\infty, \infty)$
- Range $(0, \infty)$
- Hence, $f(x) > 0$
- More in later chapter

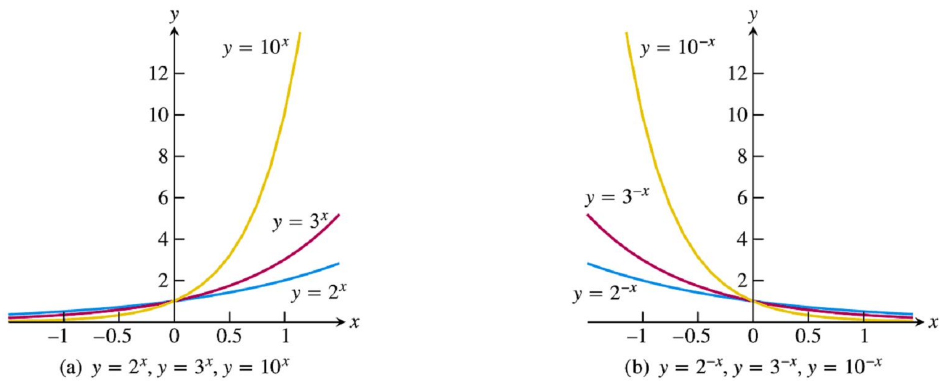


FIGURE 1.43 Graphs of exponential functions.

Note: graphs in (a) are reflections of the corresponding curves in (b) about the y-axis. This amount to the symmetry operation of $x \leftrightarrow -x$.

Logarithmic functions

- $f(x) = \log_a x$
- a is the base
- $a \neq 1$, $a > 0$
- Domain $(0, \infty)$
- Range $(-\infty, \infty)$
- They are the *inverse functions* of the exponential functions (more in later chapter)

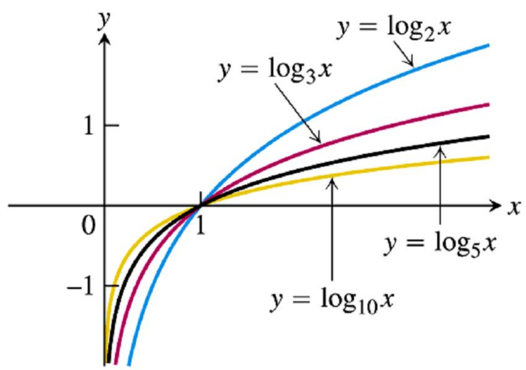


FIGURE 1.44 Graphs of four logarithmic functions.

Transcendental functions

- Functions that are not algebraic
- Include: trigonometric, inverse trigonometric, exponential, logarithmic, hyperbolic and many other functions

Example 1

- Recognizing Functions
- (a) $f(x) = 1 + x - \frac{1}{2}x^5$
- (b) $g(x) = 7^x$
- (c) $h(z) = z^7$
- (d) $y(t) = \sin(t - \pi/4)$

Increasing versus decreasing functions

- A function is said to be increasing if it rises as you move from left to right
- A function is said to be decreasing if it falls as you move from left to right

Function	Where increasing	Where decreasing
$y = x^2$	$0 \le x < \infty$	$-\infty < x \le 0$
$y = x^3$	$-\infty < x < \infty$	Nowhere
$y = 1/x$	Nowhere	$-\infty < x < 0$ and $0 < x < \infty$
$y = 1/x^2$	$-\infty < x < 0$	$0 < x < \infty$
$y = \sqrt{x}$	$0 \le x < \infty$	Nowhere
$y = x^{2/3}$	$0 \le x < \infty$	$-\infty < x \le 0$

$y=x^2$, $y=x^3$; $y=1/x$, $y=1/x^2$; $y=x^{1/2}$, $y=x^{2/3}$

DEFINITIONS **Even Function, Odd Function**

A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

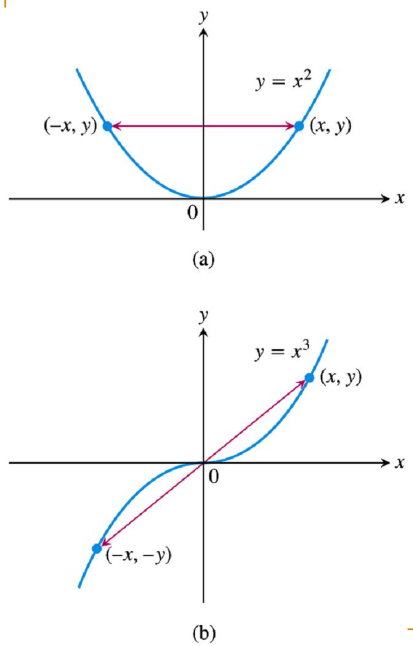


FIGURE 1.46 In part (a) the graph of $y = x^2$ (an even function) is symmetric about the y-axis. The graph of $y = x^3$ (an odd function) in part (b) is symmetric about the origin.

Recognising even and odd functions

- $f(x) = x^2$ Even function as $(-x)^2 = x^2$ for all x, symmetric about the all x, symmetric about the y-axis.
- $f(x) = x^2 + 1$ Even function as $(-x)^2 + 1 = x^2 + 1$ for all x, symmetric about the all x, symmetric about the y-axis.

Recognising even and odd functions

- $f(x) = x$. Odd function as $(-x) = -x$ for all x, symmetric about origin.
- $f(x) = x+1$. Odd function ?

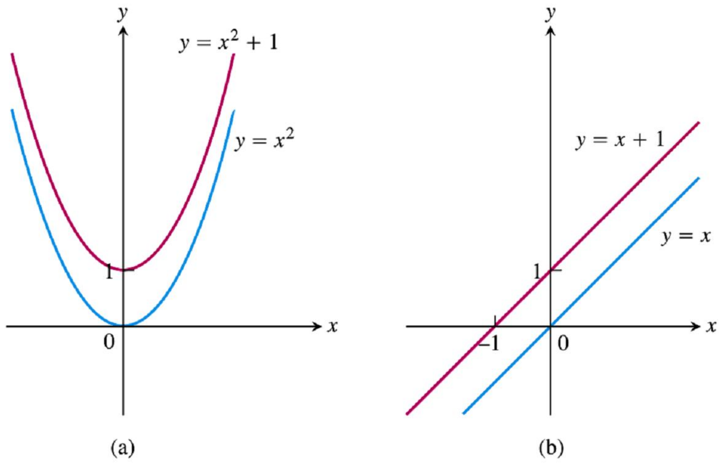


FIGURE 1.47 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd. The symmetry about the origin is lost (Example 2).

1.5

Combining Functions;
Shifting and Scaling Graphs

Sums, differences, products and quotients

- f, g are functions
- For $x \in D(f) \cap D(g)$, we can define the functions of
- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$,
- $(cf)(x) = cf(x)$, c a real number
- $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}, \quad g(x) \neq 0$

Example 1

- $f(x) = \sqrt{x}$, $g(x) = \sqrt{(1-x)}$,
- The domain common to both f,g is
- $D(f) \cap D(g) = [0,1]$ (work it out)

Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1 - x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1 - x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

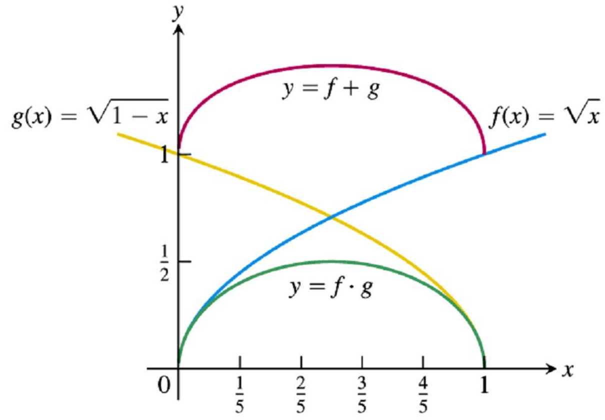


FIGURE 1.51 The domain of the function $f + g$ is the intersection of the domains of f and g, the interval $[0, 1]$ on the x-axis where these domains overlap. This interval is also the domain of the function $f \cdot g$ (Example 1).

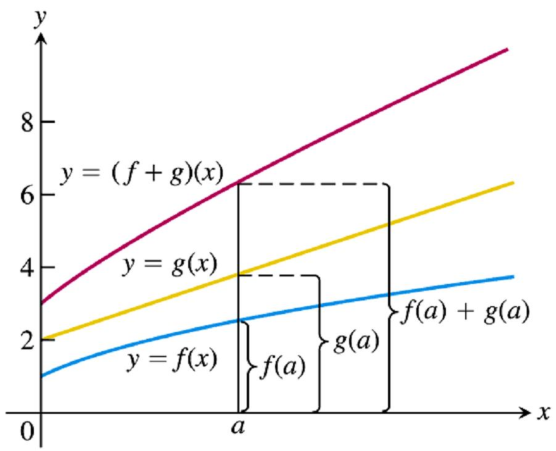


FIGURE 1.50 Graphical addition of two functions.

Composite functions

- Another way of combining functions

DEFINITION **Composition of Functions**

If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

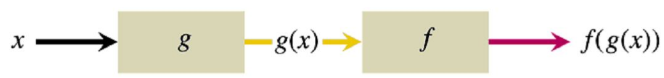


FIGURE 1.52 Two functions can be composed at x whenever the value of one function at x lies in the domain of the other. The composite is denoted by $f \circ g$.

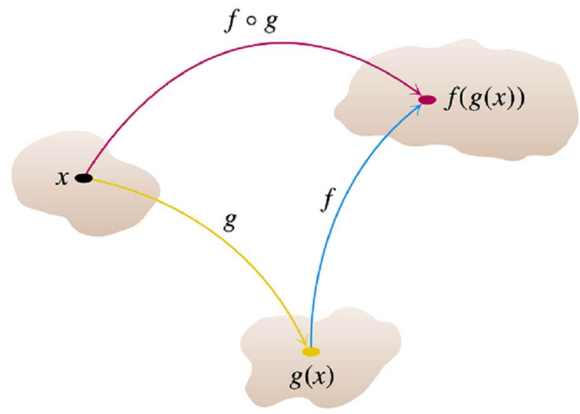


FIGURE 1.53 Arrow diagram for $f \circ g$.

Example 2

- Viewing a function as a composite
- $y(x) = \sqrt{(1 - x^2)}$ is a composite of
- $g(x) = 1 - x^2$ and $f(x) = \sqrt{x}$
- i.e. $y(x) = f\,[g(x)] = \sqrt{(1 - x^2)}$
- Domain of the composite function is $|x| \leqslant 1$, or [-1,1]
- Is $f\,[g(x)] = g\,[f(x)]$?

Example 3

- Read it yourself
- Make sure that you know how to work out the domains and ranges of each composite functions listed

Shifting a graph of a function

Shift Formulas

Vertical Shifts

$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$
	Shifts it *down* $\lvert k \rvert$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$	Shifts the graph of f *left* h units if $h > 0$
	Shifts it *right* $\lvert h \rvert$ units if $h < 0$

Example 4

- (a) $y = x^2$, $y = x^2 + 1$
- (b) $y = x^2$, $y = x^2 - 2$

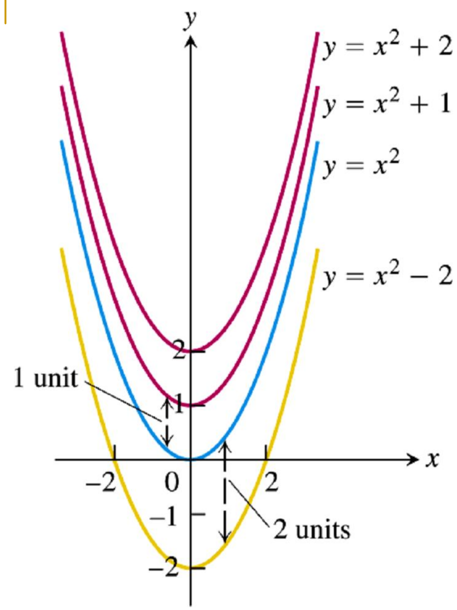


FIGURE 1.54 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Example 4a and b).

Example 4

- (c) $y = x^2$, $y = (x + 3)^2$, $y = (x - 3)^2$

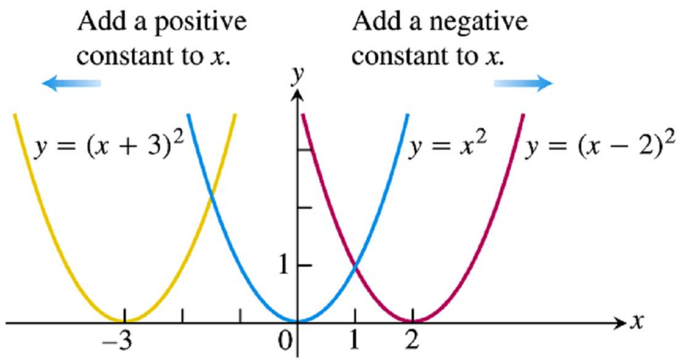


FIGURE 1.55 To shift the graph of $y = x^2$ to the left, we add a positive constant to x. To shift the graph to the right, we add a negative constant to x (Example 4c).

Example 4

- (d) $y = |x|$, $y = |x - 2| - 1$

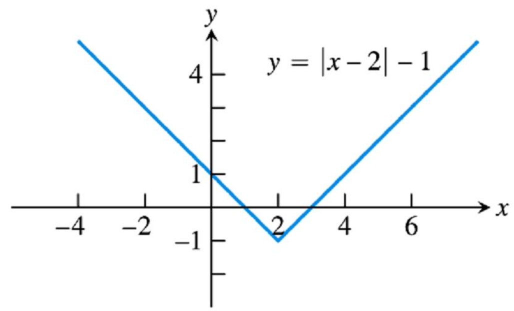


FIGURE 1.56 Shifting the graph of $y = |x|$ 2 units to the right and 1 unit down (Example 4d).

Scaling and reflecting a graph of a function

- To scale a graph of a function is to stretch or compress it, vertically or horizontally.
- This is done by multiplying a constant *c* to the function or the independent variable

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$,

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$,

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

Example 5(a)

- Vertical stretching and compression of the graph *y* = √*x* by a factor or 3

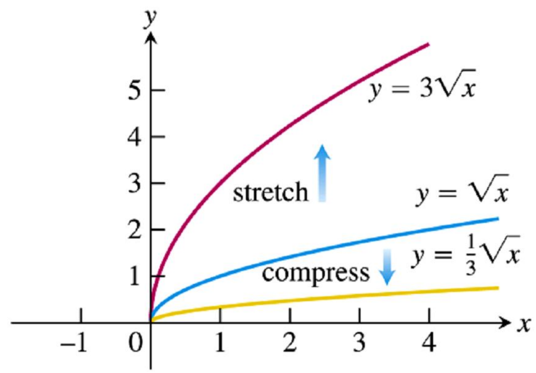


FIGURE 1.57 Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 5a).

Example 5(b)

- Horizontal stretching and compression of the graph $y = \sqrt{x}$ by a factor of 3

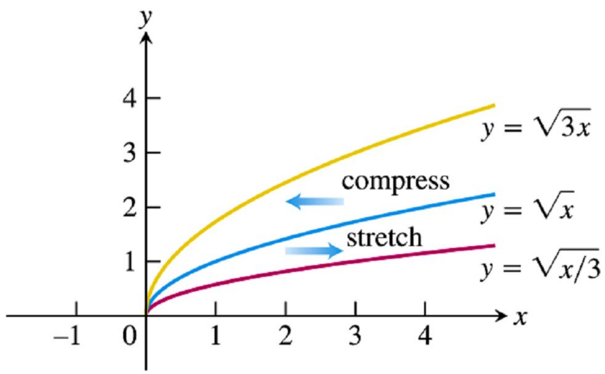


FIGURE 1.58 Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 5b).

Example 5(c)

- Reflection across the *x*- and *y*- axes
- $c = -1$

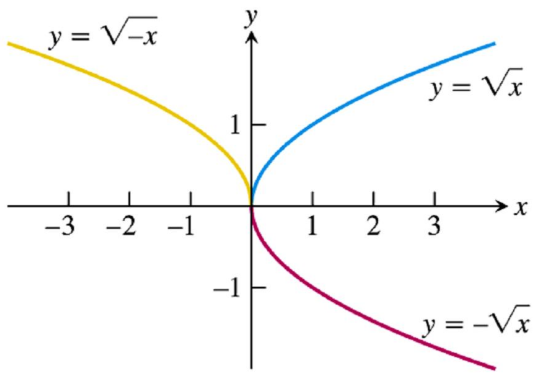


FIGURE 1.59 Reflections of the graph $y = \sqrt{x}$ across the coordinate axes (Example 5c).

Example 6

- Read it yourself

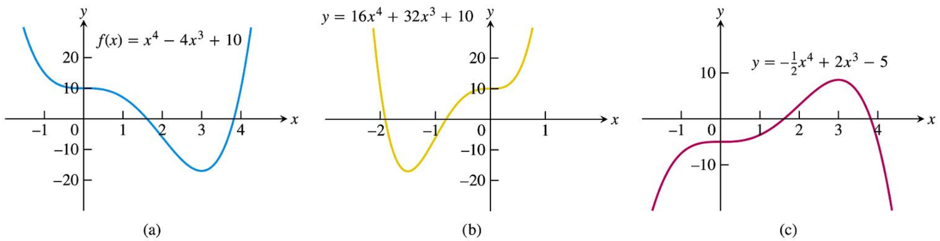


FIGURE 1.60 (a) The original graph of f. (b) The horizontal compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the y-axis. (c) The vertical compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the x-axis (Example 6).

1.6

Trigonometric Functions

Radian measure

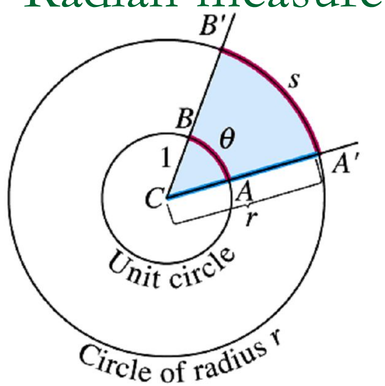


FIGURE 1.63 The radian measure of angle ACB is the length θ of arc AB on the unit circle centered at C. The value of θ can be found from any other circle, however, as the ratio s/r. Thus $s = r\theta$ is the length of arc on a circle of radius r when θ is measured in radians.

Conversion Formulas

$$1 \text{ degree} = \frac{\pi}{180} (\approx 0.02) \text{ radians}$$

Degrees to radians: multiply by $\frac{\pi}{180}$

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57) \text{ degrees}$$

Radians to degrees: multiply by $\frac{180}{\pi}$

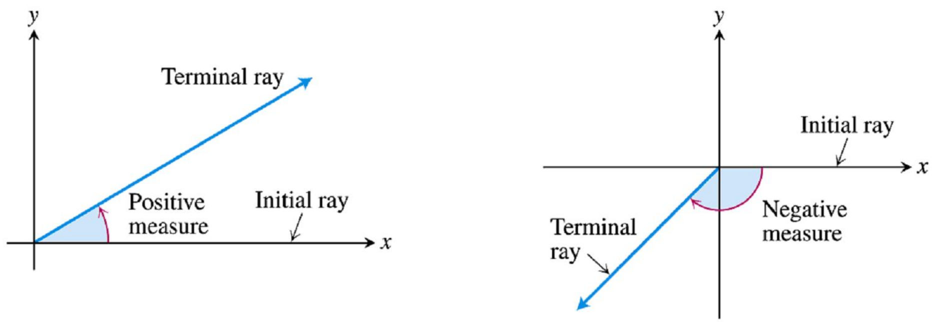


FIGURE 1.65 Angles in standard position in the xy-plane.

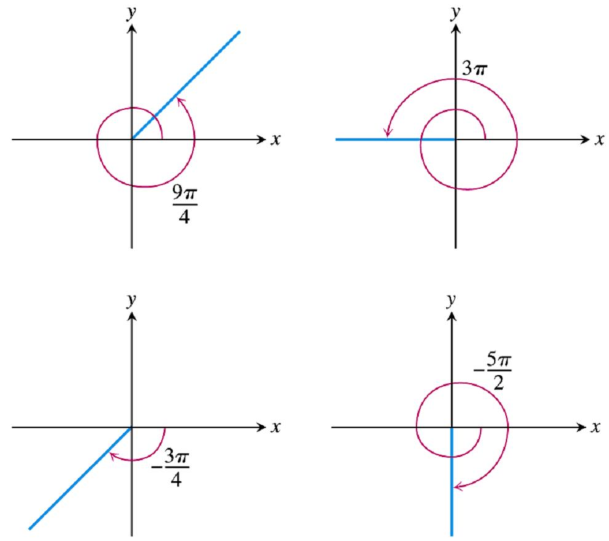
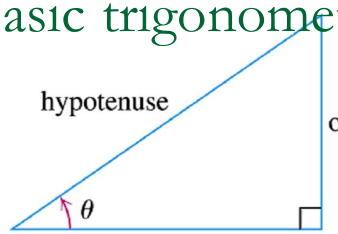


FIGURE 1.66 Nonzero radian measures can be positive or negative and can go beyond 2π.

Angle convention

- Be noted that angle will be expressed in terms of radian unless otherwise specified.
- Get used to the change of the unit

The six basic trigonometric functions



$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\csc\theta = \frac{\text{hyp}}{\text{opp}}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\sec\theta = \frac{\text{hyp}}{\text{adj}}$

$\tan\theta = \frac{\text{opp}}{\text{adj}}$ $\cot\theta = \frac{\text{adj}}{\text{opp}}$

FIGURE 1.67 Trigonometric ratios of an acute angle.

Generalised definition of the six trigo functions

- Define the trigo functios in terms of the coordinats of the point *P*(*x*,*y*) on a circle of radius *r*

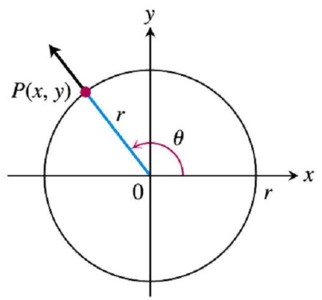


FIGURE 1.68 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.

- **sine:** $\sin\theta = y/r$
- **cosine:** $\cos\theta = x/r$
- **tangent:** $\tan\theta = y/x$
- **cosecant:** $\csc\theta = r/y$
- **secant:** $\sec\theta = r/x$
- **cotangent:** $\cot\theta = x/y$

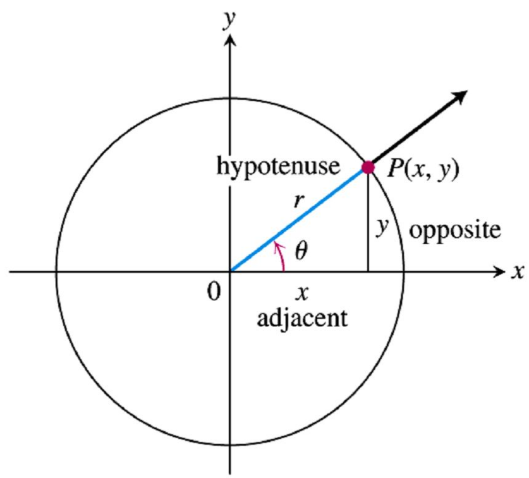


FIGURE 1.69 The new and old definitions agree for acute angles.

Mnemonic to remember when the basic trigo functions are positive or negative

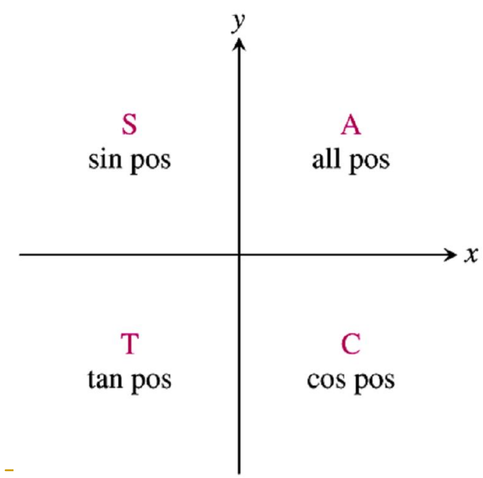


FIGURE 1.70 The CAST rule, remembered by the statement "All Students Take Calculus," tells which trigonometric functions are positive in each quadrant.

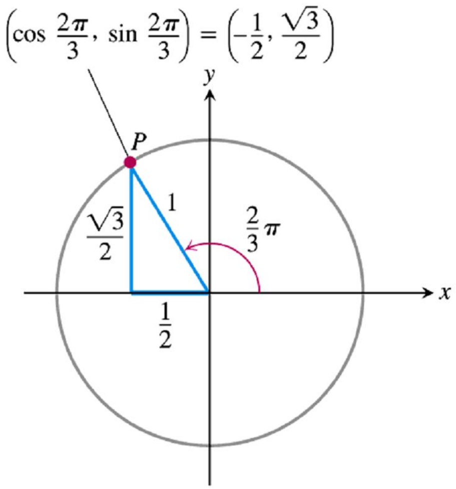


FIGURE 1.71 The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	**−180**	**−135**	**−90**	**−45**	**0**	**30**	**45**	**60**	**90**	**120**	**135**	**150**	**180**	**270**	**360**
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

Periodicity and graphs of the trigo functions

Trigo functions are also periodic.

DEFINITION Periodic Function

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

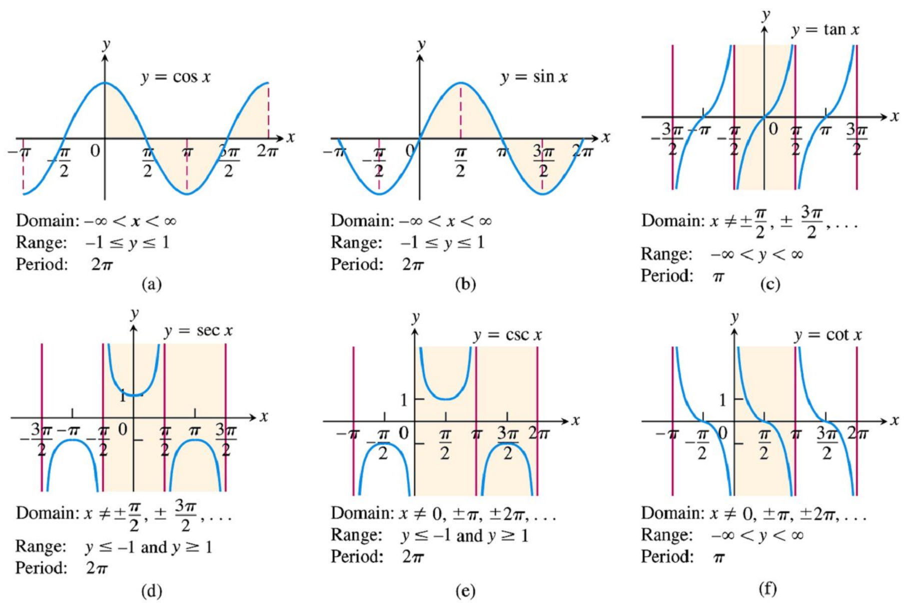


FIGURE 1.73 Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Parity of the trigo functions

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

The parity is easily deduced from the graphs.

Identities

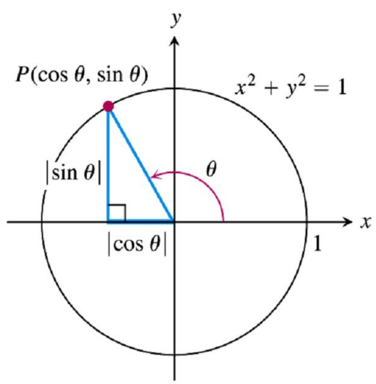


FIGURE 1.74 The reference triangle for a general angle θ.

Applying Pythagorean theorem to the right triangle leads to the identity

$$\cos^2\theta + \sin^2\theta = 1. \qquad (1)$$

Dividing identity (1) by $\cos^2\theta$ and $\sin^2\theta$ in turn gives the next two identities

$$1 + \tan^2\theta = \sec^2\theta.$$
$$1 + \cot^2\theta = \csc^2\theta.$$

Addition Formulas

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A + B) = \sin A\cos B + \cos A\sin B \qquad (2)$$

There are also similar formulas for cos (A-B) and sin (A-B). Do you know how to deduce them?

Double-Angle Formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta \qquad (3)$$

Identity (3) is derived by setting $A = B$ in (2)

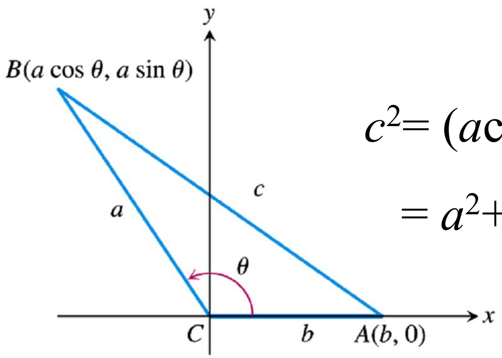
Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad (4)$$
$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \qquad (5)$$

Identities (4,5) are derived by combining (1) and (3(i))

Law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{6}$$



$$c^2 = (a\cos\theta - b)^2 + (a\sin\theta)^2$$

$$= a^2 + b^2 - 2ab\cos\theta$$

FIGURE 1.75 The square of the distance between A and B gives the law of cosines.

Chapter 2

Limits and Continuity

2.1

Rates of Change and Limits

Average Rates of change and Secant Lines

- Given an arbitrary function $y=f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y, Δy, by the length Δx

DEFINITION Average Rate of Change over an Interval
The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

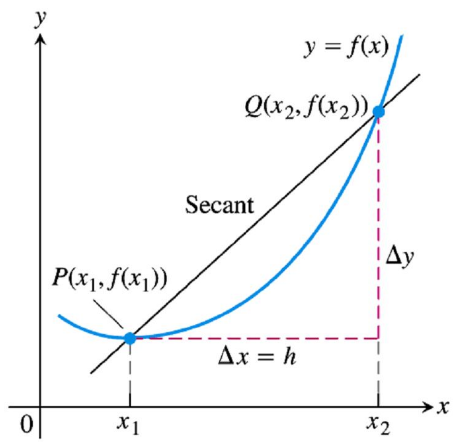


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Example 4

- Figure 2.2 shows how a population of fruit flies grew in a 50-day experiment.
- (a) Find the average growth rate from day 23 to day 45.
- (b) How fast was the number of the flies growing on day 23?

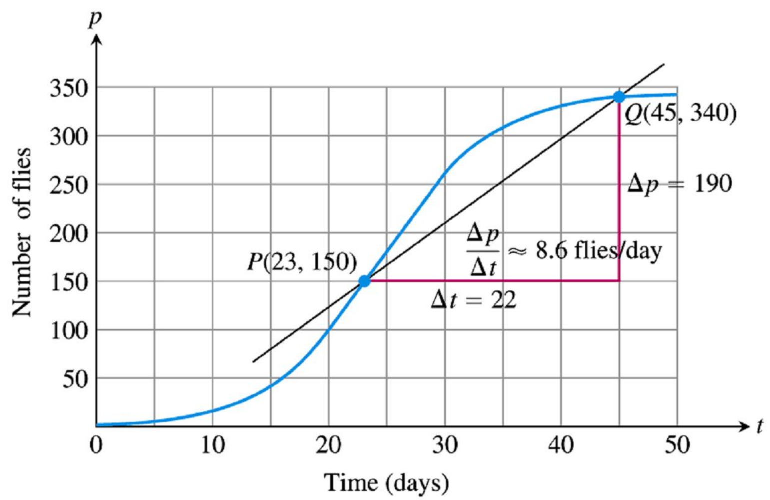


FIGURE 2.2 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line.

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\frac{340-150}{45-23} \approx 8.6$
(40, 330)	$\frac{330-150}{40-23} \approx 10.6$
(35, 310)	$\frac{310-150}{35-23} \approx 13.3$
(30, 265)	$\frac{265-150}{30-23} \approx 16.4$

Slop at $P \approx (250 - 0)/(35-14) = 16.7$ flies/day

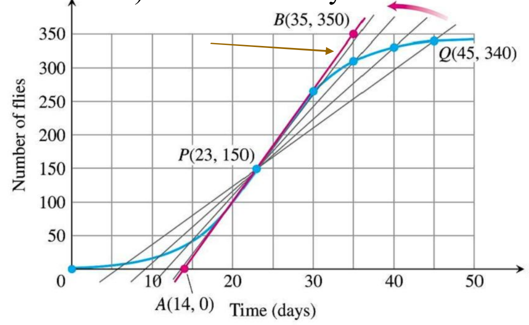


FIGURE 2.3 The positions and slopes of four secants through the point P on the fruit fly graph (Example 4).

The grow rate at day 23 is calculated by examining the average rates of change over increasingly short time intervals starting at day 23. Geometrically, this is equivalent to evaluating the slopes of secants from P to Q with Q approaching P.

Limits of function values

- Informal definition of limit:
- Let f be a function defined on an open interval about x_0, except possibly at x_0 itself.
- If f gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0

$$\lim_{x \to x_0} f(x) = L$$

- "Arbitrarily close" is not yet defined here (hence the definition is informal).

Example 5

- How does the function behave near x=1?

$$f(x)=\frac{x^2-1}{x-1}$$

- Solution:

$$f(x)=\frac{(x-1)(x+1)}{x-1}=x+1 \quad \text{for } x\neq 1$$

TABLE 2.2 The closer x gets to 1, the closer $f(x)=(x^2-1)/(x-1)$ seems to get to 2

Values of x below and above 1	$f(x)=\frac{x^2-1}{x-1}=x+1, \quad x\neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

We say that $f(x)$ approaches the limit 2 as x approaches 1, $\lim_{x\to 1} f(x)=2$ or $\lim_{x\to 1}\frac{x^2-1}{x-1}=2$

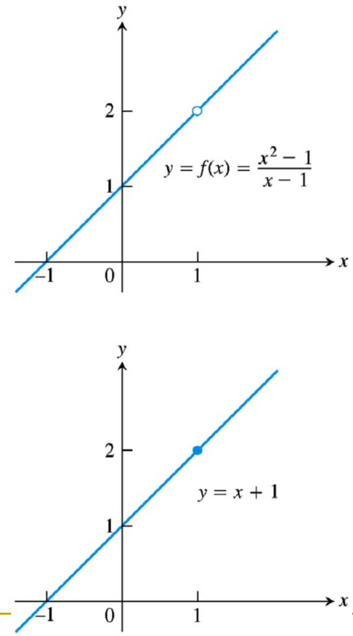


FIGURE 2.4 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 5).

Example 6

- The limit value does not depend on how the function is defined at x_0.

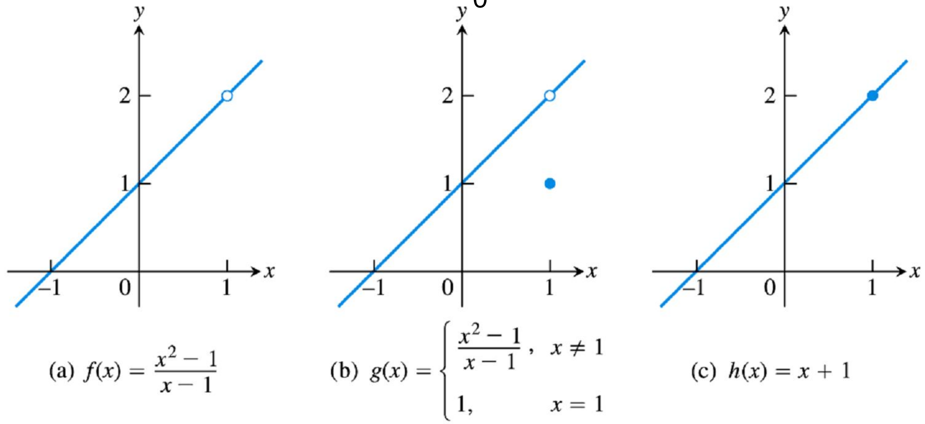
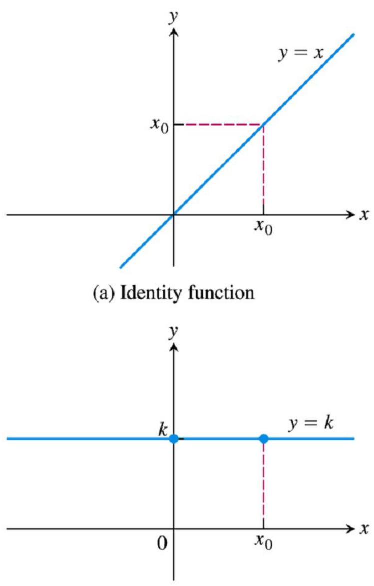


FIGURE 2.5 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 6).

Example 7

- In some special cases $\lim_{x \to x_0} f(x)$ can be evaluated by calculating $f(x_0)$. For example, constant function, rational function and identity function for which $x=x_0$ is defined
- (a) $\lim_{x \to 2} (4) = 4$ (constant function)
- (b) $\lim_{x \to -13} (4) = 4$ (constant function)
- (c) $\lim_{x \to 3} x = 3$ (identity function)
- (d) $\lim_{x \to 2} (5x-3) = 10 - 3 = 7$ (polynomial function of degree 1)
- (e) $\lim_{x \to -2} (3x+4)/(x+5) = (-6+4)/(-2+5) = -2/3$ (rational function)



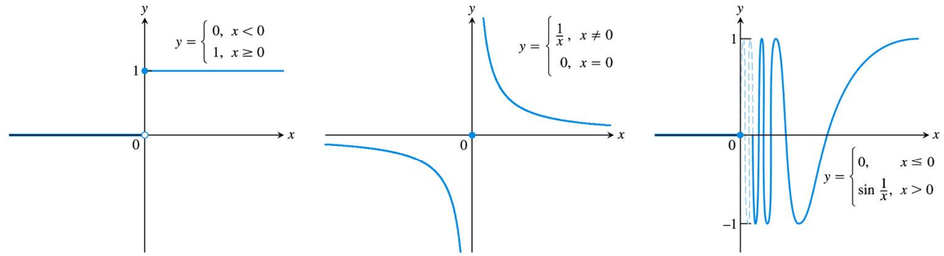
(a) Identity function

(b) Constant function

FIGURE 2.6 The functions in Example 8.

Example 9

- A function may fail to have a limit exist at a point in its domain.



(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

FIGURE 2.7 None of these functions has a limit as x approaches 0 (Example 9).

Jump — Grow to infinities — Oscillate

2.2

Calculating limits using the limits laws

The limit laws

- Theorem 1 tells how to calculate limits of functions that are arithmetic combinations of functions whose limit are already known.

THEOREM 1 Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x\to c}(f(x) + g(x)) = L + M$
The limit of the sum of two functions is the sum of their limits.
2. *Difference Rule*: $\lim_{x\to c}(f(x) - g(x)) = L - M$
The limit of the difference of two functions is the difference of their limits.
3. *Product Rule*: $\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$
The limit of a product of two functions is the product of their limits.
4. *Constant Multiple Rule*: $\lim_{x\to c}(k\cdot f(x)) = k\cdot L$
The limit of a constant times a function is the constant times the limit of the function.
5. *Quotient Rule*: $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.
6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then
$$\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$$
provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)
The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Example 1 Using the limit laws

- (a) $\lim_{x\to c}(x^3+4x^2-3)$
  $= \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3$
  (sum and difference rule)

  $= c^3 + 4c^2 - 3$
  (product and multiple rules)

Example 1

- (b) $\lim_{x\to c}(x^4+x^2-1)/(x^2+5)$
  $= \lim_{x\to c}(x^4+x^2-1) / \lim_{x\to c}(x^2+5)$

$=(\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1)/(\lim_{x\to c} x^2 + \lim_{x\to c} 5)$
$= (c^4 + c^2 - 1)/(c^2 + 5)$

Example 1

- (c) $\lim_{x\to -2} \sqrt{(4x^2-3)} = \sqrt{\lim_{x\to -2} (4x^2-3)}$
  Power rule with $r/s = \frac{1}{2}$

  $= \sqrt{[\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3]}$
  $= \sqrt{[4(-2)^2 - 3]} = \sqrt{13}$

THEOREM 2 Limits of Polynomials Can Be Found by Substitution
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero
If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example 2

- Limit of a rational function

$$\lim_{x\to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

Eliminating zero denominators algebraically

Identifying Common Factors
It can be shown that if $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.

Example 3 Canceling a common factor

- Evaluate $$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}$$
- Solution: We can't substitute x=1 since $f(x = 1)$ is not defined. Since $x\neq 1$, we can cancel the common factor of x-1:

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}=\lim_{x\to 1}\frac{(x-1)(x+2)}{x(x-1)}=\lim_{x\to 1}\frac{(x+2)}{x}=3$$

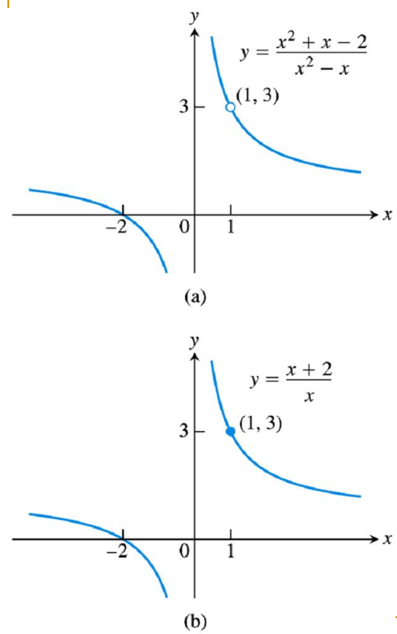


FIGURE 2.8 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$ (Example 3).

The Sandwich theorem

THEOREM 4 The Sandwich Theorem
Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.

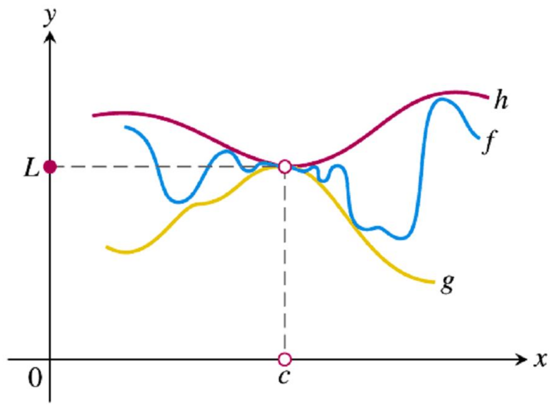


FIGURE 2.9 The graph of f is sandwiched between the graphs of g and h.

Example 6

- (a)
- The function $y = \sin\theta$ is sandwiched between $y = |\theta|$ and $y = -|\theta|$ for all values of θ. Since $\lim_{\theta\to 0}(-|\theta|) = \lim_{\theta\to 0}(|\theta|) = 0$, we have $\lim_{\theta\to 0}\sin\theta = 0$.
- (b)
- From the definition of cos θ,
  $0 \leqslant 1 - \cos\theta \leqslant |\theta|$ for all θ, and we have the limit $\lim_{x\to 0}\cos\theta = 1$

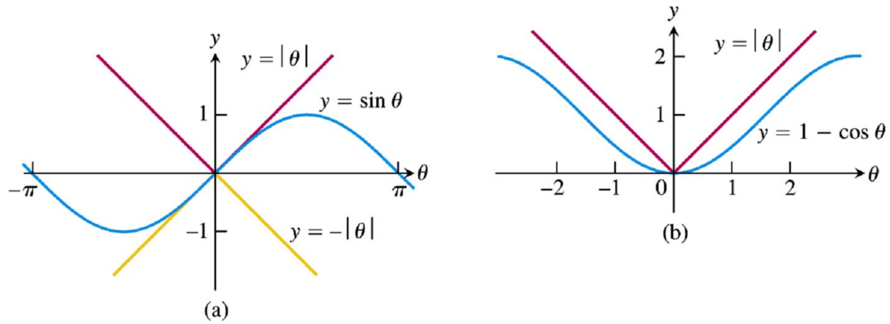


FIGURE 2.11 The Sandwich Theorem confirms that (a) $\lim_{\theta\to 0}\sin\theta = 0$ and (b) $\lim_{\theta\to 0}(1 - \cos\theta) = 0$ (Example 6).

Example 6(c)

- For any function $f(x)$, if $\lim_{x\to 0}(|f(x)|) = 0$, then $\lim_{x\to 0} f(x) = 0$ due to the sandwich theorem.
- Proof: $-|f(x)| \leqslant f(x) \leqslant |f(x)|$.
- Since $\lim_{x\to 0}(|f(x)|) = \lim_{x\to 0}(-|f(x)|) = 0$
- $\Rightarrow \lim_{x\to 0} f(x) = 0$

2.3

The Precise Definition of a Limit

Example 1 A linear function

- Consider the linear function $y = 2x - 1$ near $x_0 = 4$. Intuitively it is close to 7 when x is close to 4, so $\lim_{x \to 0} (2x-1)=7$. How close does x have to be so that $y = 2x - 1$ differs from 7 by less than 2 units?

Solution

- For what value of x is $|y-7|< 2$?
- First, find $|y-7|<2$ in terms of x:

$|y-7|<2 \equiv |2x-8|<2$
$\equiv -2< 2x-8 < 2$
$\equiv 3 < x < 5$
$\equiv -1 < x - 4 < 1$

Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0=7$.

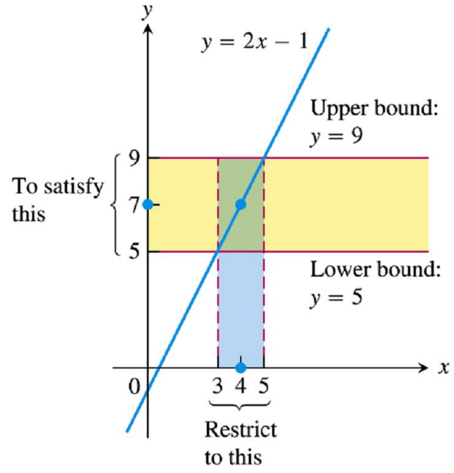


FIGURE 2.12 Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Example 1).

Definition of limit

DEFINITION Limit of a Function
Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Definition of limit

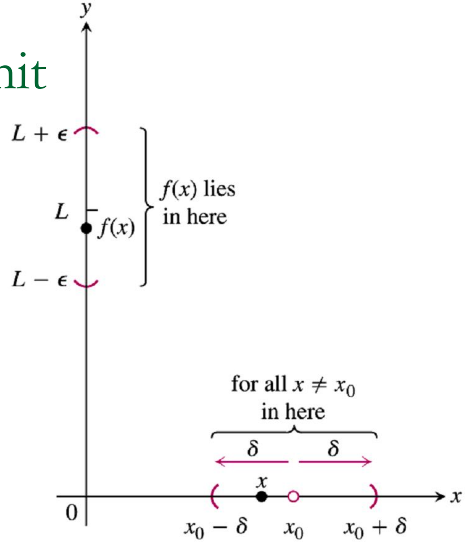


FIGURE 2.14 The relation of δ and ϵ in the definition of limit.

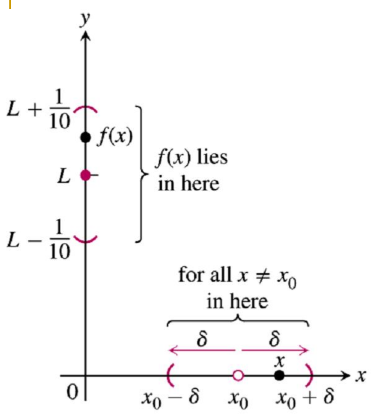
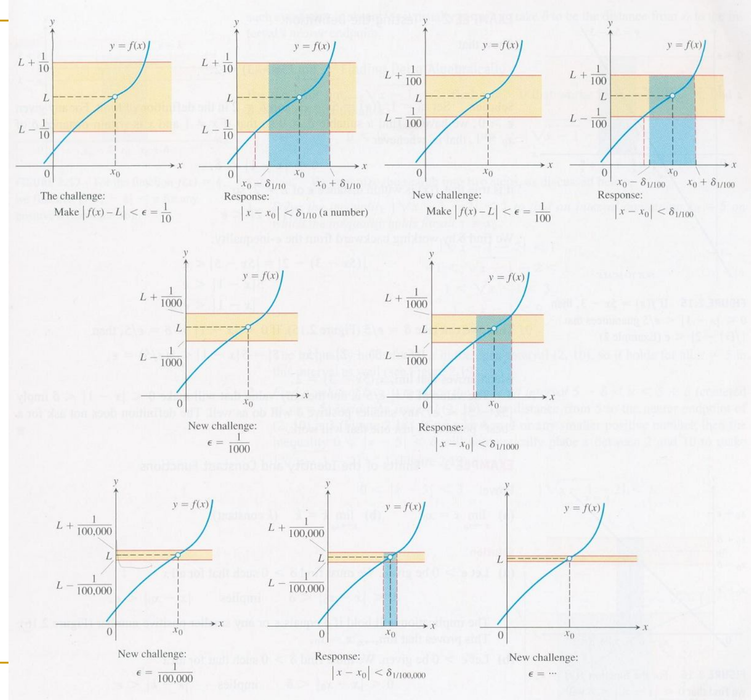


FIGURE 2.13 How should we define $\delta > 0$ so that keeping x within the interval $(x_0 - \delta, x_0 + \delta)$ will keep $f(x)$ within the interval $\left(L - \frac{1}{10}, L + \frac{1}{10}\right)$?

- The problem of proving L as the limit of $f(x)$ as x approaches x_0 is a problem of proving the existence of δ, such that whenever
- $x_0 - \delta < x < x_0 + \delta$,
- $L+\varepsilon < f(x) < L-\varepsilon$ for any arbitrarily small value of ε.
- As an example in Figure 2.13, given $\varepsilon = 1/10$, can we find a corresponding value of δ?
- How about if $\varepsilon = 1/100$? $\varepsilon = 1/1234$?
- If for any arbitrarily small value of ε we can always find a corresponding value of δ, then we has successfully proven that L is the limit of f as x approaches x_0



Example 2 Testing the definition

- Show that

$$\lim_{x \to 1} (5x - 3) = 2$$

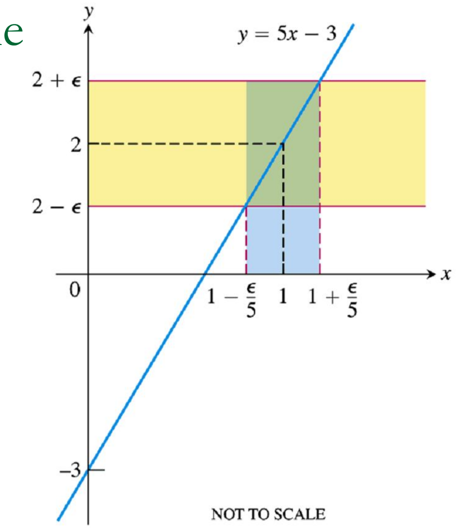


FIGURE 2.15 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

Solution

- Set x_0=1, $f(x)$=5x-3, L=2.
- For any given ε, we have to find a suitable $\delta > 0$ so that whenever

  $0<|x-1|<\delta$, $x\neq 1$,

  it is true that $f(x)$ is within distance ε of L=2, i.e.

  $|f(x)-2|<\varepsilon$.

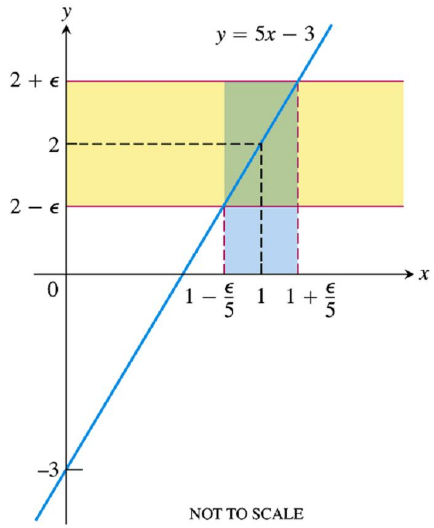
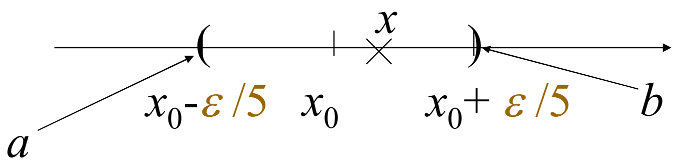


FIGURE 2.15 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

- First, obtain an open interval (a,b) in which $|f(x) - 2| < \varepsilon \equiv |5x - 5| < \varepsilon \equiv$
  $-\varepsilon/5 < x - 1 < \varepsilon/5 \equiv -\varepsilon/5 < x - x_0 < \varepsilon/5$



- choose $\delta < \varepsilon/5$. This choice will guarantee that $|f(x) - L| < \varepsilon$ whenever $x_0-\delta < x < x_0 + \delta$.

  We have shown that for any value of ε given, we can always find an corresponding value of δ that meets the "challenge" posed by an ever diminishing ε. This is an proof of existence.

  Thus we have proven that the limit for $f(x)$=5x-3 is L=2 when $x \to x_0$=1.

Example 3(a)

- Limits of the identity functions
- Prove

  $$\lim_{x\to x_0} x = x_0$$

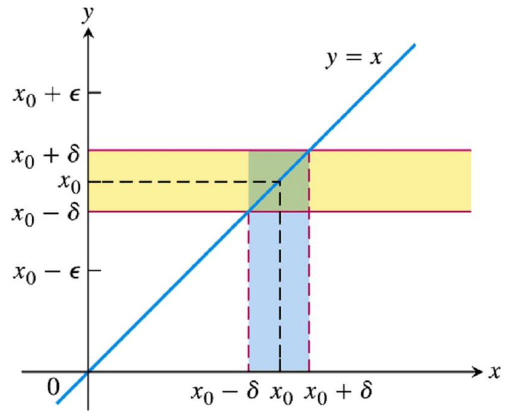


FIGURE 2.16 For the function $f(x) = x$, we find that $0 < |x - x_0| < \delta$ will guarantee $|f(x) - x_0| < \epsilon$ whenever $\delta \le \epsilon$ (Example 3a).

Solution

- Let $\varepsilon > 0$. We must find $\delta > 0$ such that for all x, $0 < |x-x_0| < \delta$ implies $|f(x)-x_0| < \varepsilon$., here, $f(x)=x$, the identity function.
- Choose $\delta < \varepsilon$ will do the job.
- The proof of the existence of δ proves

  $$\lim_{x\to x_0} x = x_0$$

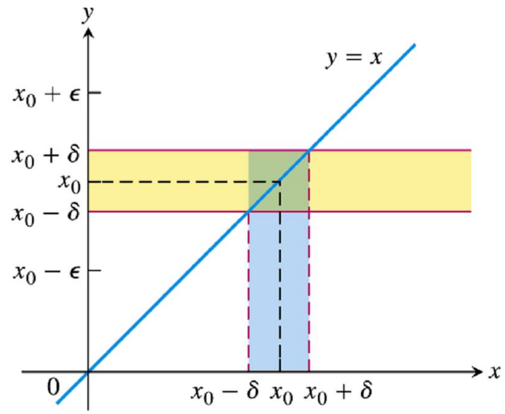


FIGURE 2.16 For the function $f(x) = x$, we find that $0 < |x - x_0| < \delta$ will guarantee $|f(x) - x_0| < \epsilon$ whenever $\delta \le \epsilon$ (Example 3a).

Example 3(b)

- Limits constant functions
- Prove

$$\lim_{x \to x_0} k = k \quad (k \text{ constant})$$

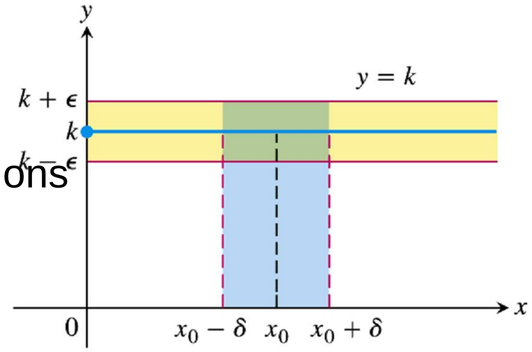


FIGURE 2.17 For the function $f(x) = k$, we find that $|f(x) - k| < \epsilon$ for any positive δ (Example 3b).

Solution

- Let $\varepsilon > 0$. We must find $\delta > 0$ such that for all x, $0 < |x-x_0| < \delta$ implies $|f(x)- k| < \varepsilon$., here, $f(x)=k$, the constant function.
- Choose any δ will do the job.
- The proof of the existence of δ proves

$$\lim_{x \to x_0} k = k$$

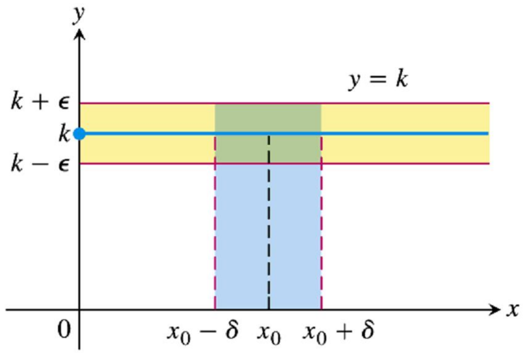


FIGURE 2.17 For the function $f(x) = k$, we find that $|f(x) - k| < \epsilon$ for any positive δ (Example 3b).

Finding delta algebraically for given epsilons

- Example 4: Finding delta algebraically
- For the limit $\lim_{x \to 5} \sqrt{x-1} = 2$

  find a $\delta > 0$ that works for $\varepsilon = 1$. That is, find a $\delta > 0$ such that for all x,

$$0 < |x-5| < \delta \Rightarrow 0 < \left|\sqrt{x-1} - 2\right| < 1$$

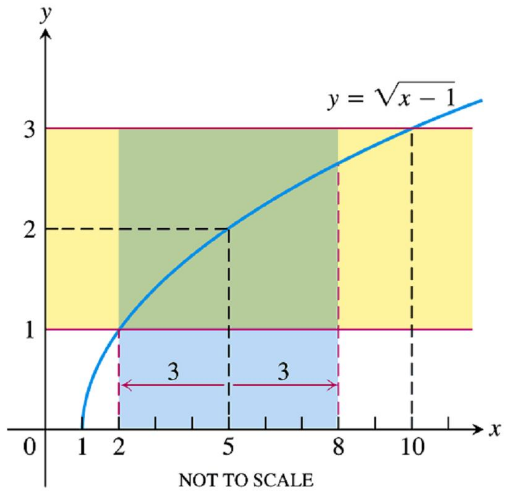


FIGURE 2.19 The function and intervals in Example 4.

Solution

- δ is found by working backward:

How to Find Algebraically a δ for a Given f, L, x_0, and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.
2. *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

Solution

- Step one: Solve the inequality $|f(x)-L|<\varepsilon$

$$0 < \left|\sqrt{x-1} - 2\right| < 1 \Rightarrow 2 < x < 10$$

- Step two: Find a value of $\delta > 0$ that places the open interval $(x_0-\delta, x_0+\delta)$ centered at x_0 inside the open interval found in step one. Hence, we choose $\delta = 3$ or a smaller number

By doing so, the inequality $0<|x - 5| < \delta$ will automatically place x between 2 and 10 to make $0 < |f(x) - 2| < 1$

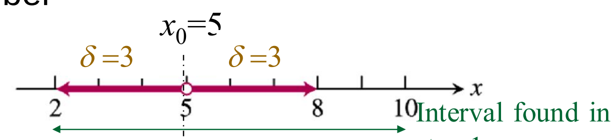


FIGURE 2.18 An open interval of radius 3 about $x_0 = 5$ will lie inside the open interval (2, 10).

Example 5

- Prove that

$$\lim_{x\to 2} f(x) = 4 \text{ if}$$

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$$

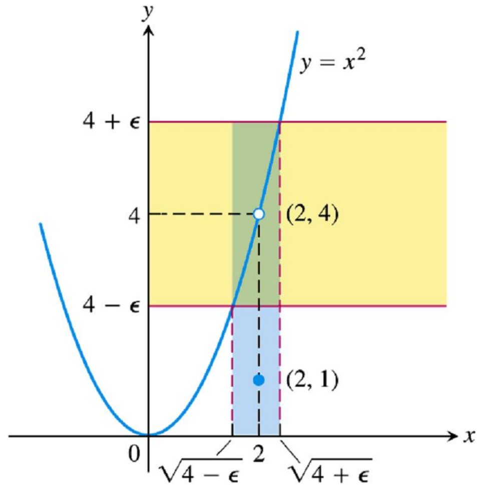


FIGURE 2.20 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \epsilon$.

Solution

- Step one: Solve the inequality $|f(x)-L|<\varepsilon$:

$$0 < \left|x^2 - 2\right| < \varepsilon \Rightarrow \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}, x \neq 2$$

- Step two: Choose
- $\delta <$ min [2-√(4-ε), √(4+ε) – 2]
- For all x,
- 0 < |x - 2| < δ ⇒ |$f(x)$-4|<ε
- This completes the proof.

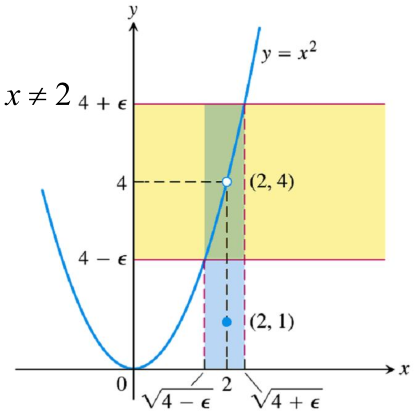


FIGURE 2.20 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \epsilon$.

2.4

One-Sided Limits and
Limits at Infinity

Two sided limit does not exist for y;

But

y does has two one-sided limits

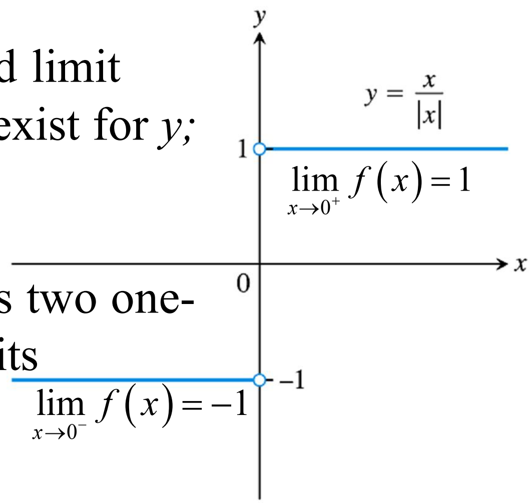


FIGURE 2.21 Different right-hand and left-hand limits at the origin.

One-sided limits

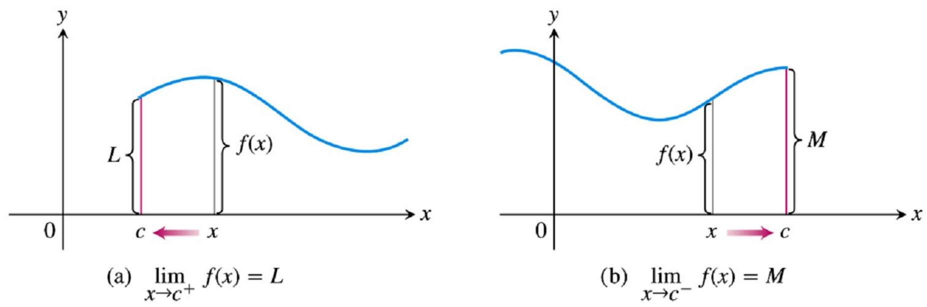


FIGURE 2.22 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

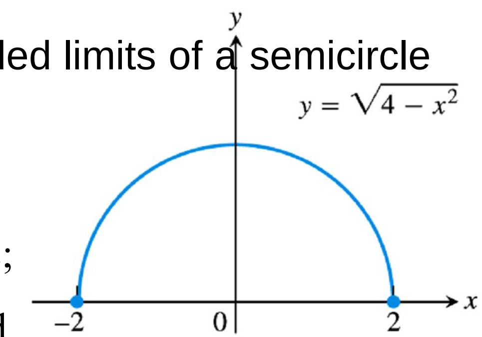
Right-hand limit Left-hand limit

Example 1

- One sided limits of a semicircle

No left hand limit at x= -2;

No two sided limit at x= -2;



No right hand limit at x=2;

No two sided limit at x= 2;

FIGURE 2.23 $\lim_{x\to 2^-} \sqrt{4 - x^2} = 0$ and $\lim_{x\to -2^+} \sqrt{4 - x^2} = 0$ (Example 1).

THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Example 2

- Limits of the function graphed in Figure 2.24
- Can you write down all the limits at *x*=0, *x*=1, *x*=2, *x*=3, *x*=4?
- What is the limit at other values of *x*?

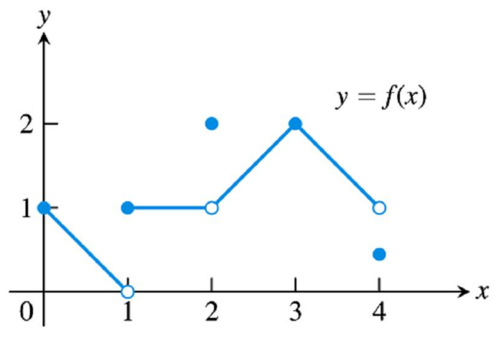


FIGURE 2.24 Graph of the function in Example 2.

Precise definition of one-sided limits

DEFINITIONS **Right-Hand, Left-Hand Limits**

We say that $f(x)$ has **right-hand limit L at x_0**, and write

$$\lim_{x \to x_0^+} f(x) = L \qquad \text{(See Figure 2.25)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit L at x_0**, and write

$$\lim_{x \to x_0^-} f(x) = L \qquad \text{(See Figure 2.26)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

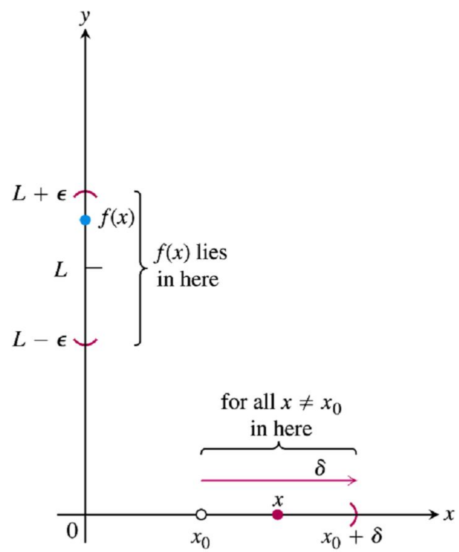


FIGURE 2.25 Intervals associated with the definition of right-hand limit.

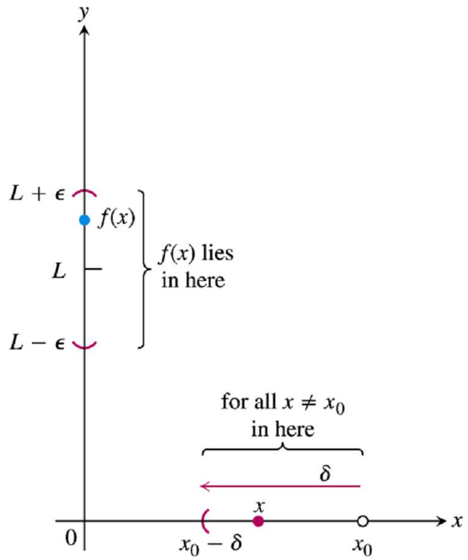


FIGURE 2.26 Intervals associated with the definition of left-hand limit.

Proof

Area $\Delta OAP = ½ \sin\theta$

Area sector $OAP = \theta/2$

Area $\Delta OAT = ½ \tan\theta$

$½ \sin\theta < \theta/2 < ½ \tan\theta$

$1 < \theta/\sin\theta < 1/\cos\theta$

$1 > \sin\theta/\theta > \cos\theta$

Taking limit $\theta \to 0^{\pm}$,

$$\lim_{\theta\to 0^{\pm}} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0} \frac{\sin\theta}{\theta}$$

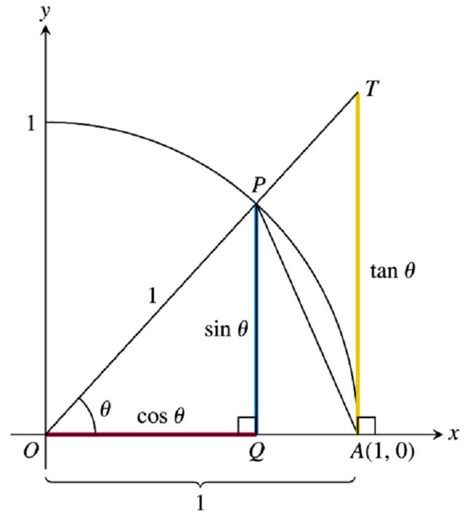


FIGURE 2.30 The figure for the proof of Theorem 7. $TA/OA = \tan\theta$, but $OA = 1$, so $TA = \tan\theta$.

Limits involving $(\sin\theta)/\theta$

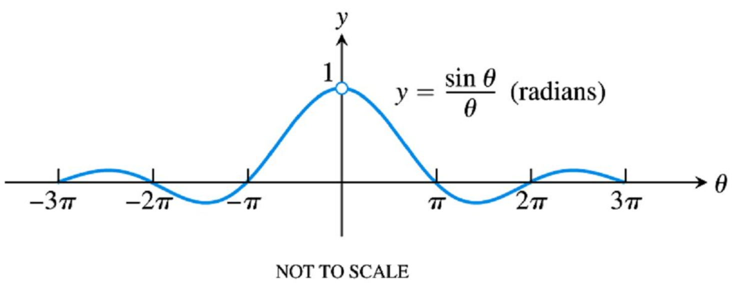


FIGURE 2.29 The graph of $f(\theta) = (\sin\theta)/\theta$.

THEOREM 7

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$

Example 5(a)

- Using theorem 7, show that

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$$

Example 5(b)

- Using theorem 7, show that

$$\lim_{x\to 0}\frac{\sin 2x}{5x}=\frac{2}{5}$$

Precise definition

DEFINITIONS Limit as x approaches ∞ or $-\infty$

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x\to-\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Finite limits as $x \to \pm\infty$

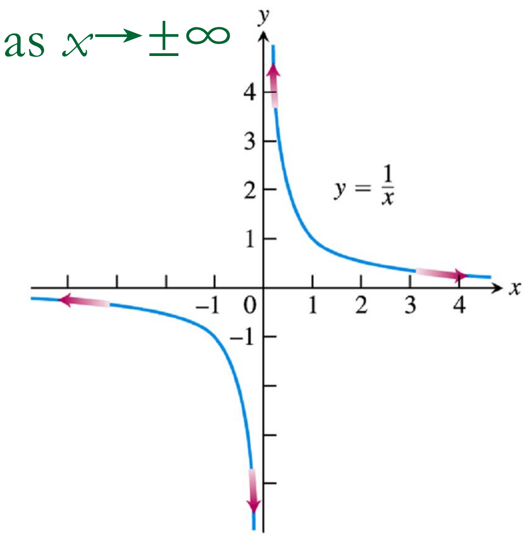


FIGURE 2.31 The graph of $y = 1/x$.

Example 6

- Limit at infinity for $f(x)=\dfrac{1}{x}$
- (a) Show that $\displaystyle\lim_{x\to\infty}\frac{1}{x}=0$
- (b) Show that $\displaystyle\lim_{x\to-\infty}\frac{1}{x}=0$

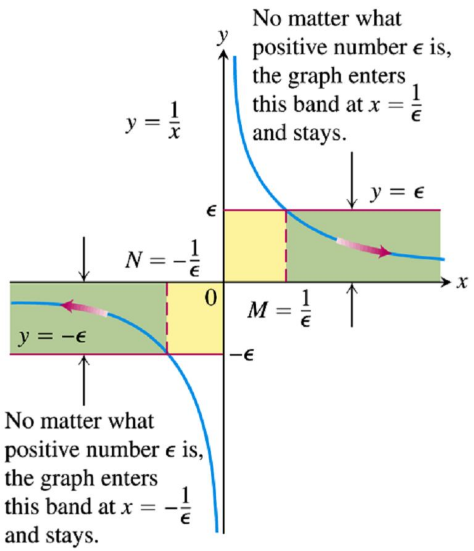


FIGURE 2.32 The geometry behind the argument in Example 6.

THEOREM 8 Limit Laws as $x \to \pm\infty$

If L, M, and k, are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \quad \text{and} \quad \lim_{x\to\pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to\pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to\pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x\to\pm\infty} (f(x)\cdot g(x)) = L\cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to\pm\infty} (k\cdot f(x)) = k\cdot L$
5. *Quotient Rule:* $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x\to\pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

Example 7(a)

- Using Theorem 8

$$\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} = 5+0=5$$

Example 7(b)

$$\lim_{x\to\infty}\frac{\pi\sqrt{3}}{x^2} = \pi\sqrt{3}\lim_{x\to\infty}\frac{1}{x^2}$$

$$=\pi\sqrt{3}\lim_{x\to\infty}\frac{1}{x}\cdot\lim_{x\to\infty}\frac{1}{x}$$

$$=\pi\sqrt{3}\cdot 0\cdot 0 = 0$$

Limits at infinity of rational functions

- Example 8

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to\infty}\frac{5+(8/x)-(3/x^2)}{3+(2/x^2)}=$$

$$=\frac{5+\lim_{x\to\infty}(8/x)-\lim_{x\to\infty}(3/x^2)}{3+\lim_{x\to\infty}(2/x^2)}=\frac{5+0-0}{3+0}=\frac{5}{3}$$

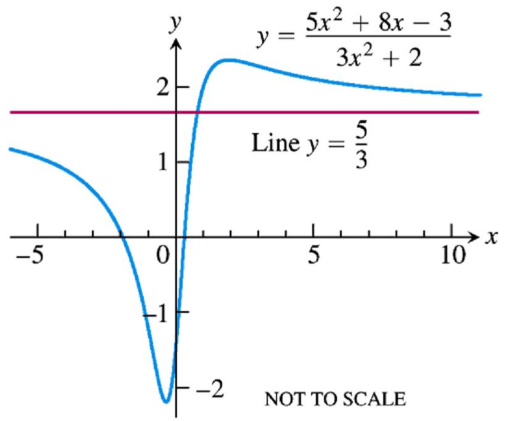


FIGURE 2.33 The graph of the function in Example 8. The graph approaches the line $y = 5/3$ as $|x|$ increases.

go back

Example 9

- Degree of numerator less than degree of denominator

$$\lim_{x\to\infty}\frac{11x+2}{2x^3-1}=\lim_{x\to\infty}\ldots=0$$

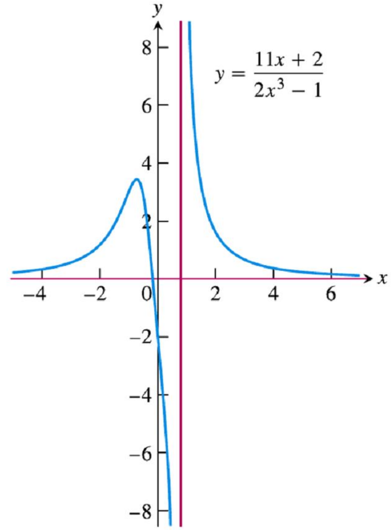


FIGURE 2.34 The graph of the function in Example 9. The graph approaches the x-axis as $|x|$ increases.

Horizontal asymptote

- x-axis is a horizontal asymptote

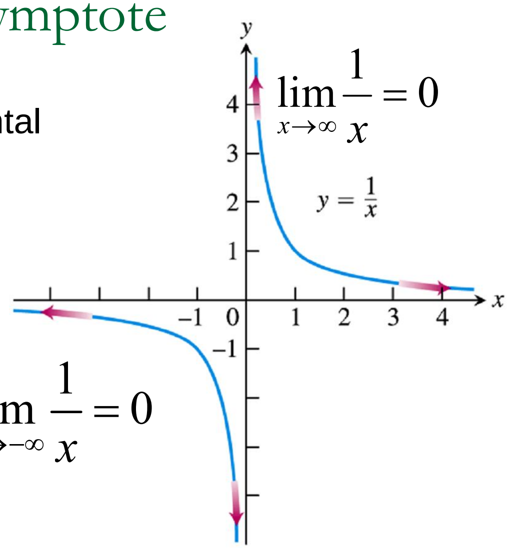


FIGURE 2.31 The graph of $y = 1/x$.

> **DEFINITION** **Horizontal Asymptote**
> A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either
>
> $$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

Figure 2.33 has the line $y=5/3$ as a horizontal asymptote on both the right and left because

$$\lim_{x\to\infty} f(x) = \frac{5}{3} \qquad \lim_{x\to-\infty} f(x) = \frac{5}{3}$$

Oblique asymptote

- Happen when the degree of the numerator polynomial is one greater than the degree of the denominator
- By long division, recast $f(x)$ into a linear function plus a remainder. The remainder shall → 0 as $x \to \pm\infty$. The linear function is the asymptote of the graph.

Example 12

- Find the oblique asymptote

$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

- Solution

$$f(x) = \frac{2x^2-3}{7x+4} = \underbrace{\left(\frac{2}{7}x - \frac{8}{49}\right)}_{\text{linear function}} + \frac{-115}{49(7x+4)}$$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(\frac{2}{7}x - \frac{8}{49}\right) + \lim_{x\to\pm\infty}\frac{-115}{49(7x+4)}$$

$$= \lim_{x\to\pm\infty}\left(\frac{2}{7}x - \frac{8}{49}\right) + 0 = \lim_{x\to\pm\infty}\left(\frac{2}{7}x - \frac{8}{49}\right)$$

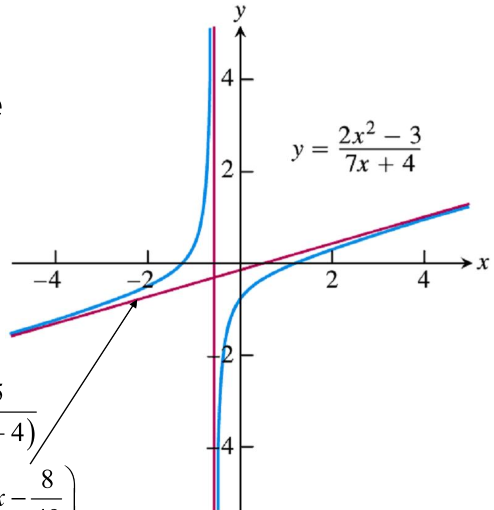


FIGURE 2.36 The function in Example 12 has an oblique asymptote.

2.5

Infinite Limits and Vertical Asymptotes

Infinite limit

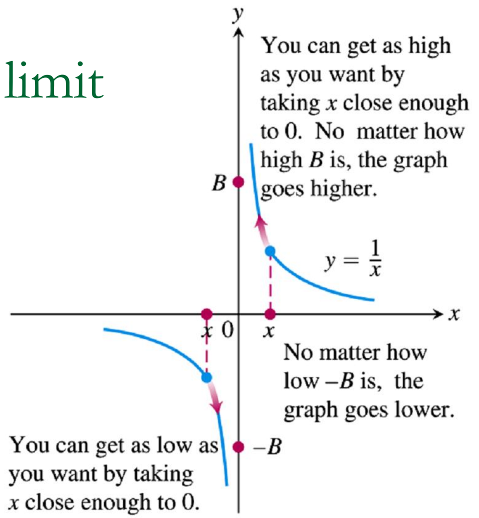


FIGURE 2.37 One-sided infinite limits:

$\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

Example 1

- Find $\lim_{x\to 1^+} \frac{1}{x-1}$ and $\lim_{x\to 1^-} \frac{1}{x-1}$

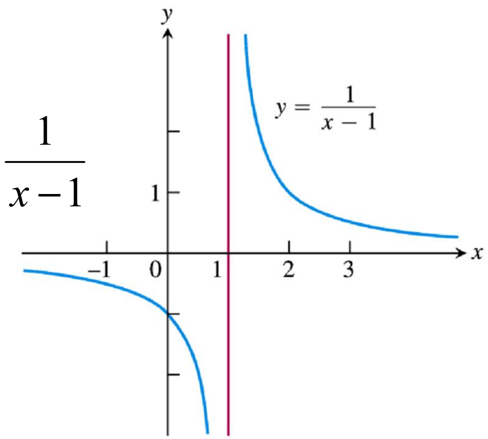


FIGURE 2.38 Near $x = 1$, the function $y = 1/(x-1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 1).

Example 2 Two-sided infinite limit

- Discuss the behavior of

(a) $f(x) = \frac{1}{x^2}$ near $x = 0$

(b) $g(x) = \frac{1}{(x+3)^2}$ near $x = -3$

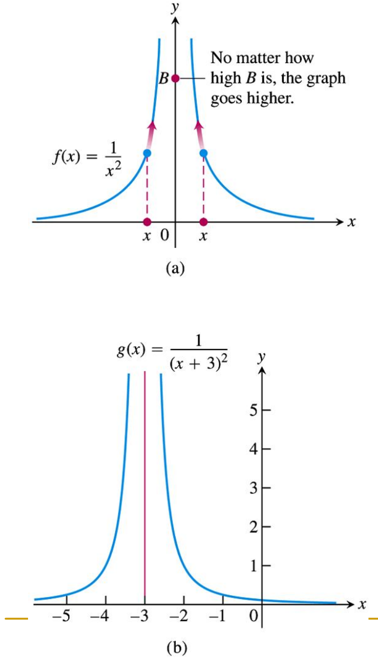


FIGURE 2.39 The graphs of the functions in Example 2. (a) $f(x)$ approaches infinity as $x \to 0$. (b) $g(x)$ approaches infinity as $x \to -3$.

Example 3

- Rational functions can behave in various ways near zeros of their denominators

$$(a)\lim_{x\to 2}\frac{(x-2)^2}{x^2-4}=\lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{(x-2)}{(x+2)}=0$$

$$(b)\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{(x+2)}=\frac{1}{4}$$

$$(c)\lim_{x\to 2^+}\frac{x-3}{x^2-4}=\lim_{x\to 2^+}\frac{x-3}{(x-2)(x+2)}=-\infty \quad \text{(note: } x>2\text{)}$$

$$(d)\lim_{x\to 2^-}\frac{x-3}{x^2-4}=\lim_{x\to 2^-}\frac{x-3}{(x-2)(x+2)}=+\infty \quad \text{(note: } x<2\text{)}$$

Example 3

$$(e)\lim_{x\to 2}\frac{x-3}{x^2-4}=\lim_{x\to 2}\frac{x-3}{(x-2)(x+2)} \quad \text{limit does not exist}$$

$$(f)\lim_{x\to 2}\frac{2-x}{(x-2)^3}=-\lim_{x\to 2}\frac{x-2}{(x-2)(x-2)^2}=-\lim_{x\to 2}\frac{1}{(x-2)^2}=-\infty$$

Precise definition of infinite limits

DEFINITIONS **Infinity, Negative Infinity as Limits**

1. We say that **$f(x)$ approaches infinity as x approaches x_0**, and write
$$\lim_{x\to x_0} f(x) = \infty,$$
if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > B.$$

2. We say that **$f(x)$ approaches negative infinity as x approaches x_0**, and write
$$\lim_{x\to x_0} f(x) = -\infty,$$
if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) < -B.$$

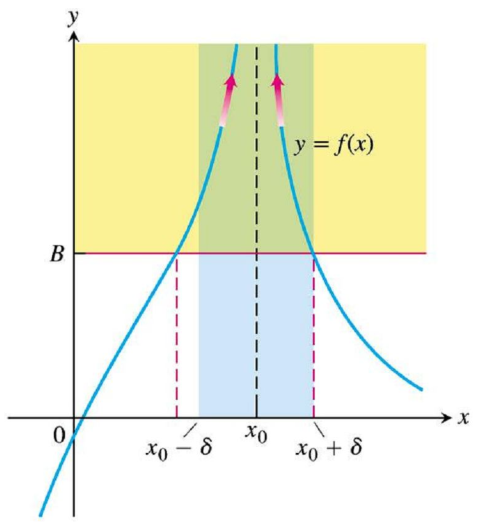


FIGURE 2.40 For $x_0 - \delta < x < x_0 + \delta$, the graph of $f(x)$ lies above the line $y = B$.

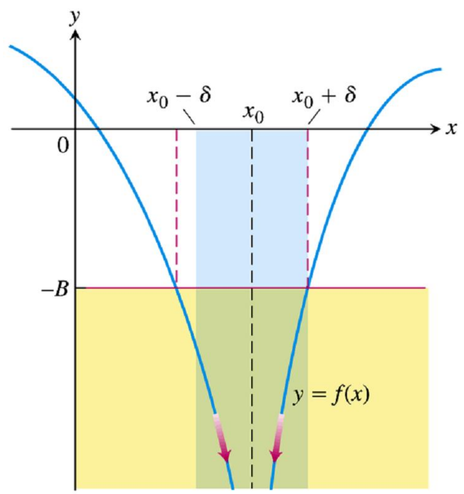


FIGURE 2.41 For $x_0 - \delta < x < x_0 + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

Example 4

- Using definition of infinite limit
- Prove that

$$\lim_{x\to 0}\frac{1}{x^2}=\infty$$

Given $B>0$, we want to find $\delta>0$ such that

$$0<|x-0|<\delta \quad \text{implies} \quad \frac{1}{x^2}>B$$

Example 4

Now

$\frac{1}{x^2}>B$ if and only if $x^2<1/B \equiv |x|<1/\sqrt{B}$

By choosing $\delta=1/\sqrt{B}$

(or any smaller positive number), we see that

$$|x|<\delta \quad \text{implies} \quad \frac{1}{x^2}>\frac{1}{\delta^2}\geq B$$

Vertical asymptotes

$$\lim_{x\to 0^+}\frac{1}{x}=\infty$$

$$\lim_{x\to 0^-}\frac{1}{x}=-\infty$$

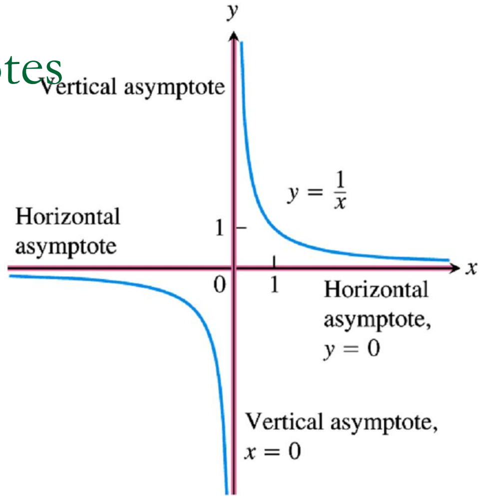


FIGURE 2.42 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

Example 5 Looking for asymptote

- Find the horizontal and vertical asymptotes of the curve

$$y=\frac{x+3}{x+2}$$

- Solution:

$$y=1+\frac{1}{x+2}$$

DEFINITION Vertical Asymptote
A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x\to a^-} f(x) = \pm\infty.$$

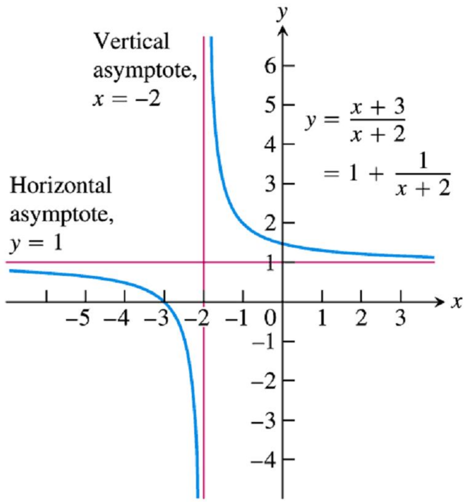


FIGURE 2.43 The lines $y = 1$ and $x = -2$ are asymptotes of the curve $y = (x + 3)/(x + 2)$ (Example 5).

Asymptote need not be two-sided

- Example 6

$$f(x) = -\frac{8}{x^2 - 2}$$

- Solution:

$$f(x) = -\frac{8}{x^2 - 2} = -\frac{8}{(x-2)(x+2)}$$

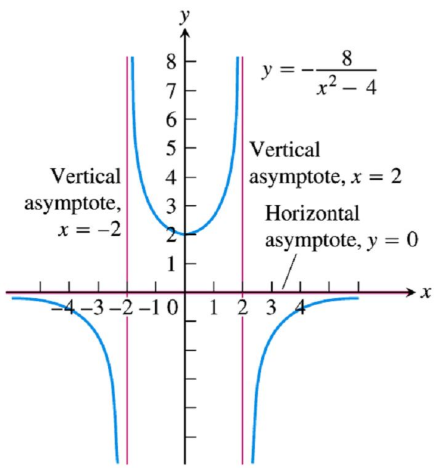


FIGURE 2.44 Graph of $y = -8/(x^2 - 4)$. Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided (Example 6).

Example 8

- A rational function with degree of freedom of numerator greater than degree of denominator

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

- Solution:

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\frac{x}{2} + 1}_{\text{linear}} + \underbrace{\frac{1}{2x - 4}}_{\text{remainder}}$$

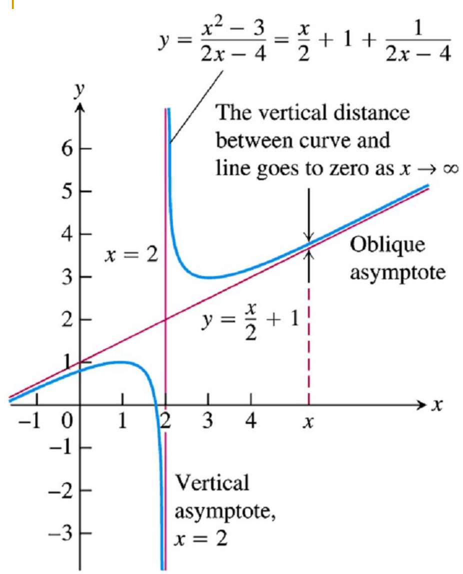


FIGURE 2.47 The graph of $f(x) = (x^2 - 3)/(2x - 4)$ has a vertical asymptote and an oblique asymptote (Example 8).

2.6

Continuity

Continuity at a point

- Example 1
- Find the points at which the function f in Figure 2.50 is continuous and the points at which f is discontinuous.

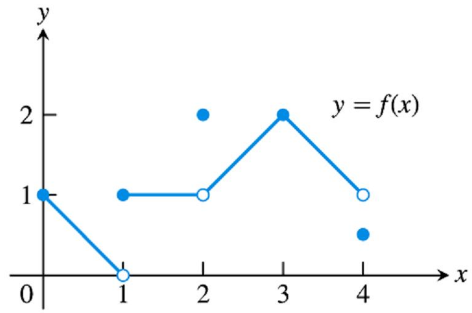


FIGURE 2.50 The function is continuous on $[0, 4]$ except at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

- f continuous:
- At $x = 0$
- At $x = 3$
- At $0 < c < 4$, $c \neq 1,2$

- f discontinuous:
- At $x = 1$
- At $x = 2$
- At $x = 4$
- $0 > c$, $c > 4$
- Why?

- To define the continuity at a point in a function's domain, we need to
- define continuity at an interior point
- define continuity at an endpoint

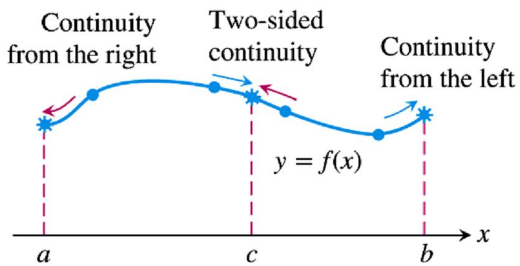


FIGURE 2.51 Continuity at points a, b, and c.

DEFINITION Continuous at a Point

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

Example 2

- A function continuous throughout its domain

$$f(x) = \sqrt{4 - x^2}$$

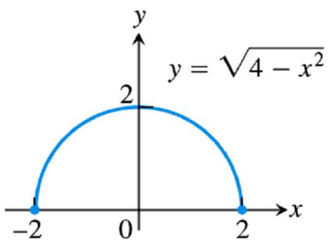


FIGURE 2.52 A function that is continuous at every domain point (Example 2).

Example 3

- The unit step function has a jump discontinuity

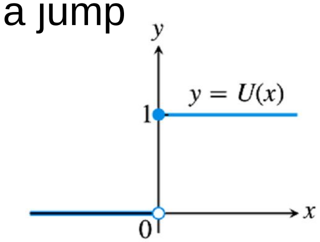


FIGURE 2.53 A function that is right-continuous, but not left-continuous, at the origin. It has a jump discontinuity there (Example 3).

Summarize continuity at a point in the form of a test

Continuity Test
A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

For one-sided continuity and continuity at an endpoint, the limits in part 2 and part 3 of the test should be replaced by the appropriate one-sided limits.

Example 4

- The greatest integer function,
- y=int x
- The function is not continuous at the integer points since limit does not exist there (left and right limits not agree)

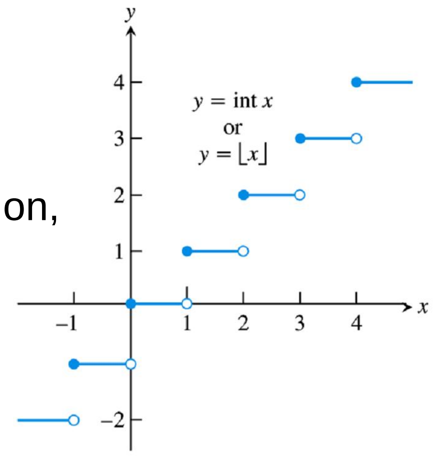


FIGURE 2.54 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

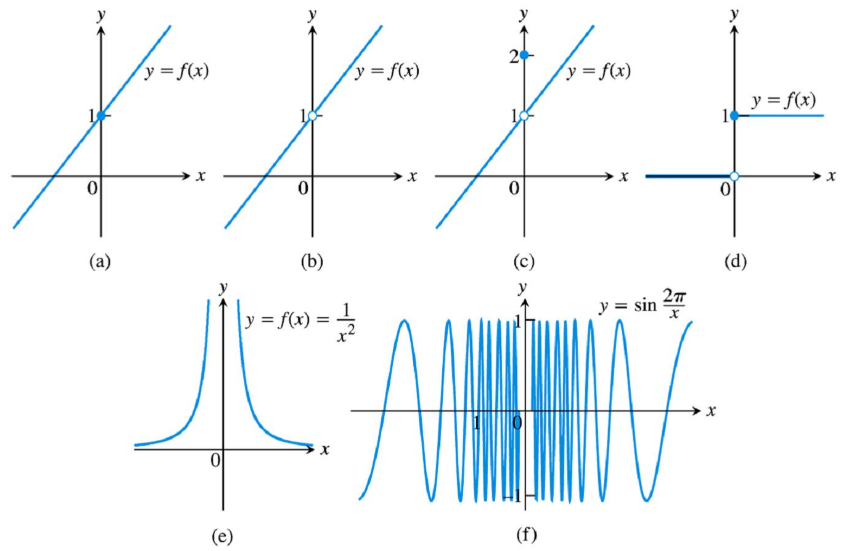


FIGURE 2.55 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

Discontinuity types

- (b), (c) removable discontinuity
- (d) jump discontinuity
- (e) infinite discontinuity
- (f) oscillating discontinuity

Continuous functions

- A function is continuous on an interval if and only if it is continuous at every point of the interval.
- Example: Figure 2.56
- $1/x$ not continuous on [-1,1] but continuous over $(-\infty,0) \cup (0, \infty)$

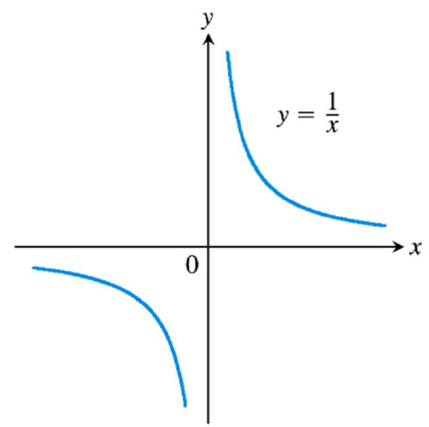


FIGURE 2.56 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$ (Example 5).

Example 5

- Identifying continuous function
- (a) $f(x)=1/x$
- (b) $f(x)= x$
- Ask: is $1/x$ continuous over its domain?

THEOREM 9 Properties of Continuous Functions
If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

Example 6

- Polynomial and rational functions are continuous
- (a) Every polynomial is continuous by
- (i) $\lim_{x \to c} P(x) = P(c)$
- (ii) Theorem 9
- (b) If $P(x)$ and $Q(x)$ are polynomial, the rational function $P(x)/Q(x)$ is continuous whenever it is defined.

Example 7

- Continuity of the absolute function
- $f(x) = |x|$ is everywhere continuous

- Continuity of the sinus and cosinus function
- $f(x) = \cos x$ and $\sin x$ is everywhere continuous

Composites

- All composites of continuous functions are continuous

THEOREM 10 Composite of Continuous Functions
If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

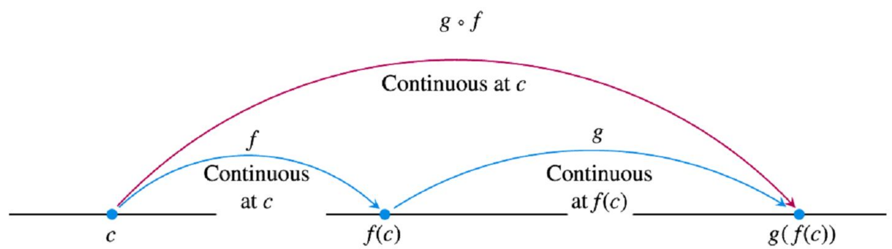


FIGURE 2.57 Composites of continuous functions are continuous.

Example 8

- Applying Theorems 9 and 10
- Show that the following functions are continuous everywhere on their respective domains.

$$(a)\ y=\sqrt{x^2-2x-5} \qquad (b)\ y=\frac{x^{2/3}}{1+x^4}$$

$$(c)\ y=\left|\frac{x-2}{x^2-2}\right| \qquad (d)\ y=\left|\frac{x\sin x}{x^2+2}\right|$$

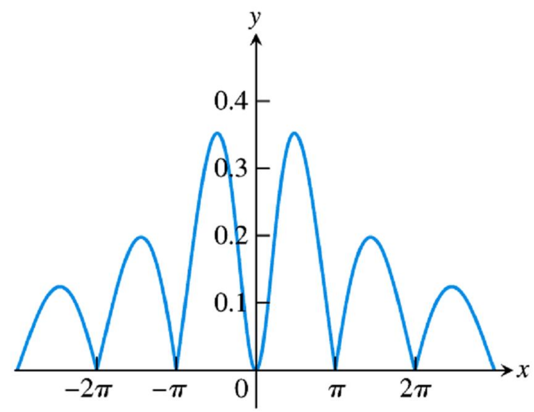
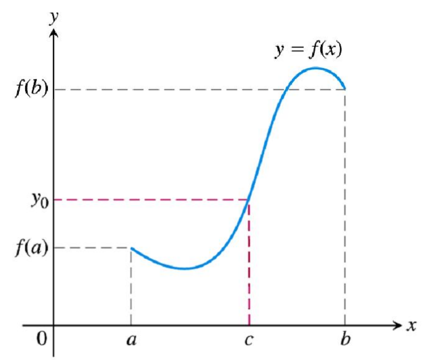


FIGURE 2.58 The graph suggests that $y = |(x\sin x)/(x^2+2)|$ is continuous (Example 8d).

THEOREM 11 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



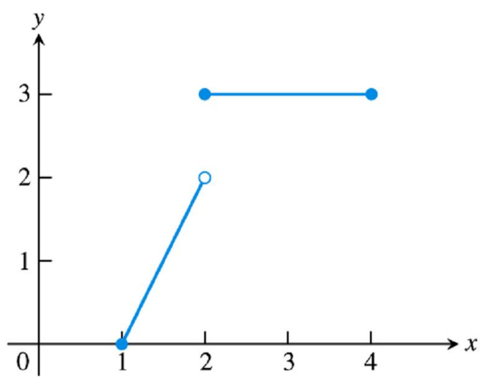


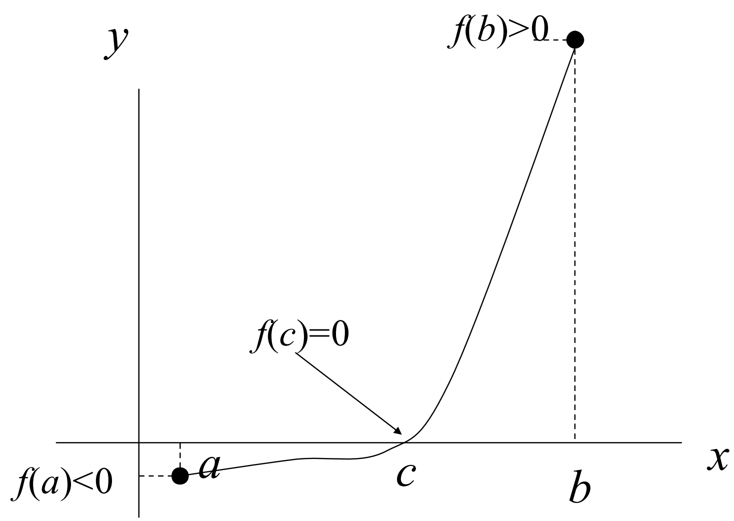
FIGURE 2.61 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

Consequence of root finding

- A solution of the equation $f(x)=0$ is called a root.
- For example, $f(x)= x^2 + x - 6$, the roots are $x=2$, $x=-3$ since $f(-3)=f(2)=0$.
- Say f is continuous over some interval.
- Say a, b (with $a < b$) are in the domain of f, such that $f(a)$ and $f(b)$ have opposite signs.
- This means either $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$
- Then, as a consequence of theorem 11, there must exist at least a point c between a and b, i.e. $a < c < b$ such that $f(c)= 0$. $x=c$ is the root.



Example

- Consider the function $f(x) = x - \cos x$
- Prove that there is at least one root for $f(x)$ in the interval $[0, \pi/2]$.

- **Solution**
- $f(x)$ is continuous on $(-\infty, \infty)$.
- Say $a = 0$, $b = \pi/2$.
- $f(x=0) = -1$; $f(x = \pi/2) = \pi/2$
- $f(a)$ and $f(b)$ have opposite signs
- Then, as a consequence of theorem 11, there must exist at least a point c between a and b, i.e. $a=0 < c < b= \pi/2$ such that $f(c)= 0$. $x=c$ is the root.

2.7

Tangents and Derivatives

What is a tangent to a curve?

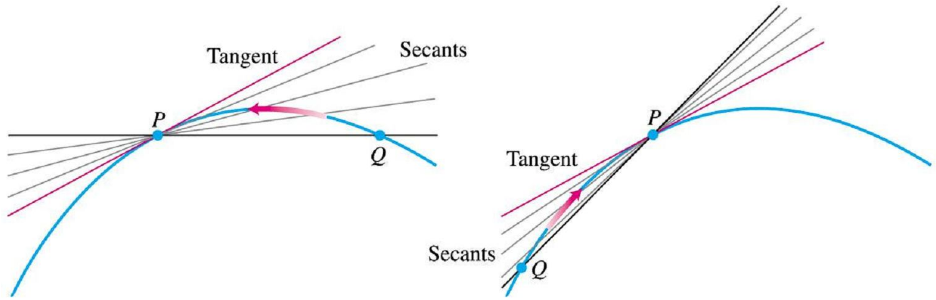


FIGURE 2.65 The dynamic approach to tangency. The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

DEFINITIONS Slope, Tangent Line

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

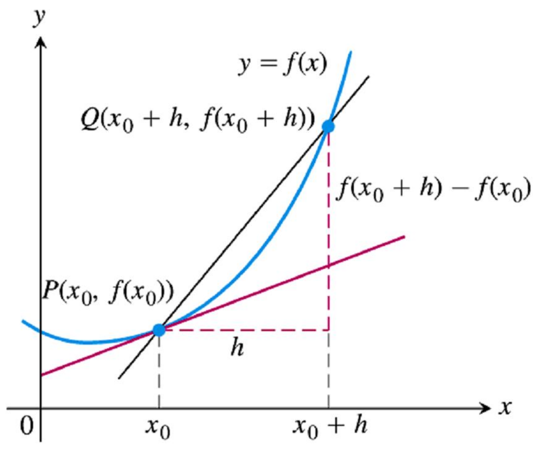


FIGURE 2.67 The slope of the tangent line at P is $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$.

Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope
$$m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$
3. If the limit exists, find the tangent line as
$$y = y_0 + m(x - x_0).$$

Example 1: Tangent to a parabola

- Find the slope of the parabola $y=x^2$ at the point $P(2,4)$. Write an equation for the tangent to the parabola at this point.

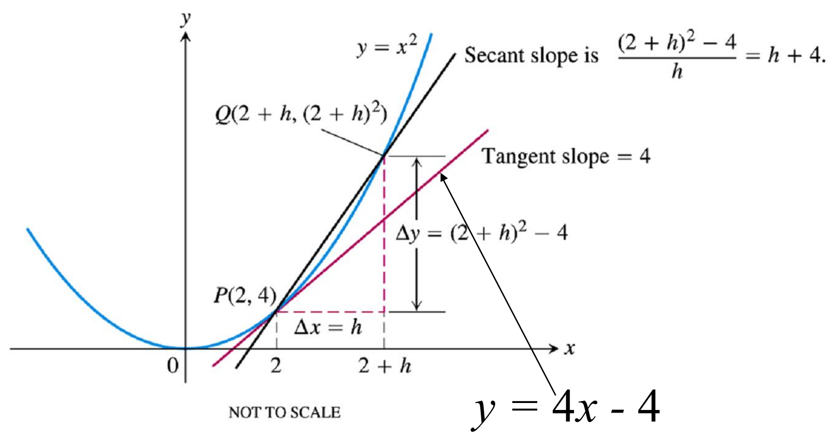


FIGURE 2.66 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ (Example 1).

Example 3

- Slope and tangent to $y=1/x$, $x\neq 0$
- (a) Find the slope of $y=1/x$ at $x = a \neq 0$
- (b) Where does the slope equal -1/4?
- (c) What happens to the tangent of the curve at the point (a, $1/a$) as a changes?

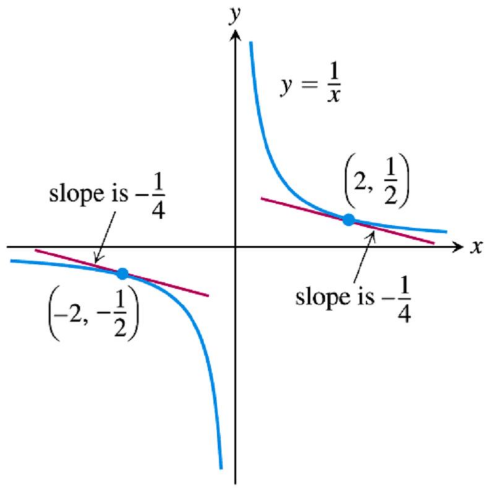


FIGURE 2.68 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 3).

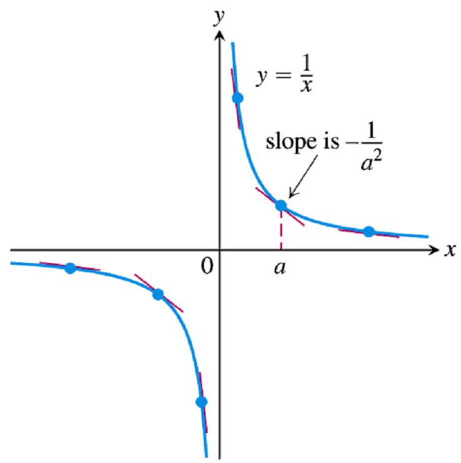


FIGURE 2.69 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away.

Chapter 3

Differentiation

3.1

The Derivative as a Function

DEFINITION **Derivative Function**
The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

- The limit

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

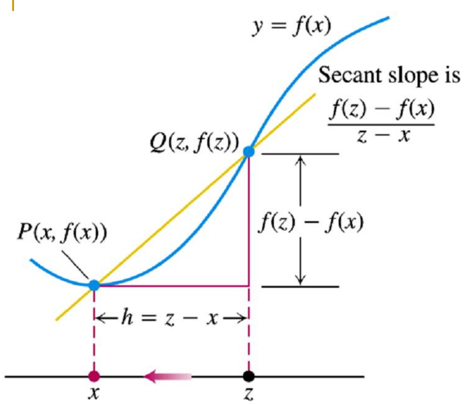
- when it existed, is called the Derivative if f at x_0.
- View derivative as a function derived from f

- If f' exists at x, f is said to be differentiable (has a derivative) at x
- If f' exists at every point in the domain of f, f is said to be differentiable.

If write $z = x + h$, then $h = z - x$

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$



Derivative of f at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

FIGURE 3.1 The way we write the difference quotient for the derivative of a function f depends on how we label the points involved.

Calculating derivatives from the definition

- Differentiation: an operation performed on a function $y = f(x)$
- d/dx operates on $f(x)$
- Write as

$$\frac{d}{dx} f(x)$$

- f' is taken as a shorthand notation for

$$\frac{d}{dx} f(x)$$

Example 1: Applying the definition

- Differentiate

$$f(x) = \frac{x}{x-1}$$

- Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{x+h}{x+h-1}\right) - \left(\frac{x}{x-1}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}$$

Example 2: Derivative of the square root function

- (a) Find the derivative of $y=\sqrt{x}$ for $x>0$
- (b) Find the tangent line to the curve $y=\sqrt{x}$ at $x = 4$

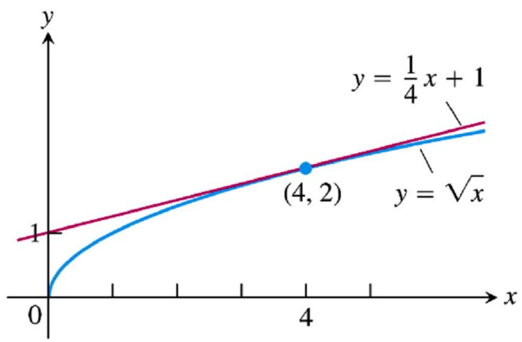


FIGURE 3.2 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

Notations

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}$$

Differentiable on an Interval; One sided derivatives

- A function *y* = *f* (*x*) is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval.
- It is differentiable on a closed interval [*a*,*b*] if it is differentiable on the interior (*a*,*b*) and if the limits

$$\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}$$

$$\lim_{h\to 0^-}\frac{f(b+h)-f(b)}{h}$$

exist at the endpoints

- A function has a derivative at a point if an only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

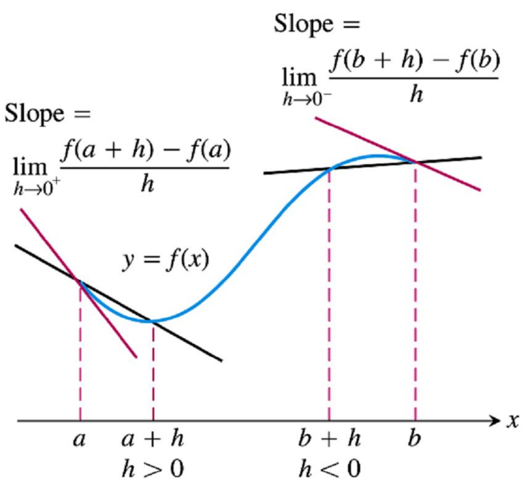


FIGURE 3.5 Derivatives at endpoints are one-sided limits.

Example 5

- $y = |x|$ is not differentiable at $x = 0$.
- Solution:
- For $x > 0$,

$$\frac{d|x|}{dx} = \frac{d}{dx}(x) = 1$$

- For $x < 0$,

$$\frac{d|x|}{dx} = \frac{d}{dx}(-x) = -1$$

- At $x = 0$, the right hand derivative and left hand derivative differ there. Hence $f(x)$ not differentiable at $x = 0$ but else where.

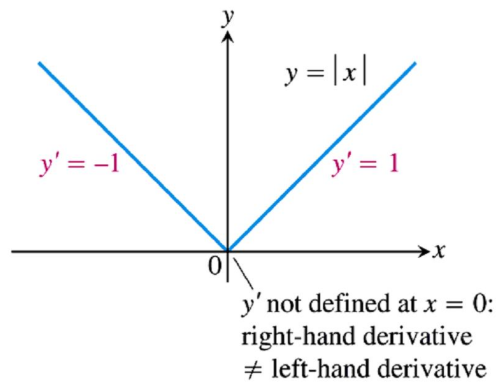


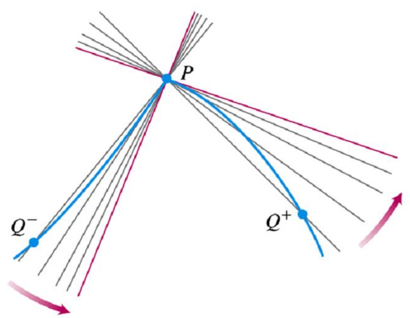
FIGURE 3.6 The function $y = |x|$ is not differentiable at the origin where the graph has a "corner."

Example 6

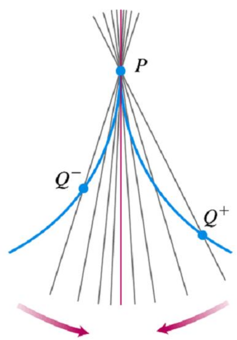
- $y = \sqrt{x}$ is not differentiable at $x = 0$
- The graph has a vertical tangent at $x = 0$

When Does a function not have a derivative at a point?

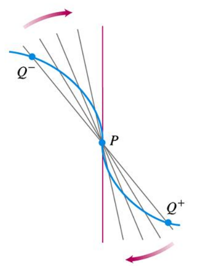
1. a *corner*, where the one-sided derivatives differ.



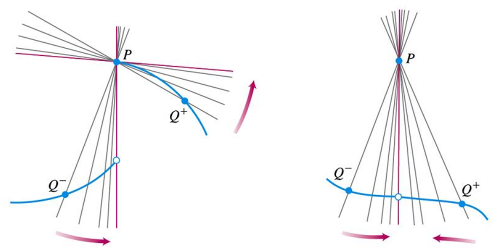
2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.



3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).



4. a *discontinuity*.



Differentiable functions are continuous

THEOREM 1 Differentiability Implies Continuity
If f has a derivative at $x = c$, then f is continuous at $x = c$.

The converse is false: continuity does not necessarily implies differentiability

Example

- $y = |x|$ is continuous everywhere, including $x = 0$, but it is not differentiable there.

The equivalent form of Theorem 1

- If f is not continuous at $x = c$, then f is not differentiable at $x = c$.
- Example: the step function is discontinuous at $x = 0$, hence not differentiable at $x = 0$.

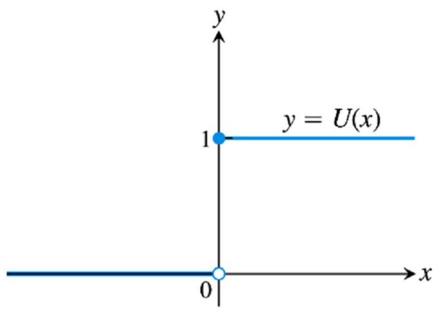


FIGURE 3.7 The unit step function does not have the Intermediate Value Property and cannot be the derivative of a function on the real line.

The intermediate value property of derivatives

THEOREM 2 Darboux's Theorem
If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

- See section 4.4

3.2

Differentiation Rules

Powers, multiples, sums and differences

RULE 1 Derivative of a Constant Function
If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

RULE 2 Power Rule for Positive Integers
If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

RULE 3 Constant Multiple Rule
If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

In particular, if $u = x^n, \frac{d}{dx}(cx^n) = cx^{n-1}$

Example 1

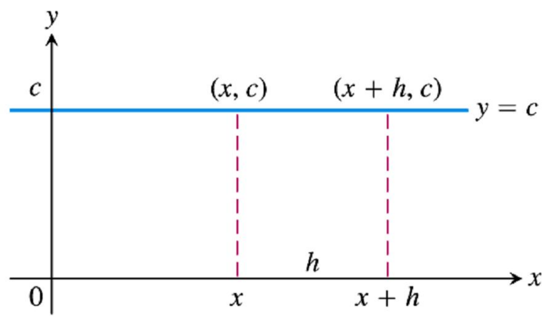


FIGURE 3.8 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

Example 3

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x^{2-1} = 6x$$

$$\frac{d}{dx}(x^2) = 2x^{2-1} = 2x$$

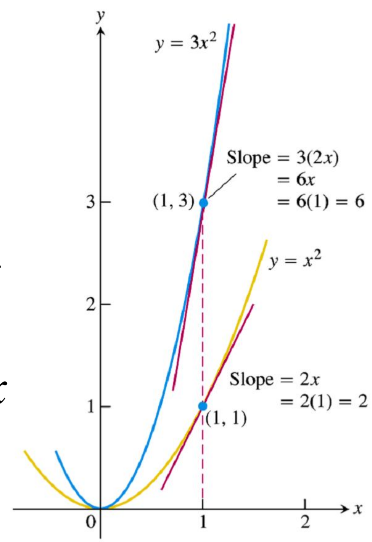


FIGURE 3.9 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinates triples the slope (Example 3).

RULE 4 Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Example 5

$$y = x^3 + \frac{4}{3}x^2 - 5x + 1$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$= 3x^2 + \frac{8}{3}x - 5$$

Example 6

- Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

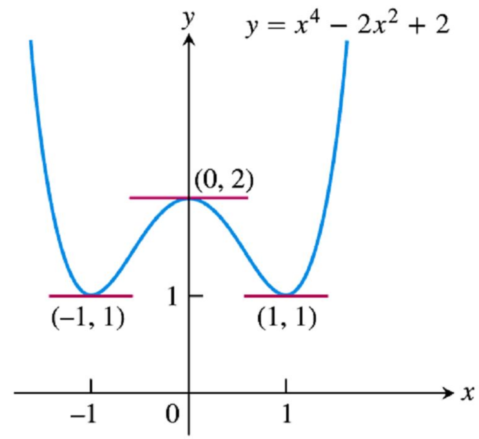


FIGURE 3.10 The curve $y = x^4 - 2x^2 + 2$ and its horizontal tangents (Example 6).

Products and quotients

- Note that

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(x \cdot x) \neq \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1$$

RULE 5 Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example 7

- Find the derivative of $y = \frac{1}{x}\left(x^2 + \frac{1}{x}\right)$

Example 8: Derivative from numerical values

- Let $y = uv$. Find $y'(2)$ if $u(2) = 3$, $u'(2) = -4$, $v(2) = 1$, $v'(2) = 2$

Example 9

- Find the derivative of $y = (x^2 + 1)(x^3 + 3)$

RULE 6 Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Negative integer powers of x

- The power rule for negative integers is the same as the rule for positive integers

RULE 7 Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Example 11

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right) = (-1)x^{-1-1} = -x^{-2}$$

$$\frac{d}{dx}\left(\frac{4}{x^3}\right) = \frac{d}{dx}\left(4x^{-3}\right) = (-4\cdot -3)x^{-3-1} = 12x^{-4}$$

Example 12: Tangent to a curve

- Find the tangent to the curve $y = x + \frac{2}{x}$ at the point (1,3)

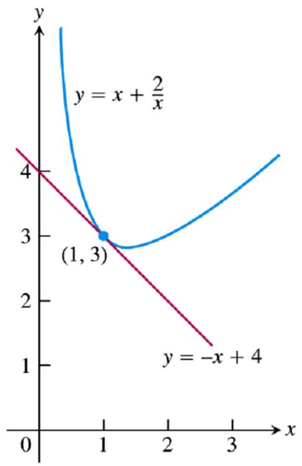


FIGURE 3.11 The tangent to the curve $y = x + (2/x)$ at (1, 3) in Example 12. The curve has a third-quadrant portion not shown here. We see how to graph functions like this one in Chapter 4.

Example 13

- Find the derivative of $y = \frac{(x-1)(x^2-2x)}{x^4}$

Second- and higher-order derivative

- Second derivative

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(y')$$
$$= y'' = D^2(f)(x) = D_x^2 f(x)$$

- *n*th derivative

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y$$

Example 14

$$y = x^3 - 3x^2 + 2$$
$$y' = 3x^2 - 6x$$
$$y'' = 6x - 6$$
$$y''' = 6$$
$$y^{(4)} = 0$$

3.3

The Derivative as a Rate of Change

Instantaneous Rates of Change

DEFINITION Instantaneous Rate of Change

The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Example 1: How a circle's area changes with its diameter

- $A = \pi D^2/4$
- How fast does the area change with respect to the diameter when the diameter is 10 m?

Motion along a line

- Position $s = f(t)$
- Displacement, $\Delta s = f(t+ \Delta t) - f(t)$
- Average velocity
- $v_{av} = \Delta s/\Delta t = [f(t+ \Delta t) - f(t)] /\Delta t$
- The instantaneous velocity is the limit of v_{av} when $\Delta t \to 0$

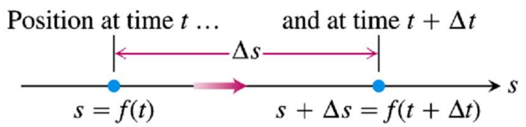


FIGURE 3.12 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$.

DEFINITION Velocity

Velocity (instantaneous velocity) is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

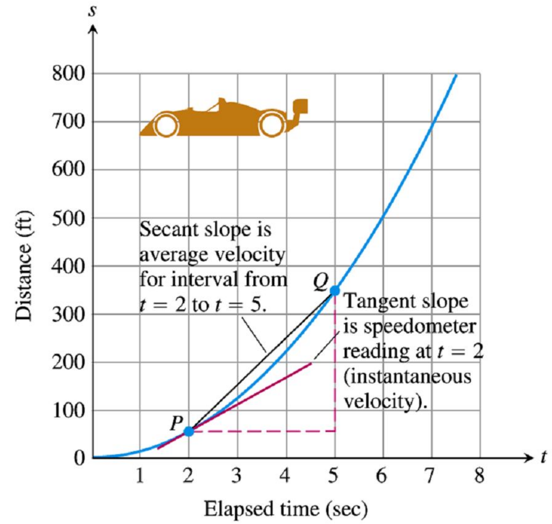


FIGURE 3.13 The time-to-distance graph for Example 2. The slope of the tangent line at P is the instantaneous velocity at $t = 2$ sec.

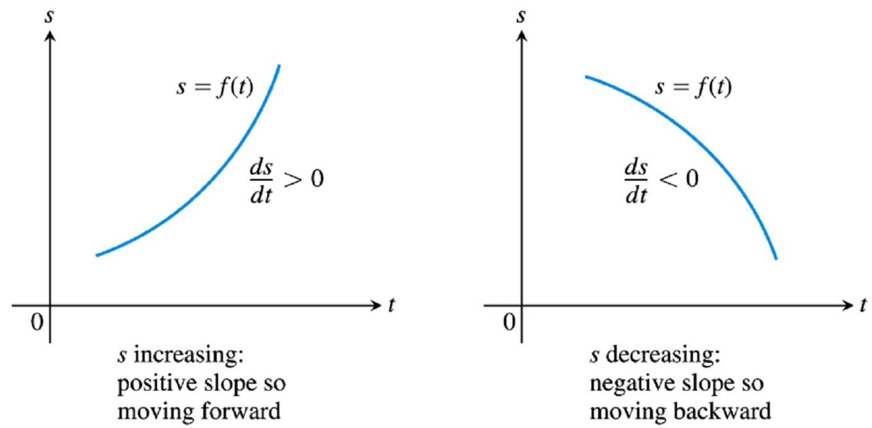


FIGURE 3.14 For motion $s = f(t)$ along a straight line, $v = ds/dt$ is positive when s increases and negative when s decreases.

DEFINITION **Speed**

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

Example 3

- Horizontal motion

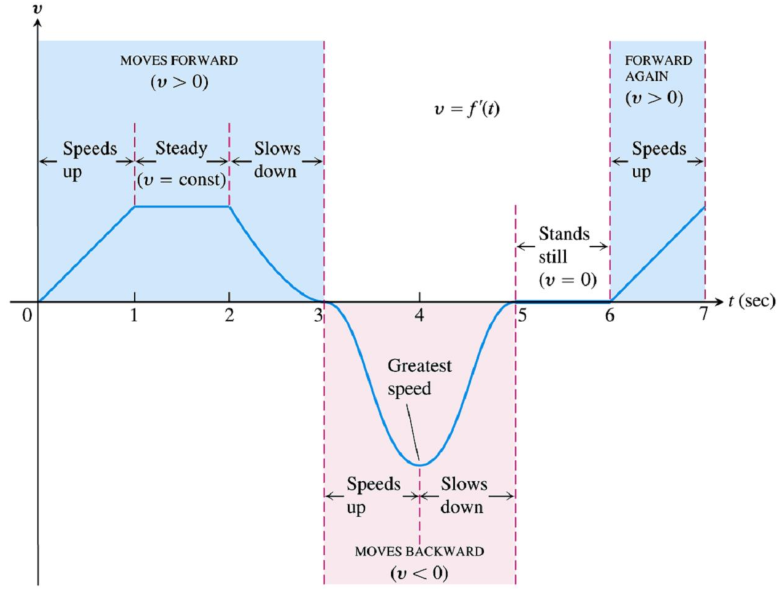


FIGURE 3.15 The velocity graph for Example 3.

DEFINITIONS **Acceleration, Jerk**

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Example 4

- Modeling free fall $s = \frac{1}{2}gt^2$
- Consider the free fall of a heavy ball released from rest at $t = 0$ sec.
- (a) How many meters does the ball fall in the first 2 sec?
- (b) What is the velocity, speed and acceleration then?

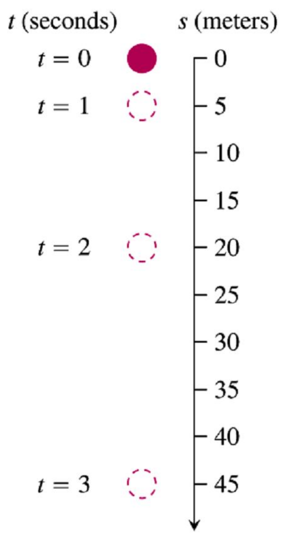


FIGURE 3.16 A ball bearing falling from rest (Example 4).

Modeling vertical motion

- A dynamite blast blows a heavy rock straight up with a launch velocity of 160 m/sec. It reaches a height of $s = 160t - 16t^2$ ft after t sec.
- (a) How high does the rock go?
- (b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?
- (c) What is the acceleration of the rock at any time t during its flight?
- (d) When does the rock hit the ground again?

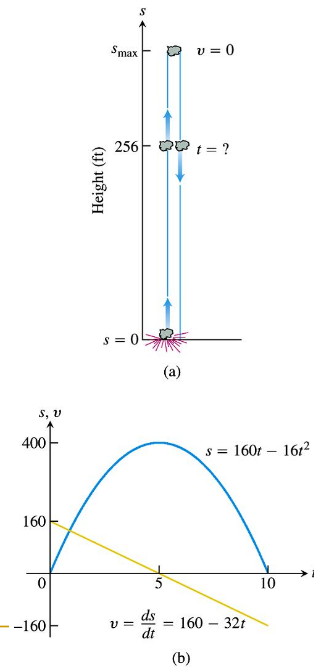


FIGURE 3.17 (a) The rock in Example 5. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

3.4

Derivatives of Trigonometric Functions

Derivative of the sine function

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \cdots$$

Derivative of the cosine function

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \cdots$$

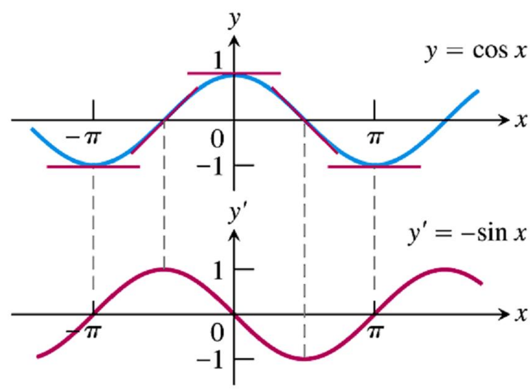


FIGURE 3.23 The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

Example 2

$$(a)\, y = 5x + \cos x$$
$$(b)\, y = \sin x \cos x$$
$$(c)\, y = \frac{\cos x}{1 - \sin x}$$

Derivative of the other basic trigonometric functions

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Example 5

- Find *d*(tan *x*)/*dx*

Example 6

- Find *y''* if *y* = sec *x*

Example 7

- Finding a trigonometric limit

$$\lim_{x\to 0}\frac{\sqrt{2+\sec x}}{\cos(\pi-\tan x)}=\frac{\sqrt{2+\sec 0}}{\cos(\pi-\tan 0)}$$
$$=\frac{\sqrt{2+1}}{\cos(\pi-0)}=\frac{\sqrt{3}}{-1}=-\sqrt{3}$$

3.5

The Chain Rule and
Parametric Equations

Differentiating composite functions

- Example:
- $y = f(u) = \sin u$
- $u = g(x) = x^2 - 4$
- How to differentiate $F(x) = f \circ g = f[g(x)]$?
- Use chain rule

Derivative of a composite function

- Example 1 Relating derivatives
- $y = (3/2)x = (1/2)(3x)$
- $= g[u(x)]$
- $g(u) = u/2$; $u(x) = 3x$
- $dy/dx = 3/2$;
- $dg/du = ½$; $du/dx = 3$;
- $dy/dx = (dy/du)\cdot(du/dx)$ (Not an accident)

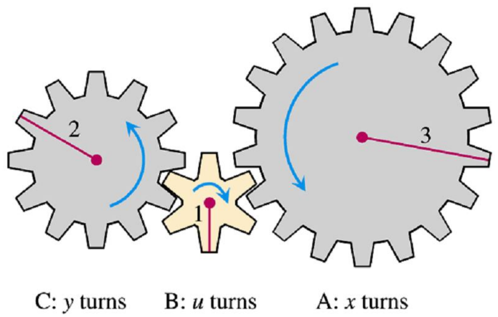


FIGURE 3.26 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ (C turns one-half turn for each B turn) and $u = 3x$ (B turns three times for A's one), so $y = 3x/2$. Thus, $dy/dx = 3/2 = (1/2)(3) = (dy/du)(du/dx)$.

Example 2

$$y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$$

$$y = u^2; u = 3x^2 + 1$$

$$\frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 6$$

$$= 2(3x^2 + 1) \cdot 6x = 36x^3 + 12x$$

c.f.

$$\frac{dy}{dx} = \frac{d}{dx}\left(9x^4 + 6x^2 + 1\right) = 36x^3 + 12x$$

THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

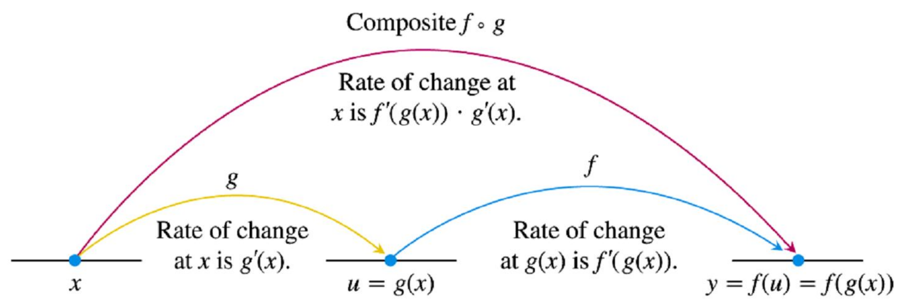


FIGURE 3.27 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x.

Example 3

- Applying the chain rule
- $x(t) = \cos(t^2 + 1)$. Find dx/dt.
- Solution:
- $x(u) = \cos(u)$; $u(t) = t^2 + 1$;
- $dx/dt = (dx/du)\cdot(du/dt) = \ldots$

Alternative form of chain rule

- If $y = f[g(x)]$, then
- $dy/dx = f'[g(x)]\cdot g'(x)$
- Think of f as 'outside function', g as 'inside-function', then
- dy/dx = differentiate the outside function and evaluate it at the inside function let alone; then multiply by the derivative of the inside function.

Example 4

- Differentiating from the outside In

$$\frac{d}{dx}\sin\underbrace{(x^2+x)}_{\substack{\text{inside}\\\text{left alone}}} = \cos\underbrace{(x^2+x)}_{\substack{\text{inside}\\\text{left alone}}}\cdot\underbrace{(2x+1)}_{\substack{\text{derivative of}\\\text{the inside}}}$$

Example 5

- A three-link 'chain'
- Find the derivative of $g(t) = \tan(5 - \sin 2t)$

Example 6

- Applying the power chain rule

$$(a)\frac{d}{dx}(5x^3 - x^4)^7$$

$$(b)\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$$

81

Example 7

- (a) Find the slope of tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$
- (b) Show that the slope of every line tangent to the curve $y = 1/(1-2x)^3$ is positive

82

Parametric equations

- A way of expressing both the coordinates of a point on a curve, (x,y) as a function of a third variable, t.
- The path or locus traced by a point particle on a curve is then well described by a set of two equations:
- $x = f(t)$, $y = g(t)$

83

DEFINITION Parametric Curve
If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

The variable t is a parameter for the curve

84

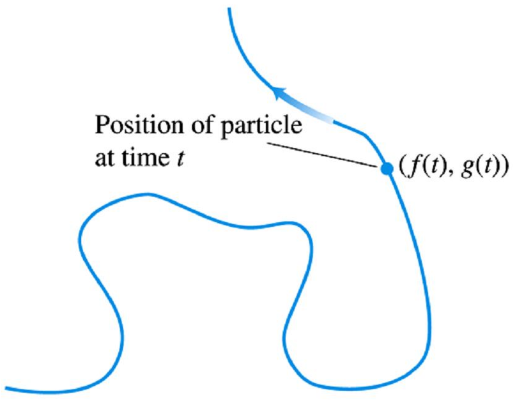


FIGURE 3.29 The path traced by a particle moving in the xy-plane is not always the graph of a function of x or a function of y.

Example 9

- Moving counterclockwise on a circle
- Graph the parametric curves
- $x = \cos t$, $y = \sin t$,
  $0 \leqslant t \leqslant 2\pi$

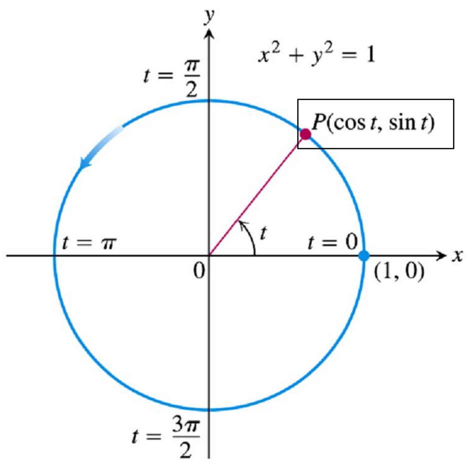


FIGURE 3.30 The equations $x = \cos t$ and $y = \sin t$ describe motion on the circle $x^2 + y^2 = 1$. The arrow shows the direction of increasing t (Example 9).

Example 10

- Moving along a parabola
- $x = \sqrt{t}$, $y = t$, $0 \leqslant t$
- Determine the relation between x and y by eliminating t.
- $y = t = (t)^2 = x^2$
- The path traced out by P (the locus) is only half the parabola, $x \geqslant 0$

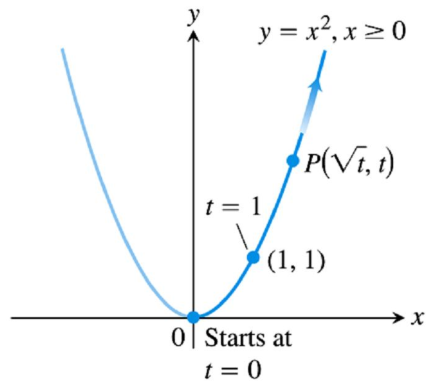


FIGURE 3.31 The equations $x = \sqrt{t}$ and $y = t$ and the interval $t \geq 0$ describe the motion of a particle that traces the right-hand half of the parabola $y = x^2$ (Example 10).

Slopes of parametrized curves

- A parametrized curved $x = f(t)$, $y = g(t)$ is differentiable at t if f and g are differentiable at t.
- At a point on a differentiable parametrised curve where y is also a differentiable function of x, i.e. $y = y(x) = y[x(t)]$,
- chain rule relates dx/dt, dy/dt, dy/dx via

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \qquad (2)$$

89

Example 12

- Differentiating with a parameter
- If $x = 2t + 3$ and $y = t^2 - 1$, find the value of dy/dx at $t = 6$.

90

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \qquad (3)$$

(3) is just the parametric formula (2) by

$$y \rightarrow dy/dx$$

91

Example 14

- Finding d^2y/dx^2 for a parametrised curve
- Find d^2y/dx^2 as a function of t if $x = t - t^2$, $y = t - t^3$.

92

3.6

Implicit Differentiation

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

Example 1: Differentiating implicitly

- Find dy/dx if $y^2 = x$

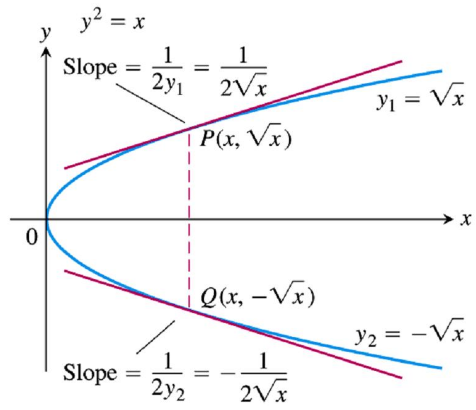


FIGURE 3.37 The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of x on the interval $x \geq 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for y.

Example 2

- Slope of a circle at a point
- Find the slope of circle $x^2 + y^2 = 25$ at (3, -4)

Example 3

- Differentiating implicitly
- Find dy/dx if $y^2 = x^2 + \sin xy$

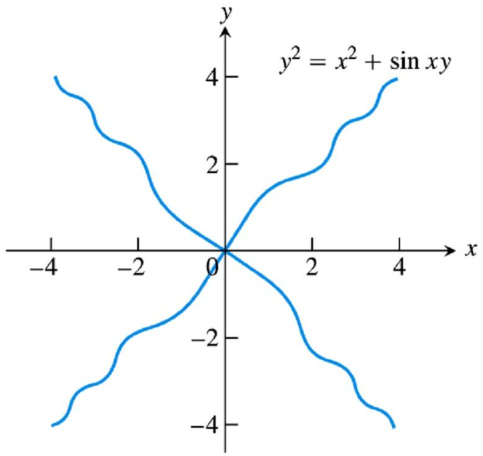


FIGURE 3.39 The graph of $y^2 = x^2 + \sin xy$ in Example 3. The example shows how to find slopes on this implicitly defined curve.

Lenses, tangents, and normal lines

If slop of tangent is m_t, the slope of normal, m_n, is given by the relation

$m_n m_t = -1$, or

$m_n = -1/m_t$

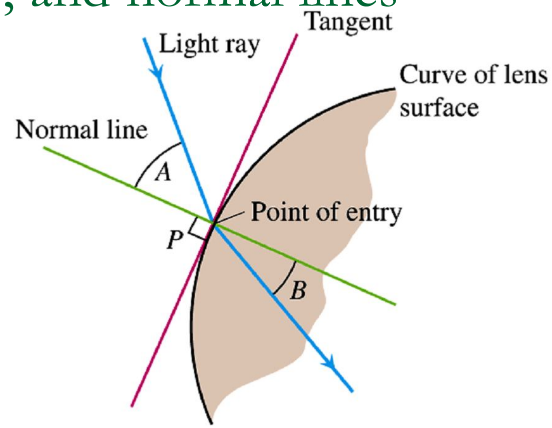


FIGURE 3.40 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

Example 4

- Tangent and normal to the folium of Descartes
- Show that the point (2,4) lies on the curve $x^2 + y^3 - 9xy = 0$. The find the tangent and normal to the curve there.

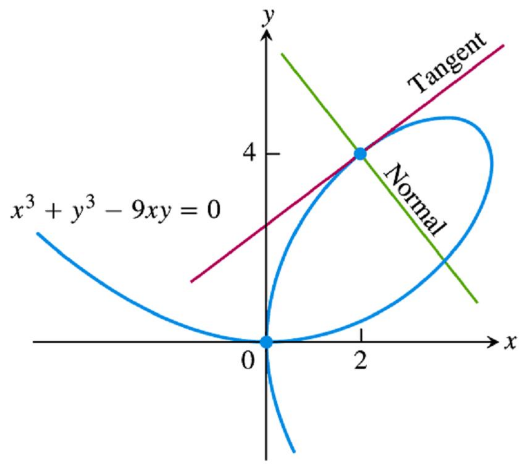


FIGURE 3.41 Example 4 shows how to find equations for the tangent and normal to the folium of Descartes at (2, 4).

Derivative of higher order

- Example 5
- Finding a second derivative implicitly
- Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Rational powers of differentiable functions

THEOREM 4 Power Rule for Rational Powers
If p/q is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$, and

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

Theorem 4 is proved based on $d/dx(x^n) = nx^{n-1}$ (where n is an integer) using implicit differentiation

- Theorem 4 provide a extension of the power chain rule to rational power:

$$\frac{d}{dx}u^{p/q} = \frac{p}{q}u^{(p/q)-1}\frac{du}{dx}$$

- $u \neq 0$ if $(p/q) < 1$, (p/q) rational number, u a differential function of x

Example 6

- Using the rational power rule
- (a) $d/dx\,(x^{1/2}) = 1/2x^{-1/2}$ for $x > 0$
- (b) $d/dx\,(x^{2/3}) = 2/3\,x^{-1/3}$ for $x \neq 0$
- (c) $d/dx\,(x^{-4/3}) = -4/3\,x^{-7/3}$ for $x \neq 0$

Proof of Theorem 4

- Let p and q be integers with $q > 0$ and

$$y = x^{p/q} \equiv y^q = x^p$$

- Explicitly differentiating both sides with respect to x...

Example 7

- Using the rational power and chain rules
- (a) Differentiate $(1-x^2)^{1/4}$
- (b) Differentiate $(\cos x)^{-1/5}$

3.8

Linearization and differentials

Linearization

- Say you have a very complicated function, $f(x) = \sin(\cot^2 x)$, and you want to calculate the value of $f(x)$ at $x = \pi/2 + \delta$, where δ is a very tiny number. The value sought can be estimated within some accuracy using linearization.

Definitions

- The tangent line $L(x) = f(a) + f'(a)(x - a)$ gives a good approximation to $f(x)$ as long as x is not to be too far away from $x=a$.
- Or in other words, we say that $L(x)$ is the linearization of f at a.
- The approximation $f(x) \approx L(x)$ of f by L is the standard linear approximation of f at a.
- The point $x = a$ is the center of the approximation.

109

Refer to graph Figure 3.47.

- The point-slope equation of the tangent line passing through the point $(a, f(a))$ on a differentiable function f at $x=a$ is
- $y = mx + c$, where c is $c = f(a) - f'(a)\,a$
- Hence the tangent line is the graph of the linear function
- $L(x) = f'(a)x + f(a) – a\,f'(a)$
  $= f(a) + f'(a)(x - a)$

110

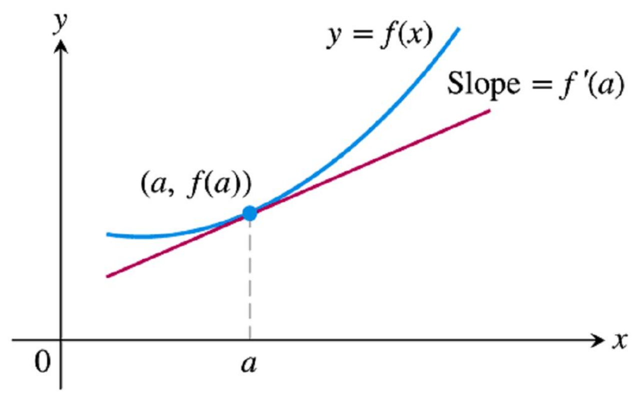


FIGURE 3.47 The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

111

Example 1 Finding Linearization

- Find the linearization of

$$f(x) = \sqrt{1+x}$$

- at $x = 0$.

112

$$f(x)=\sqrt{1+x}$$

$$f'(x)=\frac{1}{2(1+x)^{1/2}}$$

The linearization of $f(x)$ at $x=a$ is

$$f(x)=f(a)+f'(a)(x-a)=(1+a)^{1/2}+\frac{1}{2(1+a)^{1/2}}(x-a)$$

$a=0,$

$f'(0)=\frac{1}{2}; f(0)=1;$

The linearization of $f(x)$ at $x=a=0$ is $L(x)=1+x/2$

We write $f(x)\approx L(x)=1+x/2$

Accuracy of the linearized approximation

- We find that the approximation of *f*(*x*) by *L*(*x*) gets worsened as |*x* – *a*| increases (or in other words, *x* gets further away from *a*).

Approximation	True value	$\lvert$True value − approximation$\rvert$
$\sqrt{1.2}\approx 1+\frac{0.2}{2}=1.10$	1.095445	$<10^{-2}$
$\sqrt{1.05}\approx 1+\frac{0.05}{2}=1.025$	1.024695	$<10^{-3}$
$\sqrt{1.005}\approx 1+\frac{0.005}{2}=1.00250$	1.002497	$<10^{-5}$

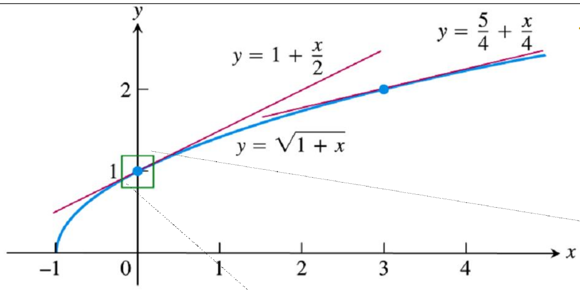


FIGURE 3.48 The graph of $y=\sqrt{1+x}$ and its linearizations at $x=0$ and $x=3$. Figure 3.49 shows a magnified view of the small window about 1 on the *y*-axis

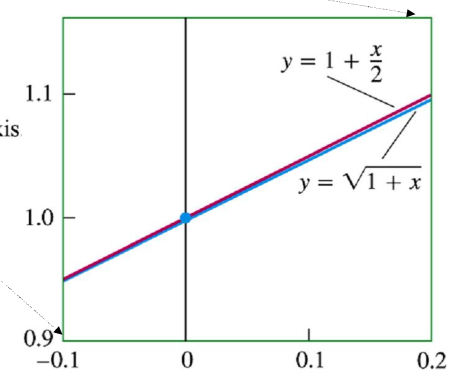


FIGURE 3.49 Magnified view of the window in Figure 3.48.

What $\frac{dy}{dx}$ is not

- Note that the derivative notation $\frac{dy}{dx}$ is not a ratio
- i.e. the derivative of the function *y* = *y*(*x*) with respect to *x*, is not to be understood as the ratio of two values, namely, *dy* and *dx*.
- d*y*/d*x* here denotes the a new quantity derived from *y* when the operation D = d/d*x* is performed on the function *y*, (d/d*x*)[*y*] = D [*y*]

Differential

- Definition:
- Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy is
  - $dy = f'(x)dx$
- dy is an dependent variable, i.e., the value of dy depends on $f'(x)$ and dx.
- Once $f'(x)$ and dx is fixed, then the value of differential dy can be calculated.

Example 4 Finding the differential dy

- (a) Find dy if $y=x^5 + 37x$.
- (b) Find the value of dy when $x=1$ and $dx =$ 0.2

$dy \div dx = f'(x)$

- Referring to the definition of the differentials dy and dx, if we take the ratio of dy and dx, i.e. $dy \div dx$, we get

  $dy \div dx = f'(x)\, dx / dx = f'(x) \equiv \frac{dy}{dx}$
- In other words, the ratio of the differential dy and dx is equal to the derivative by definition.

Differential of f, df

- We sometimes use the notation df in place of dy, so that

  $dy = f'(x)\, dx$
- is now written in terms of
  - $df = f'(x)\, dx$
- df is called the differential of f

Example of differential of f

- If $y = f(x) = 3x^2-6$, then the differential of f is

  $df = f'(x)\,dx = 6x\,dx$

- Note that in the above expression, if we take the ratio df / dx, we obtain
  - $df / dx = f'(x) = 6x$

The differential form of a function

- For every differentiable function $y=f(x)$, we can obtain its derivative, $\frac{dy}{dx}$
- Corresponds to every derivative $\frac{dy}{dx}$ there is a differential df such that

$$df = \left(\frac{dy}{dx}\right)\cdot dx$$

In addition, if $f = u + v$, then $df = \left(\frac{du}{dx}\right)\cdot dx + \left(\frac{dv}{dx}\right)\cdot dx$

Example 5

- If $y = f(x) = \tan 2x$, the derivative is

$$\frac{dy}{dx} = 2\sec^2 2x$$

- Correspond to the derivative, the differential of the function, df, is given by the product of the derivative dy/dx and the independent differential dx:

$$df = d(\tan 2x) = \left(\frac{dy}{dx}\right)\cdot dx = \left(2\sec^2 2x\right)\cdot dx$$

Example 5

- If $y = f(x) = \tan 2x$, then the differential form of the function,

$$y = f(x) = \left(\frac{x}{x+1}\right)$$

$$df = d\left(\frac{x}{x+1}\right) = \left(\frac{dy}{dx}\right)\cdot dx = \frac{(x+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x+1)}{(x+1)^2}\cdot dx$$

$$= \frac{(x+1)\left[\frac{d}{dx}(x)\cdot dx\right] - x\left[\frac{d}{dx}(x+1)\cdot dx\right]}{(x+1)^2} = \frac{(x+1)[dx] - x[d(x+1)]}{(x+1)^2}$$

$$= \frac{(x+1)dx - xd(x+1)}{(x+1)^2} = \frac{(x+1)dx - x\left[\frac{d}{dx}(x+1)\cdot dx\right]}{(x+1)^2} = \frac{(x+1)dx - x[1\cdot dx]}{(x+1)^2} = \frac{dx}{(x+1)^2}$$

dy, df : any difference?

- Sometimes for a given function, $y = f(x)$, the notation dy is used in place of the notation df.
- Operationally speaking, it does not matter whether one uses dy or df.

The derivative dy/dx is not dy divided by dx

- Due to the definition of the differentials dy, dx that their ratio, dy / dx equals to the derivative of the differentiable function $y = f(x)$, i.e. $\frac{dy}{dx} = \frac{\mathrm{d}}{\mathrm{d}x}(y) \equiv f'(x)$
- we can then move the differential dy or dx around, such as $dy = f'(x)dx$
- When we do so, we need to be reminded that dy and dx are differentials, a pair of variables, instead of thinking that the derivative $\frac{\mathrm{d}}{\mathrm{d}x}y$ is made up of a numerator "dy" and a denominator "dx" that are separable

Estimation with differential

- Referring to figure 3.51, geometrically, one can sees that if x, originally at $x=a$, changes by dx (where dx is an independent variable, the differential of x), $f(a)$ will change by

$$\Delta y = f(a+dx) - f(a)$$

- Δy can be approximated by the change of the linearization of f at $x=a$, $L(x)=f(a)+f'(a)(x-a)$,

$$\Delta y \approx \Delta L = L(a+dx)-L(a)= f'(a)dx = df(a)$$

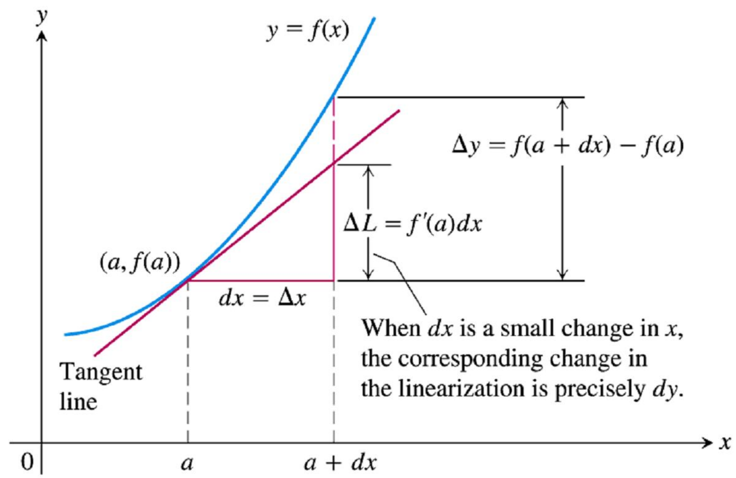


FIGURE 3.51 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

$\Delta y \approx dy$ allows estimation of $f(a+dx)$

- In other words, Δy centered around $x=a$ is approximated by $df(a)$ ($\equiv dy$, where the differential is evaluated at $x=a$):

$$\Delta y \approx dy$$

- or equivalently,

$$\Delta y = f(a+dx) - f(a) \approx dy = f'(a)dx$$

- This also allows us to estimate the value of $f(a+dx)$ if $f'(a)$, $f(a)$ are known, and dx is not too large, via

$$f(a+dx) \approx f(a) + f'(a)dx$$

Example 6

- Figure 3.52
- The radius r of a circle increases from a=10 m to 10.1 m. Use dA to estimate the increase in circle's area A. Estimate the area of the enlarged circle and compare your estimate to your true value.

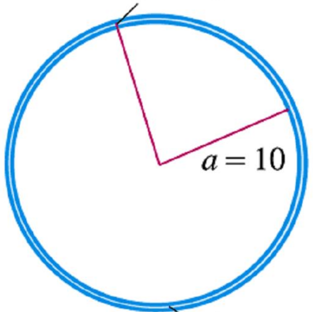
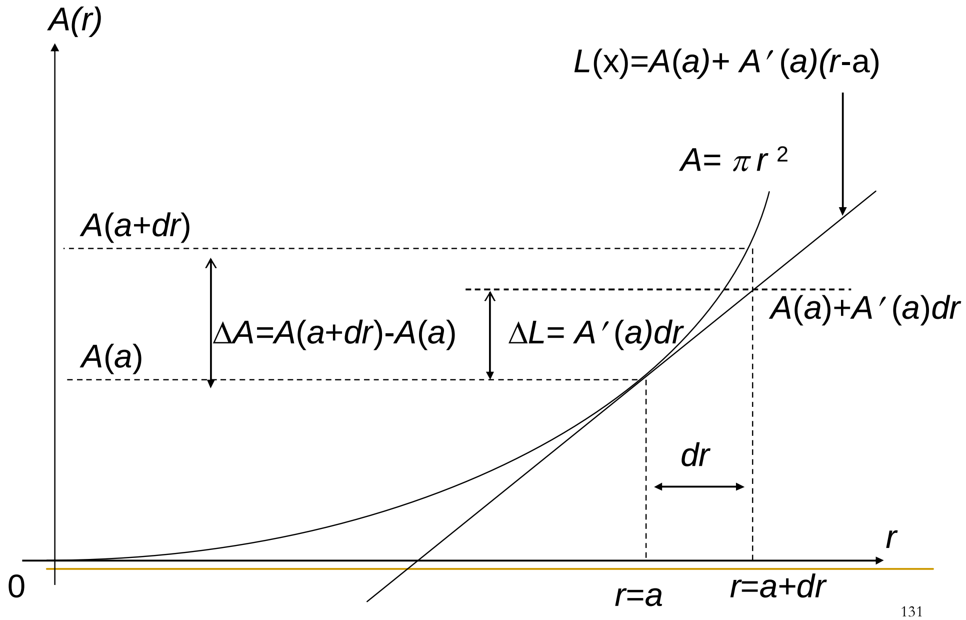


FIGURE 3.52 When dr is small compared with a, as it is when $dr = 0.1$ and $a = 10$, the differential $dA = 2\pi a\, dr$ gives a way to estimate the area of the circle with radius $r = a + dr$ (Example 6).



Solution to example 6

- Let a = 10 m, $a+dr$ = 10.1 m $\Rightarrow$ dr = 0.1m
- $A(r) = \pi r^2 \Rightarrow A(a) = \pi(10\text{ m})^2 = 100\pi\ \sigma\text{m}^2$
- $\Delta A \approx A'(a)dr = 2\pi(a)dr = 2\pi(10\text{ m})(0.1\text{ m}) =$
  $= 2\pi\ \text{m}^2$.
- $A(a+dr) = A(a) + \Delta A \approx A(a) + A'(a)dr$
  $= 102\pi\ \text{m}^2$ (this is an estimation)
- c.f the true area is $\pi(a+dr)^2 = \pi(10.1)^2 = 102.01\pi\ \text{m}^2$

Chapter 4

Applications of Derivatives

4.1

Extreme Values of Functions

DEFINITIONS Absolute Maximum, Absolute Minimum

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

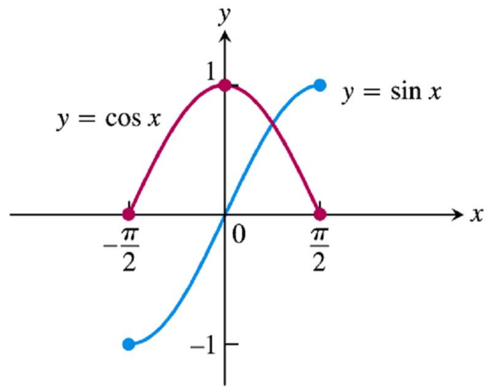


FIGURE 4.1 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

Example 1

- Exploring absolute extrema
- The absolute extrema of the following functions on their domains can be seen in Figure 4.2

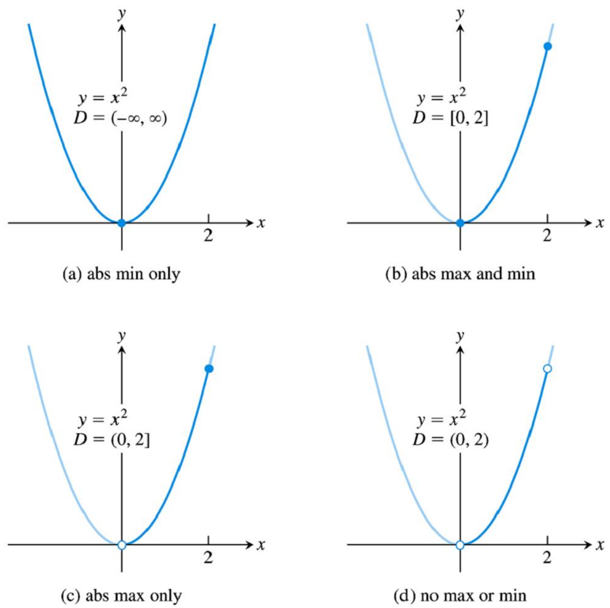


FIGURE 4.2 Graphs for Example 1.

THEOREM 1 The Extreme Value Theorem
If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$ (Figure 4.3).

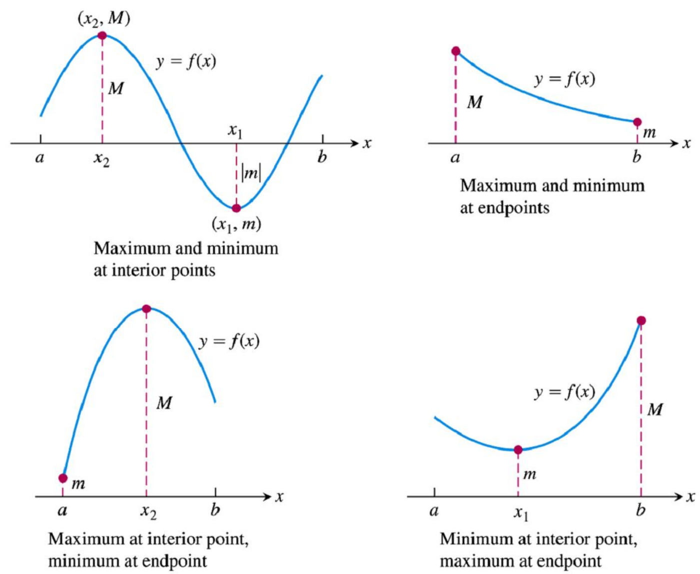


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

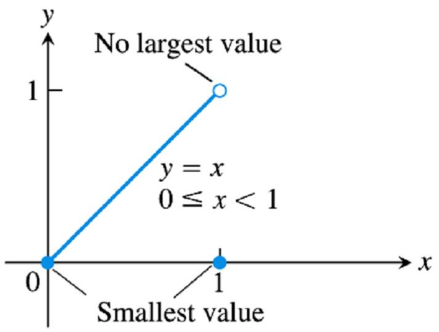


FIGURE 4.4 Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of [0, 1] except $x = 1$, yet its graph over [0, 1] does not have a highest point.

Local (relative) extreme values

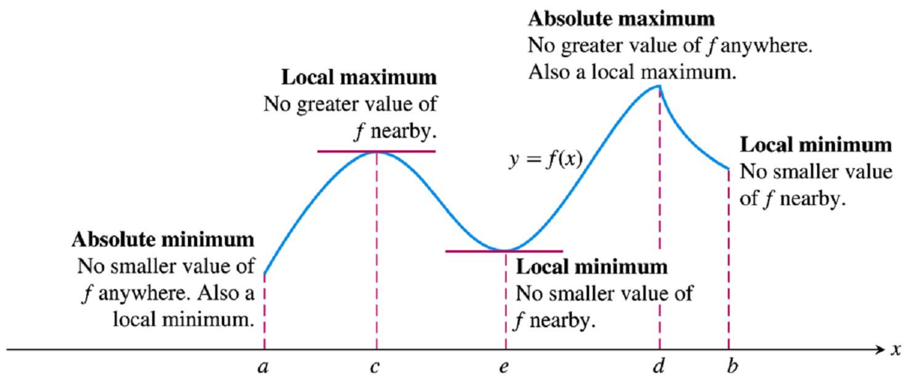


FIGURE 4.5 How to classify maxima and minima.

DEFINITIONS Local Maximum, Local Minimum

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \le f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \ge f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

Finding Extrema…with a not-always-effective method.

THEOREM 2 The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

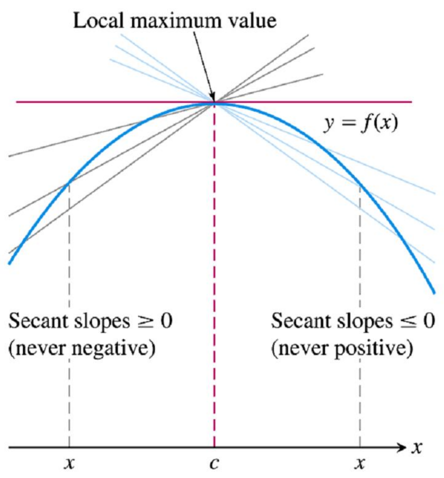


FIGURE 4.6 A curve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

DEFINITION Critical Point
An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

How to find the absolute extrema of a continuous function *f* on a finite closed interval

1. Evaluate *f* at all critical point and endpoints
2. Take the largest and smallest of these values.

Example 2: Finding absolute extrema

- Find the absolute maximum and minimum of $f(x) = x^2$ on [-2,1].

Example 3: Absolute extrema at endpoints

- Find the absolute extrema values of $g(t) = 8t - t^4$ on [-2,1].

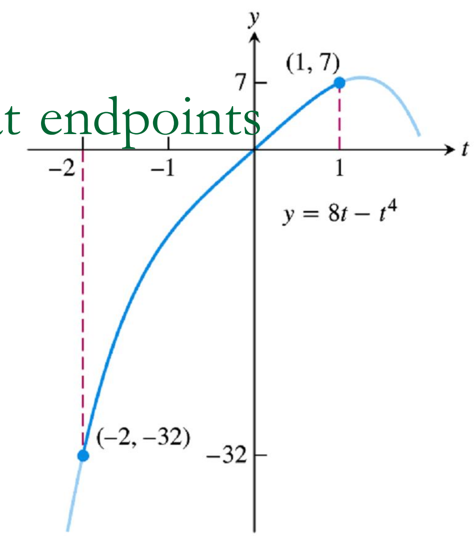


FIGURE 4.7 The extreme values of $g(t) = 8t - t^4$ on $[-2, 1]$ (Example 3).

Example 4: Finding absolute extrema on a closed interval

- Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval [-2,3].

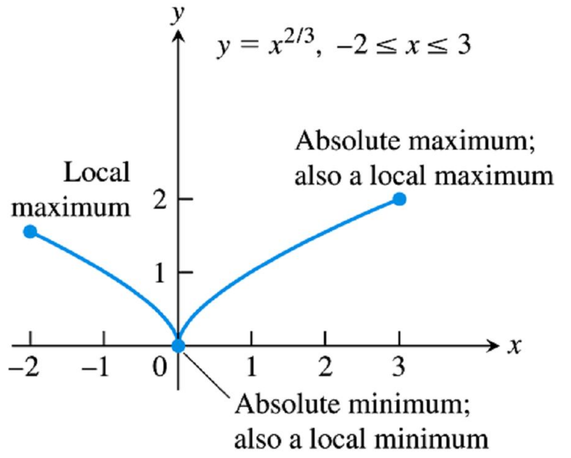
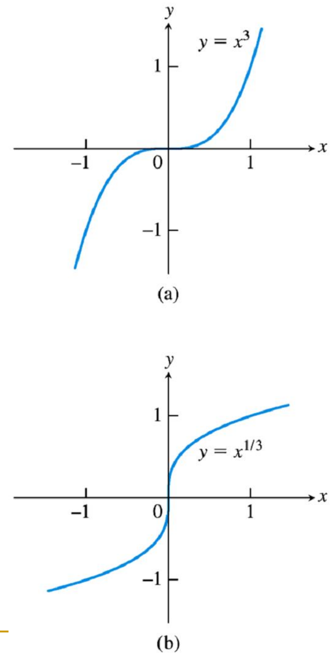


FIGURE 4.8 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).



- Not every critical point or endpoints signals the presence of an extreme value.

FIGURE 4.9 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

4.2

The Mean Value Theorem

THEOREM 3 Rolle's Theorem

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If

$$f(a) = f(b),$$

then there is at least one number c in (a, b) at which

$$f'(c) = 0.$$

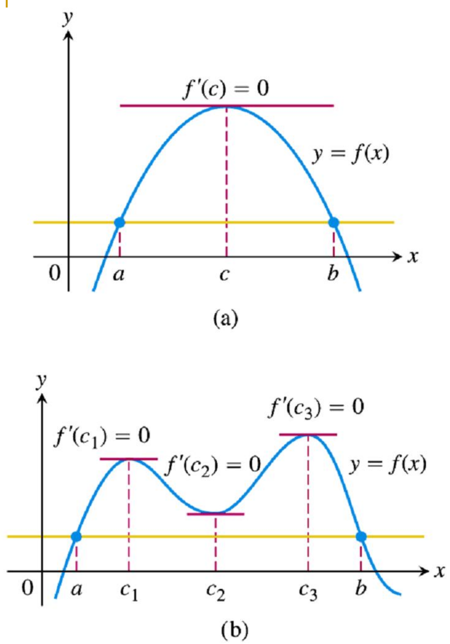


FIGURE 4.10 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

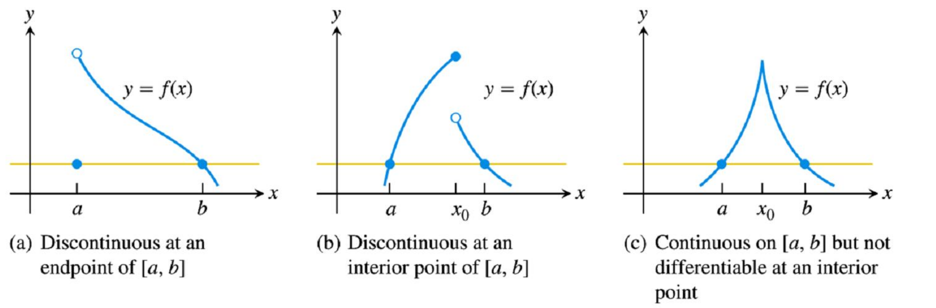


FIGURE 4.11 There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.

Example 1

- Horizontal tangents of a cubit polynomial

$$f(x) = \frac{x^3}{3} - 3x$$

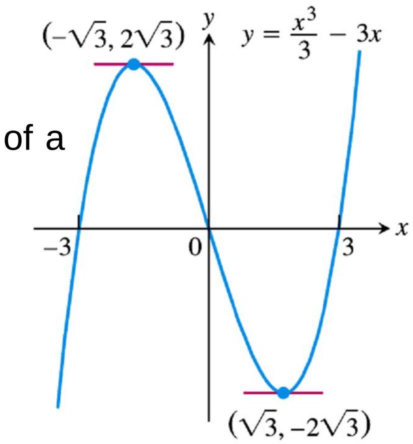


FIGURE 4.12 As predicted by Rolle's Theorem, this curve has horizontal tangents between the points where it crosses the x-axis (Example 1).

Example 2 Solution of an equation $f(x)=0$

- Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

Solution

1. Apply Intermediate value theorem to show that there exist at least one root
2. Apply Rolle's theotem to prove the uniqueness of the root.

The mean value theorem

THEOREM 4 The Mean Value Theorem
Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \qquad (1)$$

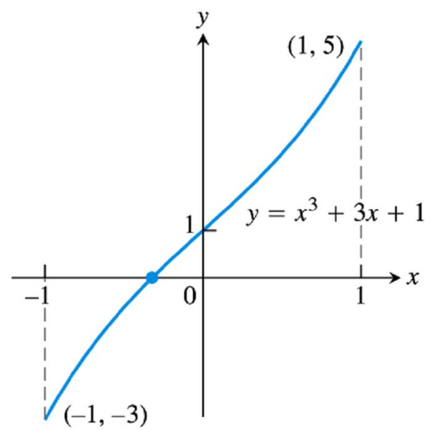


FIGURE 4.13 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the x-axis between -1 and 0 (Example 2).

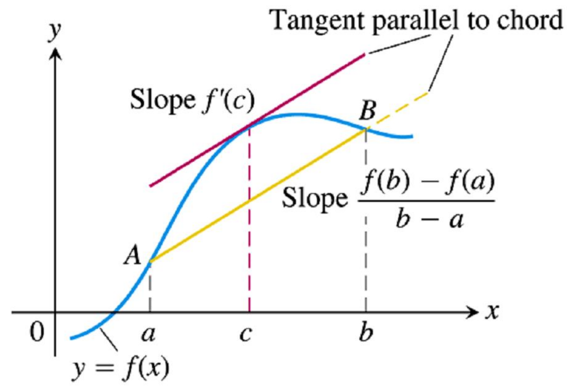


FIGURE 4.14 Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

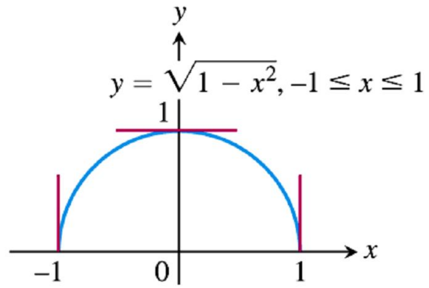


FIGURE 4.17 The function $f(x) = \sqrt{1 - x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though f is not differentiable at -1 and 1.

Example 3

- The function $f(x) = x^2$ is continuous for $0 \leqslant x \leqslant 2$ and differentiable for $0 < x < 2$.

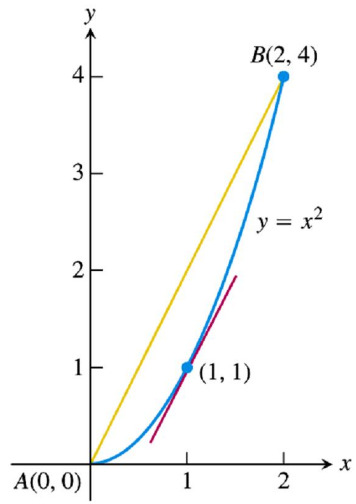


FIGURE 4.18 As we find in Example 3, $c = 1$ is where the tangent is parallel to the chord.

Mathematical consequences

COROLLARY 1 Functions with Zero Derivatives Are Constant
If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

COROLLARY 2 Functions with the Same Derivative Differ by a Constant
If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant on (a, b).

Corollary 1 can be proven using the Mean Value Theorem

- Say $x_1, x_2 \in (a,b)$ such that $x_1 < x_2$
- By the MVT on $[x_1,x_2]$ there exist some point c between x_1 and x_2 such that $f'(c) = (f(x_2) - f(x_1))/(x_2 - x_1)$
- Since $f'(c) = 0$ throughout (a,b), $f(x_2) - f(x_1) = 0$, hence $f(x_2) = f(x_1)$ for $x_1, x_2 \in (a,b)$.
- This is equivalent to $f(x)$ = a constant for $x \in (a,b)$.

Proof of Corollary 2

- At each point $x \in (a,b)$ the derivative of the difference between function $h = f - g$ is

  $h'(x) = f'(x) - g'(x) = 0$
- Thus $h(x) = C$ on (a,b) by Corollary 1. That is $f(x) - g(x) = C$ on (a,b), so $f(x) = C + g(x)$.

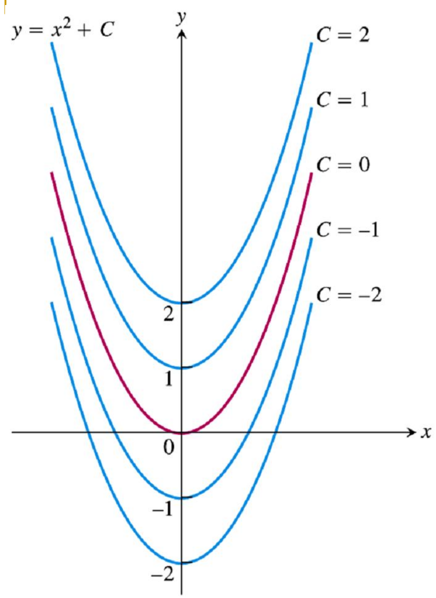


FIGURE 4.20 From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives on an interval can differ only by a vertical shift there. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for selected values of C.

Example 5

- Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0,2)$.

4.3

Monotonic Functions and The First Derivative Test

Increasing functions and decreasing functions

DEFINITIONS Increasing, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

A function that is increasing or decreasing on I is called **monotonic** on I.

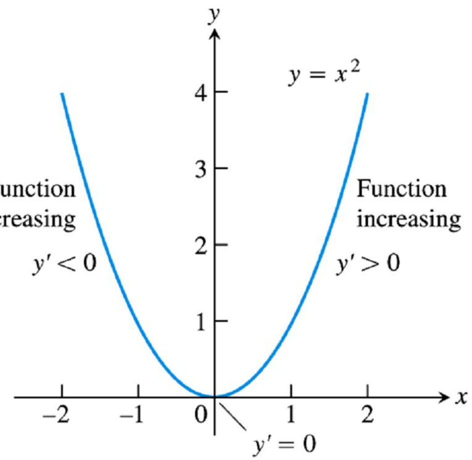


FIGURE 4.21 The function $f(x) = x^2$ is monotonic on the intervals $(-\infty, 0]$ and $[0, \infty)$, but it is not monotonic on $(-\infty, \infty)$.

COROLLARY 3 First Derivative Test for Monotonic Functions

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Mean value theorem is used to prove Corollary 3

Example 1

- Using the first derivative test for monotonic functions $f(x) = x^3 - 12x - 5$
- Find the critical point of $f(x) = x^3 - 12x - 5$ and identify the intervals on which *f* is increasing and decreasing.

Solution

$f'(x) = 3(x+2)(x-2)$

$f' +$ for $-\infty < x < -2$

$f' -12$ for $-2 < x < 2$

$f' +$ for $2 < x < \infty$

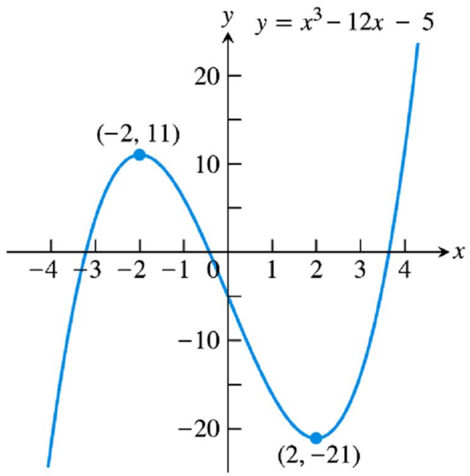


FIGURE 4.22 The function $f(x) = x^3 - 12x - 5$ is monotonic on three separate intervals (Example 1).

First derivative test for local extrema

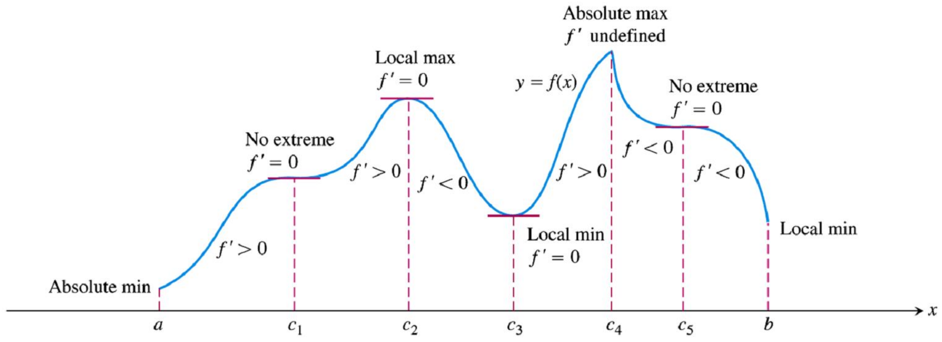


FIGURE 4.23 A function's first derivative tells how the graph rises and falls.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

Example 2: Using the first derivative test for local extrema

- Find the critical point of

$$f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$$

- Identify the intervals on which *f* is increasing and decreasing. Find the function's local and absolute extreme values.

$$f' = \frac{4(x-1)}{3x^{2/3}}; f' - \text{ve} \quad \text{for } x < 0;$$

$$f' - \text{ve} \quad \text{for } 0 < x < 1; f' + \text{ve} \quad \text{for } x > 1$$

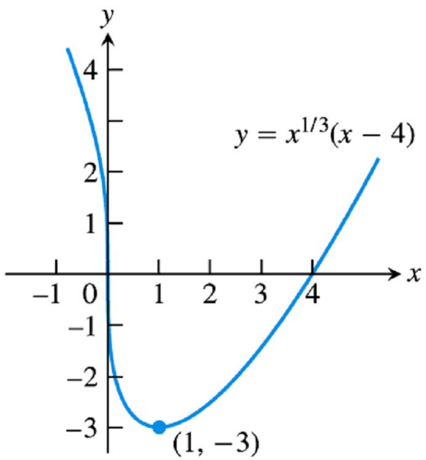


FIGURE 4.24 The function $f(x) = x^{1/3}(x-4)$ decreases when $x < 1$ and increases when $x > 1$ (Example 2).

4.4

Concavity and Curve Sketching

Concavity

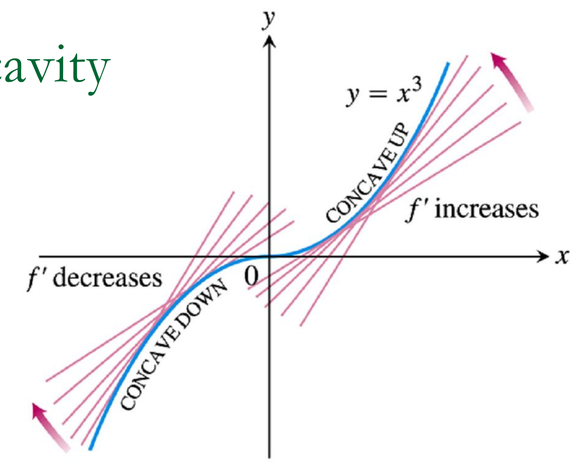


FIGURE 4.25 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

go back

DEFINITION **Concave Up, Concave Down**

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I

(b) **concave down** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Example 1(a): Applying the concavity test

- Check the concavity of the curve $y = x^3$
- Solution: $y'' = 6x$
- $y'' < 0$ for $x < 0$; $y'' > 0$ for $x > 0$;

Link to Figure 4.25

Example 1(b): Applying the concavity test

- Check the concavity of the curve $y = x^2$
- Solution: $y'' = 2 > 0$

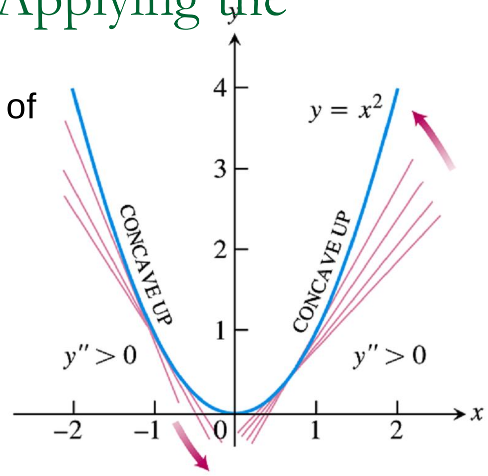


FIGURE 4.26 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

Example 2

- Determining concavity
- Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

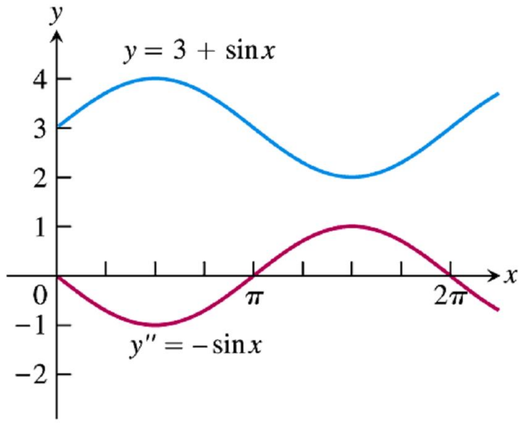


FIGURE 4.27 Using the graph of y'' to determine the concavity of y (Example 2).

Example 3

- An inflection point may not exist where An inflection point may not exist where $y'' = 0$
- The curve $y = x^4$ has no inflection point at $x=0$. Even though $y'' = 12x^2$ is zero there, it does not change sign.

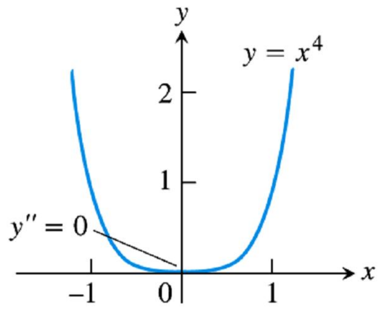


FIGURE 4.28 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 3).

Point of inflection

DEFINITION Point of Inflection
A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

Example 4

- An inflection point may occur where $y'' = 0$ does not exist
- The curve $y = x^{1/3}$ has a point of inflection at $x=0$ but y'' does not exist there.
- $y'' = -(2/9)x^{-5/3}$

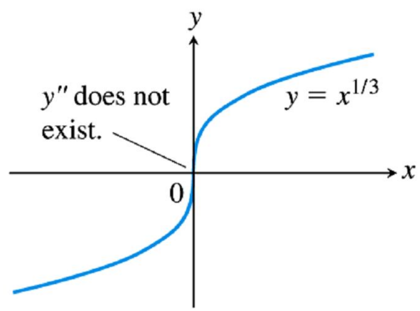


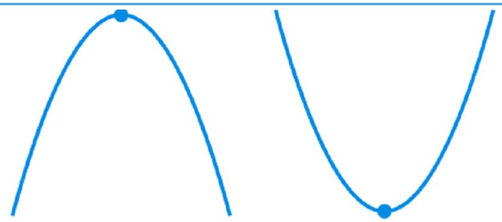
FIGURE 4.29 A point where y'' fails to exist can be a point of inflection (Example 4).

Second derivative test for local extrema

THEOREM 5 Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



$f' = 0, f'' < 0$ $\Rightarrow$ local max

$f' = 0, f'' > 0$ $\Rightarrow$ local min

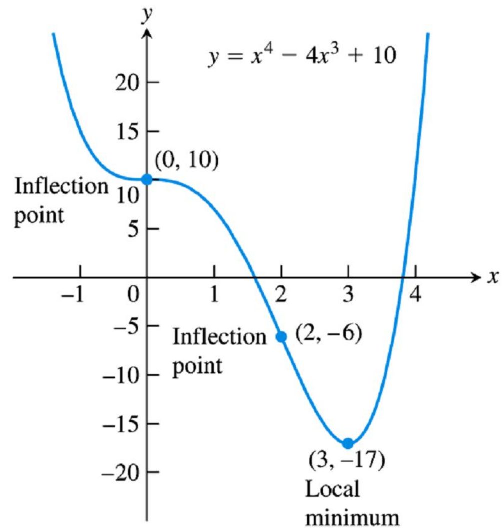


FIGURE 4.30 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 6).

Example 6: Using f' and f'' to graph f

- Sketch a graph of the function
  $f(x) = x^4 - 4x^3 + 10$
  using the following steps.

(a) Identify where the extrema of *f* occur

(b) Find the intervals on which *f* is increasing and the intervals on which *f* is decreasing

(c) Find where the graph of *f* is concave up and where it is concave down.

(d) Identify the slanted/vertical/horizontal asymtots, if there is any

(e) Sketch the general shape of the graph for *f*.

(f) Plot the specific points. Then sketch the graph.

Example

- Using the graphing strategy
- Sketch the graph of
- $f(x) = (x + 1)^2 / (x^2 + 1)$.

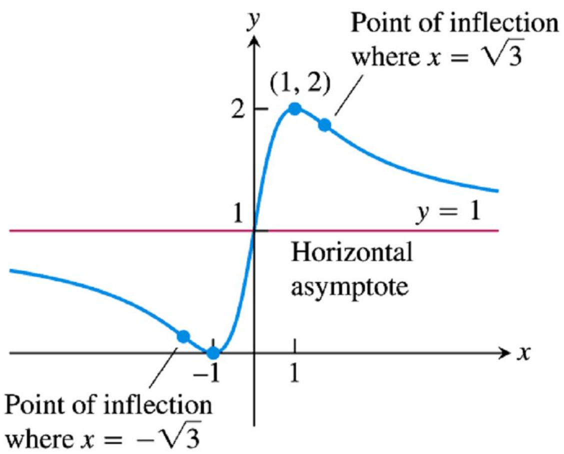
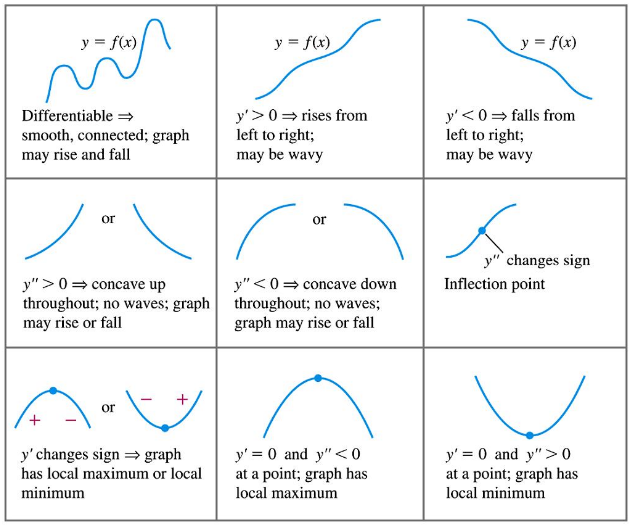


FIGURE 4.31 The graph of $y = \frac{(x+1)^2}{1+x^2}$ (Example 7).

Learning about functions from derivatives



4.5

Applied Optimization Problems

Example 1

- An open-top box is to be cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

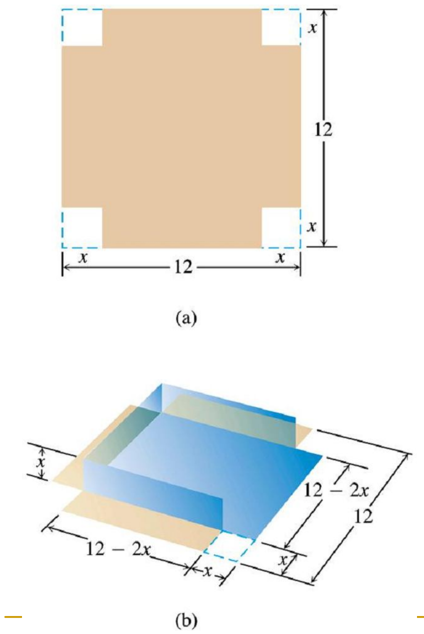


FIGURE 4.32 An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume (Example 1)?

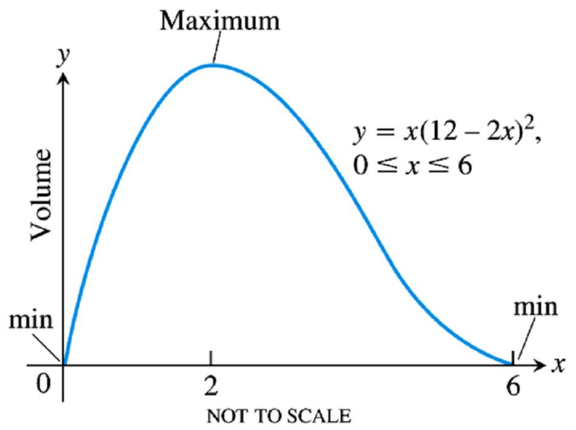


FIGURE 4.33 The volume of the box in Figure 4.32 graphed as a function of x.

Example 2

- Designing an efficient cylindrical can
- Design a 1-liter can shaped like a right circular cylinder. What dimensions will use the least material?

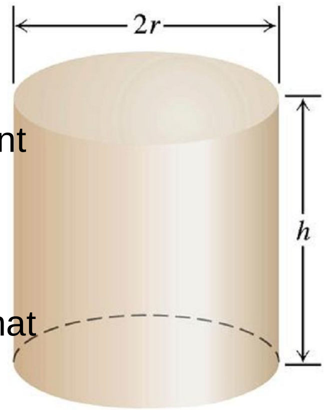


FIGURE 4.34 This 1-L can uses the least material when $h = 2r$ (Example 2).

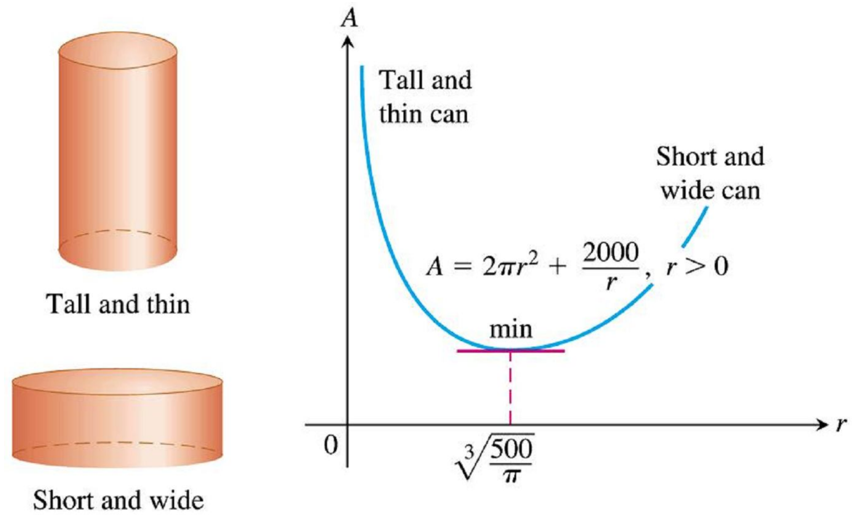


FIGURE 4.35 The graph of $A = 2\pi r^2 + 2000/r$ is concave up.

Example 3

- Inscribing rectangles
- A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what is its dimension?

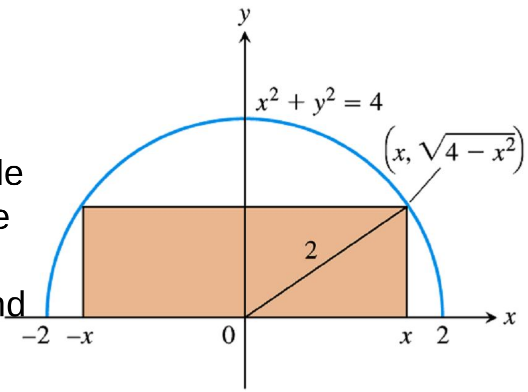


FIGURE 4.36 The rectangle inscribed in the semicircle in Example 3.

Solution

- Form the function of the area A as a function of x: $A=A(x)=x(4-x^2)^{1/2}$; $x > 0$.
- Seek the maximum of A:

4.6

Indeterminate Forms and L′ Hopital’s Rule

Indeterminate forms 0/0

THEOREM 6 L'Hôpital's Rule (First Form)
Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Example 1

- Using L′ Hopital's Rule
- (a)

$$\lim_{x\to 0}\frac{3x-\sin x}{x}=\frac{3-\cos x}{1}\bigg|_{x=0}=2$$

- (b)

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{\frac{1}{2\sqrt{1+x}}}{1}\Bigg|_{x=0}=\frac{1}{2}$$

THEOREM 7 L'Hôpital's Rule (Stronger Form)

Suppose that $f(a)=g(a)=0$, that f and g are differentiable on an open interval I containing a, and that $g'(x)\neq 0$ on I if $x\neq a$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

Example 2(a)

- Applying the stronger form of L′ Hopital's rule
- (a)

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2}=\lim_{x\to 0}\frac{(1/2)(1+x)^{-1/2}-1/2}{2x}$$

$$=\lim_{x\to 0}\frac{-(1/4)(1+x)^{-3/2}}{2}=\frac{-1}{8}$$

Example 2(b)

- Applying the stronger form of L′ Hopital's rule
- (b)

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}$$

THEOREM 8 Cauchy's Mean Value Theorem

Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Example 3

- Incorrect application of the stronger form of L' Hopital's

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2}$$

Example 4

- Using l' Hopital's rule with one-sided limits

$$(a) \lim_{x \to 0^+} \frac{\sin x}{x^2} = \lim_{x \to 0^+} \frac{\cos x}{2x} = \ldots$$

$$(b) \lim_{x \to 0^-} \frac{\sin x}{x^2} = \lim_{x \to 0^-} \frac{\cos x}{2x} = \ldots$$

Indeterminate forms ∞/∞, $\infty\cdot 0$, $\infty-\infty$

- If $f \to \pm\infty$ and $g \to \pm\infty$ as $x \to a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

- a may be finite or infinite

Example 5
Working with the indeterminate form ∞/∞

$$(a)\ \lim_{x\to\pi/2}\frac{\sec x}{1+\tan x}$$

$$\lim_{x\to(\pi/2)^-}\frac{\sec x}{1+\tan x}=\lim_{x\to(\pi/2)^-}\frac{\sec x\tan x}{\sec^2 x}=\lim_{x\to(\pi/2)^-}\sin x=1$$

$$\lim_{x\to(\pi/2)^+}\frac{\sec x}{1+\tan x}.=\ldots$$

Example 5(b)

$$(b)\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5x}=\ldots$$

Example 6

- Working with the indeterminate form $\infty\cdot 0$

$$\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$$

Example 7

- Working with the indeterminate form ∞ - ∞

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right)=\ldots$$

4.8

Antiderivatives

Finding antiderivatives

DEFINITION Antiderivative
A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example 1

- Finding antiderivatives
- Find an antiderivative for each of the following functions
- (a) $f(x) = 2x$
- (b) $f(x) = \cos x$
- (c) $h(x) = 2x + \cos x$

The most general antiderivative

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Example 2 Finding a particular antiderivative

- Find an antiderivative of $f(x) = \sin x$ that satisfies $F(0) = 3$
- Solution: $F(x)=\cos x + C$ is the most general form of the antiderivative of $f(x)$.
- We require $F(x)$ to fulfill the condition that when $x=3$ (in unit of radian), $F(x)=0$. This will fix the value of C, as per
- $F(3)= 3 = \cos 3 + C \Rightarrow 3 - \cos 3$
- Hence, $F(x)= \cos x + (3 - \cos 3)$ is the antiderivative sought

TABLE 4.2 Antiderivative formulas

	Function	General antiderivative
1.	x^n	$\frac{x^{n+1}}{n+1} + C, \quad n \neq -1$, n rational
2.	$\sin kx$	$-\frac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\frac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

Example 3 Finding antiderivatives using table 4.2

- Find the general antiderivative of each of the following functions.
- (a) $f(x) = x^5$
- (b) $g(x) = 1/x^{1/2}$
- (c) $h(x) = \sin 2x$
- (d) $i(x) = \cos(x/2)$

Example 4 Using the linearity rules for antiderivatives

- Find the general antiderivative of
- $f(x) = 3/x^{1/2} + \sin 2x$

DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,dx.$$

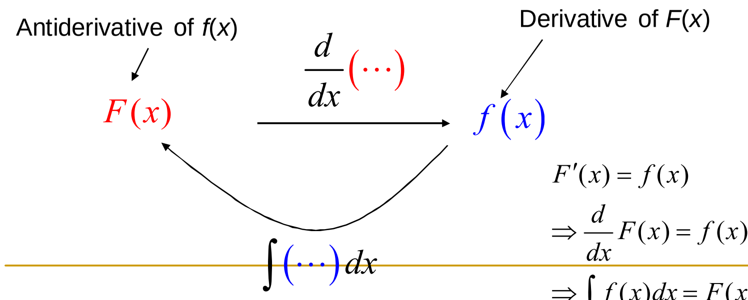
The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

- In other words, given a function $f(x)$, the most general form of its antiderivative, previously represented by the symbol $F(x) + C$, where C denotes an arbitrary constant, is now being represented in the form of an indefinite integral, namely,

$$\int f(x)dx \equiv F(x) + C$$

Operationally, the indefinite integral of $f(x)$ means

Operationally, the indefinite integral of $f(x)$ is the inverse of the operation of derivative taking of $f(x)$



$$F'(x) = f(x)$$
$$\Rightarrow \frac{d}{dx}F(x) = f(x)$$
$$\Rightarrow \int f(x)dx = F(x) + C$$

Example of indefinite integral notation

$$\int 2x\ dx = x^2 + C$$
$$\int \cos x\ dx = \sin x + C$$
$$\int (2x + \cos x)\ dx = x^2 + \sin x + C$$

Example 7 Indefinite integration done term-by term and rewriting the constant of integration

- Evaluate

$$\int \left(x^2 - 2x + 5\right) dx = \int x^2 dx - \int 2x dx + \int 5 dx = \ldots$$

Chapter 5

Integration

5.1

Estimating with Finite Sums

Riemann Sums

Approximating area bounded by the graph between [a,b]

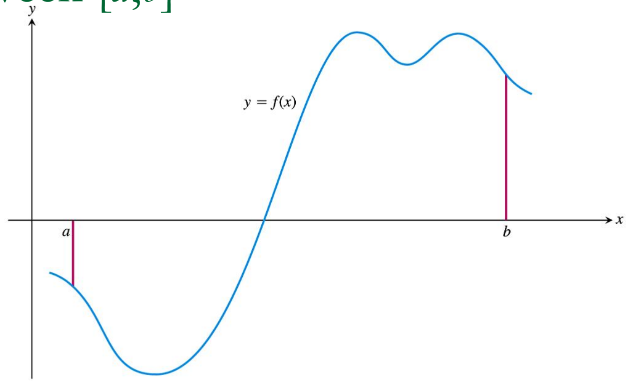


FIGURE 5.8 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

Area is approximately given by

$$f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \ldots + f(c_n)\Delta x_n$$

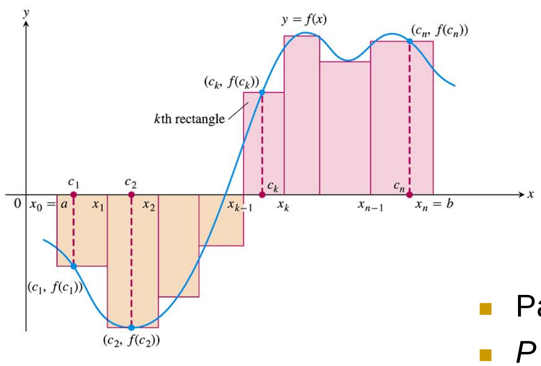
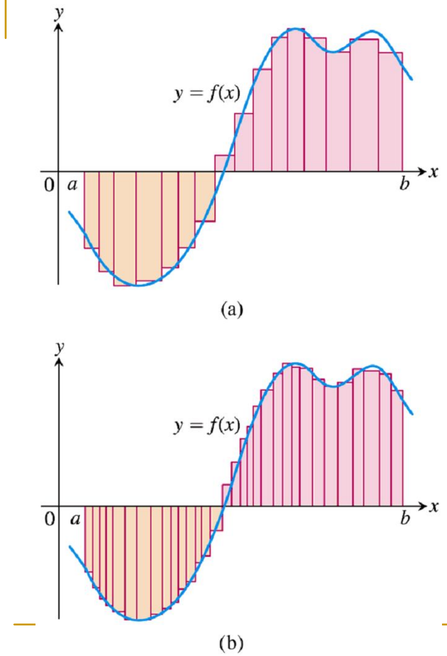


FIGURE 5.9 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis.

- Partition of [a,b] is the set of
- $P = \{x_0, x_1, x_2, \ldots x_{n-1}, x_n\}$
- $a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = b$
- $c_n \in [x_{n-1}, x_n]$
- $||P||$ = norm of P = the largest of all subinterval width

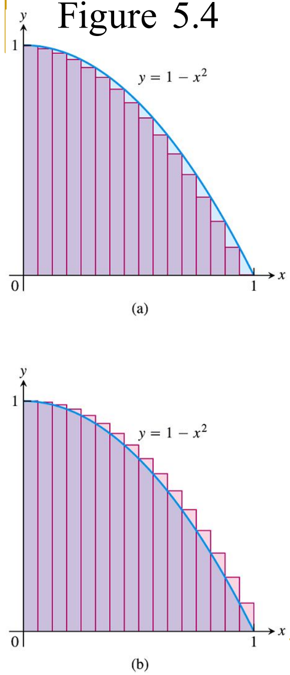


Riemann sum for f on $[a,b]$

$$R_n = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \ldots + f(c_n)\Delta x_n$$

FIGURE 5.10 The curve of Figure 5.9 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x-axis with increasing accuracy.

Figure 5.4



- Let the true value of the area is R
- Two approximations to R:
- $c_n = x_n$ corresponds to case (a). This under estimates the true value of the area R if n is finite.
- $c_n = x_{n-1}$ corresponds to case (b). This over estimates the true value of the area S if n is finite.

go back

Limits of finite sums

- Example 5 The limit of finite approximation to an area
- Find the limiting value of lower sum approximation to the area of the region R below the graphs $f(x) = 1 - x^2$ on the interval [0,1] based on Figure 5.4(a)

Solution

- $\Delta x_k = (1 - 0)/n = 1/n \equiv \Delta x$; $k = 1,2,\ldots n$
- Partition on the x-axis: $[0,1/n]$, $[1/n, 2/n]$,…, $[(n-1)/n,1]$.
- $c_k = x_k = k\Delta x = k/n$
- The sum of the stripes is

$R_n = \Delta x_1 f(c_1) + \Delta x_2 f(c_2) + \Delta x_3 f(c_3) + \ldots + \Delta x_n f(c_n)$
$= \Delta x f(1/n) + \Delta x f(2/n) + \Delta x f(3/n) + \ldots + \Delta x_n f(1)$
$= \Sigma_{k=1}^{n} \Delta x f(k\Delta x) = \Delta x \Sigma_{k=1}^{n} f(k/n)$
$= (1/n) \Sigma_{k=1}^{n} [1 - (k/n)^2]$
$= \Sigma_{k=1}^{n} 1/n - k^2/n^3 = 1 - (\Sigma_{k=1}^{n} k^2)/n^3$
$= 1 - [(n)(n+1)(2n+1)/6]/n^3 = 1 - [2n^3 + 3n^2 + n]/(6n^3)$

$\Sigma_{k=1}^{n} k^2 = (n)(n+1)(2n+1)/6$

- Taking the limit of $n \to \infty$

$$\lim_{n\to\infty} R_n = R = \left(1 - \frac{2n^3 + 3n^2 + n}{6n^3}\right) = 1 - 2/6 = 2/3$$

- The same limit is also obtained if $c_n = x_{n-1}$ is chosen instead.
- For all choice of $c_n \in [x_{n-1}, x_n]$ and partition of P, the same limit for S is obtained when $n \to \infty$

5.3

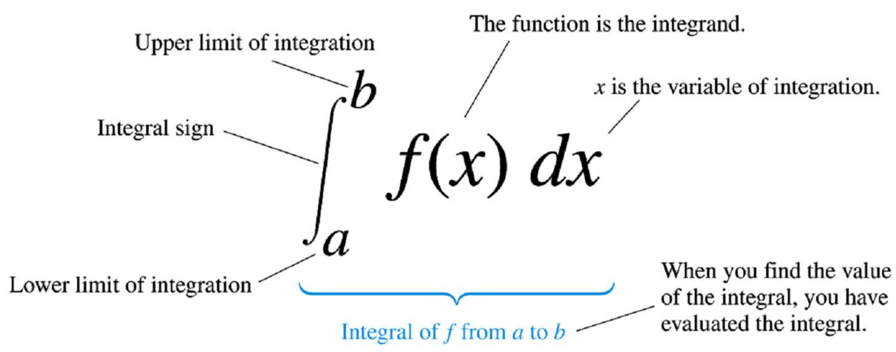
The Definite Integral

DEFINITION The Definite Integral as a Limit of Riemann Sums

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number I is the **definite integral of f over $[a, b]$** and that I is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - I \right| < \epsilon.$$



"The integral from a to b of f of x with respect to x"

- The limit of the Riemann sums of *f* on [*a*,*b*] converge to the finite integral *I*

$$\lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\Delta x_k = I = \int_a^b f(x)dx$$

- We say *f* is integrable over [*a*,*b*]
- Can also write the definite integral as

$$I = \int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du$$

$$= \int_a^b f(\text{what ever}) \quad d(\text{what ever})$$

- The variable of integration is what we call a 'dummy variable'

THEOREM 1 The Existence of Definite Integrals
A continuous function is integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

Question: is a non continuous function integrable?

Integral and nonintegrable functions

- Example 1
- A nonintegrable function on [0,1]

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

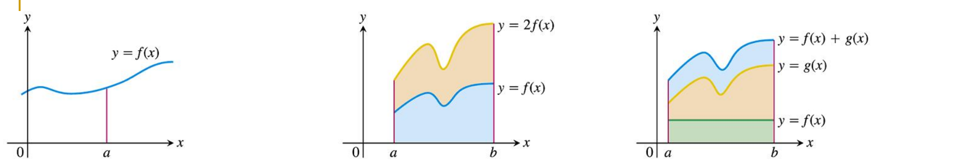
- Not integrable

Properties of definite integrals

THEOREM 2
When f and g are integrable, the definite integral satisfies Rules 1 to 7 in Table 5.3.

TABLE 5.3 Rules satisfied by definite integrals

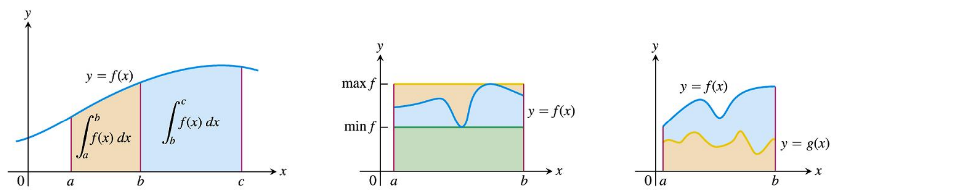
1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	Also a Definition
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any Number k
		$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5.	*Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6.	*Max-Min Inequality:*	If f has maximum value max f and minimum value min f on $[a, b]$, then $\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$	
7.	*Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
		$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special Case)



(a) *Zero Width Interval:* $\int_a^a f(x)\,dx = 0.$ (The area over a point is 0.)

(b) *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$ (Shown for $k = 2$.)

(c) *Sum:* $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ (Areas add)



(d) *Additivity for definite integrals:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

(e) *Max-Min Inequality:* $\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$

(f) *Domination:* $f(x) \ge g(x)$ on $[a, b]$ $\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

FIGURE 5.11

Example 3 Finding bounds for an integral

- Show that the value of $\int_0^1 \sqrt{1+\cos x}\,dx$ is less than 3/2

- **Solution**
- Use rule 6 Max-Min Inequality

Area under the graphs of a nonnegative function

DEFINITION **Area Under a Curve as a Definite Integral**
If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

Example 4 Area under the line $y = x$

- Compute $\int_0^b x\,dx$ (the Riemann sum) and find the area *A* under *y* = *x* over the interval [0,*b*], *b*>0

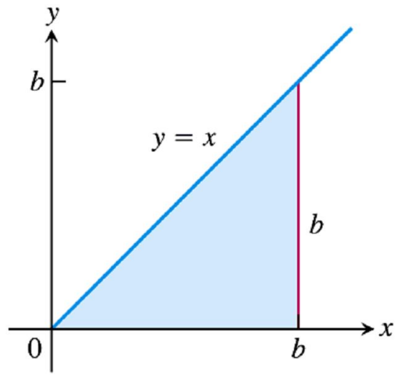


FIGURE 5.12 The region in Example 4 is a triangle.

Solution

By geometrical consideration:

A=(1/2)×high×width= (1/2)×b×b= $b^2/2$

Choose partition of n subinterval with equal width:

$$\{0 = x_0, x_1, x_2, \dots x_n = b\}, \Delta x_k = x_k - x_{k-1} = \Delta x = b/n$$

Riemann sum:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\Delta x f(c_k) = \lim_{n\to\infty}\Delta x\sum_{k=1}^{n} f(x_k)$$

$$= \lim_{n\to\infty}\Delta x\sum_{k=1}^{n} x_k = \lim_{n\to\infty}\Delta x\sum_{k=1}^{n} k\Delta x$$

$$= \lim_{n\to\infty}(\Delta x)^2\sum_{k=1}^{n} k = \lim_{n\to\infty}\left(\frac{b}{n}\right)^2\sum_{k=1}^{n} k$$

$$= \lim_{n\to\infty}\left(\frac{b}{n}\right)^2\frac{n(n+1)}{2} = \lim_{n\to\infty}\left(\frac{b}{n}\right)^2\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty}\frac{b^2}{2}\left(1+\frac{1}{n}\right) = \frac{b^2}{2}$$

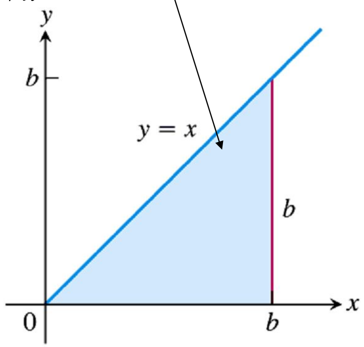


FIGURE 5.12 The region in Example 4 is a triangle.

Using the additivity rule for definite integration:

$$\int_0^b x\,dx = \int_0^a x\,dx + \int_a^b x\,dx$$

$$\rightarrow \int_a^b x\,dx = \int_0^b x\,dx - \int_0^a x\,dx = \frac{b^2}{2} - \frac{a^2}{2}, a < b$$

Using geometry, the area is the area of a trapezium

A= (1/2)(b-a)(b+a)

= $b^2/2$ - $a^2/2$

Both approaches to evaluate the area agree

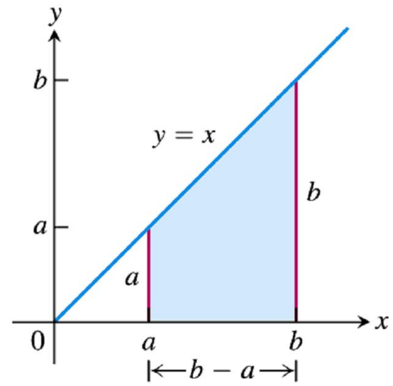


FIGURE 5.13 The area of this trapezoidal region is $A = (b^2 - a^2)/2$.

- One can prove the following Riemannian sum of the functions $f(x)=c$ and $f(x)= x^2$:

$$\int_a^b c\,dx = c(b - a), \qquad c \text{ any constant} \qquad (2)$$

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}, \qquad a < b \qquad (3)$$

Average value of a continuous function revisited

- Average value of nonnegative continuous function *f* over an interval [*a*,*b*] is

$$\frac{f(c_1)+f(c_2)+\cdots f(c_n)}{n}=\frac{1}{n}\sum_{k=1}^{n} f(c_k)$$

$$=\frac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k)=\frac{1}{b-a}\sum_{k=1}^{n}\Delta x f(c_k)$$

- In the limit of $n \rightarrow \infty$, the average =

$$\frac{1}{b-a}\int_a^b f(x)dx$$

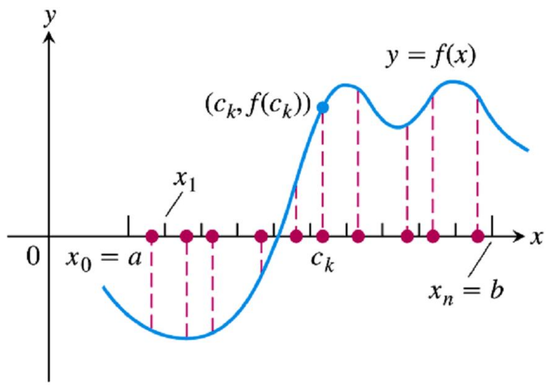


FIGURE 5.14 A sample of values of a function on an interval [*a*, *b*].

DEFINITION **The Average or Mean Value of a Function**
If f is integrable on $[a, b]$, then its **average value on** $[a, b]$, also called its **mean value**, is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example 5 Finding average value

- Find the average value of $f(x)=\sqrt{4-x^2}$ over [-2,2]

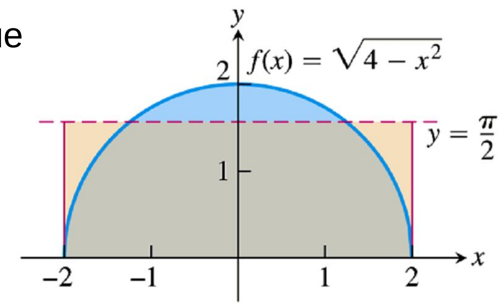


FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 5).

5.4

The Fundamental Theorem of Calculus

Mean value theorem for definite integrals

THEOREM 3 The Mean Value Theorem for Definite Integrals
If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

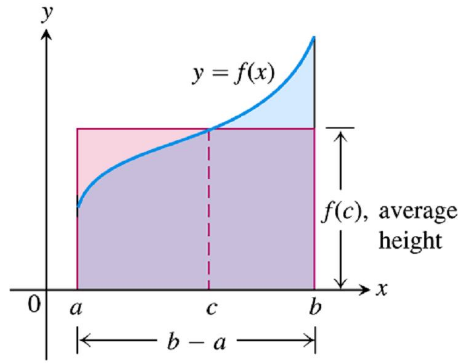


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

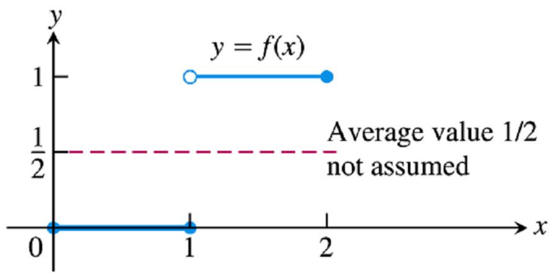


FIGURE 5.17 A discontinuous function need not assume its average value.

Example 1 Applying the mean value theorem for integrals

- Find the average value of $f(x)=4-x$ on [0,3] and where f actually takes on this value as some point in the given domain.

- **Solution**
- Average = 5/2
- Happens at $x=3/2$

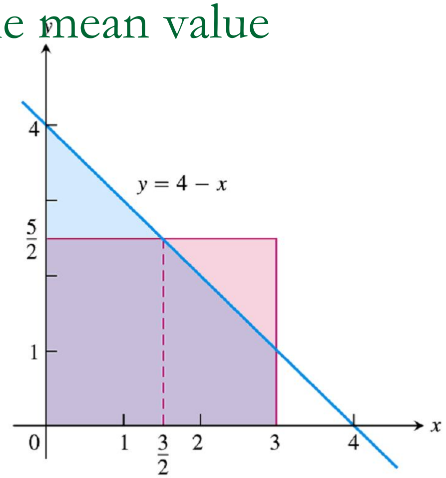


FIGURE 5.18 The area of the rectangle with base [0, 3] and height 5/2 (the average value of the function $f(x) = 4 - x$) is equal to the area between the graph of f and the x-axis from 0 to 3 (Example 1).

Fundamental theorem Part 1

- Define a function $F(x)$: $F(x)=\int_a^x f(t)dt$
- $x,a \in I$, an interval over which $f(t) > 0$ is integrable.
- The function $F(x)$ is the area under the graph of $f(t)$ over $[a,x]$, $x > a \geqslant 0$

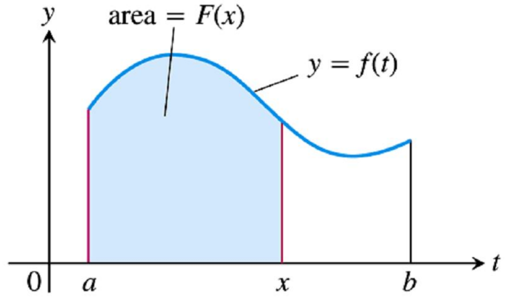


FIGURE 5.19 The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.

Fundamental theorem Part 1 (cont.)

$$F(x+h)-F(x)\approx hf(x)$$

$$\frac{F(x+h)-F(x)}{h}\approx f(x)$$

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=F'(x)=f(x)$$

The above result holds true even if f is not positive definite over $[a,b]$

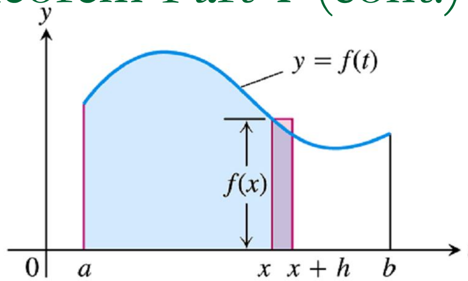


FIGURE 5.20 In Equation (1), $F(x)$ is the area to the left of x. Also, $F(x + h)$ is the area to the left of $x + h$. The difference quotient $[F(x + h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

THEOREM 4 The Fundamental Theorem of Calculus Part 1
If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \qquad (2)$$

Note: Convince yourself that

(i) $F(x)$ is an antiderivative of $f(x)$?

(ii) $f(x)$ is an derivative of $F(x)$?

d/dx

$F(x)$ → $f(x) = F'(x)$

$f(x)$ is an derivative of $F(x)$

$F(x)$ is an antiderivative of $f(x)$ because $F'(x)= f(x)$

Example 3 Applying the fundamental theorem

- Use the fundamental theorem to find

$$(a)\ \frac{d}{dx}\int_a^x \cos t\,dt \qquad (b)\ \frac{d}{dx}\int_a^x \frac{1}{1+t^2}\,dt$$

$$(c)\ \frac{dy}{dx}\ \text{if } y = \int_x^5 3t\sin t\,dt \qquad (d)\ \frac{dy}{dx}\ \text{if } y = \int_1^{x^2} \cos t\,dt$$

Example 4 Constructing a function with a given derivative and value

- Find a function $y = f(x)$ on the domain $(-\pi/2, \pi/2)$ with derivative $dy/dx = \tan x$ that satisfy $f(3)=5$.

Solution

- Consider $k(x) = \int_a^x \tan t\,dt$
- Set the constant $a = 3$, and then add to $k(3) = 0$ a value of 5, that would make $k(3) + 5 = 5$
- Hence the function that will do the job is

$$f(x) = k(x) + 5 = \int_3^x \tan t\,dt + 5$$

Fundamental theorem, part 2 (The evaluation theorem)

THEOREM 4 (Continued) **The Fundamental Theorem of Calculus Part 2**
If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

To calculate the definite integral of *f* over [*a*,*b*], do the following

- 1. Find an antiderivative *F* of *f*, and
- 2. Calculate the number

$$\int_a^b f(x)dx = F(b) - F(a)$$

To summarise

$$\frac{d}{dx}\int_a^x f(t)dt = \frac{dF(x)}{dx} = f(x)$$

$$\int_a^x \left(\frac{dF(t)}{dt}\right)dt = \int_a^x f(t)dt = F(x) - F(a)$$

Example 5 Evaluating integrals

$$(a)\int_0^{\pi} \cos x dx$$

$$(b)\int_{-\pi/4}^{0} \sec x \tan x dx$$

$$(c)\int_1^{x^2}\left(\frac{3}{2}\sqrt{t} - \frac{4}{t^2}\right)dt = \int_1^{x^2}\left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)dx$$

Example 7 Canceling areas

- Compute
- (a) the definite integral of $f(x)$ over $[0,2\pi]$
- (b) the area between the graph of $f(x)$ and the x-axis over $[0,2\pi]$

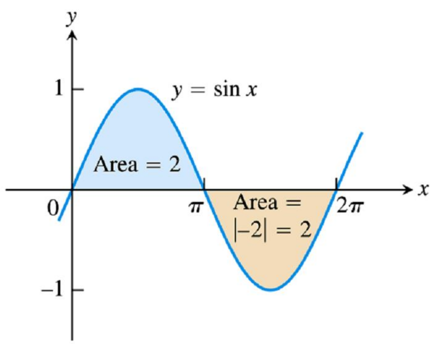


FIGURE 5.22 The total area between $y = \sin x$ and the x-axis for $0 \le x \le 2\pi$ is the sum of the absolute values of two integrals (Example 7).

Example 8 Finding area using antiderivative

- Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leqslant x \leqslant 2$.

- **Solution**
- First find the zeros of f.
- $f(x) = x(x+1)(x-2)$

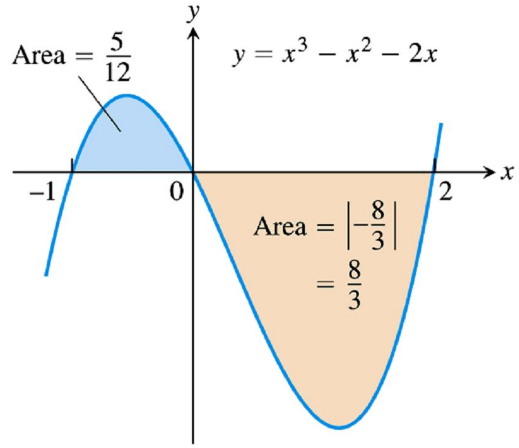


FIGURE 5.23 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 8).

5.5

Indefinite Integrals and the Substitution Rule

Note

- The indefinite integral of *f* with respect to *x*,

$$\int f(x)dx$$

  is a function plus an arbitrary constant

- A definite integral $\int_a^b f(x)dx$ is a number.

Antiderivative and indefinite integral in terms of variable x

- *If F*(*x*) is an antiderivative of *f*(*x*),

$$\frac{d}{dx}F(x)=f(x)$$

$$\Rightarrow$$

- the indefinite integral of *f*(*x*) is

$$\int f(x)dx=F(x)+C$$

Antiderivative and indefinite integral with chain rule

$$\frac{d}{dx}F(x)=f(x)$$

If $u=u(x)\rightarrow\frac{d}{du}F(u)=f(u)$

Applying chain rule:

$$\frac{d}{dx}F[u]=\frac{du(x)}{dx}\cdot\frac{dF(u)}{du}=\frac{du}{dx}\cdot f(u)\Rightarrow\frac{d}{dx}F[u]=\frac{du}{dx}\cdot f(u)$$

In other words, $F[u]$ is an antiderivative of $\frac{du}{dx}\cdot f(u)$, so that we can write

$$\int\left(\frac{du}{dx}\cdot f(u)\right)\mathrm{d}x=F[u]+C$$

The power rule in integral form

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right)=u^n\frac{du}{dx}\rightarrow\int\left(u^n\frac{du}{dx}\right)dx=\left(\frac{u^{n+1}}{n+1}\right)+C$$

$$\int\left(u^n\frac{du}{dx}\right)dx=\int u^n\left(\frac{du}{dx}dx\right)=\int u^n du$$

differential of $u(x), du$ is $du=\frac{du}{dx}dx$

If u is any differentiable function, then

$$\int u^n\,du=\frac{u^{n+1}}{n+1}+C \qquad (n\neq -1,\ n \text{ rational}). \qquad (1)$$

Example 1 Using the power rule

$$\int \sqrt{1+y^2}\cdot 2y\, dy = \int \sqrt{u}\cdot \frac{du}{dy}\cdot dy$$

$$= \int \sqrt{u}\; du = \ldots$$

Example 2 Adjusting the integrand by a constant

$$\int \sqrt{4t-1}\; dt = \int \frac{1}{4}\sqrt{4t-1}\cdot 4dt$$

$$= \frac{1}{4}\int \sqrt{u}\cdot\left(\frac{du}{dt}\right) dt = \frac{1}{4}\int \sqrt{u}\; du = \ldots$$

Substitution: Running the chain rule backwards

THEOREM 5 The Substitution Rule
If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du.$$

let $u = g(x); \int f[g(x)]\cdot g'(x)dx = \int f(u)\cdot \frac{du}{dx}dx = \int f(u)du$

Used to find the integration with the integrand in the form of the product of $f[g(x)]\cdot g'(x)dx$

$$\int \underbrace{f[g(x)]}_{f(u)}\cdot \underbrace{g'(x)dx}_{du} = \int f(u)du$$

Example 3 Using substitution

$$\int \cos\underbrace{(7x+5)}_{u}\; \underbrace{dx}_{\frac{1}{7}du} = \int \cos u\cdot\frac{du}{7} = \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7x+5) + C$$

Example 4 Using substitution

$$\int x^2 \sin x^3 \; dx = \int \sin \underbrace{x^3}_{u} \; \underbrace{x^2 dx}_{\frac{1}{3}du} =$$

Example 5 Using Identities and substitution

$$\int \frac{1}{\cos^2 2x} \; dx = \int \sec^2 2x \; dx = \int \sec^2 \underbrace{2x}_{u} \; \underbrace{dx}_{\frac{1}{2}du} =$$

$$\frac{1}{2} \int \underbrace{\sec^2 u}_{\frac{d}{du}\tan u} \; du = \frac{1}{2} \int d(\tan u) = \frac{1}{2} \tan u + C = \frac{1}{2} \tan 2x + C$$

Example 6 Using different substitutions

$$\int \frac{2z}{\sqrt[3]{z^2+1}} \; dz = \int \underbrace{\left(z^2+1\right)^{-1/3}}_{u^{-1/3}} \; \underbrace{2zdz}_{du} = \int u^{-1/3} du = \ldots$$

The integrals of $\sin^2 x$ and $\cos^2 x$

- Example 7

$$\int \sin^2 x \; dx = \frac{1}{2} \int 1 - \cos 2x \; dx$$

$$= \frac{x}{2} - \frac{1}{2} \int \cos \underbrace{2x}_{u} \underbrace{dx}_{\frac{1}{2}du}$$

$$= \frac{x}{2} - \frac{1}{4} \int \cos u du = \ldots$$

The integrals of $\sin^2 x$ and $\cos^2 x$

- Example 7(b)

$$\int \cos^2 x \, dx = \frac{1}{2}\int \cos 2x + 1 \, dx = \dots$$

5.6

Substitution and
Area Between Curves

Example 8 Area beneath the curve $y=\sin^2 x$

- For Figure 5.24, find
- (a) the definite integral of $y(x)$ over $[0,2\pi]$.
- (b) the area between the graph and the x-axis over $[0,2\pi]$.

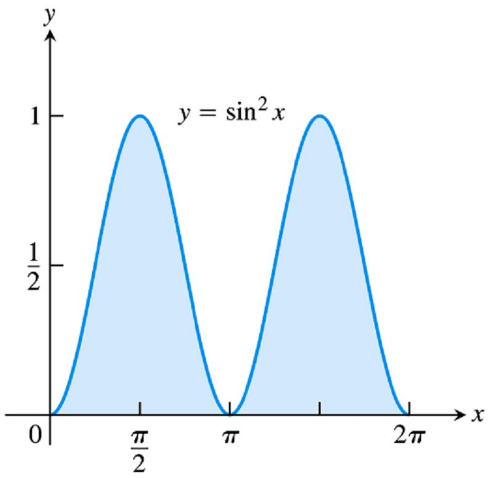


FIGURE 5.24 The area beneath the curve $y = \sin^2 x$ over $[0, 2\pi]$ equals π square units (Example 8).

Substitution formula

THEOREM 6 Substitution in Definite Integrals
If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

let $u = g(x)$; $\displaystyle\int_{x=a}^{x=b} f[g(x)]\cdot g'(x)dx = \int_{x=a}^{x=b} f[u]\cdot \frac{du}{dx}dx = \int_{u=g(a)}^{u=g(b)} f(u)du$

Example 1 Substitution

- Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$

$$\int_{x=-1}^{x=1} \underbrace{\sqrt{x^3+1}}_{u^{1/2}}\cdot\underbrace{3x^2dx}_{du} = \int_{u(x=-1)}^{u(x=1)} u^{1/2}\cdot du = \ldots$$

Example 2 Using the substitution formula

$$\int_{x=\pi/4}^{x=\pi/2} \cot x\csc^2 x dx = ?$$

$$\int \cot x \csc^2 x dx = \int \underbrace{\cot x}_{u}\cdot\underbrace{\csc^2 xdx}_{-du} = -\int udu = -\frac{u^2}{2}+c$$

$$= -\frac{\cot^2 x}{2}+c$$

$$\int_{\pi/4}^{\pi/2} \cot x\csc^2 x dx = -\frac{\cot^2 x}{2}\Bigg|_{\pi/4}^{\pi/2} = \frac{\cot^2 x}{2}\Bigg|_{\pi/2}^{\pi/4} = \frac{1}{2}\left[\underbrace{\cot^2(\pi/4)}_{1}-\underbrace{\cot^2(\pi/2)}_{0}\right] = \frac{1}{2}$$

Definite integrals of symmetric functions

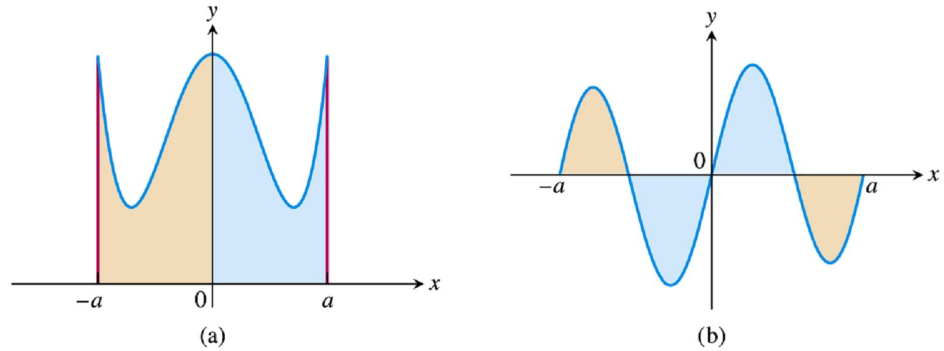


FIGURE 5.26 (a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ (b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

Theorem 7

Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Example 3 Integral of an even function

Evaluate $\int_{-2}^{2}\left(x^4-4x^2+6\right)dx$

Solution:

$f(x)=x^4-4x^2+6;$

$f(-x)=(-x)^4-4(-x)^2+6=x^4-4x^2+6=f(x)$

even function

How about integration of the same function from x=-1 to x=2

Area between curves

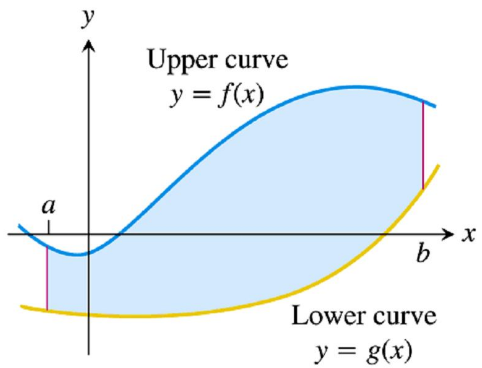


FIGURE 5.27 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

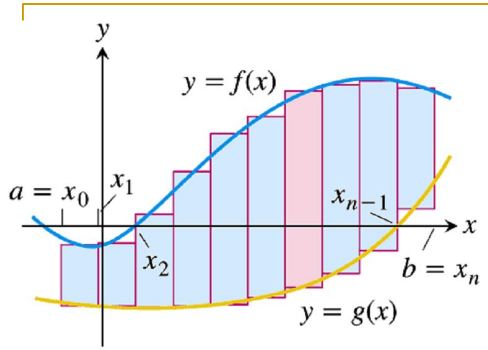


FIGURE 5.28 We approximate the region with rectangles perpendicular to the x-axis.

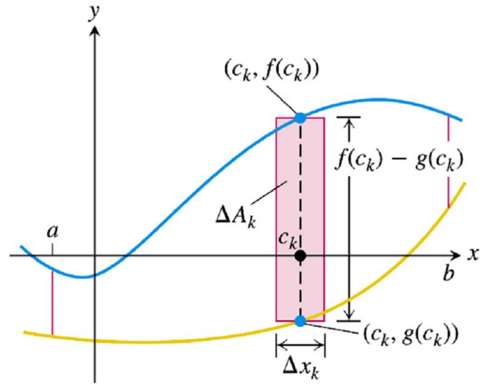


FIGURE 5.29 The area ΔA_k of the kth rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k.

$$A \approx \sum_{k=1}^{n}\Delta A_k = \sum_{k=1}^{n}\Delta x_k\left[\left(f(c_k)-g(c_k)\right]\right.$$

$$A = \lim_{\|P\|\to 0}\sum_{k=1}^{n}\Delta x_k\left[\left(f(c_k)-g(c_k)\right)\right] = \int_a^b\left[f(x)-g(x)\right]dx$$

DEFINITION **Area Between Curves**

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Example 4 Area between intersecting curves

- Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

$$\Delta A = \left(f(x) - g(x) \right) \cdot \Delta x$$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta A_k = \int_0^A dA;$$

$$A = \int_{a=-1}^{b=2} [\underbrace{f(x)}_{2-x^2} - \underbrace{g(x)}_{x}] dx$$

$$= \int_{-1}^{2} \left(2 - x^2 - x\right) dx = \ldots$$

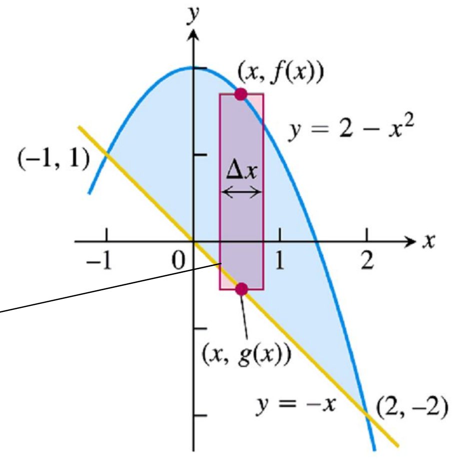


FIGURE 5.30 The region in Example 4 with a typical approximating rectangle.

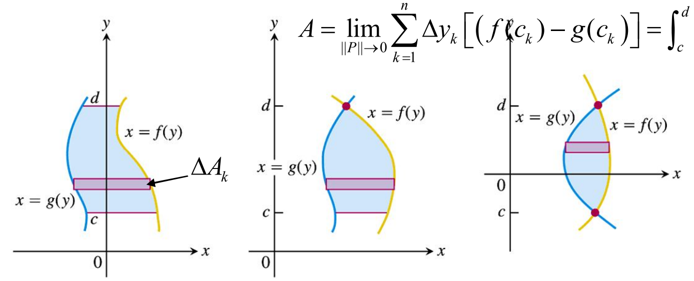
Integration with Respect to y

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these

$$A \approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} \Delta y_k \left[\left(f(c_k) - g(c_k) \right) \right]$$

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta y_k \left[\left(f(c_k) - g(c_k) \right) \right] = \int_c^d [f(y) - g(y)] dy$$



use the formula

$$A = \int_c^d [f(y) - g(y)]\, dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

Example 5 Changing the integral to match a boundary change

- Find the area of the shaded region

$$Area = A + B$$

$$A = \int_0^2 \sqrt{x} dx;$$

$$B = \int_2^4 \sqrt{x} - (x - 2) dx$$

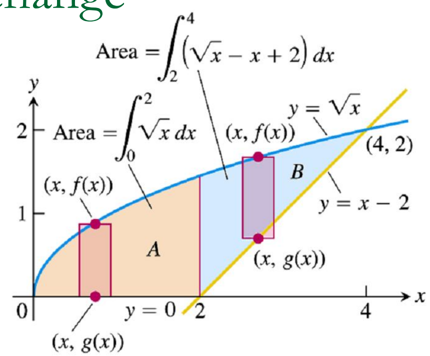


FIGURE 5.31 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 5.

Example 6 Find the area of the region in Example 5 by integrating with respect to y

$$\Delta A = \left(f(y) - g(y) \right) \cdot \Delta y$$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta A_k = \int_{y=0}^{y=4} [f(y) - g(y)] dy$$

$$= \int_0^2 (y+2) - \left(y^2\right) dy = \ldots$$

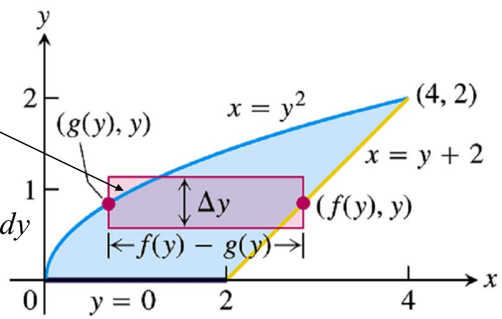


FIGURE 5.32 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y (Example 6).

6.3

Lengths of Plane Curves

Length of a parametrically defined curve

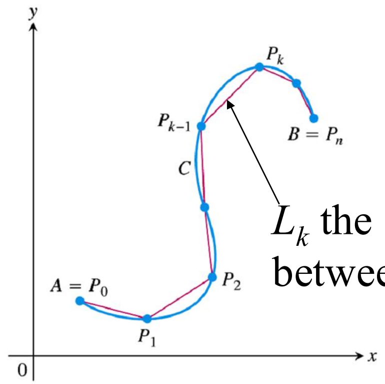


FIGURE 6.24 The curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1, and so on, ending at $B = P_n$.

$$L = \lim_{||P|| \to 0} \sum_k^n L_k$$

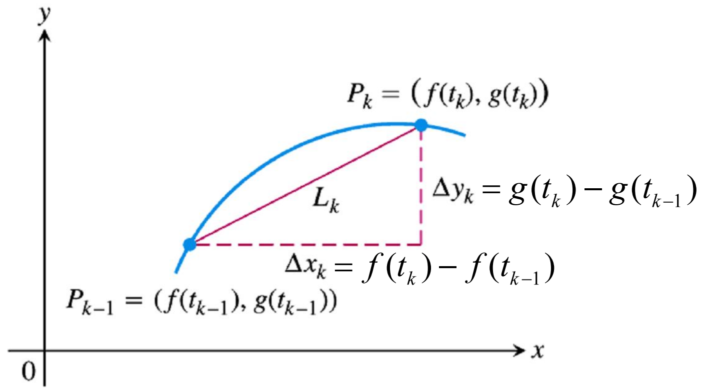


FIGURE 6.25 The arc $P_{k-1}P_k$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

y is parametried by t via $y = g(t)$;
x is parametried by t via $x = f(t)$.

$$\Delta y_k = g(t_k) - g(t_{k-1}) = g'(t_k^*) \cdot (t_k - t_{k-1}) = g'(t_k^*) \cdot \Delta t;$$
$$\Delta x_k = f(t_k) - f(t_{k-1}) = f'(t_k^{**}) \cdot (t_k - t_{k-1}) = f'(t_k^{**}) \cdot \Delta t$$

due to mean value theorem

$$L_k = \sqrt{(\Delta y_k)^2 + (\Delta x_k)^2} = \Delta t \sqrt{(g'(t_k^*))^2 + (f'(t_k^{**}))^2}$$

$$L = \lim_{n \to \infty} \sum_k^n L_k = \lim_{||P|| \to 0} \sum_k^n L_k$$

$$= \lim_{||P|| \to 0} \sum_k^n \Delta t \sqrt{(g'(t_k^*))^2 + (f'(t_k^{**}))^2}$$

$$= \int_a^b \sqrt{(g'(t))^2 + (f'(t))^2}\, dt = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

DEFINITION **Length of a Parametric Curve**

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Example 1 The circumference of a circle

- Find the length of the circle of radius *r* defined parametrically by
- $x=r\cos t$ and $y=r\sin t$, $0 \leqslant t \leqslant 2\pi$

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt \equiv \int_0^{2\pi} \sqrt{(r\cos t)^2 + (r\sin t)^2}\, dt$$

$$= r\int_0^{2\pi} dt = 2\pi r$$

Length of a curve $y = f(x)$

Assign the parameter $x = t$, the length of the curve $y = f(x)$ is then given by

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$$y = y[x(t)] \Rightarrow \frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} = \frac{dy}{dx} \qquad \left(\because \frac{dx}{dt} = 1\right)$$

$$L = \int_a^b dt \sqrt{\left(\frac{dy}{dx}\cdot\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \int_a^b dx \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}$$

$$= \int_a^b dx \sqrt{[f'(x)]^2 + 1}$$

Formula for the Length of $y = f(x)$, $a \le x \le b$

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \qquad (2)$$

Example 3 Applying the arc length formula for a graph

- Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \le x \le 1$$

Dealing with discontinuity in *dy/dx*

- At a point on a curve where *dy*/*dx* fails to exist and we may be able to find the curve's length by expressing *x* as a function of *y* and applying the following

Formula for the Length of $x = g(y)$, $c \le y \le d$
If g is continuously differentiable on $[c, d]$, the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \qquad (3)$$

Example 4 Length of a graph which has a discontinuity in *dy/dx*

- Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.
- **Solution**
- $dy/dx = (1/3)\,(2/x)^{1/3}$ is not defined at $x=0$.
- $dx/dy = 3y^{1/2}$ is continuous on $[0,1]$.

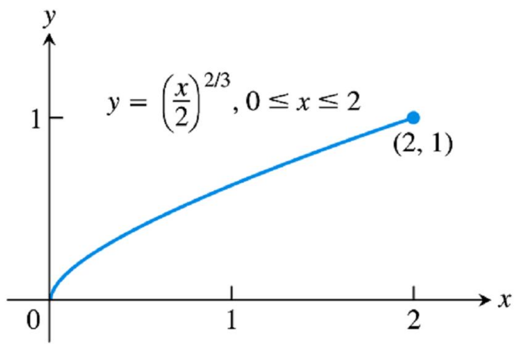


FIGURE 6.27 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Example 4).

Chapter 7

Transcendental Functions

1

7.1

Inverse Functions and Their Derivatives

2

DEFINITION One-to-One Function
A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

3

Example 1 Domains of one-to-one functions

- (a) $f(x) = x^{1/2}$ is one-to-one on any domain of nonnegative numbers
- (b) $g(x) = \sin x$ is NOT one-to-one on $[0,\pi]$ but one-to-one on $[0,\pi/2]$.

4

The Horizontal Line Test for One-to-One Functions

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

DEFINITION **Inverse Function**

Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

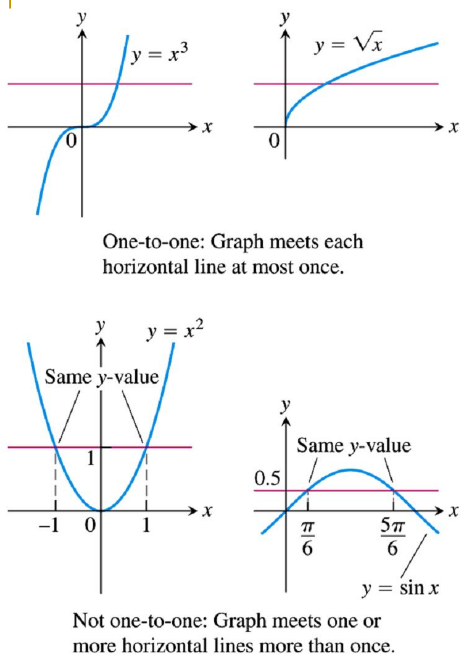


FIGURE 7.1 Using the horizontal line test, we see that $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$, but $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

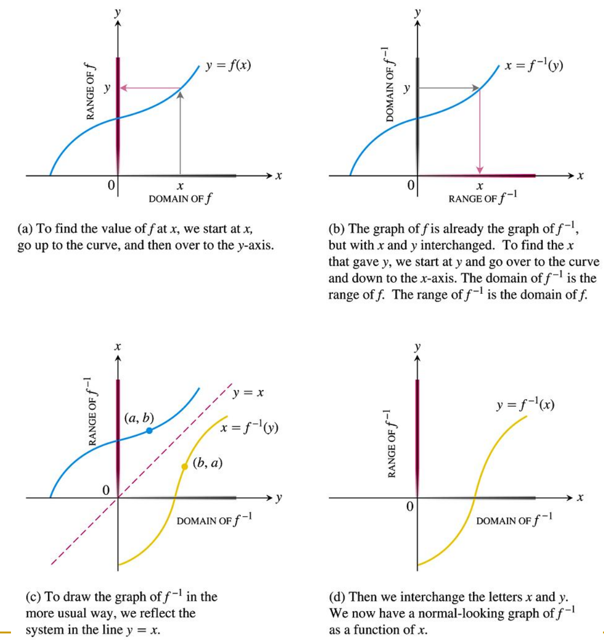


FIGURE 7.2 Determining the graph of $y = f^{-1}(x)$ from the graph of $y = f(x)$.

Finding inverses

- 1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.

2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where $f^{-1}(x)$ is expressed in the conventional format with x as the independent variable and y as the dependent variables.

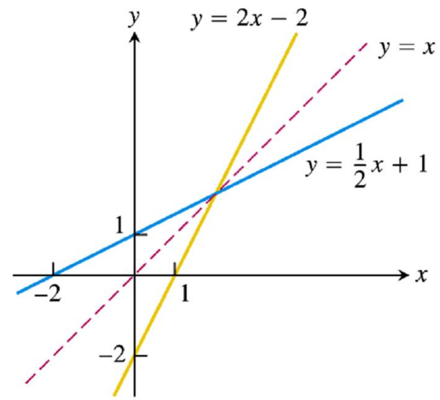


FIGURE 7.3 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$. The slopes are reciprocals of each other (Example 2).

Example 2 Finding an inverse function

- Find the inverse of $y = x/2 + 1$, expressed as a function of x.

- **Solution**
- 1. solve for x in terms of y: $x = 2(y - 1)$
- 2. interchange x and y: $y = 2(x - 1)$
- The inverse function $f^{-1}(x) = 2(x - 1)$
- Check: $f^{-1}[f(x)] = 2[f(x) - 1] = 2[(x/2 + 1) - 1] = x = f[f^{-1}(x)]$

Example 3 Finding an inverse function

- Find the inverse of $y = x^2$, $x \geqslant 0$, expressed as a function of x.
- **Solution**
- 1. solve for x in terms of y: $x = \sqrt{y}$
- 2. interchange x and y: $y = \sqrt{x}$
- The inverse function $f^{-1}(x) = \sqrt{x}$

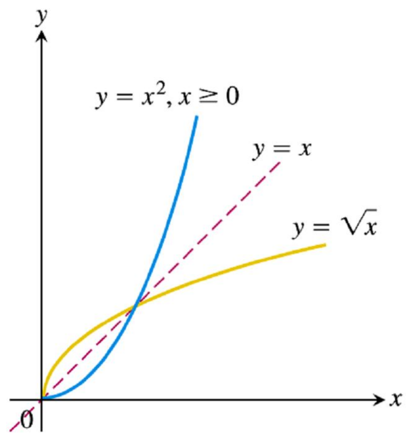


FIGURE 7.4 The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 3).

Derivatives of inverses of differentiable functions

- From example 2 (a linear function)
- $f(x) = x/2 + 1$; $f^{-1}(x) = 2(x + 1)$;
- $df(x)/dx = 1/2$; $df^{-1}(x)/dx = 2$,
- i.e. $df(x)/dx = 1/df^{-1}(x)/dx$
- Such a result is obvious because their graphs are obtained by reflecting on the $y = x$ line.
- In general, the reciprocal relationship between the slopes of f and f^{-1} holds for other functions.

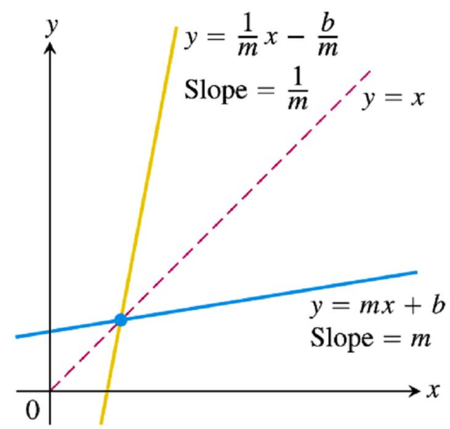
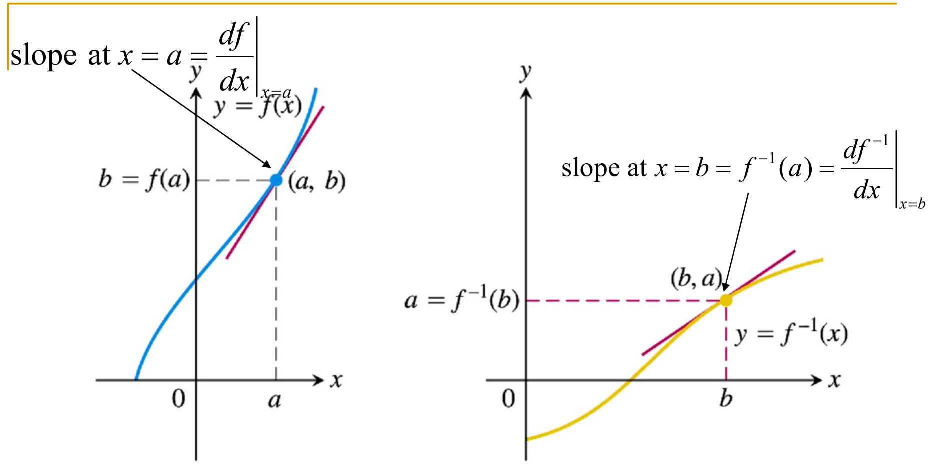


FIGURE 7.5 The slopes of nonvertical lines reflected through the line $y = x$ are reciprocals of each other.



The slopes are reciprocal: $(f^{-1})'(b) = \frac{1}{f'(a)}$ or $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$

FIGURE 7.6 The graphs of inverse functions have reciprocal slopes at corresponding points.

THEOREM 1 The Derivative Rule for Inverses
If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}} \qquad (1)$$

Example 4 Applying theorem 1

- The function $f(x) = x^2$, $x \geqslant 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$, and $(f^{-1})'(x) = 1/(2\sqrt{x})$.
- Theorem 1 predicts that the derivative of

$f^{-1}(x)$ is

$(f^{-1})'(x) = 1/f'[f^{-1}(x)] = 1/f'[\sqrt{x}]$
$= 1/(2\sqrt{x})$

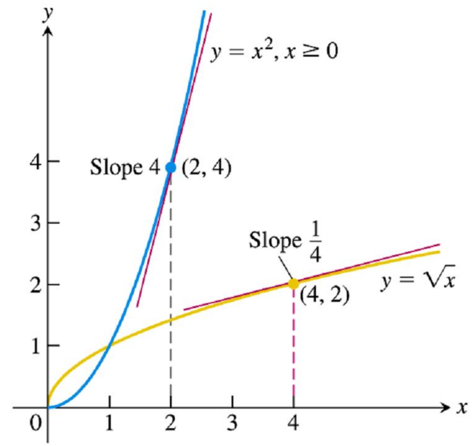


FIGURE 7.7 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 4).

Example 5 Finding a value of the inverse derivative

- Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without a formula for f^{-1}.
- The point for f is (2,6); The corresponding point for f^{-1} is (6,2).
- **Solution**
- $df/dx = 3x^2$
- $df^{-1}/dx|_{x=6} = 1/(df/dx|_{x=2}) = 1/(df/dx|_{x=2})$
$= 1/3x^2|_{x=2} = 1/12$

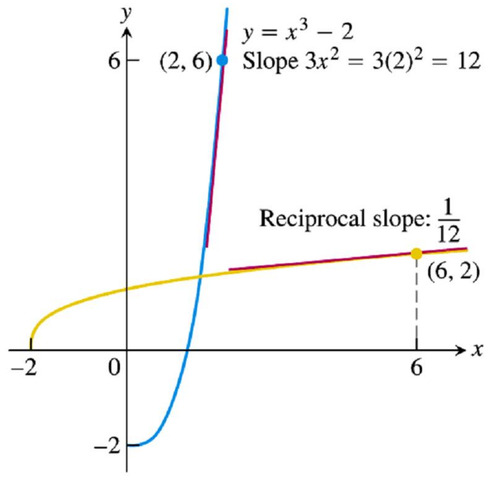


FIGURE 7.8 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 5).

7.2

Natural Logarithms

Definition of natural logarithmic fuction

DEFINITION **The Natural Logarithm Function**

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$

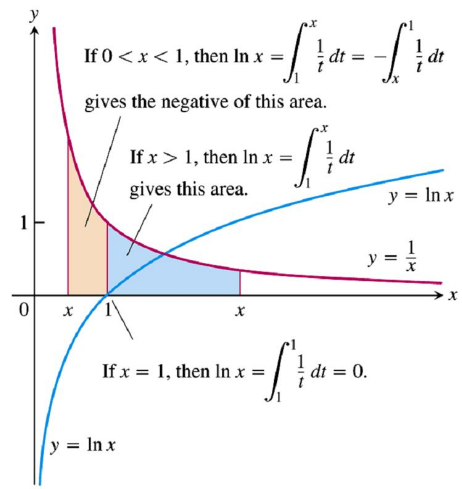


FIGURE 7.9 The graph of $y = \ln x$ and its relation to the function $y = 1/x, x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left.

- Domain of ln x = $(0,\infty)$
- Range of ln x = $(-\infty,\infty)$
- ln x is an increasing function since

  $dy/dx = 1/x > 0$

$$\ln x = \int_1^x \frac{1}{x}\,dx$$

TABLE 7.1 Typical 2-place values of ln x

x	ln x
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

e lies between 2 and 3

$\ln x = 1$

DEFINITION **The Number e**

The number e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1$$

By definition, the antiderivative of ln x is just $1/x$

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0 \qquad (1)$$

Let $u = u(x)$. By chain rule,

$d/dx\,[\ln u(x)] = d/du(\ln u)\cdot du(x)/dx$

$=(1/u)\cdot du(x)/dx$

Example 1 Derivatives of natural logarithms

$$(a)\ \frac{d}{dx}\ln 2x =$$

$$(b)\quad u = x^2 + 3;\ \frac{d}{dx}\ln u = \frac{du}{dx}\frac{1}{u} =$$

Properties of logarithms

THEOREM 2 Properties of Logarithms

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln ax = \ln a + \ln x$
2. *Quotient Rule*: $\ln \frac{a}{x} = \ln a - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ — Rule 2 with $a = 1$
4. *Power Rule*: $\ln x^r = r \ln x$ — r rational

Example 2 Interpreting the properties of logarithms

$$(a)\ \ln 6 = \ln(2\cdot 3) = \ln 2 + \ln 3;$$

$$(b)\ \ln 4 - \ln 5 = \ln(4/5) = \ln 0.8$$

$$(c)\ \ln(1/8) = \ln 1 - \ln 2^3 = -3\ln 2$$

Example 3 Applying the properties to function formulas

$$(a)\ \ln 4 + \ln\sin x = \ln(4\sin x);$$

$$(b)\ \ln\frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$$

$$(c)\ \ln(\sec x) = \ln\frac{1}{\cos x} = -\ln\cos x$$

$$(d)\ \ln\sqrt[3]{x+1} = \ln(x+1)^{1/3} = (1/3)\ln(x+1)$$

Proof of ln ax = ln a + ln x

- ln *ax* and ln *x* have the same derivative:

$$\frac{d}{dx}\ln ax = \frac{d(ax)}{dx}\frac{1}{ax} = a\frac{1}{ax} = \frac{1}{x} = \frac{d}{dx}\ln x$$

- Hence, by the corollary 2 of the mean value theorem, they differs by a constant *C*

$$\ln ax = \ln x + C$$

- We will prove that *C* = ln *a* by applying the definition ln *x* at *x* = 1.

Estimate the value of ln 2

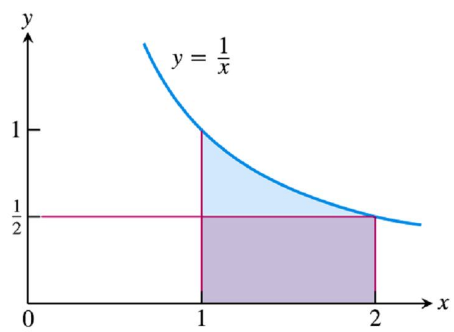


FIGURE 7.10 The rectangle of height $y = 1/2$ fits beneath the graph of $y = 1/x$ for the interval $1 \le x \le 2$.

$$\ln 2 = \int_1^2 \frac{1}{x}dx$$

$$\frac{1}{2}\cdot(2-1) < \int_1^2 \frac{1}{x}dx < 1\cdot(2-1) = 1$$

$$\frac{1}{2} < \ln 2 < 1$$

The integral $\int (1/u)\, du$

From $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

For $u > 0$

Taking the integration on both sides gives

$\int \frac{d}{dx}\ln u dx = \int \frac{1}{u}\frac{du}{dx}dx.$

Let $y = \ln u \to \frac{d}{dx}\ln u dx = \frac{dy}{dx}dx = dy \to \int \frac{d}{dx}\ln u dx = \int dy = \int d\ln u$

$\int d\ln u = \int \frac{du}{u} \to \ln u + C' = \int \frac{du}{u};$

For $u < 0$:

$-u > 0,$

$\int \frac{d}{dx}\ln(-u)dx = \int \frac{1}{(-u)}\frac{d(-u)}{dx}dx$

$\int d\ln(-u) = \int \frac{du}{u} \to \ln(-u) + C'' = \int \frac{du}{u}$

Combining both cases of $u > 0, u < 0$,

$\int \frac{du}{u} = \ln|u| + C$

recall: $\int u^n du = \frac{u^{n+1}}{n+1} + C, n$ rational, $\neq -1$

> If u is a differentiable function that is never zero,
>
> $$\int \frac{1}{u}du = \ln|u| + C. \quad (5)$$

From $\int u^{-1}du = \ln|u| + C.$

let $u = f(x).$

$$\int u^{-1}du = \int \frac{du}{u} = \int \frac{df(x)}{f(x)} = \int \frac{\frac{df(x)}{dx}dx}{f(x)}$$

$$\Rightarrow \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

Example 4 Applying equation (5)

$$\text{(a)} \int \frac{2x\,dx}{x^2-5} = \int \frac{d(x^2-5)}{x^2-5} = \ln|x^2-5| + C$$

$$\text{(b)} \int_{-\pi/2}^{\pi/2} \frac{4\cos x}{3+2\sin x}\,dx = \ldots$$

The integrals of tan x and cot x

$$\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc x| + C$$

Example 5

$$\int \tan 2x\,dx = \int \frac{\sin 2x}{\cos 2x}dx = \int \frac{-\frac{1}{2}\frac{d}{dx}\cos 2x}{\cos 2x}dx$$

$$= -\frac{1}{2}\int \frac{d\cos 2x}{\cos 2x} = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C$$

$$= -\frac{1}{2}\ln|\cos 2x| + C$$

$$= \frac{1}{2}\ln|\sec 2x| + C$$

Example 6 Using logarithmic differentiation

- Find *dy/dx* if $y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, x > 1$

$$\ln y = \ln(x^2+1) + (1/2)\ln(x+3) - \ln(x-1)$$

$$\frac{d}{dy}\ln y = \frac{d}{dy}\ln(x^2+1) + \frac{1}{2}\frac{d}{dy}\ln(x+3) - \frac{d}{dy}\ln(x-1)$$

$$= \ldots$$

7.3

The Exponential Function

The inverse of ln x and the number e

- ln x is one-to-one, hence it has an inverse. We name the inverse of ln x, $\ln^{-1} x$ as exp (x)

$$\lim_{x\to\infty} \ln^{-1} x = \infty, \lim_{x\to-\infty} \ln^{-1} x = 0$$

The graph of the inverse of ln x

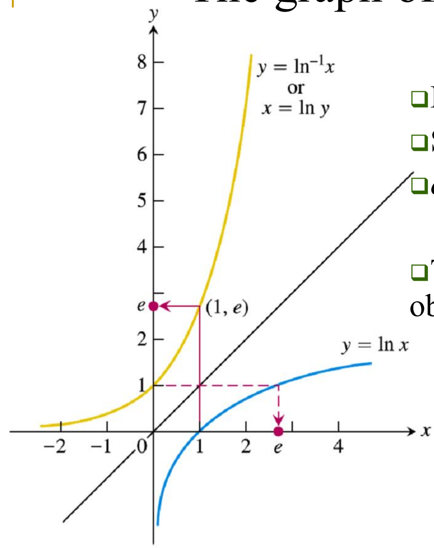


FIGURE 7.11 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

- Definition of e as $\ln e = 1$.
- So, $e = \ln^{-1}(1) = \exp(1)$
- $e = 2.718281828459045\ldots$ (an irrational number)
- The approximate value for e is obtained numerically (later).

The function $y = e^x$

- We can raise the number e to a rational power r, e^r
- e^r is positive since e is positive, hence e^r has a logarithm (recall that logarithm is defied only for positive number).
- From the power rule of theorem 2 on the properties of natural logarithm, $\ln x^r = r \ln x$, where r is rational, we have

$$\ln e^r = r$$

- We take the inverse to obtain

$$\ln^{-1} (\ln e^r) = \ln^{-1} (r)$$

$$e^r = \ln^{-1} (r) = \exp r, \text{ for } r \text{ rational.}$$

The number e to a real (possibly irrational) power x

- How do we define e^x where x is irrational?
- This can be defined by assigning e^x as exp x since $\ln^{-1}(x)$ is defined (as the inverse function of ln x is defined for all real x).

DEFINITION The Natural Exponential Function
For every real number x, $e^x = \ln^{-1} x = \exp x$.

Note: please do make a distinction between e^x and exp x. They have different definitions.

e^x is the number e raised to the power of real number x.

exp x is defined as the inverse of the logarithmic function, $\exp x = \ln^{-1} x$

Typical values of e^x

x	e^x (rounded)
−1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

Inverse Equations for e^x and ln x

$$e^{\ln x} = x \quad (\text{all } x > 0) \qquad (2)$$

$$\ln(e^x) = x \quad (\text{all } x) \qquad (3)$$

- (2) follows from the definition of the exponent function:
- From $e^x = \exp x$, let $x \rightarrow \ln x$
- $e^{\ln x} = \exp[\ln x] = x$ (by definition).
- For (3): From $e^x = \exp x$, take logarithm both sides, $\rightarrow \ln e^x = \ln[\exp x] = x$ (by definition)

Example 1 Using inverse equations

$(a)\ \ln e^2 = \ldots$

$(b)\ \ln e^{-1} = \ldots$

$(c)\ \ln \sqrt{e} = \ln e^{1/2} = \ldots$

$(d)\ \ln e^{\sin x} = \ldots$

$(f)\ e^{\ln 2} = \ldots$

$(g)\ e^{\ln(x^2+1)} = \ldots$

$(h)\ e^{3\ln 2} = e^{\ln 2^3} = \ldots$

$(i)\ e^{3\ln 2} = e^{3\cdot\ln 2} = \left(e^{\ln 2}\right)^3 = \ldots$

Example 2 Solving for an exponent

- Find k if $e^{2k}=10$.

The general exponential function a^x

- Since $a = e^{\ln a}$ for any positive number a
- $a^x = (e^{\ln a})^x = e^{x\ln a}$

DEFINITION General Exponential Functions
For any numbers $a > 0$ and x, the exponential function with base a is

$$a^x = e^{x \ln a}.$$

For the first time we have a precise meaning for an irrational exponent. (previously a^x is defined for only rational x and a)

Example 3 Evaluating exponential functions

$(a)\ 2^{\sqrt{3}} = \left(e^{\ln 2}\right)^{\sqrt{3}} = e^{\sqrt{3}\ln 2} \approx e^{1.20} \approx 3.32$

$(b)\ 2^{\pi} = \left(e^{\ln 2}\right)^{\pi} = e^{\pi \ln 2} \approx e^{2.18} \approx 8.8$

Laws of exponents

THEOREM 3 Laws of Exponents for e^x

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \frac{1}{e^x}$
3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1x_2} = (e^{x_2})^{x_1}$

Theorem 3 also valid for a^x

Proof of law 1

$$y_1 = e^{x_1}, y_2 = e^{x_2}$$
$$\Rightarrow x_1 = \ln y_1, x_2 = \ln y_2$$
$$\Rightarrow x_1 + x_2 = \ln y_1 + \ln y_2 = \ln y_1y_2$$
$$\Rightarrow \exp(x_1 + x_2) = \exp(\ln y_1y_2)$$
$$e^{x_1+x_2} = y_1y_2 = e^{x_1}e^{x2}$$

Example 4 Applying the exponent laws

$$(a)e^{x+\ln 2} =$$
$$(b)e^{-\ln x} =$$
$$(c)\frac{e^{2x}}{e} =$$
$$(d)\left(e^3\right)^x =$$

The derivative and integral of e^x

$$f(x) = \ln x, y = e^x = \ln^{-1} x = f^{-1}(x)$$
$$\frac{dy}{dx} = \frac{d}{dx}e^x = \frac{d}{dx}f^{-1}(x) = \frac{1}{\left.\frac{df(x)}{dx}\right|_{x=f^{-1}(x)}}$$
$$= \frac{1}{(1/x)|_{x=f^{-1}(x)}} = \frac{1}{(1/x)|_{x=y}} = y = e^x$$

$$\frac{d}{dx}e^x = e^x \qquad (5)$$

Example 5 Differentiating an exponential

$$\frac{d}{dx}\left(5e^{x}\right)=$$

By the virtue of the chain rule, we obtain

If u is any differentiable function of x, then

$$\frac{d}{dx}e^{u}=e^{u}\frac{du}{dx}. \qquad (6)$$

$$f(u)=e^{u};u=u(x);$$

$$\frac{d}{dx}\left(e^{u(x)}\right)=\frac{d}{dx}f(u)=\frac{df(u)}{du}\frac{du(x)}{dx}=e^{u}\frac{du}{dx}$$

$$\int e^{u}\,du=e^{u}+C.$$

This is the integral equivalent of (6)

Example 7 Integrating exponentials

$$(a)\int_{0}^{\ln 2} e^{3x}dx=$$

$$(b)\int_{0}^{\pi/2} e^{\sin x}\cos x\,dx=\int_{0}^{\pi/2}\underbrace{e^{\sin x}}_{e^{u}}\underbrace{\cos x dx}_{du}$$

$$=\int_{u(0)}^{u(\pi/2)} e^{u}du=\int_{u(0)}^{u(\pi/2)} du$$

$$=e^{u}\Big|_{u(0)}^{u(\pi/2)}=e^{u(\pi/2)}-e^{u(0)}=e^{\sin(\pi/2)}-e^{\sin(0)}=e-1$$

The number *e* expressed as a limit

THEOREM 4 The Number *e* as a Limit
The number e can be calculated as the limit

$$e=\lim_{x\to 0}(1+x)^{1/x}.$$

Proof

- If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$.
  But by definition of derivative,
- $$f'(y) = \lim_{h\to 0}\frac{f(y+h)-f(y)}{h}$$
  $$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{x\to 0}\frac{f(1+x)-f(x)}{x}$$
  $$= \lim_{x\to 0}\frac{\ln(1+x)-\ln(1)}{x} = \lim_{x\to 0}\frac{\ln(1+x)}{x}$$
  $$= \lim_{x\to 0}[\ln(1+x)]^{\frac{1}{x}} = \ln\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right] = 1 \quad (\text{since } f'(1)=1)$$
  $$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}(1+\frac{1}{y})^{y} = e$$

Define x^n for any real $x > 0$ as $x^n = e^{n\ln x}$.

Here n need not be rational but can be any real number as long as x is positive.

Then we can take the logarithm of x^n:

$$\ln x^n = \ln\left(e^{n\ln x}\right) = n\ln x.$$

Note: *c.f* the power rule in theorem 2.

Can you tell the difference?

Once x^n is defined via $x^n = e^{n\ln x}$, we can take its differentiation:

$$\frac{d}{dx}x^n = \frac{d}{dx}\left(e^{\overbrace{n\ln x}^{u(x)}}\right) = \frac{du}{dx}\frac{de^u}{du} = \frac{n}{x}e^{n\ln x} = \frac{n}{x}x^n = nx^{n-1}$$

$$\Rightarrow \frac{d}{dx}x^n = nx^{n-1}$$

Note: Can you tell the difference between this formula and the one we discussed in earlier chapters?

- By virtue of chain rule,

$$u = u(x);$$

$$\frac{d}{dx}u^n = \frac{du(x)}{dx}\frac{du^n}{du} = \frac{du(x)}{dx}nu^{n-1}$$

Power Rule (General Form)
If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

Example 9 using the power rule with irrational powers

$$(a)\frac{d}{dx}x^{\sqrt{2}} \equiv \frac{du^n}{dx} = \frac{du}{dx}nu^{n-1}$$

$$\frac{du}{dx}nu^{n-1} \equiv \frac{dx}{dx}\sqrt{2}x^{\sqrt{2}-1} = \sqrt{2}x^{\sqrt{2}-1}$$

$$(b)\frac{d}{dx}(2+\sin 3x)^{\pi} \equiv \frac{du^n}{dx} = \frac{du}{dx}nu^{n-1}$$

$$\frac{du}{dx}nu^{n-1} \equiv \frac{d(2+\sin 3x)}{dx}\pi u^{\pi-1} = 3\pi(2+\sin 3x)^{\pi-1}\cos 3x$$

The derivative of a^x

$$a^x = e^{x\ln a}$$

$$\frac{d}{dx}a^x = \frac{d}{dx}\left(e^{\overbrace{x\ln a}^{u}}\right) = \frac{d}{dx}(x\ln a)\frac{d}{du}(e^u)$$

$$= e^u \ln a = e^{x\ln a}\ln a = a^x \ln a$$

By virtue of the chain rule,

$$\frac{d}{dx}a^{u(x)} = \frac{du}{dx}\frac{d}{du}(a^u) = a^u \ln a \frac{du}{dx}$$

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}. \quad (1)$$

7.4

a^x and $\log_a x$

Example 1 Differentiating general exponential functions

$$(a)\frac{d}{dx}3^x = \frac{d}{dx}\left(e^{\overbrace{x\ln 3}^{u}}\right) = \frac{d}{dx}(x\ln 3)\frac{d}{du}(e^u)$$

$$= \ln 3 \cdot e^{x\ln 3} = 3^x \ln 3$$

$$(b)\frac{d}{dx}3^{-x} = -\frac{d}{d(-x)}3^{\overbrace{(-x)}^{u}} = -\frac{d}{du}3^u = -\frac{d}{du}3^u$$

$$= -3^u \ln 3 = -3^{(-x)}\ln 3 = -\ln 3/3^x$$

$$(c)\frac{d}{dx}3^{\overbrace{\sin x}^{u}} = \frac{du}{dx}\frac{d}{du}3^u = \frac{d(\sin x)}{dx}3^u \ln 3 = 3^{\sin x}\ln 3 \cdot \cos x$$

Other power functions

- Example 2 Differentiating a general power function
- Find dy/dx if $y = x^x$, $x > 0$.
- **Solution:** Write x^x as a power of e
- $x^x = e^{x\ln x}$

$$\frac{d}{dx}\left(e^{\overbrace{x\ln x}^{u}}\right) = \frac{du}{dx}\frac{d}{du}\left(e^u\right) = \frac{d}{dx}(x\ln x)\cdot\left(e^u\right) = \ldots$$

Example 3 Integrating general exponential functions

$$(a)\int 2^x\,dx = \frac{2^x}{\ln 2} + C$$

$$(b)\int 2^{\overbrace{\sin x}^{u}}\overbrace{\cos dx}^{du} = \int 2^u\,du = \ldots$$

Integral of a^u

From $\frac{d}{dx}a^{u(x)} = a^u \ln a \frac{du}{dx}$, devide by $\ln a$:

$$\Rightarrow \frac{1}{\ln a}\frac{d}{dx}a^{u(x)} = a^u\frac{du}{dx}$$

$\Rightarrow \frac{d}{dx}a^{u(x)} = a^u \ln a\frac{du}{dx}$, integrate both sides wrp to dx:

$$\Rightarrow \int\left(\frac{d}{dx}a^u\right)dx = \int\left(a^u \ln a\frac{du}{dx}\right)dx:$$

$$\Rightarrow \int da^u = \ln a\int a^u du + C$$

$$\Rightarrow \int a^u du = \frac{1}{\ln a}\int da^u = \frac{a^u}{\ln a}$$

$$\int a^u\,du = \frac{a^u}{\ln a} + C. \qquad (2)$$

Logarithm with base a

DEFINITION $\log_a x$

For any positive number $a \neq 1$,

$\log_a x$ is the inverse function of a^x.

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0) \qquad (3)$$

$$\log_a(a^x) = x \qquad (\text{all } x) \qquad (4)$$

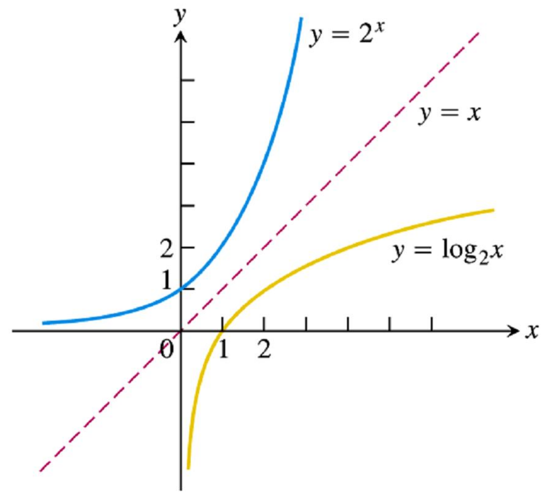


FIGURE 7.13 The graph of 2^x and its inverse, $\log_2 x$.

Example 4 Applying the inverse equations

$$(a)\log_2 2^5 = 5$$
$$(b)2^{\log_2 3} = 3$$
$$(c)\log_{10} 10^{(-7)} = -7$$
$$(d)10^{\log_{10} 4} = 4$$

Evaluation of $\log_a x$

Taking ln on both sides of $a^{\log_a x} = x$ gives

$\ln(a^{\log_a x}) = \ln x$

LHS, $\ln(a^{\log_a x}) = \log_a x \ln a$.

Equating LHS to RHS yields

$\log_a x \ln a = \ln x$

$$\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a} \quad (5)$$

- Example: $\log_{10}2 = \ln 2 / \ln 10$

TABLE 7.2 Rules for base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:*
   $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule:*
   $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Reciprocal Rule:*
   $\log_a \frac{1}{y} = -\log_a y$
4. *Power Rule:*
   $\log_a x^y = y \log_a x$

- Proof of rule 1:

$\ln xy = \ln x + \ln y$

divide both sides by $\ln a$

$$\frac{\ln(xy)}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a}$$

$$\log_a(xy) = \log_a x + \log_a y$$

Derivatives and integrals involving $\log_a x$

$$\frac{d}{dx}(\log_a u) = \frac{du}{dx}\frac{d(\log_a u)}{du}$$

$$\frac{d}{du}(\log_a u) = \frac{d}{du}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{du}(\ln u) = \frac{1}{\ln a}\frac{1}{u}$$

$$\frac{d}{dx}(\log_a u) = \frac{du}{dx}\cdot\left(\frac{1}{\ln a}\frac{1}{u}\right) = \frac{1}{\ln a}\left(\frac{1}{u}\right)\cdot\frac{du}{dx}$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}$$

Example 5

$$(a)\frac{d}{dx}\left(\log_{10}\overbrace{(3x+1)}^{u}\right) = \frac{du}{dx}\frac{d(\log_{10} u)}{du}$$

$$= \frac{d}{dx}(3x+1)\frac{1}{\ln 10}\frac{d(\ln u)}{du} = \frac{3}{\ln 10}\frac{1}{(3x+1)}$$

$$(b)\int\frac{\log_2 x}{x}dx = \frac{1}{\ln 2}\int\underbrace{\ln x}_{u}\underbrace{\frac{dx}{x}}_{d(\ln x)\equiv du} = \frac{1}{\ln 2}\int u du = \ldots$$

7.5

Exponential Growth and Decay

The law of exponential change

- For a quantity *y* increases or decreases at a rate proportional to it size at a give time *t* follows the law of exponential change, as per

$$\frac{dy}{dt} \propto y(t) \Rightarrow \frac{dy}{dt} = ky(t).$$

k is the proportional constant.

Very often we have to specify the value of y at some specified time, for example the initial condition

$y(t=0) = y_0$

Rearrange the equation $\frac{dy}{dt} = ky$:

$$\frac{1}{y}\frac{dy}{dt} = k \rightarrow \int \frac{1}{y}\frac{dy}{dt}dt = \int k dt$$

$$\rightarrow \int \frac{1}{y}dy = k\int dt = kt \rightarrow \ln|y| = kt + \ln C$$

$$\rightarrow y = \pm Ce^{kt} = Ae^{kt}, A = \pm C.$$

Put in the initial value of y at $t = 0$ is y_0:

$$\rightarrow y(0) = y_0 = Ae^{k \cdot 0} = A \rightarrow y = y_0 e^{kt}$$

The Law of Exponential Change

$$y = y_0 e^{kt} \qquad (2)$$

Growth: $k > 0$ Decay: $k < 0$

The number k is the **rate constant** of the equation.

Example 1 Reducing the cases of infectious disease

- Suppose that in the course of any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1000? Assume the law of exponential change applies.

Example 3 Half-life of a radioactive element

- The effective radioactive lifetime of polonium-210 is very short (in days). The number of radioactive atoms remaining after t days in a sample that starts with y_0 radioactive atoms is $y = y_0 \exp(-5\times10^{-3}t)$. Find the element's half life.

Solution

- Radioactive elements decay according to the exponential law of change. The half life of a given radioactive element can be expressed in term of the rate constant k that is specific to a given radioactive species. Here $k=-5\times10^{-3}$.
- At the half-life, $t = t_{1/2}$,

  $y(t_{1/2}) = y_0/2 = y_0 \exp(-5\times10^{-3}\, t_{1/2})$

  $\exp(-5\times10^{-3}\, t_{1/2}) = 1/2$

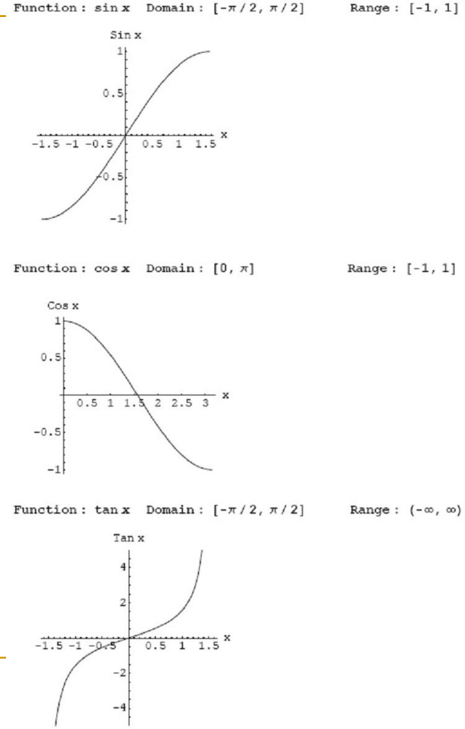
$\rightarrow \ln(1/2) = -5\times10^{-3}\, t_{1/2}$

$\rightarrow t_{1/2} = -\ln(1/2)/5\times10^{-3} = \ln(2)/5\times10^{-3} = \ldots$

7.7

Inverse Trigonometric Functions

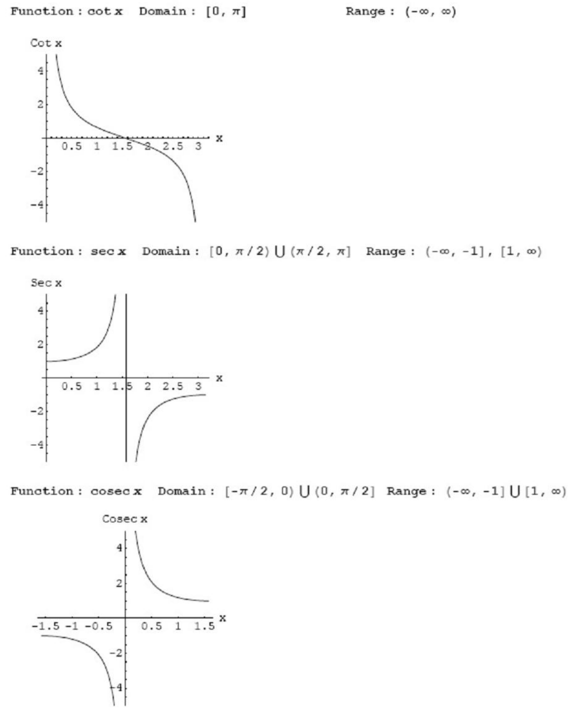
Domain restriction that makes the trigonometric functions one-to-one



Defining the inverses

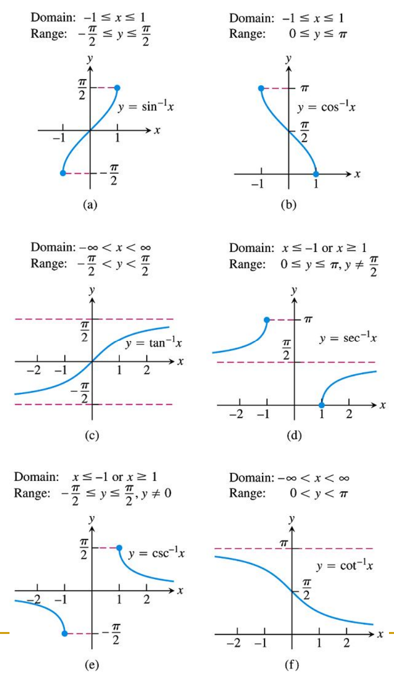
- Trigo functions are periodic, hence not one-to-one in the their domains.
- If we restrict the trigonometric functions to intervals on which they are one-to-one, then we can define their inverses.

Domain restriction that makes the trigonometric functions one-to-one



Inverses for the restricted trigo functions

$$y = \sin^{-1} x = \arcsin x$$
$$y = \cos^{-1} x = \arccos x$$
$$y = \tan^{-1} x = \arctan x$$
$$y = \cot^{-1} x = \operatorname{arccot} x$$
$$y = \sec^{-1} x = \operatorname{arcsec} x$$
$$y = \csc^{-1} x = \operatorname{arccsc} x$$



- The graphs of the inverse trigonometric functions can be obtained by reflecting the graphs of the restricted trigo functions through the line $y = x$.

FIGURE 7.17 Graphs of the six basic inverse trigonometric functions.

DEFINITION **Arcsine and Arccosine Functions**

$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.

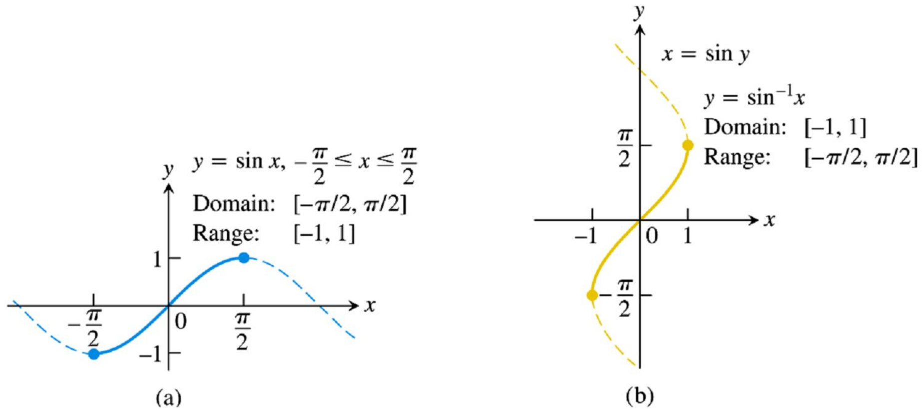


FIGURE 7.18 The graphs of (a) $y = \sin x, -\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

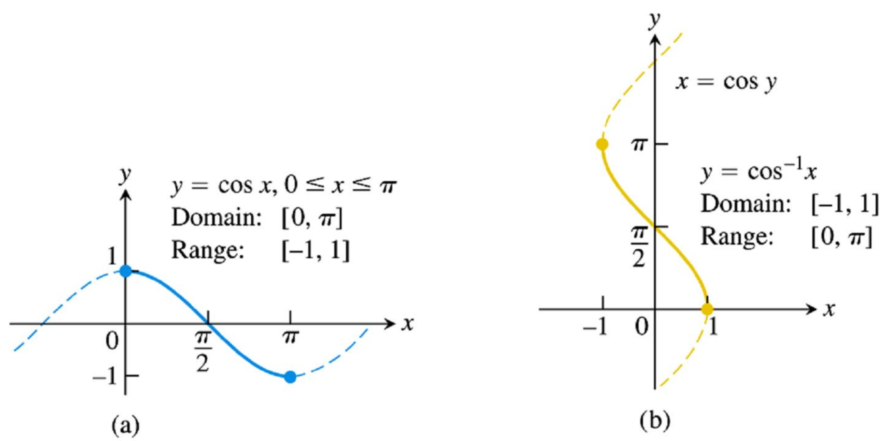


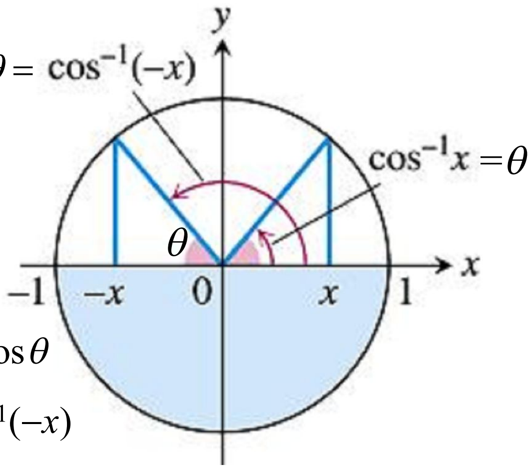
FIGURE 7.19 The graphs of (a) $y = \cos x$, $0 \le x \le \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

Some specific values of $\sin^{-1} x$ and $\cos^{-1} x$

x	$\sin^{-1} x$
$\sqrt{3}/2$	$\pi/3$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/6$
$-1/2$	$-\pi/6$
$-\sqrt{2}/2$	$-\pi/4$
$-\sqrt{3}/2$	$-\pi/3$

x	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/3$
$-1/2$	$2\pi/3$
$-\sqrt{2}/2$	$3\pi/4$
$-\sqrt{3}/2$	$5\pi/6$

$\phi = \pi - \theta = \cos^{-1}(-x)$



$\theta = \cos^{-1} x;$

$\cos\phi = \cos(\pi - \theta) = -\cos\theta$

$\phi = \cos^{-1}(-\cos\theta) = \cos^{-1}(-x)$

Add up θ and ϕ:

$\theta + \phi = \cos^{-1} x + \cos^{-1}(-x)$

FIGURE 7.20 $\cos^{-1} x$ and $\cos^{-1}(-x)$ are supplementary angles (so their sum is π).

$\pi = \cos^{-1} x + \cos^{-1}(-x)$

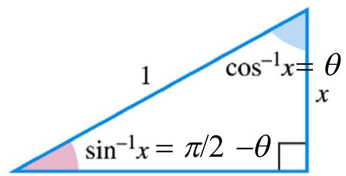


FIGURE 7.21 $\sin^{-1} x$ and $\cos^{-1} x$ are complementary angles (so their sum is $\pi/2$).

$$\cos^{-1} x = \theta; \sin^{-1} x = \left(\frac{\pi}{2} - \theta\right);$$

$$\cos^{-1} x + \sin^{-1} x + = \theta + \left(\frac{\pi}{2} - \theta\right) = \frac{\pi}{2}$$

link to slide derivatives of the other three

DEFINITION **Arctangent and Arccotangent Functions**

$y = \tan^{-1} x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$.

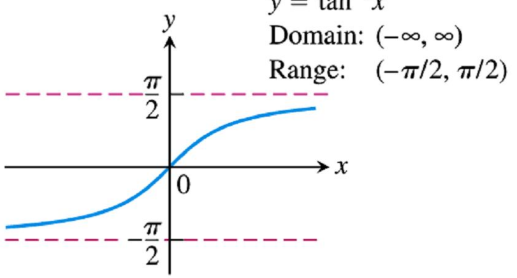


FIGURE 7.22 The graph of $y = \tan^{-1} x$.

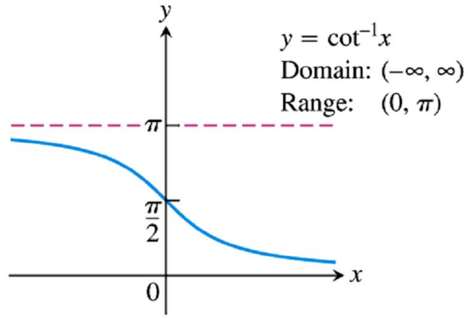


FIGURE 7.23 The graph of $y = \cot^{-1} x$.

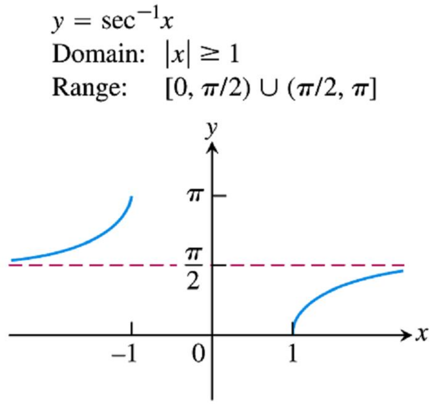


FIGURE 7.24 The graph of $y = \sec^{-1} x$.

Domain: $|x| \geq 1$
Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$

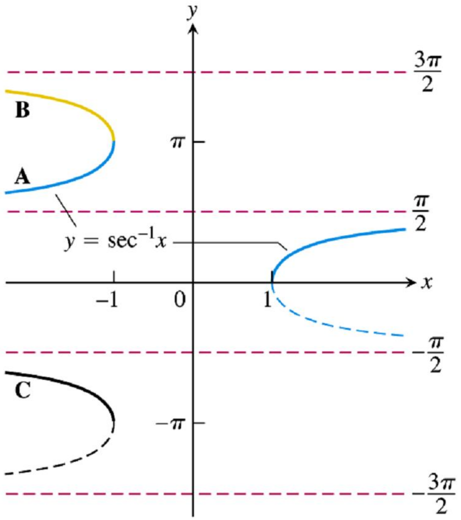


FIGURE 7.26 There are several logical choices for the left-hand branch of $y = \sec^{-1} x$. With choice **A**, $\sec^{-1} x = \cos^{-1}(1/x)$, a useful identity employed by many calculators.

$y = \csc^{-1} x$
Domain: $|x| \geq 1$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$

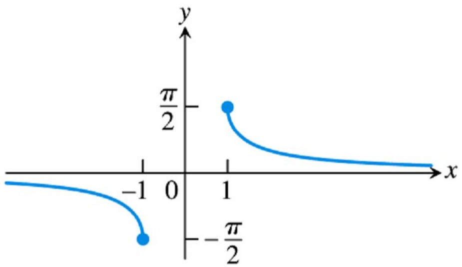


FIGURE 7.25 The graph of $y = \csc^{-1} x$.

Some specific values of $\tan^{-1} x$

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

Example 4

- Find cos α, tan α, sec α, csc α if $\alpha = \sin^{-1}(2/3)$.

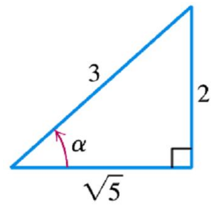


FIGURE 7.27 If $\alpha = \sin^{-1}(2/3)$, then the values of the other basic trigonometric functions of α can be read from this triangle (Example 4).

- sin $\alpha = 2/3$
- ...

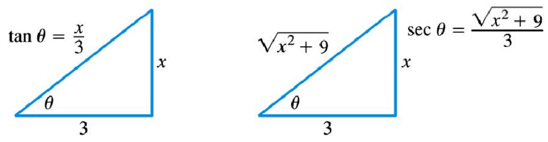
EXAMPLE 5 Find $\sec\left(\tan^{-1}\frac{x}{3}\right)$.

Solution We let $\theta = \tan^{-1}(x/3)$ (to give the angle a name) and picture θ in a right triangle with

$$\tan\theta = \text{opposite/adjacent} = x/3.$$

The length of the triangle's hypotenuse is

$$\sqrt{x^2 + 3^2} = \sqrt{x^2 + 9}.$$



Thus,

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \sec\theta$$

$$= \frac{\sqrt{x^2+9}}{3}. \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

The derivative of $y = \sin^{-1} x$

$$f(x) = \sin^{-1} x \Rightarrow f^{-1}(x) = \sin x;$$

$$\frac{df(x)}{dx} = \frac{1}{\left.\frac{df^{-1}(x)}{dx}\right|_{x=f(x)}} = \frac{1}{\cos x|_{x=f(x)}} = \frac{1}{\cos f(x)}$$

$$\text{Let } y = f(x) = \sin^{-1} x \to x = \sin y \Rightarrow \cos y = \sqrt{1-x^2}$$

$$\frac{1}{\cos(f(x))} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

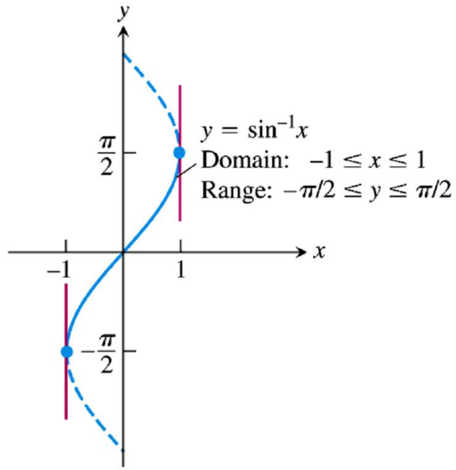


FIGURE 7.29 The graph of $y = \sin^{-1} x$.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

Note that the graph is not differentiable at the end points of $x=\pm1$ because the tangents at these points are vertical.

The derivative of $y = \sin^{-1} u$

If $u = u(x)$ is an diffrentiable function of x,

$$\frac{d}{dx}\sin^{-1} u = ?$$

Use chain rule: Let $y = \sin^{-1} u$

$$\frac{d}{dx}\sin^{-1} u = \frac{du}{dx}\frac{d}{du}\left(\sin^{-1} u\right) = \frac{du}{dx}\frac{1}{\sqrt{1-u^2}}$$

Note that $|u|<1$ for the formula to apply

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1.$$

Example 7 Applying the derivative formula

$$\frac{d}{dx}\sin^{-1}x^2 = \dots$$

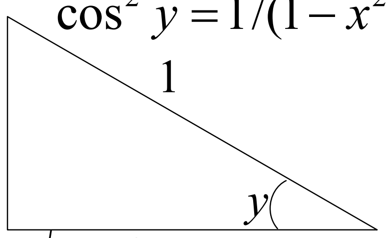
The derivative of $y = \tan^{-1} u$

$$\cos^2 y = 1/(1-x^2)$$

$y = \tan^{-1}x \Rightarrow x = \tan y$

$1 = \frac{d}{dx}(\tan y) = \frac{dy}{dx}\sec^2 y$

$\frac{dy}{dx} = \cos^2 y = 1/(1-x^2)$



By virtue of chain rule, we obtain

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}.$$

Example 8

$$x(t) = \tan^{-1}\sqrt{t}.$$

$$\left.\frac{dx}{dt}\right|_{t=16} = ?$$

The derivative of $y = \sec^{-1} x$

$y = \sec^{-1}x \Rightarrow x = \sec y$

$1 = \frac{d}{dx}(\sec y) = \frac{dy}{dx}\sec y \tan y$

$\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2-1}$

$\frac{d}{dx}\sec^{-1}x = \cos y \cot y = \pm\frac{1}{x}\frac{1}{\sqrt{(x^2-1)}}$

$\because \frac{dy}{dx} > 0$ (from Figure 7.30),

$\frac{dy}{dx} = \frac{1}{|x|}\frac{1}{\sqrt{(x^2-1)}}$

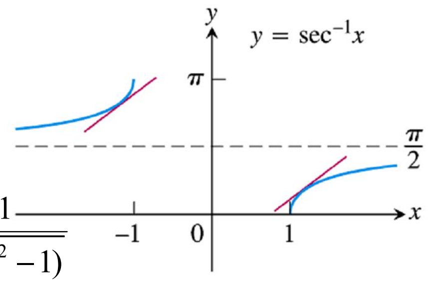


FIGURE 7.30 The slope of the curve $y = \sec^{-1}x$ is positive for both $x < -1$ and $x > 1$.

The derivative of $y = \sec^{-1} u$

By virtue of chain rule, we obtain

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \qquad |u| > 1.$$

Example 5 Using the formula

$$\frac{d}{dx}\sec^{-1}\left(5x^4\right) = \ldots$$

Derivatives of the other three

- The derivative of $\cos^{-1}x$, $\cot^{-1}x$, $\csc^{-1}x$ can be easily obtained thanks to the following identities:

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

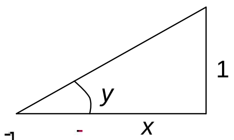
Link to fig. 7.21

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$
2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$
3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{du/dx}{1+u^2}$
4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{du/dx}{1+u^2}$
5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$
6. $\dfrac{d(\csc^{-1} u)}{dx} = \dfrac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

Example 10 A tangent line to the arccotangent curve

- Find an equation for the tangent to the graph of $y = \cot^{-1} x$ at $x = -1$.
- Use either $\frac{df^{-1}(x)}{dx} = \frac{1}{\left.\frac{df(x)}{dx}\right|_{x=f^{-1}(x)}}$
- Or



Ans = $-\frac{1}{1 + x^2}$.

Integration formula

- By integrating both sides of the derivative formulas in Table 7.3, we obtain three useful integration formulas in Table 7.4.

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1.	$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$	(Valid for $u^2 < a^2$)
2.	$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$	(Valid for all u)
3.	$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$	(Valid for $\lvert u\rvert > a > 0$)

Example 11 Using the integral formulas

$$(a)\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} =$$

$$(b)\int_0^1 \frac{dx}{1+x^2} =$$

$$(c)\int_{2/\sqrt{3}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} =$$

Example 13 Completing the square

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-4x)}} = \int \frac{dx}{\sqrt{-[(x-2)^2-4]}}$$

$$= \int \frac{dx}{\sqrt{4-(x-2)^2}} = \int \frac{du}{\sqrt{2^2-u^2}} = \dots$$

Example 15 Using substitution

$$\int \frac{dx}{\sqrt{e^{2x}-6}} = \int \frac{dx}{\sqrt{(e^x)^2-(\sqrt{6})^2}} =$$

$$\int \frac{1}{\sqrt{(e^x)^2-(\sqrt{6})^2}} \frac{de^x}{e^x} = \int \frac{1}{\sqrt{u^2-(\sqrt{6})^2}} \frac{du}{u} = \ldots$$

7.8

Hyperbolic Functions

Even and odd parts of the exponential function

- In general:
- $f(x) = ½ [f(x) + f(-x)] + ½ [f(x) - f(-x)]$
- $½ [f(x) + f(-x)]$ is the even part
- $½ [f(x) - f(-x)]$ is the odd part

- Specifically:
- $f(x) = e^x = ½ (e^x + e^{-x}) + ½ (e^x – e^{-x})$
- The odd part $½ (e^x - e^{-x}) \equiv \cosh x$ (hyperbolic cosine of x)
- The even part $½ (e^x + e^{-x}) \equiv \sinh x$ (hyperbolic sine of x)

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Proof of

$$\sinh 2x = 2\cosh x \sinh x$$

$$\sinh 2x = \frac{1}{2}(e^{2x} - e^{-2x}) = \frac{1}{2}\frac{(e^{4x}-1)}{e^{2x}}$$

$$= \frac{1}{2}\frac{(e^{2x}-1)}{e^{x}}\frac{(e^{2x}+1)}{e^{x}} = \frac{2}{2}\cdot\frac{1}{2}(e^{x}-e^{-x})(e^{x}+e^{-x})$$

$$= 2\cdot\frac{1}{2}(e^{x}-e^{-x})\cdot\frac{1}{2}(e^{x}+e^{-x}) = 2\sinh x\cosh x$$

TABLE 7.5 The six basic hyperbolic functions

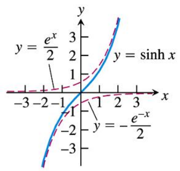
Hyperbolic sine of x: $\sinh x = \frac{e^x - e^{-x}}{2}$

Hyperbolic cosine of x: $\cosh x = \frac{e^x + e^{-x}}{2}$

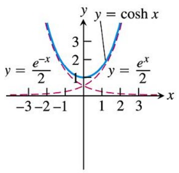
Hyperbolic tangent: $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent: $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

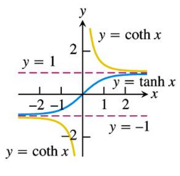
FIGURE 7.31



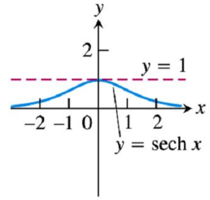
(a)



(b)

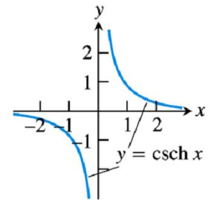


Hyperbolic secant: $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$



(d)

Hyperbolic cosecant: $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$



(e)

Derivatives and integrals

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.8 Integral formulas for hyperbolic functions

$$\int \sinh u\, du = \cosh u + C$$

$$\int \cosh u\, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

$$\frac{d}{dx}\sinh u = \frac{du}{dx}\frac{d}{dx}\sinh x$$
$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{1}{2}(e^{x}-e^{-x}) = \frac{1}{2}(e^{x}+e^{-x}) = \cosh x$$
$$\therefore \frac{d}{dx}\sinh u = \frac{du}{dx}\cosh x$$

Example 1 Finding derivatives and integrals

$$(a)\frac{d}{dx}\tanh\overbrace{\sqrt{1+t^2}}^{u} = \frac{du}{dx}\frac{d}{du}\tanh u$$

$$(b)\int \coth \overbrace{5x}^{u} dx = \frac{1}{5}\int \coth u du = \frac{1}{5}\int \frac{\cosh u \; du}{\sinh u}$$

$$= \frac{1}{5}\int \frac{\overbrace{d(\sinh u)}^{dv}}{\underbrace{\sinh u}_{v}} = \frac{1}{5}\int \frac{dv}{v} = \frac{1}{5}\ln|v| + C = \frac{1}{5}\ln|\sinh 5x| + C$$

$$(c)\int \sinh^2 x \; dx = \frac{1}{2}\int (\cosh 2x - 1)\; dx = \ldots$$

$$(d)\int 4e^{x}\sinh x \; dx = 4\int \frac{\overbrace{e^{x}}^{u} - e^{-x}}{2}\; de^{x} = 2\int u - u^{-1}\; du$$

$$= 2\left(\frac{u^2}{2} - \ln|u|\right) + C = (e^{x})^2 - \ln e^{2x} + C = e^{2x} - 2x + C$$

Inverse hyperbolic functions

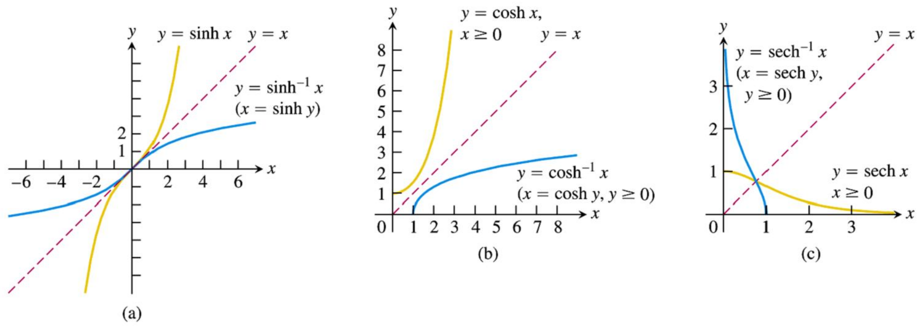


FIGURE 7.32 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

The inverse is useful in integration.

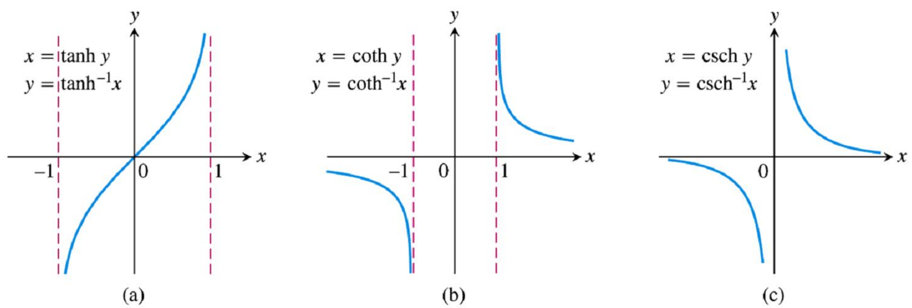


FIGURE 7.33 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

Useful Identities

TABLE 7.9 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1}\frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1}\frac{1}{x}$$

Proof

$\operatorname{sech}^{-1}x = \cosh^{-1}\frac{1}{x}$.

Take sech of $\cosh^{-1}\frac{1}{x}$.

$$\operatorname{sech}\left(\cosh^{-1}\frac{1}{x}\right) = \frac{1}{\cosh\left(\cosh^{-1}\frac{1}{x}\right)} = \frac{1}{\frac{1}{x}} = x$$

$$\operatorname{sech}\left(\cosh^{-1}\frac{1}{x}\right) = x$$

Take sech^{-1} on both sides:

$$\operatorname{sech}^{-1}\left(\operatorname{sech}\left(\cosh^{-1}\frac{1}{x}\right)\right) = \operatorname{sech}^{-1}x \Rightarrow \left(\cosh^{-1}\frac{1}{x}\right) = \operatorname{sech}^{-1}x$$

TABLE 7.10 Derivatives of inverse hyperbolic functions

$\frac{d(\sinh^{-1}u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$		Integrating these formulas will allows us to obtain a list of useful integration formula involving hyperbolic functions
$\frac{d(\cosh^{-1}u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$,	$u > 1$	
$\frac{d(\tanh^{-1}u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$,	$\lvert u\rvert < 1$	*e.g.*
$\frac{d(\coth^{-1}u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$,	$\lvert u\rvert > 1$	$\frac{1}{\sqrt{1+x^2}} = \frac{d}{dx}\sinh^{-1}x$
$\frac{d(\operatorname{sech}^{-1}u)}{dx} = \frac{-du/dx}{u\sqrt{1-u^2}}$,	$0 < u < 1$	$\to \int\frac{1}{\sqrt{1+x^2}}dx = \int\frac{d}{dx}\sinh^{-1}x\,dx$
$\frac{d(\operatorname{csch}^{-1}u)}{dx} = \frac{-du/dx}{\lvert u\rvert\sqrt{1+u^2}}$,	$u \neq 0$	$\int\frac{1}{\sqrt{1+x^2}}dx = \sinh^{-1}x + C$

Proof

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}.$$

let $y = \sinh^{-1}x$

$$x = \sinh y \to \frac{d}{dx}x = \frac{d}{dx}\sinh y = \frac{dy}{dx}\cosh y$$

$$\to \frac{dy}{dx} = \operatorname{sech} y = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

$\Rightarrow$ By virtue of chain rule,

$$\frac{d}{dx}\sinh^{-1}u = \frac{du}{dx}\frac{1}{\sqrt{1+u^2}}$$

Example 2 Derivative of the inverse hyperbolic cosine

- Show that

$$\frac{d}{dx}\cosh^{-1}u = \frac{1}{\sqrt{u^2-1}}.$$

Let $y = \cosh^{-1} x$...

Example 3 Using table 7.11

$$\int_0^1 \frac{2dx}{\sqrt{3+4x^2}}$$

Let $y = 2x$

$$\int_0^1 \frac{2dx}{\sqrt{3+4x^2}} = \int_0^2 \frac{dy}{\sqrt{3+y^2}}$$

Scale it again to normalise the constant 3 to 1

$$\text{Let } z = \frac{y}{\sqrt{3}} \to \int_0^2 \frac{dy}{\sqrt{3+y^2}} = \int_0^{2/\sqrt{3}} \frac{\sqrt{3}dz}{\sqrt{3+3z^2}} = \int_0^{2/\sqrt{3}} \frac{dz}{\sqrt{1+z^2}}$$

$$= \sinh^{-1} z\Big|_0^{2/\sqrt{3}} = \sinh^{-1}(2/\sqrt{3}) - \sinh^{-1}(0) = \sinh^{-1}(2/\sqrt{3}) - 0$$

$$= \sinh^{-1}(2/\sqrt{3})$$

$$\sinh^{-1}(2/\sqrt{3}) = ?$$

$$\text{Let } q = \sinh^{-1}(2/\sqrt{3})$$

$$\sinh q = 2/\sqrt{3} \to \frac{1}{2}\left(e^q - e^{-q}\right) = \frac{2}{\sqrt{3}}$$

$$e^{2q} - \frac{4}{\sqrt{3}}e^q - 1 = 0$$

$$e^q = \frac{\frac{4}{\sqrt{3}} + \sqrt{\left(-\frac{4}{\sqrt{3}}\right)^2 - 4(-1)}}{2} = \frac{\frac{4}{\sqrt{3}} + \sqrt{\frac{296}{9}}}{2} = 2.682$$

$$\sinh^{-1}(2/\sqrt{3}) = q = \ln 2.682 = 0.9866$$

TABLE 7.11 Integrals leading to inverse hyperbolic functions

1.	$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C,$	$a > 0$
2.	$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C,$	$u > a > 0$
3.	$\int \frac{du}{a^2-u^2} = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C$	if $u^2 < a^2$
	$\int \frac{du}{a^2-u^2} = \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C,$	if $u^2 > a^2$
4.	$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C,$	$0 < u < a$
5.	$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left\lvert\frac{u}{a}\right\rvert + C,$	$u \neq 0$ and $a > 0$

Chapter 8

Techniques of Integration

8.1

Basic Integration Formulas

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C$ (any number k)
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C$ $(n \neq -1)$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \frac{a^u}{\ln a} + C$ $(a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$
21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C$ $(a > 0)$
22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C$ $(u > a > 0)$

Example 1 Making a simplifying substitution

$$\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \int \frac{d(\overbrace{x^2-9x}^{u})}{\sqrt{x^2-9x+1}}$$

$$= \int \frac{du}{\sqrt{u+1}} = \int \frac{d(u+1)}{\sqrt{u+1}} = \int \frac{dv}{\sqrt{v}} = 2v^{1/2} + C$$

$$= 2(u+1)^{1/2} + C = 2\left(x^2-9x+1\right)^{1/2} + C$$

Example 2 Completing the square

$$\int \frac{dx}{\sqrt{8x-x^2}} = \int \frac{dx}{\sqrt{16-(x-4)^2}} =$$

$$\int \frac{d(x-4)}{\sqrt{16-(x-4)^2}} = \int \frac{du}{\sqrt{4^2-u^2}}$$

$$= \sin^{-1}\frac{u}{4} + C = \sin^{-1}\left(\frac{x-4}{4}\right) + C$$

Example 3 Expanding a power and using a trigonometric identity

$$\int (\sec x + \tan x)^2 dx$$

$$= \int (\sec^2 x + \tan^2 x + 2\sec x \tan x) dx.$$

$$\text{Racall:} \tan^2 x = \sec^2 x - 1; \frac{d}{dx}\tan x = \sec^2 x; \frac{d}{dx}\sec x = \tan x \sec x;$$

$$= \int (2\sec^2 x - 1 + 2\sec x \tan x) dx$$

$$= 2\tan x + -x + 2\sec x + C$$

Example 4 Eliminating a square root

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx =$$

$$\cos 4x = \cos 2(2x) = 2\cos^2(2x) - 1$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, x$$

$$= \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx = \ldots$$

Example 5 Reducing an improper fraction

$$\int \frac{3x^2-7x}{3x+2} dx$$

$$= \int x - 3 + \frac{6}{3x+2} dx$$

$$= \int x - 3 + \frac{2}{x+2/3} dx$$

$$= \frac{1}{2}x^2 - 3x + 2\ln|x + \frac{2}{3}| + C$$

Example 6 Separating a fraction

$$\int \frac{3x+2}{\sqrt{1-x^2}}dx$$
$$=3\int \frac{x}{\sqrt{1-x^2}}dx+\int \frac{2}{\sqrt{1-x^2}}dx$$
$$=3\int \frac{\frac{1}{2}d(x^2)}{\sqrt{1-x^2}}+2\int \frac{1}{\sqrt{1-x^2}}dx$$
$$=\frac{3}{2}\int \frac{du}{\sqrt{1-u}}+2\sin^{-1}x+C \qquad \int \frac{du}{(1-u)^{1/2}}=-2(1-u)^{1/2}+C'$$
$$=\frac{3}{2}[-2(1-u)^{1/2}]+2\sin^{-1}x+C''$$
$$=-3\sqrt{(1-x^2)}+2\sin^{-1}x+C''$$

Example 7 Integral of $y = \sec x$

$$\int \sec x dx = ?$$

$$d\sec x=\sec x\tan x dx$$
$$d\tan x=\sec^2 x dx=\sec x\sec x dx$$
$$d(\sec x+\tan x)=\sec x(\sec x+\tan x)dx$$
$$\sec x dx=\frac{d(\sec x+\tan x)}{\sec x+\tan x}$$

$$\int \sec x dx=\int \frac{d(\sec x+\tan x)}{\sec x+\tan x}=\ln|\sec x+\tan x|+C$$

TABLE 8.2 The secant and cosecant integrals

1. $\int \sec u\, du = \ln|\sec u + \tan u| + C$
2. $\int \csc u\, du = -\ln|\csc u + \cot u| + C$

Procedures for Matching Integrals to Basic Formulas

Procedure	Example
Making a simplifying substitution	$\frac{2x-9}{\sqrt{x^2-9x+1}}dx = \frac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$
Using a trigonometric identity	$(\sec x+\tan x)^2 = \sec^2 x + 2\sec x\tan x + \tan^2 x$ $= \sec^2 x + 2\sec x\tan x + (\sec^2 x - 1)$ $= 2\sec^2 x + 2\sec x\tan x - 1$
Eliminating a square root	$\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$
Reducing an improper fraction	$\frac{3x^2-7x}{3x+2} = x - 3 + \frac{6}{3x+2}$
Separating a fraction	$\frac{3x+2}{\sqrt{1-x^2}} = \frac{3x}{\sqrt{1-x^2}} + \frac{2}{\sqrt{1-x^2}}$
Multiplying by a form of 1	$\sec x = \sec x \cdot \frac{\sec x+\tan x}{\sec x+\tan x}$ $= \frac{\sec^2 x + \sec x\tan x}{\sec x+\tan x}$

8.2

Integration by Parts

Product rule in integral form

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$$

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int g(x)\frac{d}{dx}[f(x)]dx + \int f(x)\frac{d}{dx}[g(x)]dx$$

$$f(x)g(x) = \int g(x)f'(x)dx + \int f(x)g'(x)dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Integration by parts formula

Alternative form of Eq. (1)

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

$$g'(x) \equiv \frac{dg(x)}{dx} = h(x) \rightarrow \int g'(x)dx = \int h(x)dx$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

$$\rightarrow \int f(x)h(x)dx = f(x)\left[\int h(x)dx\right] - \int \left\{ f'(x)\left[\int h(x)dx\right] \right\} dx$$

Alternative form of the integration by parts formula

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$$

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int g(x)\frac{d}{dx}[f(x)]dx + \int f(x)\frac{d}{dx}[g(x)]dx$$

$$f(x)g(x) = \int g(x)df(x) + \int f(x)dg(x)$$

Let $u = f(x); v = g(x)$.The above formular is recast into the form

$$uv = \int vdu + \int udv$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

Example 4 Repeated use of integration by parts

$$\int x^2 e^x dx = ?$$

Example 5 Solving for the unknown integral

$$\int e^x \cos x dx = ?$$

Evaluating by parts for definite integrals

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \qquad (3)$$

or, equivalently

$$\int_a^b f(x)h(x)dx = f'(x)h(x)\Big]_a^b - \int_a^b \left\{ f'(x)\left[\int h(x)dx\right]\right\} dx$$

Example 6 Finding area

- Find the area of the region in Figure 8.1

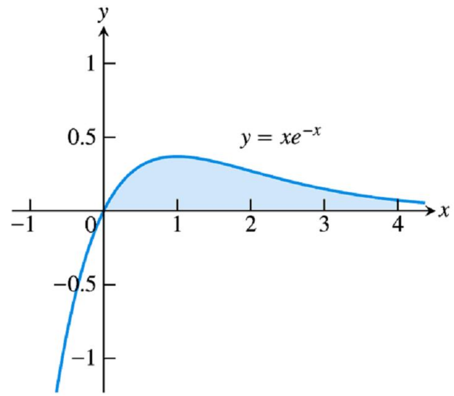


FIGURE 8.1 The region in Example 6.

Solution

$$\int_0^4 xe^{-x}dx = \dots$$

Example 9 Using a reduction formula

- Evaluate $\int \cos^3 x dx$
- Use

$$\int \cos^n x dx = \int \overbrace{\cos^{n-1} x}^{u} \cdot \overbrace{\cos x dx}^{dv}$$

$$= \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$$

8.3

Integration of Rational Functions by Partial Fractions

General description of the method

- A rational function *f*(*x*)/*g*(*x*) can be written as a sum of partial fractions. To do so:
- (a) The degree of *f*(*x*) must be less than the degree of *g*(*x*). That is, the fraction must be proper. If it isn't, divide *f*(*x*) by *g*(*x*) and work with the remainder term.
- We must know the factors of *g*(*x*). In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors.

Reducibility of a polynomial

- A polynomial is said to be **reducible** if it is the product of two polynomials of lower degree.
- A polynomial is **irreducible** if it is not the product of two polynomials of lower degree.

- THEOREM (Ayers, Schaum's series, pg. 305)
- Consider a polynomial $g(x)$ of order $n \geqslant 2$ (with leading coefficient 1). Two possibilities:

1. $g(x) = (x-r)\, h_1(x)$, where $h_1(x)$ is a polynomial of degree n-1, or
2. $g(x) = (x^2+px+q)\, h_2(x)$, where $h_2(x)$ is a polynomial of degree n-2, and (x^2+px+q) is the irreducible quadratic factor.

Example

$$g(x) = x^3 - 4x = \underbrace{(x-2)}_{\text{linear factor}} \cdot \underbrace{x(x+2)}_{\text{poly. of degree 2}}$$

$$g(x) = x^3 + 4x = \underbrace{(x^2+4)}_{\text{irreducible quadratic factor}} \cdot \underbrace{x}_{\text{poly. of degree 1}}$$

$$g(x) = x^4 - 9 = \underbrace{(x^2+3)}_{\text{irreducible quadratic factor}} \cdot \underbrace{(x+\sqrt{3})(x-\sqrt{3})}_{\text{poly. or degree 2}}$$

$$g(x) = x^3 - 3x^2 - x + 3 = \underbrace{(x+1)}_{\text{linear factor}} \underbrace{(x-2)^2}_{\text{poly. or degree 2}}$$

Quadratic polynomial

- A quadratic polynomial (polynomial or order n = 2) is either reducible or not reducible.
- Consider: $g(x)= x^2+px+q$.
- If $(p^2-4q) \geqslant 0$, $g(x)$ is reducible, i.e. $g(x) = (x+r_1)(x+r_2)$.
- If $(p^2-4q) < 0$, $g(x)$ is irreducible.

- In general, a polynomial of degree n can always be expressed as the product of linear factors and irreducible quadratic factors:

$$P_n(x) = (x-r_1)^{n_1}(x-r_2)^{n_2}\ldots(x-r_l)^{n_l} \times$$
$$(x^2+p_1x+q_1)^{m_1}(x^2+p_2x+q_2)^{m_2}\ldots(x^2+p_kx+q_k)^{m_k}$$

$$n = (n_1+n_2+\ldots+n_l) + 2(m_1+m_2+\ldots+m_l)$$

Integration of rational functions by partial fractions

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:
$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$
Do this for each distinct linear factor of $g(x)$.
2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2+px+q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:
$$\frac{B_1x + C_1}{x^2+px+q} + \frac{B_2x + C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx + C_n}{(x^2+px+q)^n}.$$
Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.
3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example 1 Distinct linear factors

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}dx = \ldots$$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)} = \ldots$$

Example 2 A repeated linear factor

$$\int \frac{6x+7}{(x+2)^2}dx = \ldots$$

$$\frac{6x+7}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

Example 3 Integrating an improper fraction

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx = \ldots$$

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)} = \ldots$$

Example 4 Integrating with an irreducible quadratic factor in the denominator

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx = \ldots$$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2} = \ldots$$

Example 5 A repeated irreducible quadratic factor

$$\int \frac{1}{x(x^2+1)^2}dx = ?$$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2} = \ldots$$

Other ways to determine the coefficients

- Example 8 Using differentiation
- Find *A*, *B* and *C* in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$\frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3} = \frac{x-1}{(x+1)^3}$$

$$\Rightarrow A(x+1)^2 + B(x+1) + C = x-1$$

$$x = -1 \rightarrow C = -2$$

$$\Rightarrow A(x+1)^2 + B(x+1) = x+1$$

$$\Rightarrow A(x+1) + B = 1$$

$$\frac{d}{dx}[A(x+1)+B] = \frac{d}{dx}(1) = 0$$

$$A = 0$$

$$B = 1$$

Example 9 Assigning numerical values to x

- Find *A*, *B* and *C* in

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \equiv f(x)$$

$$= x^2 + 1$$

$$f(1) = 2A+ = 1^2 + 1 = 2 \Rightarrow A = 1$$

$$f(2) = -B = 2^2 + 1 = 5; \Rightarrow B = -5$$

$$f(3) = 2C = 3^2 + 1 = 10; \Rightarrow C = 5$$

8.4

Trigonometric Integrals

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the work into three cases.

Case 1 If m is odd, we write m as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \quad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x \, dx$, we write n as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \quad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Example 1 *m* is odd

$$\int \sin^3 x \cos^2 x \; dx = ?$$

$$\int \sin^3 x \cos^2 x \; dx = -\int \sin^2 x \cos^2 x \; d(\cos x)$$

$$= \int (\cos^2 x - 1) \cos^2 x \; d(\cos x)$$

$$= \int (u^2 - 1) u^2 du = \ldots$$

Example 2 *m* is even and *n* is odd

$$\int \cos^5 x \; dx = ?$$

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (\cos^2 x)^2 \; d(\sin x) =$$

$$= \int (1\text{-}\sin^2 x)^2 \; d \underbrace{\sin x}_{u}$$

$$= \int (1\text{-}u^2)^2 \; du = \int 1 + u^4 - 2u^2 \; du = \ldots$$

Example 3 *m* and *n* are both even

$$\int \cos^2 x \sin^4 x \; dx = ?$$

$$\int \cos^2 x \sin^4 x \, dx =$$

$$\int \left(\frac{1\text{-}\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1\text{-}\cos 2x)(1+\cos 2x)^2 \; dx$$

$$= \frac{1}{4}\int \left(1+\cos^2 2x - \cos 2x - \cos^3 2x\right) dx = \ldots$$

Example 6 Integrals of powers of tan x and sec x

$$\int \sec^3 x dx = ?$$

Use integration by parts.

$$\int \sec^3 x dx = \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x dx}_{dv};$$

$$dv = \sec^2 x dx \to v = \int \sec^2 x dx = \tan x$$

$$u = \sec x \to du = \sec x \tan x dx$$

$$\int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x dx}_{dv}$$

$$= \sec x \tan x - \int \tan x \cdot \underbrace{\sec x \tan x dx}_{du}$$

$$= \sec x \tan x - \int \tan^2 x \sec x dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx$$

$$\int \sec^3 x dx = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \ldots$$

$$\int \sec x dx = \int \sec x \frac{(\tan x + \sec x)}{\tan x + \sec x} dx$$

$$= \int \frac{(\sec x \tan x + \sec^2 x)}{\tan x + \sec x} dx$$

$$= \int \frac{d(\sec x + \tan x)}{\tan x + \sec x}$$

$$= \ln|\sec x + \tan x| + C$$

Example 7 Products of sines and cosines

$$\int \cos 5x \sin 3x dx = ?$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x];$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x];$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

$$\int \cos 5x \sin 3x dx$$

$$= \frac{1}{2}\int [\sin(-2x) + \sin 8x] dx$$

$$= \ldots$$

8.5

Trigonometric Substitutions

Three basic substitutions

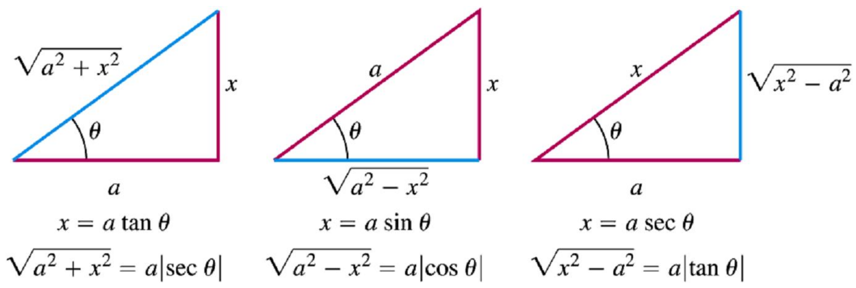


FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

Useful for integrals involving $\sqrt{a^2-x^2}, \sqrt{a^2+x^2}, \sqrt{x^2-a^2}$

Example 1 Using the substitution $x=a\tan\theta$

$$\int \frac{dx}{\sqrt{4+x^2}} = ?$$

$$x = 2\tan y \to dx = 2\sec^2 y dy = 2(\tan^2 y+1)dy$$

$$\int \frac{dx}{\sqrt{4+4\tan^2 y}} = \int \frac{2(\tan^2 y+1)}{\sqrt{4+4\tan^2 y}} dy$$

$$= \int \frac{(\tan^2 y+1)}{\sqrt{1+\tan^2 y}} dy = \int \sqrt{\sec^2 y} dy = \int |\sec y| dy$$

$$= \ln|\sec y + \tan y| + C$$

Example 2 Using the substitution $x = a\sin\theta$

$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = ?$$

$$x = 3\sin y \to dx = 3\cos y\ dy$$

$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2 y \cdot 3\cos y\ dy}{\sqrt{9-9\sin^2 y}} =$$

$$= 9\int \frac{\sin^2 y \cdot \cos y\ dy}{\sqrt{1-\sin^2 y}}$$

$$= 9\int \sin^2 y dy = \ldots$$

Example 3 Using the substitution $x = a\sec\theta$

$$\int \frac{dx}{\sqrt{25x^2-4}} = ?$$

$$x = \frac{2}{5}\sec y \to dx = \frac{2}{5}\sec y \tan y\ dy$$

$$\int \frac{dx}{\sqrt{25x^2-4}} = \frac{2}{5}\int \frac{\sec y \tan y\ dy}{\sqrt{4\sec^2 y - 4}} = \frac{1}{5}\int \frac{\sec y \tan y\ dy}{\sqrt{\sec^2 y - 1}}$$

$$= \frac{1}{5}\int \frac{\sec y \tan y\ dy}{\sqrt{\sec^2 y - 1}} = \frac{1}{5}\int \sec y\ dy$$

$$= \frac{1}{5}\ln|\sec y + \tan y| + C = \ldots$$

Example 4 Finding the volume of a solid of revolution

$$V = 16\pi \int_0^2 \frac{dx}{\left(x^2+4\right)^2} = ?$$

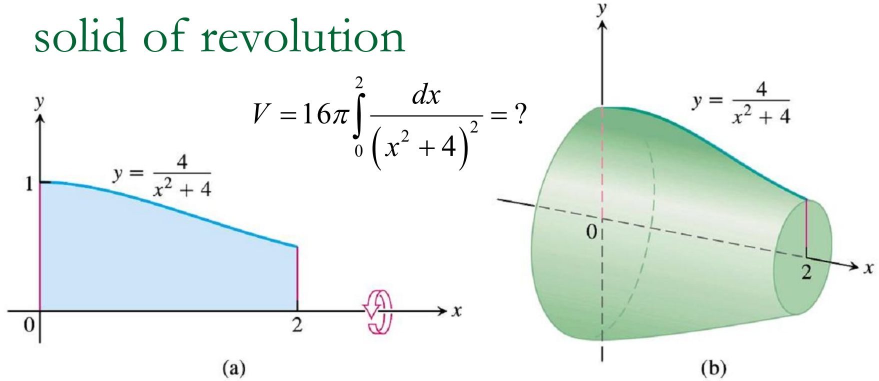


FIGURE 8.7 The region (a) and solid (b) in Example 4.

8.6

Integral Tables

Solution

$$V = 16\pi \int_0^2 \frac{dx}{\left(x^2+4\right)^2} = ?$$

Let $x = 2\tan y \rightarrow dx = 2\sec^2 y dy$

$$V = \pi \int_0^{\pi/4} \frac{2\sec^2 y dy}{\left(\tan^2 y + 1\right)^2} = \pi \int_0^{\pi/4} \frac{2\sec^2 y dy}{\left(\sec^2 y\right)^2}$$

$$= 2\pi \int_0^{\pi/4} \cos^2 y dy = \ldots$$

Integral tables is provided at the back of Thomas'

- T-4 A brief tables of integrals
- Integration can be evaluated using the tables of integral.

EXAMPLE 1 Find

$$\int x(2x+5)^{-1}\,dx.$$

Solution We use Formula 8 (not 7, which requires $n \neq -1$):

$$\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C.$$

With $a = 2$ and $b = 5$, we have

$$\int x(2x+5)^{-1}\,dx = \frac{x}{2} - \frac{5}{4}\ln|2x+5| + C.$$

EXAMPLE 2 Find

$$\int \frac{dx}{x\sqrt{2x+4}}.$$

Solution We use Formula 13(b):

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C, \qquad \text{if } b > 0.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x+4}} = \frac{1}{\sqrt{4}}\ln\left|\frac{\sqrt{2x+4}-\sqrt{4}}{\sqrt{2x+4}+\sqrt{4}}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{2x+4}-2}{\sqrt{2x+4}+2}\right| + C.$$

EXAMPLE 3 Find

$$\int \frac{dx}{x\sqrt{2x-4}}.$$

Solution We use Formula 13(a):

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}}\tan^{-1}\sqrt{\frac{ax-b}{b}} + C.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x-4}} = \frac{2}{\sqrt{4}}\tan^{-1}\sqrt{\frac{2x-4}{4}} + C = \tan^{-1}\sqrt{\frac{x-2}{2}} + C.$$

EXAMPLE 4 Find

$$\int \frac{dx}{x^2\sqrt{2x-4}}.$$

Solution We begin with Formula 15:

$$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}} + C.$$

With $a = 2$ and $b = -4$, we have

$$\int \frac{dx}{x^2\sqrt{2x-4}} = -\frac{\sqrt{2x-4}}{-4x} + \frac{2}{2\cdot 4}\int \frac{dx}{x\sqrt{2x-4}} + C.$$

We then use Formula 13(a) to evaluate the integral on the right (Example 3) to obtain

$$\int \frac{dx}{x^2\sqrt{2x-4}} = \frac{\sqrt{2x-4}}{4x} + \frac{1}{4}\tan^{-1}\sqrt{\frac{x-2}{2}} + C.$$

EXAMPLE 5 Find

$$\int x \sin^{-1} x \, dx.$$

Solution We use Formula 99:

$$\int x^n \sin^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \sin^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1}\, dx}{\sqrt{1 - a^2x^2}}, \qquad n \neq -1.$$

With $n = 1$ and $a = 1$, we have

$$\int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2\, dx}{\sqrt{1 - x^2}}.$$

The integral on the right is found in the table as Formula 33:

$$\int \frac{x^2}{\sqrt{a^2 - x^2}} dx = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2} x \sqrt{a^2 - x^2} + C.$$

With $a = 1$,

$$\int \frac{x^2\, dx}{\sqrt{1 - x^2}} = \frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1 - x^2} + C.$$

The combined result is

$$\int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2}\left(\frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1 - x^2} + C\right)$$

$$= \left(\frac{x^2}{2} - \frac{1}{4}\right) \sin^{-1} x + \frac{1}{4} x \sqrt{1 - x^2} + C'.$$

8.8

Improper Integrals

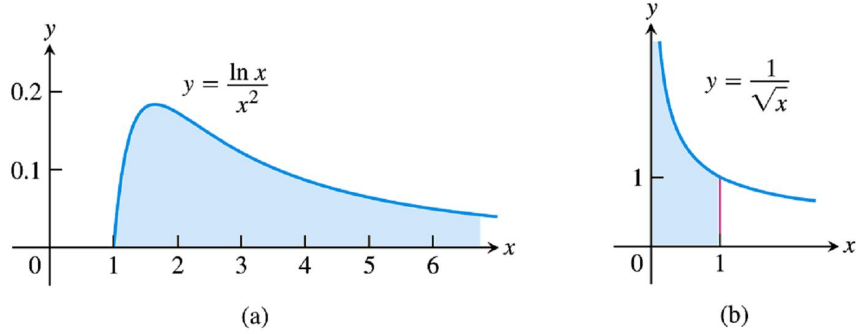


FIGURE 8.17 Are the areas under these infinite curves finite?

Infinite limits of integration

$$A(a) = \lim_{b\to\infty} A(b) = \lim_{b\to\infty} 2 - 2e^{-b/2} = 2$$

$$A(b) = \int_0^b e^{-x/2} dx = \ldots = 2 - 2e^{-b/2}$$

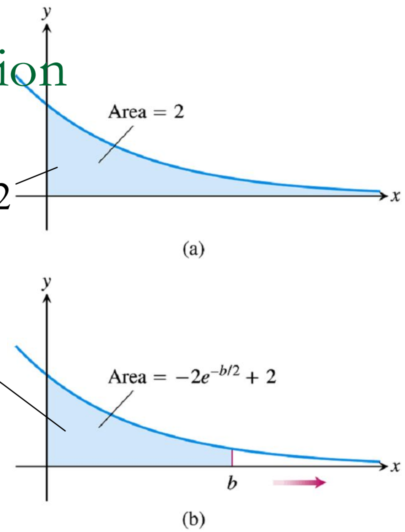


FIGURE 8.18 (a) The area in the first quadrant under the curve $y = e^{-x/2}$ is (b) an improper integral of the first type.

DEFINITION Type I Improper Integrals

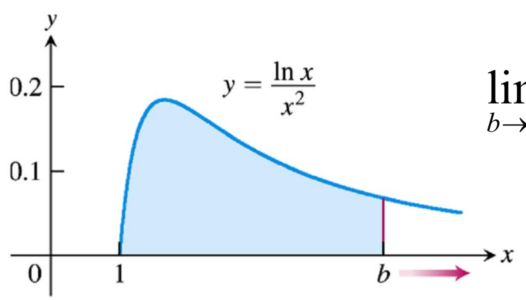
Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then
$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$
2. If $f(x)$ is continuous on $(-\infty, b]$, then
$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$
3. If $f(x)$ is continuous on $(-\infty, \infty)$, then
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$$
where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Example 1 Evaluating an improper integral on [1,∞]

- Is the area under the curve $y=(\ln x)/x^2$ from 1 to ∞ finite? If so, what is it?



$$\lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}dx = ?$$

FIGURE 8.19 The area under this curve is an improper integral (Example 1).

Solution

$$\int_1^b \frac{\ln x}{x}\frac{dx}{x} = \int_1^b \frac{\ln x}{x} d(\ln x) = \int_{\ln 1}^{\ln b} \frac{u}{e^u}du \qquad ;u=\ln x, x=e^u$$

$$\int_0^{\ln b} u\underbrace{e^{-u}du}_{dw} = u\underbrace{(-e^{-u})}_{w}\Big|_0^{\ln b} - \int_0^{\ln b}\underbrace{(-e^{-u})}_{w}du$$

$$= ue^{-u}\Big|_{\ln b}^{0} + \int_0^{\ln b} e^{-u}du = ue^{-u}\Big|_{\ln b}^{0} - e^{-u}\Big|_0^{\ln b}$$

$$= -\ln b\cdot e^{-\ln b} - (e^{-\ln b} - 1) = -\frac{1}{b}\ln b - \frac{1}{b} + 1$$

$$\lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{b}\ln b - \frac{1}{b} + 1\right] = 1$$

Example 2 Evaluating an integral on [-∞,∞]

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = ?$$

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \lim_{b\to\infty}\int_{-b}^{0}\frac{dx}{1+x^2} + \lim_{b\to\infty}\int_0^b\frac{dx}{1+x^2}$$

$$= 2\lim_{b\to\infty}\int_0^b\frac{dx}{1+x^2}$$

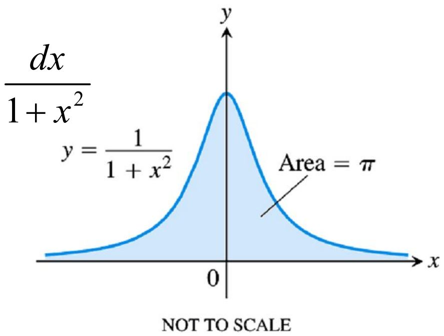


FIGURE 8.20 The area under this curve is finite (Example 2).

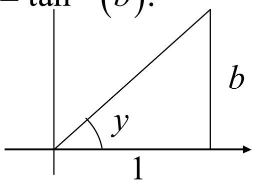
Solution

Using the integral table (Eq. 16)

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int_0^b \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^b = \tan^{-1}(b) - \tan^{-1}0 = \tan^{-1}(b).$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\lim_{b\to\infty}\tan^{-1}b = 2\cdot\frac{\pi}{2} = \pi$$



$$y = \tan^{-1}b \Rightarrow b = \tan y$$

$$\lim_{b\to\infty}\tan^{-1}b = \frac{\pi}{2}$$

DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then
$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

Example 3 Integrands with vertical asymptotes

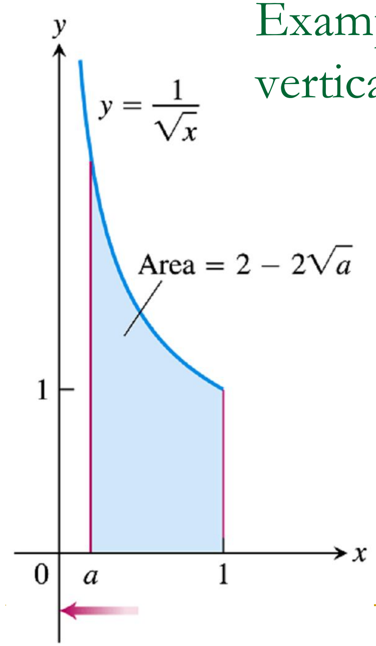
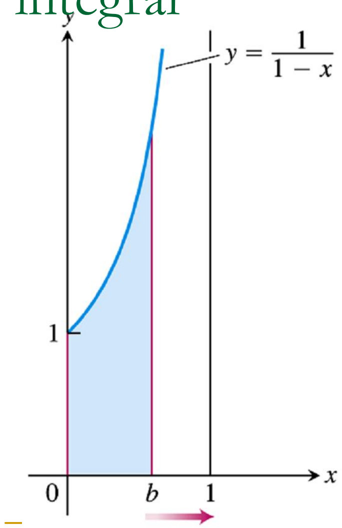


FIGURE 8.21 The area under this curve is

$$\lim_{a\to 0^+}\int_a^1 \left(\frac{1}{\sqrt{x}}\right)dx = 2,$$

an improper integral of the second kind.

Example 4 A divergent improper integral



- Investigate the convergence of
$$\int_0^1 \frac{dx}{1-x}$$

FIGURE 8.22 The limit does not exist:

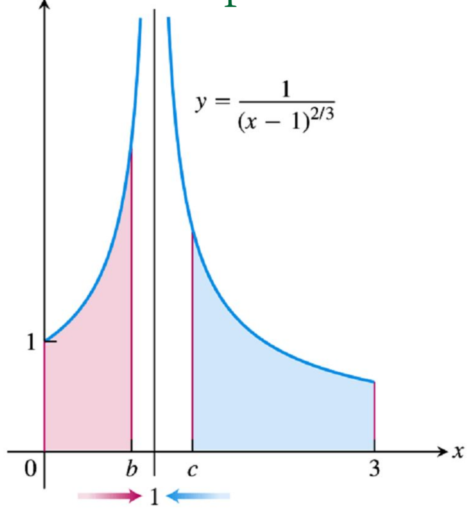
$$\int_0^1\left(\frac{1}{1-x}\right)dx = \lim_{b\to 1^-}\int_0^b \frac{1}{1-x}\,dx = \infty$$

The area beneath the curve and above the x-axis for $[0, 1)$ is not a real number (Example 4).

Solution

$$\int_0^1 \frac{dx}{1-x} = \lim_{b\to 1^-}\int_0^b \frac{dx}{1-x} = -\lim_{b\to 1^-}\left[\ln|x-1|\right]_0^b$$
$$= -\lim_{b\to 1^-}\left[\ln|b-1| - \ln|0-1|\right]$$
$$= -\lim_{b\to 1^-}\left[\ln|b-1| - \ln|0-1|\right] = \lim_{b\to 1^-}\left[\ln|b-1|^{-1}\right]$$
$$= \lim_{\varepsilon\to 0}\left[\ln\frac{1}{\varepsilon}\right] = \infty$$

Example 5 Vertical asymptote at an interior point



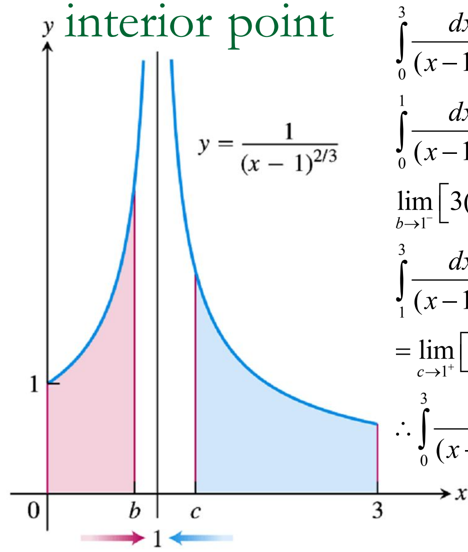
$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = ?$$

FIGURE 8.23 Example 5 shows the convergence of

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = 3 + 3\sqrt[3]{2},$$

so the area under the curve exists (so it is a real number).

Example 5 Vertical asymptote at an interior point



$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$
$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-}\left[3(x-1)^{1/3}\right]_0^b =$$
$$\lim_{b\to 1^-}\left[3(b-1)^{1/3} - 3(-1)^{1/3}\right] = \lim_{b\to 1^-}\left[0+3\right] = 3;$$
$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1+}\int_c^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+}\left[3(x-1)^{1/3}\right]_c^3$$
$$= \lim_{c\to 1^+}\left[3(3-1)^{1/3} - 3(c-1)^{1/3}\right] = 3\cdot 2^{2/3}$$
$$\therefore \int_0^3 \frac{dx}{(x-1)^{2/3}} = 3(1+2^{2/3})$$

Example 7 Finding the volume of an infinite solid

- The cross section of the solid in Figure 8.24 perpendicular to the *x*-axis are circular disks with diameters reaching from the *x*-axis to the curve *y* = e^x, $-\infty < x < \ln 2$. Find the volume of the horn.

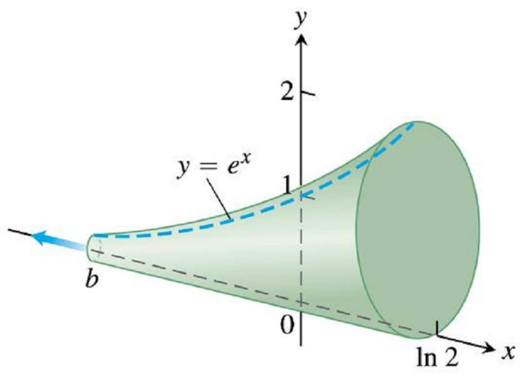


FIGURE 8.24 The calculation in Example 7 shows that this infinite horn has a finite volume.

Example 7 Finding the volume of an infinite solid

volume of a slice of disk of thickness dx, diameter y

$$dV = \pi (y/2)^2 \, dx$$

$$V = \int_0^V dV = \frac{1}{4} \lim_{b \to -\infty} \int_b^{\ln 2} \pi y(x)^2 \, dx$$

$$= \frac{1}{4} \lim_{b \to -\infty} \int_b^{\ln 2} \pi e^{2x} \, dx$$

$$= \frac{1}{8} \lim_{b \to -\infty} \left[\pi e^{2x} \right]_b^{\ln 2}$$

$$= \frac{1}{8} \lim_{b \to -\infty} \left[4\pi - \pi e^{2b} \right]$$

$$= \frac{1}{8} \pi \lim_{b \to -\infty} (4 - e^{2b}) = \frac{\pi}{2}$$

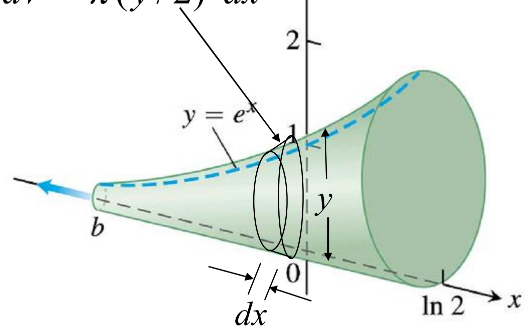


FIGURE 8.24 The calculation in Example 7 shows that this infinite horn has a finite volume.

Chapter 11

Infinite Sequences and Series

11.1

Sequences

What is a sequence

- A sequence is a list of numbers

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

  in a given order.
- Each *a* is a ***term*** of the sequence.
- Example of a sequence:
- 2,4,6,8,10,12,…,2*n*,…
- *n* is called the ***index*** of a_n

DEFINITION **Infinite Sequence**
An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

- In the previous example, a general term a_n of index *n* in the sequence is described by the formula

$$a_n = 2n.$$

- We denote the sequence in the previous example by $\{a_n\} = \{2, 4,6,8,\ldots\}$
- In a sequence the order is important:
- 2,4,6,8,… and …,8,6,4,2 are not the same

Other example of sequences

$$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \cdots, \sqrt{n}, \cdots\}, a_n = \sqrt{n};$$

$$\{b_n\} = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots, (-1)^{n+1}\frac{1}{n}, \cdots\}; b_n = (-1)^{n+1}\frac{1}{n};$$

$$\{c_n\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots, \frac{n-1}{n}, \cdots\}; c_n = \frac{n-1}{n};$$

$$\{d_n\} = \{1, -1, 1, -1, 1, \cdots, (-1)^{n+1}, \cdots\}; d_n = (-1)^{n+1};$$

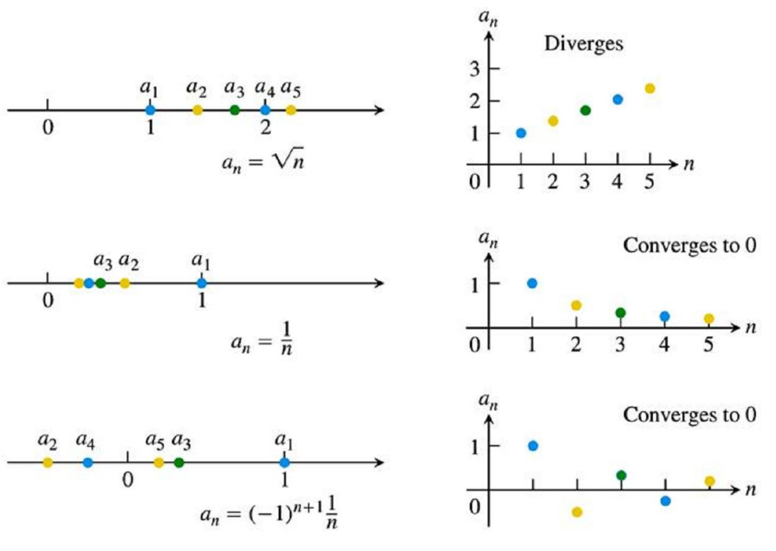


FIGURE 11.1 Sequences can be represented as points on the real line or as points in the plane where the horizontal axis n is the index number of the term and the vertical axis a_n is its value.

DEFINITIONS Converges, Diverges, Limit

The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Figure 11.2).

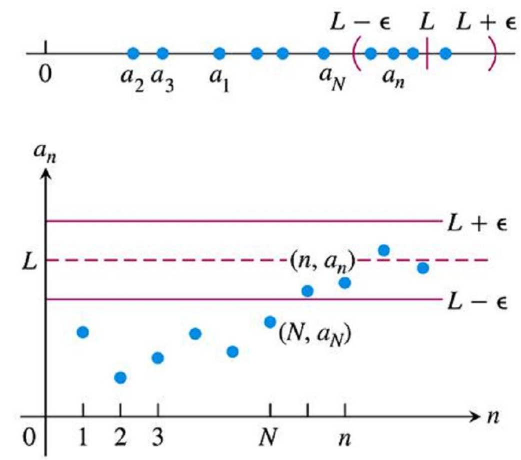


FIGURE 11.2 $a_n \to L$ if $y = L$ is a horizontal asymptote of the sequence of points $\{(n, a_n)\}$. In this figure, all the a_n's after a_N lie within ϵ of L.

DEFINITION **Diverges to Infinity**

The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty .$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \quad \text{or} \quad a_n \to -\infty .$$

THEOREM 1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

EXAMPLE 3 Applying Theorem 1

By combining Theorem 1 with the limits of Example 1, we have:

(a) $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = -1 \cdot \lim_{n\to\infty}\frac{1}{n} = -1 \cdot 0 = 0$ — Constant Multiple Rule and Example 1a

(b) $\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n} = 1 - 0 = 1$ — Difference Rule and Example 1a

(c) $\lim_{n\to\infty}\frac{5}{n^2} = 5 \cdot \lim_{n\to\infty}\frac{1}{n} \cdot \lim_{n\to\infty}\frac{1}{n} = 5 \cdot 0 \cdot 0 = 0$ — Product Rule

(d) $\lim_{n\to\infty}\frac{4 - 7n^6}{n^6 + 3} = \lim_{n\to\infty}\frac{(4/n^6) - 7}{1 + (3/n^6)} = \frac{0 - 7}{1 + 0} = -7.$ — Sum and Quotient Rules

THEOREM 2 **The Sandwich Theorem for Sequences**

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

EXAMPLE 4 Applying the Sandwich Theorem

Since $1/n \to 0$, we know that

(a) $\frac{\cos n}{n} \to 0$ because $-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$;

(b) $\frac{1}{2^n} \to 0$ because $0 \le \frac{1}{2^n} \le \frac{1}{n}$;

(c) $(-1)^n \frac{1}{n} \to 0$ because $-\frac{1}{n} \le (-1)^n \frac{1}{n} \le \frac{1}{n}$.

THEOREM 3 The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

- Example 6: Applying theorem 3 to show that the sequence $\{2^{1/n}\}$ converges to 0.

- Taking $a_n = 1/n$, $\Rightarrow \lim_{n\to\infty} a_n = 0 \equiv L$
- Define $f(x)=2^x$. Note that $f(x)$ is continuous on $x=L$, and is defined for all $x = a_n = 1/n$
- According to Theorem 3,
- $\lim_{n\to\infty} f(a_n) = f(L)$
- LHS: $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(1/n) = \lim_{n\to\infty} 2^{1/n}$
- RHS = $f(L) = 2^L = 2^0 = 1$
- Equating LHS = RHS, we have $\lim_{n\to\infty} 2^{1/n} = 1$
- $\Rightarrow$ the sequence $\{2^{1/n}\}$ converges to 1

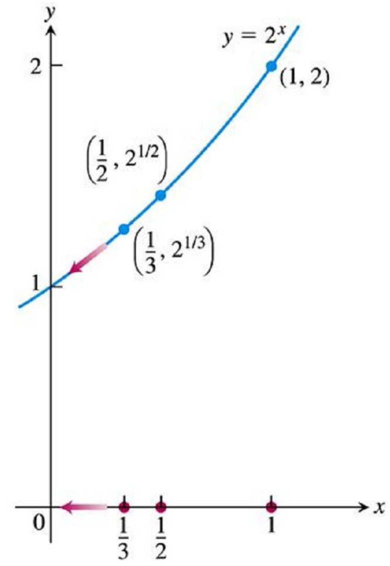


FIGURE 11.3 As $n \to \infty$, $1/n \to 0$ and $2^{1/n} \to 2^0$ (Example 6).

THEOREM 4

Suppose that $f(x)$ is a function defined for all $x \ge n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \ge n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L.$$

- Example 7: Applying l'Hopital rule
- Show that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
- Solution: The function $f(x) = \frac{\ln x}{x}$ is defined for $x \geqslant 1$ and agrees with the sequence $\{a_n = (\ln n)/n\}$ for $n \geqslant 1$.
- Applying l'Hopital rule on $f(x)$:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

- By virtue of Theorem 4,

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$$

Example 9 Applying l'Hopital rule to determine convergence

Does the sequence whose nth term is $a_n = \left(\frac{n+1}{n-1}\right)^n$ converge?

If so, find $\lim_{n\to\infty} a_n$.

Solution: Use l'Hopital rule

Let $f(x) = \left(\frac{x+1}{x-1}\right)^x$ so that $f(n) = a_n$ for $n \geq 1$.

$$\to \ln f(x) = x \ln\left(\frac{x+1}{x-1}\right)$$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} x \ln\left(\frac{x+1}{x-1}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{x+1}{x-1}\right)}{1/x}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{-2}{x^2-1}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{2x^2}{x^2-1} = 2$$

By virtue of Theorem 4, $\lim_{x\to\infty} \ln f(x) = 2 \Rightarrow$

$$\lim_{x\to\infty} f(x) = \exp(2) \Rightarrow \lim_{n\to\infty} a_n = \exp(2)$$

THEOREM 5
The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Example 10

- (a) (ln n^2)/n = 2 (ln n) / n → 2·0 = 0
- (b) $\sqrt[n]{n^2} = n^{2/n} = \left(n^{1/n}\right)^2 \to (1)^2$
- (c) $\sqrt[n]{3n} = \sqrt[n]{3} \cdot \sqrt[n]{n} = 3^{1/n} \cdot n^{1/n} \to 1 \cdot 1 = 1$
- (d) $\left(-\frac{1}{2}\right)^n \to 0$
- (e) $\left(\frac{n-2}{n}\right)^n = \left(1 + \frac{(-2)}{n}\right)^n \to e^{-2}$
- (f) $\frac{100^n}{n!} \to 0$

DEFINITION Nondecreasing Sequence
A sequence $\{a_n\}$ with the property that $a_n \le a_{n+1}$ for all n is called a **nondecreasing sequence**.

- Example 12 Nondecreasing sequence
- (a) 1,2,3,4,…,n,…
- (b) ½, 2/3, ¾, 4/5 , …,n/(n+1),…
  (nondecreasing because a_{n+1}-$a_n \geqslant 0$)
- (c) {3} = {3,3,3,…}

- Two kinds of nondecreasing sequences: bounded and non-bounded.

DEFINITIONS Bounded, Upper Bound, Least Upper Bound
A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \le M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

- Example 13 Applying the definition for boundedness
- (a) 1,2,3,…,n,…has no upper bound
- (b) ½, 2/3, ¾, 4/5 , …,n/(n+1),…is bounded from above by M = 1.
- Since no number less than 1 is an upper bound for the sequence, so 1 is the least upper bound.

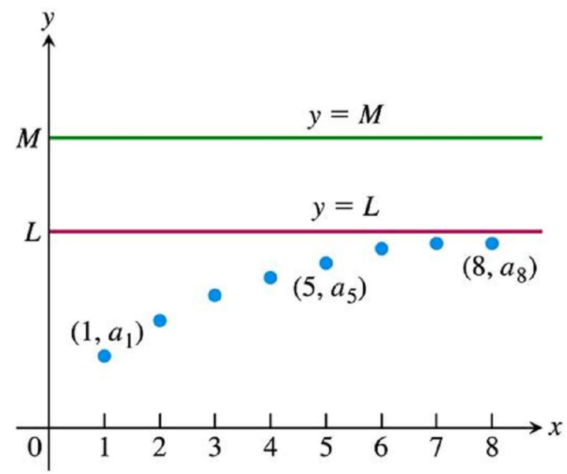


FIGURE 11.4 If the terms of a nondecreasing sequence have an upper bound M, they have a limit $L \le M$.

THEOREM 6 The Nondecreasing Sequence Theorem
A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

- If a non-decreasing sequence converges it is bounded from above.
- If a non-decreasing sequence is bounded from above it converges.
- In Example 13 (b) {½, 2/3, ¾, 4/5 , …,$n/(n+1)$,…} is bounded by the least upper bound $M = 1$. Hence according to Theorem 6, the sequence converges, and the limit of convergence is the least upper bound 1.

11.2

Infinite Series

DEFINITIONS Infinite Series, nth Term, Partial Sum, Converges, Sum
Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$\vdots$$
$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$
$$\vdots$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Example of a partial sum formed by a sequence $\{a_n = 1/2^{n-1}\}$

Partial sum		Suggestive expression for partial sum	Value
First:	$s_1 = 1$	$2 - 1$	1
Second:	$s_2 = 1 + \frac{1}{2}$	$2 - \frac{1}{2}$	$\frac{3}{2}$
Third:	$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$2 - \frac{1}{4}$	$\frac{7}{4}$
⋮	⋮	⋮	⋮
*n*th:	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$

FIGURE 11.5 As the lengths 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, . . . are added one by one, the sum approaches 2.

Geometric series

- Geometric series are the series of the form $a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$
- a and $r = a_{n+1}/a_n$ are fixed numbers and $a \neq 0$. r is called the ratio.
- Three cases can be classified: $r < 1$, $r > 1$, $r = 1$.

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1-r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Short hand notation for infinite series

$$\sum_{n}^{\infty} a_n, \sum_{k=1}^{\infty} a_k \text{ or } \sum a_n$$

- The infinite series is either converge or diverge

Proof of $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ for $|r|<1$

Assume $r \neq 1$.

$$s_n = \sum_{k=1}^{k=n} ar^{k-1} \qquad = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$rs_n = r\left(a + ar + ar^2 + \ldots + ar^{n-1}\right) = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$$

$$s_n - rs_n = a - ar^n = a\left(1 - r^n\right)$$

$$s_n = a\left(1 - r^n\right)/\left(1 - r\right)$$

If $|r|<1$: $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a\left(1-r^n\right)}{1-r} = \frac{a}{1-r}$ (By theorem 5.4, $\lim_{n\to\infty} r^n = 1$ for $|r|<1$)

For cases $|r| \geqslant 1$

$$\text{If } |r| > 1: \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \infty \text{ (Because } |r^n| \to \infty \text{ if } |r|>1$$

$$\text{If } r = 1: s_n = a + ar + ar^2 + \ldots + ar^{n-1} = (n+1)a$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} a(n+1) = a \lim_{n\to\infty}(n+1) = \infty$$

EXAMPLE 1 Index Starts with $n = 1$

The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1-(1/3)} = \frac{1}{6}.$$

Example 2 Index starts with n=0

- The series $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \frac{5}{4^0} - \frac{5}{4^1} + \frac{5}{4^2} - \frac{5}{4^3} + \ldots$

  is a geometric series with a=5, r=-(1/4).
- It converges to $s_\infty = a/(1-r) = 5/(1+1/4) = 4$

- Note: Be reminded that no matter how complicated the expression of a geometric series is, the series is simply completely specified by *r* and *a.* In other words, if you know *r* and *a* of a geometric series, you know almost everything about the series.

Example 4

$$5.232323\ldots = 5.\overline{23} = 5.\dot{2}\dot{3}$$

Express the above decimal as a ratio of two integers.

$$5.\dot{2}\dot{3} = 5 + [\ldots]$$

$$[\ldots] = 0.23 + 0.0023 + 0.000023 + \ldots$$

$$= 0.23(\ldots) = \frac{23}{100}(\ldots)$$

$$(\ldots) = 1 + 0.01 + 0.0001 + \ldots = \frac{a}{1-r} = \frac{1}{1-0.01} = \frac{1}{1-\frac{1}{100}} = \frac{1}{\frac{99}{100}} = \frac{100}{99}$$

$$5.\dot{2}\dot{3} = \frac{23}{100}\frac{100}{99} = \frac{23}{99}$$

Example 5 A nongeometric but telescopic series

- Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$
- **Solution**

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)}$$

$$s_k = \sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \frac{1}{n} - \frac{1}{(n+1)} =$$

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots\left(\frac{1}{k-1}-\frac{1}{k}\right)+\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

$$= 1 - \frac{1}{k+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} s_k = 1$$

Divergent series

- Example 6

$$\sum n^2 = 1+2+4+16+\ldots n^2+\ldots$$

diverges because the partial sums s_n grows beyond every number L

$$\sum \frac{n+1}{n} = \frac{2}{1}+\frac{3}{2}+\frac{4}{3}+\ldots+\frac{n+1}{n}+\ldots$$

diverges because each term is greater than 1,

$$\Rightarrow \frac{2}{1}+\frac{3}{2}+\frac{4}{3}+\ldots+\frac{n+1}{n}+\ldots > \sum_{n=1}^{\infty} 1 \to \infty$$

Note

- In general, when we deal with a series, there are two questions we would like to answer:
- (1) the existence of the limit of the series $s_\infty = \sum_{k=1}^{\infty} a_k$
- (2) In the case where the limit of the series exists, what is the value of this limit?

- The tests that will be discussed in the following only provide the answer to question (1) but not necessarily question (2).

The *n*th-term test for divergence

THEOREM 7

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

- Let S be the convergent limit of the series, i.e. $\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n = S$
- When n is large, s_n and s_{n-1} are close to S
- This means $a_n = s_n - s_{n-1} \rightarrow a_n = S - S = 0$ as $n \rightarrow \infty$

The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

- Question: will the series converge if $a_n \rightarrow 0$?

Example 7 Applying the nth-term test

$(a)\sum_{n=1}^{\infty} n^2$ diverges because $\lim_{n\to\infty} n^2 = \infty$, i.e. $\lim_{n\to\infty} a_n$ fail to exist.

$(b)\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0.$

$(c)\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ fail to exist.

$(d)\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = \frac{-1}{2} \neq 0$ (l'Hopital rule)

Example 8 $a_n \rightarrow 0$ but the series diverges

$$1+\underbrace{\frac{1}{2}+\frac{1}{2}}_{2\text{ terms}}+\underbrace{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}_{4\text{ terms}}+\ldots+\underbrace{\frac{1}{2^n}+\frac{1}{2^n}+\frac{1}{2^n}+\ldots+\frac{1}{2^n}}_{2^n\text{ terms}}+\ldots$$

- The terms are grouped into clusters that add up to 1, so the partial sum increases without bound → the series diverges
- Yet $a_n = 2^{-n} \rightarrow 0$

THEOREM 8

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k\sum a_n = kA$ (Any number k).

- Corollary:
- Every nonzero constant multiple of a divergent series diverges
- If Σa_n converges and Σb_n diverges, then $\Sigma(a_n+b_n)$ and $\Sigma(a_n- b_n)$ both diverges.

- Question:
- If Σa_n and Σb_n both diverges, must $\Sigma(a_n \pm b_n)$ diverge?

EXAMPLE 9 Find the sums of the following series.

(a)

$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} \quad \text{Difference Rule}$$

$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/6)} \quad \text{Geometric series with } a = 1 \text{ and } r = 1/2, 1/6$$

$$= 2 - \frac{6}{5}$$

$$= \frac{4}{5}$$

(b)

$$\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n} \quad \text{Constant Multiple Rule}$$

$$= 4\left(\frac{1}{1-(1/2)}\right) \quad \text{Geometric series with } a = 1, r = 1/2$$

$$= 8$$

11.3

The Integral Test

Nondecreasing partial sums

- Suppose $\{a_n\}$ is a sequence with $a_n > 0$ for all n
- Then, the partial sum $s_{n+1} = s_n + a_n \geqslant s_n$
- $\Rightarrow$ The partial sum form a nondecreasing sequence

$$\{s_n = \sum_{k=1}^{n} a_k\} = \{s_1, s_2, s_2, \ldots, s_n, \ldots\}$$

- Theorem 6, the Nondecreasing Sequence Theorem tells us that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the partial sums are bounded from above.

Corollary of Theorem 6
A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

Example 1 The harmonic series

- The series $\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\frac{2}{4}=\frac{1}{2}}+\underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>\frac{4}{8}=\frac{1}{2}}+\underbrace{\left(\frac{1}{9}+\frac{1}{10}+...+\frac{1}{16}\right)}_{>\frac{8}{16}=\frac{1}{2}}+...$ diverges.
- Consider the sequence of partial sum $\{s_1, s_2, s_4, s_{16}, \cdots, s_{2^k}, \cdots\}$

$s_1 = 1$

$s_2 = s_1 + 1/2 > 1\cdot(1/2)$

$s_4 = s_2 + (1/3+1/4) > 2\cdot(1/2)$

$s_8 = s_4 + (1/5+1/6+1/7+1/8) > 3\cdot(1/2)$

...

$s_{2^k} > k\cdot(1/2)$

- The partial sum of the first 2^k term in the series, $s_n > k/2$, where k=0,1,2,3…
- This means the partial sum, s_n, is not bounded from above.
- Hence, by the virtue of Corollary 6, the harmonic series diverges

THEOREM 9 The Integral Test
Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

EXAMPLE 3 The p-Series

Show that the **p-series**

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

Solution If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x. Since

$$\int_1^{\infty}\frac{1}{x^p}dx = \int_1^{\infty} x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \frac{1}{1-p}\lim_{b\to\infty}\left(\frac{1}{b^{p-1}}-1\right)$$

$$= \frac{1}{1-p}(0-1) = \frac{1}{p-1},$$

$b^{p-1}\to\infty$ as $b\to\infty$ because $p-1>0$.

the series converges by the Integral Test. We emphasize that the sum of the p-series is *not* $1/(p-1)$. The series converges, but we don't know the value it converges to.

If $p < 1$, then $1 - p > 0$ and

$$\int_1^{\infty}\frac{1}{x^p}dx = \frac{1}{1-p}\lim_{b\to\infty}(b^{1-p}-1) = \infty.$$

The series diverges by the Integral Test.

If $p = 1$, we have the (divergent) harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots.$$

We have convergence for $p > 1$ but divergence for every other value of p.

Example 4 A convergent series

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent by the integral test:

Let $f(x) = \frac{1}{x^2+1}$, so that $f(n) = a_n = \frac{1}{n^2+1}$. $f(x)$ is continuos, positive, decreasing for all $x \geq 1$.

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x^2+1} dx = \ldots = \lim_{b\to\infty} \tan^{-1} x \Big]_1^b = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by the integral test.

Caution

- The integral test only tells us whether a given series converges or otherwise
- The test DOES NOT tell us what the convergent limit of the series is (in the case where the series converges), as the series and the integral need not have the same value in the convergent case.

11.4

Comparison Tests

THEOREM 10 The Comparison Test
Let $\sum a_n$ be a series with no negative terms.

(a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N.

(b) $\sum a_n$ diverges if there is a divergent series of nonnegative terms $\sum d_n$ with $a_n \geq d_n$ for all $n > N$, for some integer N.

EXAMPLE 1 Applying the Comparison Test

(a) The series

$$\sum_{n=1}^{\infty} \frac{5}{5n-1}$$

diverges because its nth term

$$\frac{5}{5n-1} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n}$$

is greater than the nth term of the divergent harmonic series.

(b) The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

converges because its terms are all positive and less than or equal to the corresponding terms of

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots.$$

The geometric series on the left converges and we have

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1-(1/2)} = 3.$$

The fact that 3 is an upper bound for the partial sums of $\sum_{n=0}^{\infty}(1/n!)$ does not mean that the series converges to 3. As we will see in Section 11.9, the series converges to e.

Caution

- The comparison test only tell us whether a given series converges or otherwise
- The test DOES NOT tell us what the convergent limit of the series is (in the case where the series converges), as the two series need not have the same value in the convergent case

THEOREM 11 **Limit Comparison Test**

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

EXAMPLE 2 Using the Limit Comparison Test

Which of the following series converge, and which diverge?

(a) $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

(b) $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

Solution

(a) Let $a_n = (2n+1)/(n^2+2n+1)$. For large n, we expect a_n to behave like $2n/n^2 = 2/n$ since the leading terms dominate for large n, so we let $b_n = 1/n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2,$$

$\sum a_n$ diverges by Part 1 of the Limit Comparison Test. We could just as well have taken $b_n = 2/n$, but $1/n$ is simpler.

Example 2 continued

(b) Let $a_n = 1/(2^n - 1)$. For large n, we expect a_n to behave like $1/2^n$, so we let $b_n = 1/2^n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - (1/2^n)} = 1,$$

$\sum a_n$ converges by Part 1 of the Limit Comparison Test.

Caution

- The limit comparison test only tell us whether a given series converges or otherwise
- The test DOES NOT tell us what the convergent limit of the series is (in the case where the series converges)

11.5

The Ratio and Root Tests

THEOREM 12 The Ratio Test

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

EXAMPLE 1 Applying the Ratio Test

Investigate the convergence of the following series.

(a) $\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$ **(b)** $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

Solution

(a) For the series $\sum_{n=0}^{\infty}(2^n + 5)/3^n$,

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1}+5)/3^{n+1}}{(2^n+5)/3^n} = \frac{1}{3}\cdot\frac{2^{n+1}+5}{2^n+5} = \frac{1}{3}\cdot\left(\frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}}\right) \to \frac{1}{3}\cdot\frac{2}{1} = \frac{2}{3}.$$

The series converges because $\rho = 2/3$ is less than 1. This does *not* mean that 2/3 is the sum of the series. In fact,

$$\sum_{n=0}^{\infty}\frac{2^n+5}{3^n} = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty}\frac{5}{3^n} = \frac{1}{1-(2/3)} + \frac{5}{1-(1/3)} = \frac{21}{2}.$$

(b) If $a_n = \frac{(2n)!}{n!n!}$, then $a_{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{n!n!(2n+2)(2n+1)(2n)!}{(n+1)!(n+1)!(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{4n+2}{n+1} \to 4.$$

The series diverges because $\rho = 4$ is greater than 1.

THEOREM 13 **The Root Test**

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

Caution

- The ratio test only tell us whether a given series converges or otherwise
- The test DOES NOT tell us what the convergent limit of the series is (in the case where the series converges)

EXAMPLE 3 Applying the Root Test

Which of the following series converges, and which diverges?

(a) $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ **(b)** $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$ **(c)** $\sum_{n=1}^{\infty}\left(\frac{1}{1+n}\right)^n$

Solution

(a) $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ converges because $\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{\left(\sqrt[n]{n}\right)^2}{2} \to \frac{1}{2} < 1.$

(b) $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$ diverges because $\sqrt[n]{\frac{2^n}{n^2}} = \frac{2}{\left(\sqrt[n]{n}\right)^2} \to \frac{2}{1} > 1.$

11.6

Alternating Series, Absolute and Conditional Convergence

Alternating series

- A series in which the terms are alternately positive and negative

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\cdots+\frac{(-1)^{n+1}}{n}+\cdots$$

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+\frac{(-1)^{n}4}{2^{n}}+\cdots$$

$$1-2+3-4+5-6+\cdots(-1)^{n+1}n+\cdots$$

THEOREM 14 The Alternating Series Test (Leibniz's Theorem)
The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

The alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges because it satisfies the three requirements of Leibniz's theorem.

Reminder

- Tutorial class during revisio week
- Wednesday, 11 am, Bilik Tutorial BT 144 (tentative – announced on moodel)

THEOREM 15 The Alternating Series Estimation Theorem

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 14, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the numerical value of the first unused term. Furthermore, the remainder, $L - s_n$, has the same sign as the first unused term.

EXAMPLE 2 We try Theorem 15 on a series whose sum we know:

$$\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} \;\vdots + \frac{1}{256} - \cdots.$$

The theorem says that if we truncate the series after the eighth term, we throw away a total that is positive and less than 1/256. The sum of the first eight terms is 0.6640625. The sum of the series is

$$\frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}.$$

The difference, $(2/3) - 0.6640625 = 0.0026041666\ldots$, is positive and less than $(1/256) = 0.00390625$.

DEFINITION Absolutely Convergent

A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

Example:

The geometric series

$$\sum_{n=1}^{\infty} 1\left(-\frac{1}{2}\right)^{n-1} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$$ converges absolutely since

the corresponding absolute series

$$\sum_{n=1}^{\infty}\left|1\left(-\frac{1}{2}\right)^{n-1}\right| = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$ converges

DEFINITION Conditionally Convergent

A series that converges but does not converge absolutely **converges conditionally**.

Example:

The alternative harmonic series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$ converges (by virture of Leibniz Theorem)

But the corresponding absolute series

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$ diverges (a harmonic series)

Hence, by definition, the alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges conditionally.

THEOREM 16 The Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

In other words, if a series converges absolutely, it converges.

In the previous example, we shown that the geometric series $\sum_{n=1}^{\infty} 1\left(-\frac{1}{2}\right)^{n-1}$ converges absolutely. Hence, by virtue of the absolute convergent test, the series $\sum_{n=1}^{\infty} 1\left(-\frac{1}{2}\right)^{n-1}$ converges.

Caution

- All series that are absolutely convergent converges.
- But the converse is not true, namely, not all convergent series are absolutely convergent.
- Think of series that is conditionally convergent. These are convergent series that are not absolutely convergent.

EXAMPLE 3 Applying the Absolute Convergence Test

(a) For $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$, the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \text{. } p \text{ series with } p=2$$

The original series converges because it converges absolutely.

(b) For $\sum_{n=1}^{\infty}\frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$, the corresponding series of absolute values is

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \cdots,$$

which converges by comparison with $\sum_{n=1}^{\infty}(1/n^2)$ because $|\sin n| \le 1$ for every n. The original series converges absolutely; therefore it converges.

THEOREM 17 The Rearrangement Theorem for Absolutely Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

1. **The nth-Term Test:** Unless $a_n \to 0$, the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. **p-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge? If yes, so does $\sum a_n$, since absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

11.7

Power Series

DEFINITIONS Power Series, Center, Coefficients

A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \qquad (1)$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \qquad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

EXAMPLE 1 A Geometric Series

Taking all the coefficients to be 1 in Equation (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This is the geometric series with first term 1 and ratio x. It converges to $1/(1-x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1. \qquad (3)$$

Mathematica simulation

EXAMPLE 3 Testing for Convergence Using the Ratio Test

For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(b) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

(c) $\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

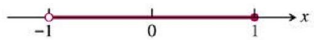
(d) $\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

Note: To test the convergence of an alternating series, check the convergence of the absolute version of the series using ratio test.

Solution Apply the Ratio Test to the series $\sum|u_n|$, where u_n is the nth term of the series in question.

(a) $\left|\frac{u_{n+1}}{u_n}\right| = \frac{n}{n+1}|x| \to |x|.$

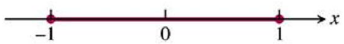
The series converges absolutely for $|x| < 1$. It diverges if $|x| > 1$ because the nth term does not converge to zero. At $x = 1$, we get the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges. At $x = -1$ we get $-1 - 1/2 - 1/3 - 1/4 - \cdots$, the negative of the harmonic series; it diverges. Series (a) converges for $-1 < x \le 1$ and diverges elsewhere.



Continued on next slide

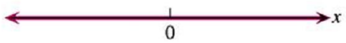
(b) $\left|\frac{u_{n+1}}{u_n}\right| = \frac{2n-1}{2n+1}x^2 \to x^2.$

The series converges absolutely for $x^2 < 1$. It diverges for $x^2 > 1$ because the nth term does not converge to zero. At $x = 1$ the series becomes $1 - 1/3 + 1/5 - 1/7 + \cdots$, which converges by the Alternating Series Theorem. It also converges at $x = -1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x = -1$ is the negative of the value at $x = 1$. Series (b) converges for $-1 \le x \le 1$ and diverges elsewhere.



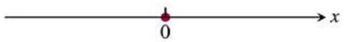
(c) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0$ for every x.

The series converges absolutely for all x.



(d) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \to \infty$ unless $x = 0$.

The series diverges for all values of x except $x = 0$.



The radius of convergence of a power series

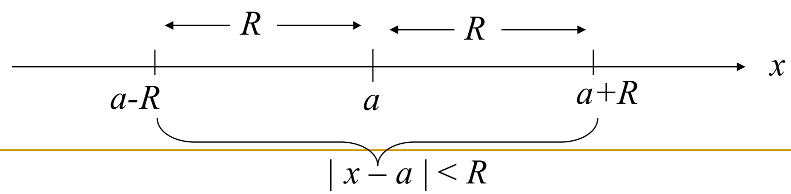
THEOREM 18 **The Convergence Theorem for Power Series**

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges for $x = c \ne 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.

COROLLARY TO THEOREM 18

The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three possibilities:

1. There is a positive number R such that the series diverges for x with $|x - a| > R$ but converges absolutely for x with $|x - a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).



- R is called the radius of convergence of the power series
- The interval of radius R centered at $x = a$ is called the interval of convergence
- The interval of convergence may be open, closed, or half-open: $[a-R, a+R]$, $(a-R, a+R)$, $[a-R, a+R)$ or $(a-R, a+R]$
- A power series converges for all x that lies within the interval of convergence.

THEOREM 19 The Term-by-Term Differentiation Theorem

If $\sum c_n(x-a)^n$ converges for $a - R < x < a + R$ for some $R > 0$, it defines a function f:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \qquad a - R < x < a + R.$$

Such a function f has derivatives of all orders inside the interval of convergence. We can obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at every interior point of the interval of convergence of the original series.

How to Test a Power Series for Convergence

1. *Use the Ratio Test (or nth-Root Test) to find the interval where the series converges absolutely.* Ordinarily, this is an open interval

$$|x-a| < R \qquad \text{or} \qquad a - R < x < a + R.$$

2. *If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint*, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. *If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x-a| > R$* (it does not even converge conditionally), because the nth term does not approach zero for those values of x.

See example 3 (previous slides and determine their interval of convergence)

EXAMPLE 4 Applying Term-by-Term Differentiation

Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

$$= \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1$$

Solution

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1$$

$$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1$$

Caution

- Power series is term-by-term differentiable
- However, in general, not all series is term-by-term differentiable, e.g. the trigonometric series $\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$ is not (it's not a power series)

A power series can be integrated term by term throughout its interval of convergence

THEOREM 20 The Term-by-Term Integration Theorem
Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a - R < x < a + R$ $(R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

EXAMPLE 5 A Series for $\tan^{-1} x$, $-1 \le x \le 1$

Identify the function

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad -1 \le x \le 1.$$

Solution We differentiate the original series term by term and get

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots, \qquad -1 < x < 1.$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

We can now integrate $f'(x) = 1/(1+x^2)$ to get

$$\int f'(x)\,dx = \int \frac{dx}{1+x^2} = \tan^{-1} x + C.$$

The series for $f(x)$ is zero when $x = 0$, so $C = 0$. Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x, \qquad -1 < x < 1. \qquad (7)$$

In Section 11.10, we will see that the series also converges to $\tan^{-1} x$ at $x = \pm 1$.

EXAMPLE 6 A Series for $\ln(1+x)$, $-1 < x \le 1$

The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x \qquad \text{Theorem 20}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x < 1.$$

It can also be shown that the series converges at $x = 1$ to the number ln 2, but that was not guaranteed by the theorem.

THEOREM 21 The Series Multiplication Theorem for Power Series
If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

EXAMPLE 7 Multiply the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}, \qquad \text{for } |x| < 1,$$

by itself to get a power series for $1/(1-x)^2$, for $|x| < 1$.

Solution Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + \cdots + x^n + \cdots = 1/(1-x)$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = 1 + x + x^2 + \cdots + x^n + \cdots = 1/(1-x)$$

and

$$c_n = \underbrace{a_0 b_n + a_1 b_{n-1} + \cdots + a_k b_{n-k} + \cdots + a_n b_0}_{n+1 \text{ terms}}$$

$$= \underbrace{1 + 1 + \cdots + 1}_{n+1 \text{ ones}} = n + 1.$$

Then, by the Series Multiplication Theorem,

$$A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1) x^n$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots$$

is the series for $1/(1-x)^2$. The series all converge absolutely for $|x| < 1$.

Notice that Example 4 gives the same answer because

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

11.8

Taylor and Maclaurin Series

Series Representation

- In the previous topic we see that an infinite series represents a function. The converse is also true, namely:

  A function that is infinitely differentiable *f*(*x*) can be expressed as a power series $\sum_{n=1}^{\infty} b_n (x-a)^n$

- We say: The function *f*(*x*) generates the power series $\sum_{n=1}^{\infty} b_n (x-a)^n$
- The power series generated by the infinitely differentiable function is called Taylor series.
- The Taylor series provide useful polynomial approximations of the generating functions

Finding the Taylor series representation

- In short, given an infinitely differentiable function *f*(*x*), we would like to find out what is the Taylor series representation of *f*(*x*), i.e. what is the coefficients of b_n in $\sum_{n=1}^{\infty} b_n (x-a)^n$
- In addition, we would also need to work out the interval of *x* in which the Taylor series representation of *f*(*x*) converges.
- In generating the Taylor series representation of a generating function, we need to specify the point *x*=*a* at which the Taylor series is to be generated.

DEFINITIONS **Taylor Series, Maclaurin Series**

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

Note: Maclaurin series is effectively a special case of Taylor series with $a = 0$.

Example 1 Finding a Taylor series

- Find the Taylor series generated by *f*(*x*)=1/*x* at *a*= 2. Where, if anywhere, does the series converge to 1/*x*?
- $f(x) = x^{-1}$; $f'(x) = -x^{-2}$; $f^{(n)}(x) = (-1)^n n!\, x^{(n+1)}$
- The Taylor series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k k! x^{-(k+1)}}{k!}\Bigg|_{x=2} (x-2)^k =$$

$$(-1)^0 2^{-1}(x-2)^0 + (-1)^1 2^{-2}(x-2)^1 + (-1)^2 2^{-3}(x-2)^2 + \ldots (-1)^k 2^{-(k+1)}(x-2)^k + \ldots =$$

$$1/2 - (x-2)/4 + (x-2)^2/8 + \ldots (-1)^k (x-2)^k / 2^{(k+1)} + \ldots$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k = 1/2 - (x-2)/4 + (x-2)^2/8 + \ldots (-1)^k (x-2)^k / 2^{(k+1)} + \ldots$$

This is a geometric series with $r = -(x-2)/2$,

Hence, the Taylor series converges for $|r| = |(x-2)/2| < 1$,

or equivalently, $0 < x < 4$.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k = \frac{a}{1-r} = \frac{1/2}{1-(-(x-2)/2)} = \frac{1}{x}$$

$\Rightarrow$ the Taylor series $1/2 - (x-2)/4 + (x-2)^2/8 + \ldots (-1)^k (x-2)^k / 2^{(k+1)} + \ldots$

converges to $\frac{1}{x}$ for $0 < x < 4$.

**Mathematica simulation*

Taylor polynomials

- Given an infinitely differentiable function f, we can approximate $f(x)$ at values of x near a by the Taylor polynomial of f, i.e. $f(x)$ can be approximated by $f(x) \approx P_n(x)$, where

$$P_n(x) = \sum_{k=0}^{k=n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$
$$= \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- $P_n(x)$ = Taylor polynomial of degree n of f generated at $x=a$.
- $P_n(x)$ is simply the first n terms in the Taylor series of f.
- The remainder, $|R_n(x)| = | f(x) - P_n(x)|$ becomes smaller if higher order approximation is used
- In other words, the higher the order n, the better is the approximation of $f(x)$ by $P_n(x)$
- In addition, the Taylor polynomial gives a close fit to f near the point $x = a$, but the error in the approximation can be large at points that are far away.

Example 2 Finding Taylor polynomial for e^x at $x = 0$

$$f(x) = e^x \rightarrow f^{(n)}(x) = e^x$$

$$P_n(x) = \sum_{k=0}^{k=n} \frac{f^{(k)}(x)}{k!}\Bigg|_{x=0} x^k = \frac{e^0}{0!}x^0 + \frac{e^0}{1!}x^1 + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \ldots \frac{e^0}{n!}x^n$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!}$$ This is the Taylor polynomial of order n for e^x

If the limit $n \rightarrow \infty$ is taken, $P_n(x) \rightarrow$ Taylor series.

The Taylor series for e^x is $1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$,

In this special case, the Taylor series for e^x converges to e^x for all x.
(To be proven later)

DEFINITION **Taylor Polynomial of Order *n***

Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order *n*** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

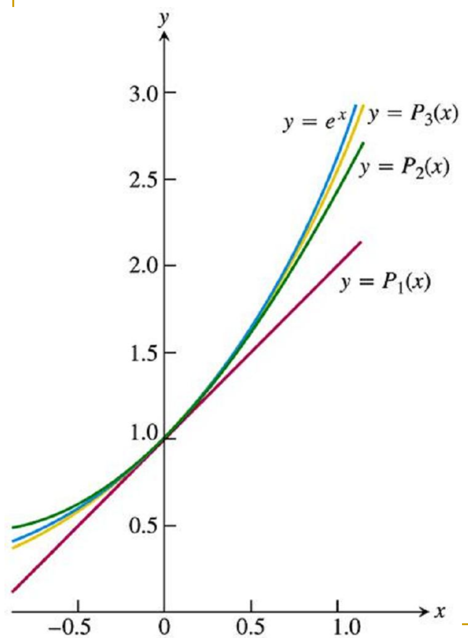


FIGURE 11.12 The graph of $f(x) = e^x$ and its Taylor polynomials
$P_1(x) = 1 + x$
$P_2(x) = 1 + x + (x^2/2!)$
$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!)$.
Notice the very close agreement near the center $x = 0$ (Example 2).

*Mathematica simulation

EXAMPLE 3 Finding Taylor Polynomials for cos x

Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

Solution The cosine and its derivatives are

$$\begin{aligned} f(x) &= \cos x, & f'(x) &= -\sin x, \\ f''(x) &= -\cos x, & f^{(3)}(x) &= \sin x, \\ &\vdots & &\vdots \\ f^{(2n)}(x) &= (-1)^n \cos x, & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x. \end{aligned}$$

At $x = 0$, the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0.$$

The Taylor series generated by f at 0 is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ = 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}. \end{aligned}$$

This is also the Maclaurin series for cos x. In Section 11.9, we will see that the series converges to cos x at every x.

Because $f^{(2n+1)}(0) = 0$, the Taylor polynomials of orders $2n$ and $2n + 1$ are identical:

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Figure 11.13 shows how well these polynomials approximate $f(x) = \cos x$ near $x = 0$. Only the right-hand portions of the graphs are given because the graphs are symmetric about the y-axis.

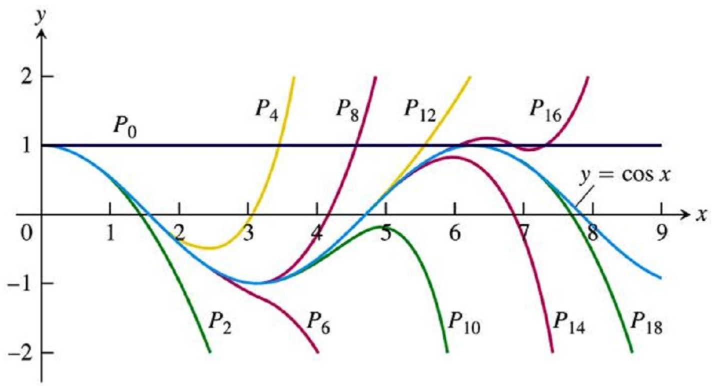


FIGURE 11.13 The polynomials

$$P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$$

converge to cos x as $n \to \infty$. We can deduce the behavior of cos x arbitrarily far away solely from knowing the values of the cosine and its derivatives at $x = 0$ (Example 3).

*Mathematica simulation

11.9

Convergence of Taylor Series; Error Estimates

- When does a Taylor series converge to its generating function?
- ANS:
  The Taylor series converge to its generating function if the |remainder| =
  $|R_n(x)| = |f(x)-P_n(x)| \rightarrow 0$ as $n \rightarrow \infty$

Taylor's Formula

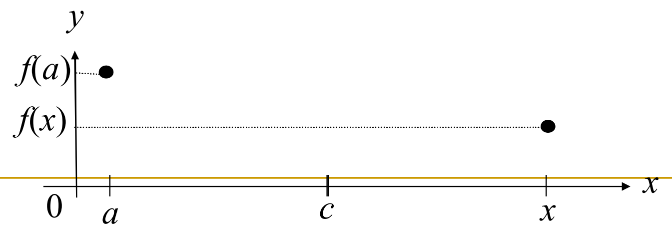
If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \quad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \quad (2)$$

$R_n(x)$ is called the remainder of order n



$f(x) = P_n(x) + R_n(x)$ for each x in I.

If $R_n(x) \to 0$ as $n \to \infty$, $P_n(x)$ converges to $f(x)$, then we can write

$$f(x) = \lim_{n\to\infty} P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Example 1 The Taylor series for e^x revisited

- Show that the Taylor series generated by $f(x)=e^x$ at $x=0$ converges to $f(x)$ for every value of x.
- Note: This can be proven by showing that $|R_n| \to 0$ when $n \to \infty$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!} + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \text{ for some } c \text{ between 0 and } x$$

$$|R_n(x)| = \left|\frac{e^c}{(n+1)!}x^{n+1}\right|.$$

If $x > 0, 0 < c < x$

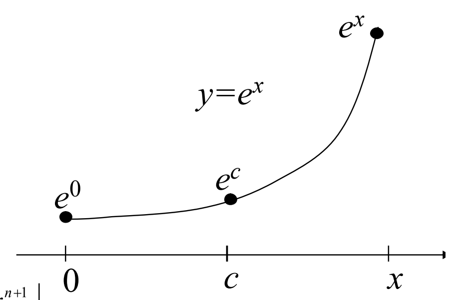
$$\Rightarrow 1 = e^0 < e^c < e^x \to \left|\frac{x^{n+1}}{(n+1)!}\right| < \left|\frac{e^c}{(n+1)!}x^{n+1}\right| < \left|\frac{e^x x^{n+1}}{(n+1)!}\right|$$

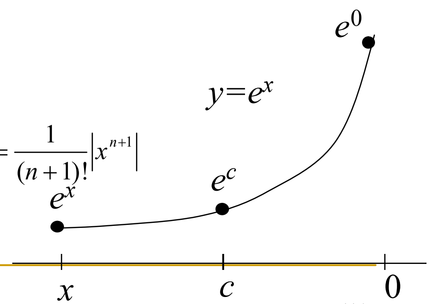
$$\to |R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \text{ for } x > 0.$$

If $x < 0, x < c < 0$

$$\Rightarrow e^x < e^c < e^0 \to \left|\frac{e^c}{(n+1)!}x^{n+1}\right| < \left|\frac{e^0 x^{n+1}}{(n+1)!}\right| = \left|\frac{x^{n+1}}{(n+1)!}\right| = \frac{1}{(n+1)!}\left|x^{n+1}\right|$$

$$\to |R_n(x)| < \frac{\left|x^{n+1}\right|}{(n+1)!} \text{ for } x < 0$$





Combining the result of both $x > 0$ and $x < 0$,

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \text{ when } x > 0,$$

$$|R_n(x)| < \frac{|x^{n+1}|}{(n+1)!} \text{ when } x < 0$$

Hence, irrespective of the sign of x, $\lim_{n\to\infty} |R_n(x)| = 0$ and the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge to e^x for every x.

THEOREM 5
The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

THEOREM 1
Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

THEOREM 23 The Remainder Estimation Theorem

If there is a positive constant M such that $|f^{(n+1)}(t)| \le M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!}.$$

If this condition holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

EXAMPLE 3 The Taylor Series for $\cos x$ at $x = 0$ Revisited

Show that the Taylor series for $\cos x$ at $x = 0$ converges to $\cos x$ for every value of x.

Solution We add the remainder term to the Taylor polynomial for $\cos x$ (Section 11.8, Example 3) to obtain Taylor's formula for $\cos x$ with $n = 2k$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + R_{2k}(x).$$

Because the derivatives of the cosine have absolute value less than or equal to 1, the Remainder Estimation Theorem with $M = 1$ gives

$$|R_{2k}(x)| \le 1 \cdot \frac{|x|^{2k+1}}{(2k+1)!}.$$

For every value of x, $R_{2k} \to 0$ as $k \to \infty$. Therefore, the series converges to $\cos x$ for every value of x. Thus,

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \quad (5)$$

EXAMPLE 6 Calculate e with an error of less than 10^{-6}.

Solution We can use the result of Example 1 with $x = 1$ to write

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1),$$

with

$$R_n(1) = e^c \frac{1}{(n+1)!} \quad \text{for some } c \text{ between 0 and 1.}$$

For the purposes of this example, we assume that we know that $e < 3$. Hence, we are certain that

$$\frac{1}{(n+1)!} < R_n(1) < \frac{3}{(n+1)!}$$

because $1 < e^c < 3$ for $0 < c < 1$.

By experiment we find that $1/9! > 10^{-6}$, while $3/10! < 10^{-6}$. Thus we should take $(n+1)$ to be at least 10, or n to be at least 9. With an error of less than 10^{-6},

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{9!} \approx 2.718282.$$

EXAMPLE 7 For what values of x can we replace $\sin x$ by $x - (x^3/3!)$ with an error of magnitude no greater than 3×10^{-4}?

Solution Here we can take advantage of the fact that the Taylor series for $\sin x$ is an alternating series for every nonzero value of x. According to the Alternating Series Estimation Theorem (Section 11.6), the error in truncating

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

after $(x^3/3!)$ is no greater than

$$\left|\frac{x^5}{5!}\right| = \frac{|x|^5}{120}.$$

Therefore the error will be less than or equal to 3×10^{-4} if

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \quad \text{or} \quad |x| < \sqrt[5]{360 \times 10^{-4}} \approx 0.514.$$

Rounded down, to be safe

The Alternating Series Estimation Theorem tells us something that the Remainder Estimation Theorem does not: namely, that the estimate $x - (x^3/3!)$ for $\sin x$ is an underestimate when x is positive because then $x^5/120$ is positive.

Figure 11.15 shows the graph of $\sin x$, along with the graphs of a number of its approximating Taylor polynomials. The graph of $P_3(x) = x - (x^3/3!)$ is almost indistinguishable from the sine curve when $-1 \le x \le 1$.

11.10

Applications of Power Series

The binomial series for powers and roots

- Consider the Taylor series generated by $f(x) = (1+x)^m$, where m is a constant:

$$f(x) = (1+x)^m$$
$$f'(x) = m(1+x)^{m-1}, f''(x) = m(m-1)(1+x)^{m-2},$$
$$f'''(x) = m(m-1)(m-2)(1+x)^{m-3},$$
$$\vdots$$
$$f^{(k)}(x) = m(m-1)(m-2)\ldots(m-k+1)(1+x)^{m-k};$$
$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\infty}\frac{m(m-1)(m-2)\ldots(m-k+1)}{k!}x^k$$
$$= 1 + mx + m(m-1)x^2 + m(m-1)(m-2)x^3 + \ldots + \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!}x^k + \ldots$$

The binomial series for powers and roots

$$f(x) = (1+x)^m$$
$$= 1 + mx + m(m-1)x^2 + m(m-1)(m-2)x^3 + \ldots + \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!}x^k + \ldots$$

- This series is called the binomial series, converges absolutely for $|x| < 1$. (The convergence can be determined by using Ratio test, $\left|\frac{u_{k+1}}{u_k}\right| = \left|\frac{m-k}{k+1}x\right| \to |x|$

In short, the binomial series is the Taylor series for $f(x) = (1+x)^m$, where m a constant

The Binomial Series

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k \geq 3.$$

Taylor series representation of ln x at $x = 1$

- $f(x) = \ln x$; $f'(x) = x^{-1}$;
- $f''(x) = (-1)(1)x^{-2}$; $f'''(x) = (-1)^2(2)(1)x^{-3}$ …
- $f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$;

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}\bigg|_{x=1} (x-1)^n = \frac{f^{(0)}(x)}{0!}\bigg|_{x=1} (x-1)^0 + \sum_{n=1}^{\infty} \frac{f^{(n)}(x)}{n!}\bigg|_{x=1} (x-1)^n$$

$$= \frac{\ln 1}{0!} + \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}(n-1)!x^{-n}}{n!}\bigg|_{x=1} (x-1)^n = 0 + \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}(1)^{-n}}{n}(x-1)^n$$

$$= \frac{(-1)^0}{1}(x-1)^1 + \frac{(-1)^1}{2}(x-1)^2 + \frac{(-1)^2}{3}(x-1)^3 + \ldots$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots + (-1)^n \frac{1}{n}(x-1)^n + \ldots$$

*Mathematica simulation

EXAMPLE 2 Using the Binomial Series

We know from Section 3.8, Example 1, that $\sqrt{1+x} \approx 1 + (x/2)$ for $|x|$ small. With $m = 1/2$, the binomial series gives quadratic and higher-order approximations as well, along with error estimates that come from the Alternating Series Estimation Theorem:

$$(1+x)^{1/2} = 1 + \frac{x}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots.$$

Substitution for x gives still other approximations. For example,

$$\sqrt{1-x^2} \approx 1 - \frac{x^2}{2} - \frac{x^4}{8} \qquad \text{for } |x^2| \text{ small}$$

$$\sqrt{1-\frac{1}{x}} \approx 1 - \frac{1}{2x} - \frac{1}{8x^2} \qquad \text{for } \left|\frac{1}{x}\right| \text{ small, that is, } |x| \text{ large.}$$

EXAMPLE 7 Limits Using Power Series

Evaluate

$$\lim_{x \to 1} \frac{\ln x}{x-1}.$$

Solution We represent ln x as a Taylor series in powers of $x - 1$. This can be accomplished by calculating the Taylor series generated by ln x at $x = 1$ directly or by replacing x by $x - 1$ in the series for ln $(1 + x)$ in Section 11.7, Example 6. Either way, we obtain

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \cdots,$$

from which we find that

$$\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1}\left(1 - \frac{1}{2}(x-1) + \cdots\right) = 1.$$

EXAMPLE 8 Limits Using Power Series

Evaluate

$$\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}.$$

Solution The Taylor series for $\sin x$ and $\tan x$, to terms in x^5, are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$$

Hence,

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \cdots = x^3\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

and

$$\lim_{x\to 0} \frac{\sin x - \tan x}{x^3} = \lim_{x\to 0}\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

$$= -\frac{1}{2}.$$

TABLE 11.1 Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \le 1$$

$$\ln\frac{1+x}{1-x} = 2\tanh^{-1} x = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}, \quad |x| < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \le 1$$

Binomial Series

$$(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \cdots + \frac{m(m-1)(m-2)\cdots(m-k+1)x^k}{k!} + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k, \quad |x| < 1,$$

where

$$\binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2!}, \quad \binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!} \quad \text{for } k \ge 3.$$

Note: To write the binomial series compactly, it is customary to define $\binom{m}{0}$ to be 1 and to take $x^0 = 1$ (even in the usually excluded case where $x = 0$), yielding $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$. If m is a *positive integer*, the series terminates at x^m and the result converges for all x.

11.11

Fourier Series

‘Weakness’ of power series approximation

- In the previous lesson, we have learnt to approximate a given function using power series approximation, which give good fit if the approximated power series representation is evaluated near the point it is generated
- For point far away from the point the power series being generated, the approximation becomes poor
- In addition, the series approximation works only within the interval of convergence. Outside the interval of convergence, the series representation fails to represent the generating function
- Furthermore, power series approximation can not represent satisfactorily a function that has a jump discontinuity.
- Fourier series, our next topic, provide an alternative to overcome such shortage

Suppose we wish to approximate a function f on the interval $[0, 2\pi]$ by a sum of sine and cosine functions,

$$f_n(x) = a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots + (a_n \cos nx + b_n \sin nx)$$

or, in sigma notation,

$$f_n(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx). \qquad (1)$$

We would like to choose values for the constants $a_0, a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots, b_n$ that make $f_n(x)$ a "best possible" approximation to $f(x)$. The notion of "best possible" is defined as follows:

1. $f_n(x)$ and $f(x)$ give the same value when integrated from 0 to 2π.
2. $f_n(x) \cos kx$ and $f(x) \cos kx$ give the same value when integrated from 0 to 2π $(k = 1, \ldots, n)$.
3. $f_n(x) \sin kx$ and $f(x) \sin kx$ give the same value when integrated from 0 to 2π $(k = 1, \ldots, n)$.

We chose f_n so that the integrals on the left remain the same when f_n is replaced by f, so we can use these equations to find $a_0, a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots, b_n$ from f:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \qquad (2)$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx, \qquad k = 1, \ldots, n \qquad (3)$$

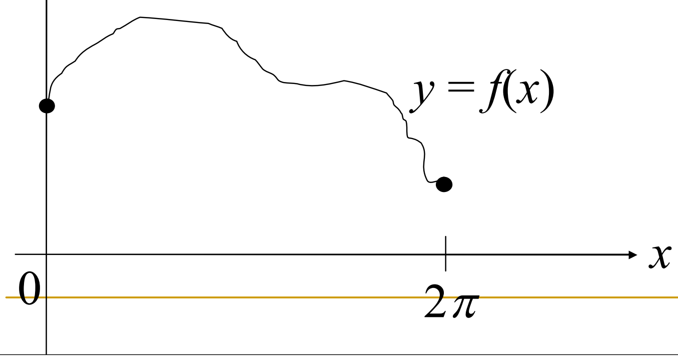
$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx, \qquad k = 1, \ldots, n \qquad (4)$$

A function $f(x)$ defined on $[0, 2\pi]$ can be represented by a Fourier series

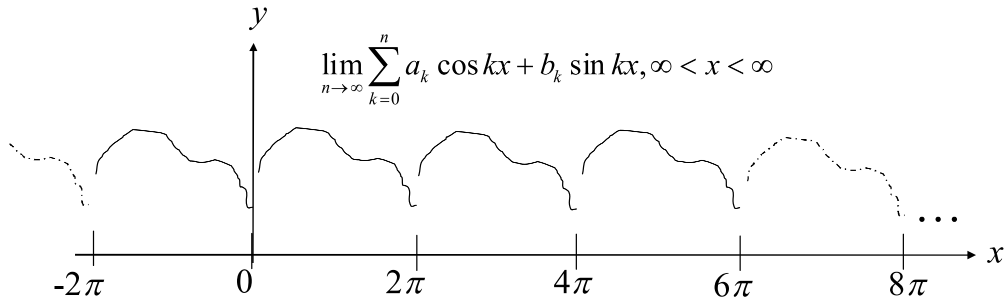
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\sum_{k=0}^{n} f_k(x) = \lim_{n\to\infty}\sum_{k=0}^{n} a_k \cos kx + b_k \sin kx$$

$$= a_0 + \lim_{n\to\infty}\sum_{k=1}^{n} a_k \cos kx + b_k \sin kx,$$ ← Fourier series representation of $f(x)$

$0 \le x \le 2\pi$.



$$\lim_{n\to\infty}\sum_{k=0}^{n} a_k \cos kx + b_k \sin kx, \infty < x < \infty$$



If $-\infty < x < \infty$, the Fourier series $\lim_{n\to\infty}\sum_{k=0}^{n} f_k(x) = \lim_{n\to\infty}\sum_{k=0}^{n} a_k \cos kx + b_k \sin kx$ acutally represents a periodic function $f(x)$ of a period of $L = 2\pi$,

Orthogonality of sinusoidal functions

m,k nonzero integer.

If $m=k$,

$$\int_0^{2\pi}\cos mx\cos kx dx=\int_0^{2\pi}\cos mx\cos mx dx=\frac{1}{2}\int_0^{2\pi}1+\cos(2mx)dx=\frac{1}{2}\left[x+\frac{\sin 2mx}{2m}\right]_0^{2\pi}=\pi.$$

$$\int_0^{2\pi}\sin mx\sin kx dx=\int_0^{2\pi}\sin^2 mx dx=\pi$$

If $m\neq k$,

$\int_0^{2\pi}\cos mx\cos kx dx=0, \int_0^{2\pi}\sin mx\sin kx dx=0$.(can be proven using, say, integration by parts or formula for the product of two sinusoidal functions).

In addtion, $\int_0^{2\pi}\sin mx dx=\int_0^{2\pi}\cos mx dx=0$.

Also, $\int_0^{2\pi}\sin mx\cos kx dx=0$ for all m,k. We say sin and cos functions are orthogonal to each other.

Derivation of a_k, $k \geqslant 1$

$$f_n(x)=a_0+\sum_{k=1}^{n}a_k\cos kx+b_k\sin kx$$

Multiply both sides by $\cos mx$ (m nonzero integer), and integrate with respect to x from $x=0$ to $x=2\pi$. By doing so, the integral $\int_0^{2\pi}\cos mx\sin kx dx$ get 'killed off' due to the orthogality property of the sinusoidal functions.

In addtion, $\int_0^{2\pi}\cos mx\cos kx dx$ will also gets 'killed off' except for the case $m=k$.

$$\int_0^{2\pi}f(x)\cos mx dx=$$

$$\int_0^{2\pi}a_0\cos mx dx+\sum_{k=1}^{n}a_k\int_0^{2\pi}\cos kx\cos mx dx+\sum_{k=1}^{n}b_k\int_0^{2\pi}\sin kx\cos mx dx$$

$$=0+a_m\int_0^{2\pi}\cos mx\cos mx dx+0=\pi a_m$$

$$\Rightarrow a_m=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos mx\, dx.$$

Derivation of a_0

$$f_n(x)=a_0+\sum_{k=1}^{n}a_k\cos kx+b_k\sin kx$$

Integrate both sides with respect to x from $x=0$ to $x=2\pi$

$$\int_0^{2\pi}f_n(x)dx=\int_0^{2\pi}a_0 dx+\sum_{k=1}^{n}\int_0^{2\pi}a_k\cos kx dx+b_k\sin kx dx=$$

$$\int_0^{2\pi}a_0 dx+\sum_{k=1}^{n}a_k\int_0^{2\pi}\cos kx dx+\sum_{k=1}^{n}b_k\int_0^{2\pi}\sin kx dx$$

$$=2\pi a_0+0+0=2\pi a_0$$

$$\Rightarrow 2\pi a_0=\int_0^{2\pi}f_n(x)dx.$$

For large enough n, f_n gives a good representation of f, hence we can replace f_n by f:

$$\Rightarrow a_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)dx$$

Derivation of b_k, $k \geqslant 1$

b_k is simularly derived by multiplying both sides by $\sin mx$ (m nonzero integer), and integrate with respect to x from $x=0$ to $x=2\pi$.

$$\int_0^{2\pi}f(x)\sin mx dx=$$

$$\int_0^{2\pi}a_0\sin mx dx+\sum_{k=1}^{n}a_k\int_0^{2\pi}\cos kx\sin mx dx+\sum_{k=1}^{n}b_k\int_0^{2\pi}\sin kx\sin mx dx$$

$$=0+0+b_m\int_0^{2\pi}\sin mx\sin mx dx=\pi b_m$$

$$\Rightarrow b_m=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin mx\, dx.$$

- Fourier series can represent some functions that cannot be represented by Taylor series, e.g. step function such as

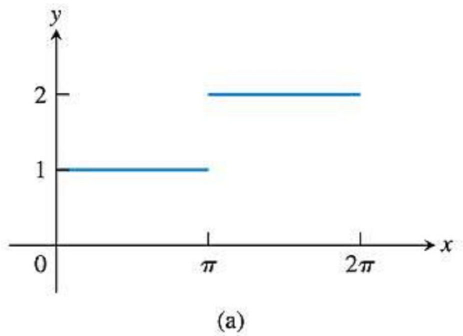


FIGURE 11.16 (a) The step function

$$f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 2, & \pi < x \le 2\pi \end{cases}$$

EXAMPLE 1 Finding a Fourier Series Expansion

Fourier series can be used to represent some functions that cannot be represented by Taylor series; for example, the step function f shown in Figure 11.16a.

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x \le \pi \\ 2, & \text{if } \pi < x \le 2\pi. \end{cases}$$

The coefficients of the Fourier series of f are computed using Equations (2), (3), and (4).

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$
$$= \frac{1}{2\pi}\left(\int_0^{\pi} 1\,dx + \int_{\pi}^{2\pi} 2\,dx\right) = \frac{3}{2}$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx$$
$$= \frac{1}{\pi}\left(\int_0^{\pi}\cos kx\,dx + \int_{\pi}^{2\pi} 2\cos kx\,dx\right)$$
$$= \frac{1}{\pi}\left(\left[\frac{\sin kx}{k}\right]_0^{\pi} + \left[\frac{2\sin kx}{k}\right]_{\pi}^{2\pi}\right) = 0, \quad k \ge 1$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx$$
$$= \frac{1}{\pi}\left(\int_0^{\pi}\sin kx\,dx + \int_{\pi}^{2\pi} 2\sin kx\,dx\right)$$
$$= \frac{1}{\pi}\left(\left[-\frac{\cos kx}{k}\right]_0^{\pi} + \left[-\frac{2\cos kx}{k}\right]_{\pi}^{2\pi}\right)$$
$$= \frac{\cos k\pi - 1}{k\pi} = \frac{(-1)^k - 1}{k\pi}.$$

So

$$a_0 = \frac{3}{2}, \quad a_1 = a_2 = \cdots = 0,$$

and

$$b_1 = -\frac{2}{\pi}, \quad b_2 = 0, \quad b_3 = -\frac{2}{3\pi}, \quad b_4 = 0, \quad b_5 = -\frac{2}{5\pi}, \quad b_6 = 0, \ldots$$

The Fourier series is

$$\frac{3}{2} - \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

Notice that at $x = \pi$, where the function $f(x)$ jumps from 1 to 2, all the sine terms vanish, leaving 3/2 as the value of the series. This is not the value of f at π, since $f(\pi) = 1$. The Fourier series also sums to 3/2 at $x = 0$ and $x = 2\pi$. In fact, all terms in the Fourier series are periodic, of period 2π, and the value of the series at $x + 2\pi$ is the same as its value at x. The series we obtained represents the periodic function graphed in Figure 11.16b, with domain the entire real line and a pattern that repeats over every interval of width 2π. The function jumps discontinuously at $x = n\pi, n = 0, \pm 1, \pm 2, \ldots$ and at these points has value 3/2, the average value of the one-sided limits from each side. The convergence of the Fourier series of f is indicated in Figure 11.17.

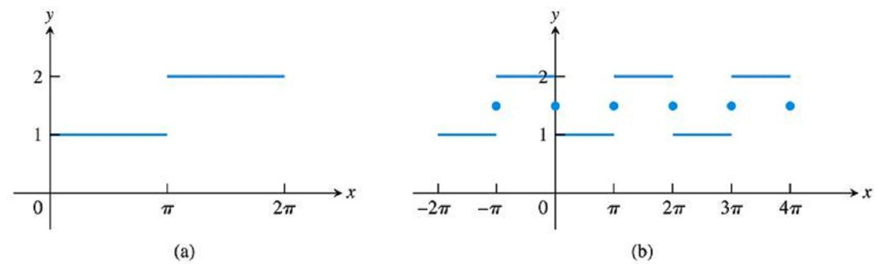


FIGURE 11.16 (a) The step function

$$f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 2, & \pi < x \le 2\pi \end{cases}$$

(b) The graph of the Fourier series for f is periodic and has the value 3/2 at each point of discontinuity (Example 1).

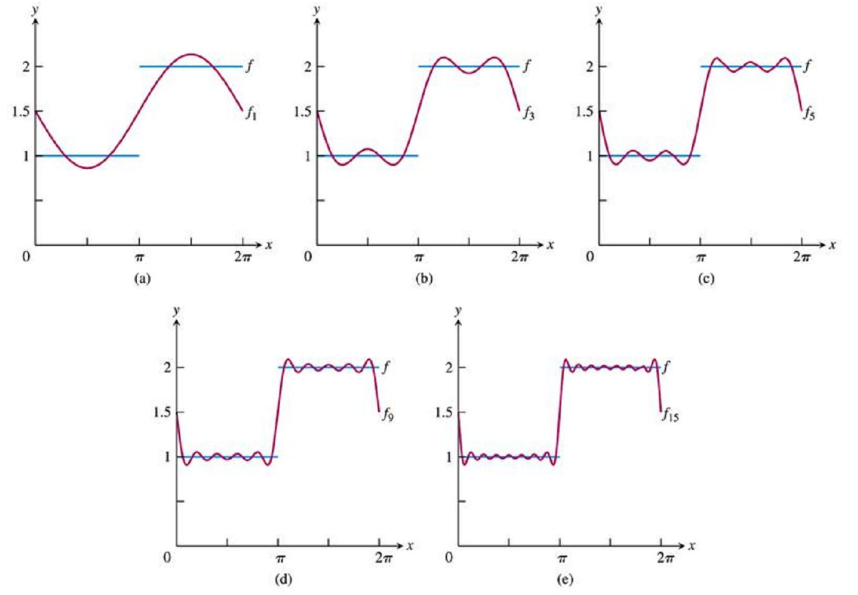


FIGURE 11.17 The Fourier approximation functions f_1, f_3, f_5, f_9, and f_{15} of the function $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 2, & \pi < x \le 2\pi \end{cases}$ in Example 1.

THEOREM 24 Let $f(x)$ be a function such that f and f' are piecewise continuous on the interval $[0, 2\pi]$. Then f is equal to its Fourier series at all points where f is continuous. At a point c where f has a discontinuity, the Fourier series converges to

$$\frac{f(c^+) + f(c^-)}{2}$$

where $f(c^+)$ and $f(c^-)$ are the right- and left-hand limits of f at c.

Fourier series representation of a function defined on the general interval $[a,b]$

- For a function defined on the interval [0,2π], the Fourier series representation of *f*(*x*) is defined as $f(x) = a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$
- How about a function defined on an general interval of [*a*,*b*] where the period is *L*=*b*-*a* instead of 2π? Can we still use $a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$ to represent *f*(*x*) on [*a*,*b*]?

Fourier series representation of a function defined on the general interval $[a,b]$

- For a function defined on the interval of [*a*,*b*] the Fourier series representation on [*a*,*b*] is actually

$$a_0 + \sum_{k=1}^{n} a_k \cos\frac{2\pi kx}{L} + b_k \sin\frac{2\pi kx}{L} x$$

$$a_0 = \frac{1}{L}\int_a^b f(x)dx$$

$$a_m = \frac{2}{L}\int_a^b f(x)\cos\frac{2\pi mx}{L}dx$$

$$b_m = \frac{2}{L}\int_a^b f(x)\sin\frac{2\pi mx}{L}dx, m \text{ positive integer}$$

- *L*=*b* - *a*

Derivation of a_0

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cos\frac{2\pi kx}{L} + b_k \sin\frac{2\pi kx}{L} x$$

$$\int_a^b f(x)dx = \int_a^b a_0 dx + \sum_{k=1}^{n} \int_a^b a_k \cos\frac{2\pi kx}{L} dx + b_k \sin\frac{2\pi kx}{L} dx =$$

$$\int_a^b a_0 dx + \sum_{k=1}^{n} a_k \int_a^b \cos\frac{2\pi kx}{L} dx + \sum_{k=1}^{n} b_k \int_a^b \sin\frac{2\pi kx}{L} dx$$

$$= a_0 (b-a)$$

$$\Rightarrow a_0 = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{L}\int_a^b f(x)dx$$

Derivation of a_k

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cos\frac{2\pi kx}{L} + b_k \sin\frac{2\pi kx}{L} x$$

$$\int_a^b f(x)\cos\frac{2\pi mx}{L} dx$$

$$= \int_a^b a_0 \cos\frac{2\pi mx}{L} dx + \sum_{k=1}^{n} \int_a^b \left(a_k \cos\frac{2\pi kx}{L} \cos\frac{2\pi mx}{L} dx + b_k \sin\frac{2\pi kx}{L} \cos\frac{2\pi mx}{L} \right) dx$$

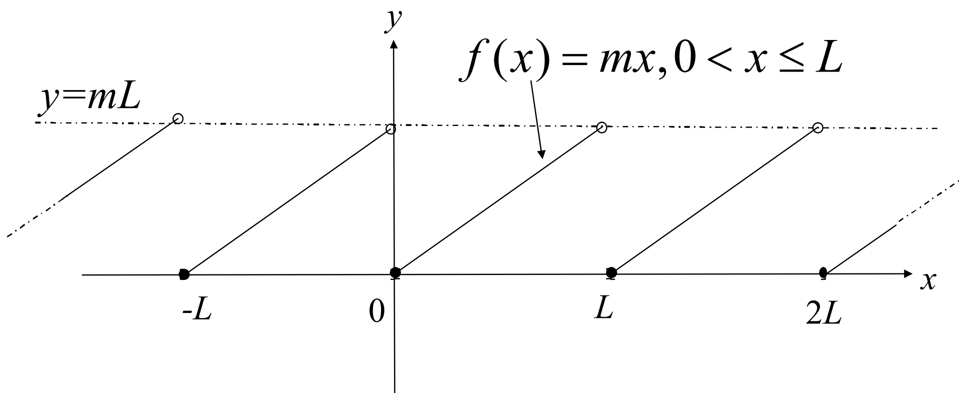
$$= 0 + a_m \int_a^b \cos^2\frac{2\pi mx}{L} dx + 0 = a_m \frac{L}{2}$$

$$\Rightarrow a_m = \frac{2}{L}\int_a^b f(x)\cos\frac{2\pi mx}{L} dx$$

Similarly,

$$b_m = \frac{2}{L}\int_a^b f(x)\sin\frac{2\pi mx}{L} dx$$

Example:



$a=0,\ b=L$

$$a_0 = \frac{1}{L}\int_a^b f(x)dx = \frac{1}{L}\int_a^b mxdx = \frac{m}{2L}\left(b^2 - a^2\right) = \frac{mL}{2}$$

$$a_k = \frac{2}{L}\int_a^b mx\cos\frac{2\pi kx}{L} dx = \frac{2m}{L}\int_a^b x\cos\frac{2\pi kx}{L} dx = \frac{2m}{L}\frac{L^2\left(\cos 2k\pi - 1\right)}{4k^2\pi^2} = 0;$$

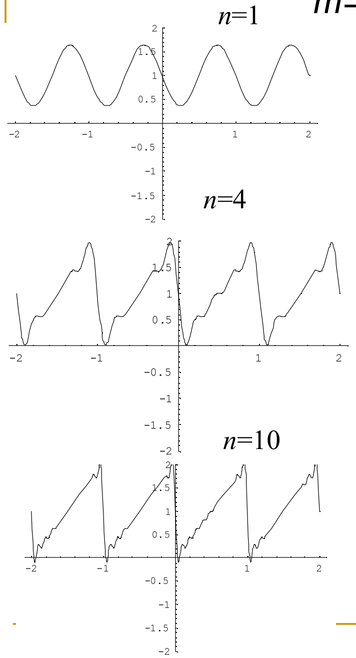
$$b_k = \frac{2}{L}\int_a^b f(x)\sin\frac{2\pi kx}{L} dx = \frac{2m}{L}\int_0^L x\sin\frac{2\pi kx}{L} dx$$

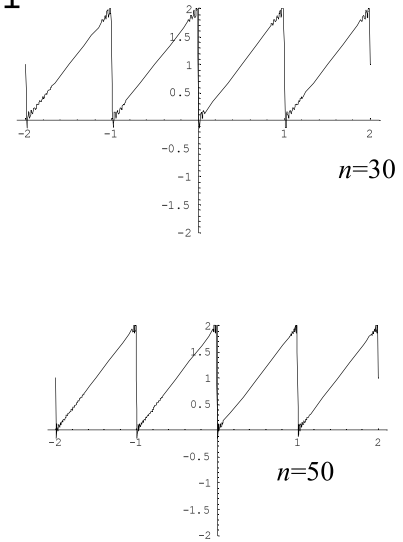
$$= \frac{2m}{L}\cdot L^2\left(\frac{-2k\pi\cos(2k\pi) + \sin 2k\pi}{4k^2\pi^2}\right) = \frac{-mL}{k\pi};$$

$$f(x) = mx = \frac{mL}{2} - \frac{mL}{\pi}\sum_{k=1}^{n}\frac{\sin 2\pi kx}{k}$$

$$= mL\left(\frac{1}{2} - \frac{\sin 2\pi x}{\pi} - \frac{\sin 4\pi x}{2\pi} - \frac{\sin 6\pi x}{3\pi} - \ldots - \frac{\sin 2n\pi x}{n\pi} + \ldots\right)$$

m=2, L = 1





- *mathematica simulation

Tutorial 1

Q1 (a) The matrix A is under a sequence of elementary row operation. Obtain the resultant matrices after each operation as indicated below.

$$A=\begin{pmatrix}1 & 2 & -3 & 0\\ 2 & 4 & -2 & 2\\ 3 & 6 & -4 & 3\end{pmatrix}\xrightarrow{R_3^1(-3)}\begin{pmatrix}1 & 2 & -3 & 0\\ 2 & 4 & -2 & 2\\ 0 & 0 & 5 & 3\end{pmatrix}$$

$$\xrightarrow{R_2^1(-2)}\begin{pmatrix} & & & \\ & & & \\ & & & \end{pmatrix}\xrightarrow{R_3^2(-5/4)}\begin{pmatrix} & & & \\ & & & \\ & & & \end{pmatrix}\xrightarrow{R_1^2(3/4)}\begin{pmatrix} & & & \\ & & & \\ & & & \end{pmatrix}$$

(b) Perform further operation to obtain the row reduced echelon form for matrix A, $\tilde{A}$

(c) What is the ratio, $|A|/|\tilde{A}|$?

Q2 Consider the 3×3 matrix A as given below.

$$A=\begin{pmatrix}1 & 2 & -3\\ 2 & 4 & -2\\ 3 & 6 & -4\end{pmatrix}$$

(a) What is the adjoint of matrix A? Show your working.
(b) What is the determinant of matrix A? Show your working.
(c) What is the inverse of A?

Q3 Given the set of three 3-vectors: $K_1 = (1,\sqrt{2}, \sqrt{2})^T$, $K_2 = (1,\sqrt{2}, 1)^T$, $K_3 = (\sqrt{2}, 1,\sqrt{2})^T$. Determine if they are linearly independent. You must show your working and argument clearly.

Q4 (a) Propose an example of a basis set for $V_3(R)$ other than the elementary vectors. Your answer must be stated in the form of column vector.
(b) Given a basis set X = $\{X_1= (1,\sqrt{2})^T, X_2= (\sqrt{2},1)^T\}$ in $V_2(R)$. Find the coordinate vector of the vector $v = (4,3)^T$ relative to the basis X.

1. Given the 3-square matrix $A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{pmatrix}$.

(i) Use a sequence of elementary row operation to transform A into its equivalent matrix in reduced row echelon form (RREF). Show your steps clearly.

[4 marks]

(ii) What is the rank of A?

[2 marks]

(iii) Given as set of three 3-vectors, $v_1=[1,2,3]$, $v_2=[4,5,6]$, $v_3=[6,7,8]$. Based on your answer from (*i*), (*ii*), are these vectors linearly dependent? Explain your answer.

[2 marks]

(iv) Find the determinant of A.

[3 marks]

(v) Does the inverse of A exist? Explain your answer.

[2 marks]

Consider the homogeneous equation system $AX=0$, where $X=[x_1, x_2, x_3]^T$ is the column vectors of 3 unknowns.

(vi) Has the homogeneous equation system any non-trivial solutions? Explain your answer.

[2 marks]

(vii) How many linearly independent solution are there for $AX=0$?

[2 marks]

(viii) Obtain the solution for $AX=0$.

[3 marks]

<u>Tutorial Questions [ZCA 110(C)]</u>

Thomas' Calculus (11 edition)

Chapter 1 Practice Exercises
(pg. 69 – pg. 72)

Functions and graphs
29. 31. 35. 40. 41. 45. 49.
Piecewise-defined functions
51. 53.
Composition of functions
55. 57. 63. 67.
Trigonometry
69. 73.

Chapter 2 Practice Exercises
(pg. 142 – pg. 143)

Limits and continuity
5. 7.
Finding limits
13. 15. 17. 19.
Limits at infinity
25. 27. 29.
Roots
37

Chapter 3 Practice Exercises
(pg. 235 – pg. 240)

Derivatives of functions
23. 27. 35. 39.
Implicit differentiation
45. 49. 53.
Derivative definition
63. 67.
Slopes, tangents and normals
71. 76. 77.
Tangents and normals to implicitly defined curves
85. 87.
Tangents to parametrised curves
89.
Analyzing graphs
93.
Trigonometric limits
97. 104.
Related rates
107. 117.

Chapter 4 Practice Exercises
(pg. 318 – pg. 321)

Existence of extreme values
1. 5. 8.
The mean value theorem
11. 17.
Conclusions from graph
19.
Graphs and graphing
23. 27. 33. 39.
Applying l' Hôpotal' s rule
53. 61.
Optimisation
67. 69.
Finding indefinite integrals
75. 83. 89.

Chapter 5 Practice Exercises
(pg. 388 – pg. 391)

Finite sums and estimates
3.
Definite integrals
5. 9.
Area
11. 17. 19.
Evaluating indefinite integrals
37. 41. 43.
Evaluating definite integrals
45. 51. 57. 65.
Average values
71.
Differentiating integrals
77. 79.
Theory and examples
83.

Chapter 6 Practice Exercises
(pg. 461 – pg. 464)

Length of curves
19. 21. 23.

Chapter 7 Practice Exercises
(pg. 547 – pg. 550)

Differentiation
3. 19. 23. 24.
Logarithmic differentiation
27. 29.
Integration
31. 39. 59. 69.
Solving equations with logarithmic or exponential terms
83.
Evaluating limits
85. 91.
Theory and applications
101. 103. 107.

Chapter 8 Practice Exercises
(pg. 634 – pg. 638)

Integration using substitutions
1. 7. 15. 35. 43. 63.
Integration by parts
85. 89.
Partial fractions
97. 109.
Trigonometric substitutions
111. 113.
Quadratic terms
115. 117.
Trigonometric integrals
119. 125.
Improper integrals
135. 143.
Convergence or divergence
145. 149.
Assorted integrations
151. 185.

Chapter 11 Practice Exercises
(pg. 840 – pg. 842)

Convergent or divergent sequences
5. 11.
Convergent series
19. 23.
Convergent or divergent series
25. 33.
Power series
45. 47.
Maclaurin series
51. 55.
Taylor series
65. 67.
Theory and examples
91. 93.
Fourier series
105. 107.

Chapter 1 Practice Exercises

Inequalities

In Exercises 1–4, solve the inequalities and show the solution sets on the real line.

1. $7 + 2x \geq 3$

2. $-3x < 10$

3. $\frac{1}{5}(x - 1) < \frac{1}{4}(x - 2)$

4. $\frac{x - 3}{2} \geq -\frac{4 + x}{3}$

Absolute Value

Solve the equations or inequalities in Exercises 5–8.

5. $|x + 1| = 7$

6. $|y - 3| < 4$

7. $\left|1 - \frac{x}{2}\right| > \frac{3}{2}$

8. $\left|\frac{2x + 7}{3}\right| \leq 5$

Coordinates

9. A particle in the plane moved from $A(-2, 5)$ to the y-axis in such a way that Δy equaled $3\Delta x$. What were the particle's new coordinates?

10. **a.** Plot the points $A(8, 1)$, $B(2, 10)$, $C(-4, 6)$, $D(2, -3)$, and $E(14/3, 6)$.

b. Find the slopes of the lines AB, BC, CD, DA, CE, and BD.

c. Do any four of the five points A, B, C, D, and E form a parallelogram?

d. Are any three of the five points collinear? How do you know?

e. Which of the lines determined by the five points pass through the origin?

11. Do the points $A(6, 4)$, $B(4, -3)$, and $C(-2, 3)$ form an isosceles triangle? A right triangle? How do you know?

12. Find the coordinates of the point on the line $y = 3x + 1$ that is equidistant from $(0, 0)$ and $(-3, 4)$.

Lines

In Exercises 13–24, write an equation for the specified line.

13. through $(1, -6)$ with slope 3

14. through $(-1, 2)$ with slope $-1/2$

15. the vertical line through $(0, -3)$

16. through $(-3, 6)$ and $(1, -2)$

17. the horizontal line through $(0, 2)$

18. through $(3, 3)$ and $(-2, 5)$

19. with slope -3 and y-intercept 3

20. through $(3, 1)$ and parallel to $2x - y = -2$

21. through $(4, -12)$ and parallel to $4x + 3y = 12$

22. through $(-2, -3)$ and perpendicular to $3x - 5y = 1$

23. through $(-1, 2)$ and perpendicular to $(1/2)x + (1/3)y = 1$

24. with x-intercept 3 and y-intercept -5

Functions and Graphs

25. Express the area and circumference of a circle as functions of the circle's radius. Then express the area as a function of the circumference.

26. Express the radius of a sphere as a function of the sphere's surface area. Then express the surface area as a function of the volume.

27. A point P in the first quadrant lies on the parabola $y = x^2$. Express the coordinates of P as functions of the angle of inclination of the line joining P to the origin.

28. A hot-air balloon rising straight up from a level field is tracked by a range finder located 500 ft from the point of liftoff. Express the balloon's height as a function of the angle the line from the range finder to the balloon makes with the ground.

In Exercises 29–32, determine whether the graph of the function is symmetric about the y-axis, the origin, or neither.

29. $y = x^{1/5}$

30. $y = x^{2/5}$

31. $y = x^2 - 2x - 1$

32. $y = e^{-x^2}$

In Exercises 33–40, determine whether the function is even, odd, or neither.

33. $y = x^2 + 1$

34. $y = x^5 - x^3 - x$

35. $y = 1 - \cos x$

36. $y = \sec x \tan x$

37. $y = \frac{x^4 + 1}{x^3 - 2x}$

38. $y = 1 - \sin x$

39. $y = x + \cos x$

40. $y = \sqrt{x^4 - 1}$

In Exercises 41–50, find the **(a)** domain and **(b)** range.

41. $y = |x| - 2$
42. $y = -2 + \sqrt{1 - x}$
43. $y = \sqrt{16 - x^2}$
44. $y = 3^{2-x} + 1$
45. $y = 2e^{-x} - 3$
46. $y = \tan(2x - \pi)$
47. $y = 2\sin(3x + \pi) - 1$
48. $y = x^{2/5}$
49. $y = \ln(x - 3) + 1$
50. $y = -1 + \sqrt[3]{2 - x}$

Piecewise-Defined Functions

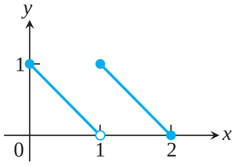
In Exercises 51 and 52, find the **(a)** domain and **(b)** range.

51. $y = \begin{cases} \sqrt{-x}, & -4 \le x \le 0 \\ \sqrt{x}, & 0 < x \le 4 \end{cases}$

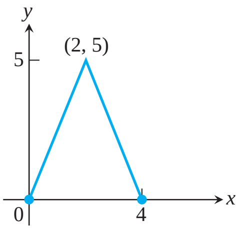
52. $y = \begin{cases} -x - 2, & -2 \le x \le -1 \\ x, & -1 < x \le 1 \\ -x + 2, & 1 < x \le 2 \end{cases}$

In Exercises 53 and 54, write a piecewise formula for the function.

53.



54.



Composition of Functions

In Exercises 55 and 56, find

a. $(f \circ g)(-1)$.
b. $(g \circ f)(2)$.
c. $(f \circ f)(x)$.
d. $(g \circ g)(x)$.

55. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{\sqrt{x + 2}}$

56. $f(x) = 2 - x, \quad g(x) = \sqrt[3]{x + 1}$

In Exercises 57 and 58, **(a)** write a formula for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

57. $f(x) = 2 - x^2, \quad g(x) = \sqrt{x + 2}$

58. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{1 - x}$

Composition with absolute values In Exercises 59–64, graph f_1 and f_2 together. Then describe how applying the absolute value function before applying f_1 affects the graph.

	$f_1(x)$	$f_2(x) = f_1(\lvert x \rvert)$
59.	x	$\lvert x \rvert$
60.	x^3	$\lvert x \rvert^3$
61.	x^2	$\lvert x \rvert^2$
62.	$\dfrac{1}{x}$	$\dfrac{1}{\lvert x \rvert}$
63.	$\sqrt{x}$	$\sqrt{\lvert x \rvert}$
64.	$\sin x$	$\sin \lvert x \rvert$

Composition with absolute values In Exercises 65–68, graph g_1 and g_2 together. Then describe how taking absolute values after applying g_1 affects the graph.

	$g_1(x)$	$g_2(x) = \lvert g_1(x) \rvert$
65.	x^3	$\lvert x^3 \rvert$
66.	$\sqrt{x}$	$\lvert \sqrt{x} \rvert$
67.	$4 - x^2$	$\lvert 4 - x^2 \rvert$
68.	$x^2 + x$	$\lvert x^2 + x \rvert$

Trigonometry

In Exercises 69–72, sketch the graph of the given function. What is the period of the function?

69. $y = \cos 2x$
70. $y = \sin \dfrac{x}{2}$
71. $y = \sin \pi x$
72. $y = \cos \dfrac{\pi x}{2}$

73. Sketch the graph $y = 2\cos\left(x - \dfrac{\pi}{3}\right)$.

74. Sketch the graph $y = 1 + \sin\left(x + \dfrac{\pi}{4}\right)$.

In Exercises 75–78, ABC is a right triangle with the right angle at C. The sides opposite angles A, B, and C are a, b, and c, respectively.

75. a. Find a and b if $c = 2$, $B = \pi/3$.
b. Find a and c if $b = 2$, $B = \pi/3$.

76. a. Express a in terms of A and c.
b. Express a in terms of A and b.

77. a. Express a in terms of B and b.
b. Express c in terms of A and a.

78. a. Express $\sin A$ in terms of a and c.
b. Express $\sin A$ in terms of b and c.

79. Height of a pole Two wires stretch from the top T of a vertical pole to points B and C on the ground, where C is 10 m closer to the base of the pole than is B. If wire BT makes an angle of 35° with the horizontal and wire CT makes an angle of 50° with the horizontal, how high is the pole?

80. Height of a weather balloon Observers at positions A and B 2 km apart simultaneously measure the angle of elevation of a weather balloon to be 40° and 70°, respectively. If the balloon is directly above a point on the line segment between A and B, find the height of the balloon.

T **81. a.** Graph the function $f(x) = \sin x + \cos(x/2)$.
b. What appears to be the period of this function?
c. Confirm your finding in part (b) algebraically.

T **82. a.** Graph $f(x) = \sin(1/x)$.
b. What are the domain and range of f?
c. Is f periodic? Give reasons for your answer.

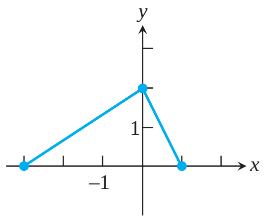
Chapter 1 Additional and Advanced Exercises

Functions and Graphs

1. The graph of f is shown. Draw the graph of each function.

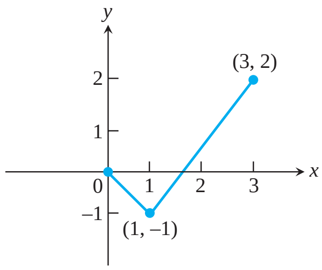
a. $y = f(-x)$ **b.** $y = -f(x)$

c. $y = -2f(x + 1) + 1$ **d.** $y = 3f(x - 2) - 2$



2. A portion of the graph of a function defined on $[-3, 3]$ is shown. Complete the graph assuming that the function is

a. even. **b.** odd.



3. Are there two functions f and g such that $f \circ g = g \circ f$? Give reasons for your answer.

4. Are there two functions f and g with the following property? The graphs of f and g are not straight lines but the graph of $f \circ g$ is a straight line. Give reasons for your answer.

5. If $f(x)$ is odd, can anything be said of $g(x) = f(x) - 2$? What if f is even instead? Give reasons for your answer.

6. If $g(x)$ is an odd function defined for all values of x, can anything be said about $g(0)$? Give reasons for your answer.

7. Graph the equation $|x| + |y| = 1 + x$.

8. Graph the equation $y + |y| = x + |x|$.

Trigonometry

In Exercises 9–14, ABC is an arbitrary triangle with sides a, b, and c opposite angles A, B, and C, respectively.

9. Find b if $a = \sqrt{3}, A = \pi/3, B = \pi/4$.

10. Find $\sin B$ if $a = 4, b = 3, A = \pi/4$.

11. Find $\cos A$ if $a = 2, b = 2, c = 3$.

12. Find c if $a = 2, b = 3, C = \pi/4$.

13. Find $\sin B$ if $a = 2, b = 3, c = 4$.

14. Find $\sin C$ if $a = 2, b = 4, c = 5$.

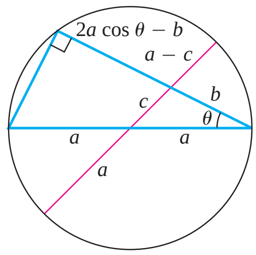
Derivations and Proofs

15. Prove the following identities.

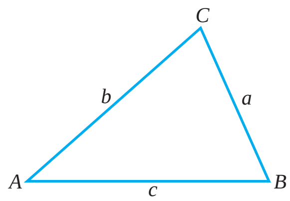
a. $\dfrac{1 - \cos x}{\sin x} = \dfrac{\sin x}{1 + \cos x}$

b. $\dfrac{1 - \cos x}{1 + \cos x} = \tan^2 \dfrac{x}{2}$

16. Explain the following "proof without words" of the law of cosines. (Source: "Proof without Words: The Law of Cosines," Sidney H. Kung, *Mathematics Magazine*, Vol. 63, No. 5, Dec. 1990, p. 342.)



17. Show that the area of triangle ABC is given by $(1/2)ab \sin C = (1/2)bc \sin A = (1/2)ca \sin B$.



18. Show that the area of triangle ABC is given by $\sqrt{s(s - a)(s - b)(s - c)}$ where $s = (a + b + c)/2$ is the semiperimeter of the triangle.

19. Properties of inequalities If a and b are real numbers, we say that ***a* is less than *b*** and write $a < b$ if (and only if) $b - a$ is positive. Use this definition to prove the following properties of inequalities.

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$

2. $a < b \Rightarrow a - c < b - c$

3. $a < b$ and $c > 0 \Rightarrow ac < bc$

4. $a < b$ and $c < 0 \Rightarrow bc < ac$
(Special case: $a < b \Rightarrow -b < -a$)

5. $a > 0 \Rightarrow \frac{1}{a} > 0$

6. $0 < a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

7. $a < b < 0 \Rightarrow \frac{1}{b} < \frac{1}{a}$

20. Prove that the following inequalities hold for any real numbers a and b.

a. $|a| < |b|$ if and only if $a^2 < b^2$

b. $|a - b| \geq ||a| - |b||$

Generalizing the triangle inequality Prove by mathematical induction that the inequalities in Exercises 21 and 22 hold for any n real numbers $a_1, a_2, \ldots, a_n$. (Mathematical induction is reviewed in Appendix 1.)

21. $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$

22. $|a_1 + a_2 + \cdots + a_n| \geq |a_1| - |a_2| - \cdots - |a_n|$

23. Show that if f is both even and odd, then $f(x) = 0$ for every x in the domain of f.

24. a. Even-odd decompositions Let f be a function whose domain is symmetric about the origin, that is, $-x$ belongs to the domain whenever x does. Show that f is the sum of an even function and an odd function:

$$f(x) = E(x) + O(x),$$

where E is an even function and O is an odd function. (*Hint:* Let $E(x) = (f(x) + f(-x))/2$. Show that $E(-x) = E(x)$, so that E is even. Then show that $O(x) = f(x) - E(x)$ is odd.)

b. Uniqueness Show that there is only one way to write f as the sum of an even and an odd function. (*Hint:* One way is given in part (a). If also $f(x) = E_1(x) + O_1(x)$ where E_1 is even and O_1 is odd, show that $E - E_1 = O_1 - O$. Then use Exercise 23 to show that $E = E_1$ and $O = O_1$.)

Grapher Explorations—Effects of Parameters

25. What happens to the graph of $y = ax^2 + bx + c$ as

a. a changes while b and c remain fixed?

b. b changes (a and c fixed, $a \neq 0$)?

c. c changes (a and b fixed, $a \neq 0$)?

26. What happens to the graph of $y = a(x + b)^3 + c$ as

a. a changes while b and c remain fixed?

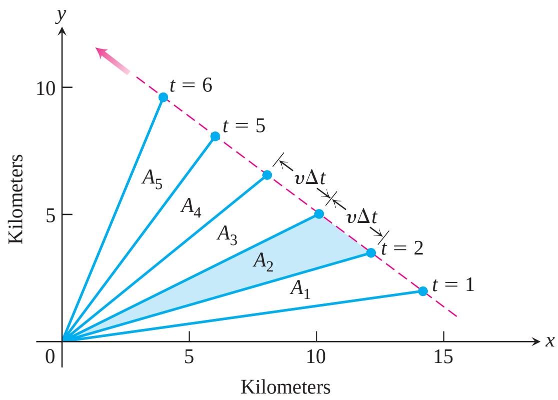
b. b changes (a and c fixed, $a \neq 0$)?

c. c changes (a and b fixed, $a \neq 0$)?

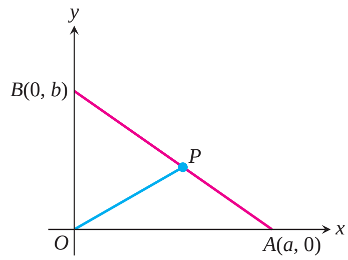
27. Find all values of the slope of the line $y = mx + 2$ for which the x-intercept exceeds $1/2$.

Geometry

28. An object's center of mass moves at a constant velocity v along a straight line past the origin. The accompanying figure shows the coordinate system and the line of motion. The dots show positions that are 1 sec apart. Why are the areas $A_1, A_2, \ldots, A_5$ in the figure all equal? As in Kepler's equal area law (see Section 13.6), the line that joins the object's center of mass to the origin sweeps out equal areas in equal times.



29. a. Find the slope of the line from the origin to the midpoint P, of side AB in the triangle in the accompanying figure $(a, b > 0)$.



b. When is OP perpendicular to AB?

Chapter 2 Practice Exercises

Limits and Continuity

1. Graph the function

$$f(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \geq 1. \end{cases}$$

Then discuss, in detail, limits, one-sided limits, continuity, and one-sided continuity of f at $x = -1, 0$, and 1. Are any of the discontinuities removable? Explain.

2. Repeat the instructions of Exercise 1 for

$$f(x) = \begin{cases} 0, & x \leq -1 \\ 1/x, & 0 < |x| < 1 \\ 0, & x = 1 \\ 1, & x > 1. \end{cases}$$

3. Suppose that $f(t)$ and $g(t)$ are defined for all t and that $\lim_{t \to t_0} f(t) = -7$ and $\lim_{t \to t_0} g(t) = 0$. Find the limit as $t \to t_0$ of the following functions.

a. $3f(t)$ **b.** $(f(t))^2$

c. $f(t) \cdot g(t)$ **d.** $\dfrac{f(t)}{g(t) - 7}$

e. $\cos(g(t))$ **f.** $|f(t)|$

g. $f(t) + g(t)$ **h.** $1/f(t)$

4. Suppose that $f(x)$ and $g(x)$ are defined for all x and that $\lim_{x\to 0} f(x) = 1/2$ and $\lim_{x\to 0} g(x) = \sqrt{2}$. Find the limits as $x \to 0$ of the following functions.

a. $-g(x)$ **b.** $g(x)\cdot f(x)$

c. $f(x) + g(x)$ **d.** $1/f(x)$

e. $x + f(x)$ **f.** $\dfrac{f(x)\cdot \cos x}{x - 1}$

In Exercises 5 and 6, find the value that $\lim_{x\to 0} g(x)$ must have if the given limit statements hold.

5. $\lim_{x\to 0}\left(\dfrac{4 - g(x)}{x}\right) = 1$ **6.** $\lim_{x\to -4}\left(x \lim_{x\to 0} g(x)\right) = 2$

7. On what intervals are the following functions continuous?

a. $f(x) = x^{1/3}$ **b.** $g(x) = x^{3/4}$

c. $h(x) = x^{-2/3}$ **d.** $k(x) = x^{-1/6}$

8. On what intervals are the following functions continuous?

a. $f(x) = \tan x$ **b.** $g(x) = \csc x$

c. $h(x) = \dfrac{\cos x}{x - \pi}$ **d.** $k(x) = \dfrac{\sin x}{x}$

Finding Limits

In Exercises 9–16, find the limit or explain why it does not exist.

9. $\lim \dfrac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$

a. as $x \to 0$ **b.** as $x \to 2$

10. $\lim \dfrac{x^2 + x}{x^5 + 2x^4 + x^3}$

a. as $x \to 0$ **b.** as $x \to -1$

11. $\lim_{x\to 1} \dfrac{1 - \sqrt{x}}{1 - x}$ **12.** $\lim_{x\to a} \dfrac{x^2 - a^2}{x^4 - a^4}$

13. $\lim_{h\to 0} \dfrac{(x + h)^2 - x^2}{h}$ **14.** $\lim_{x\to 0} \dfrac{(x + h)^2 - x^2}{h}$

15. $\lim_{x\to 0} \dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x}$ **16.** $\lim_{x\to 0} \dfrac{(2 + x)^3 - 8}{x}$

In Exercises 17–20, find the limit of $g(x)$ as x approaches the indicated value.

17. $\lim_{x\to 0^+} (4g(x))^{1/3} = 2$ **18.** $\lim_{x\to \sqrt{5}} \dfrac{1}{x + g(x)} = 2$

19. $\lim_{x\to 1} \dfrac{3x^2 + 1}{g(x)} = \infty$ **20.** $\lim_{x\to -2} \dfrac{5 - x^2}{\sqrt{g(x)}} = 0$

Limits at Infinity

Find the limits in Exercises 21–30.

21. $\lim_{x\to\infty} \dfrac{2x + 3}{5x + 7}$ **22.** $\lim_{x\to -\infty} \dfrac{2x^2 + 3}{5x^2 + 7}$

23. $\lim_{x\to -\infty} \dfrac{x^2 - 4x + 8}{3x^3}$ **24.** $\lim_{x\to\infty} \dfrac{1}{x^2 - 7x + 1}$

25. $\lim_{x\to -\infty} \dfrac{x^2 - 7x}{x + 1}$ **26.** $\lim_{x\to\infty} \dfrac{x^4 + x^3}{12x^3 + 128}$

27. $\lim_{x\to\infty} \dfrac{\sin x}{\lfloor x \rfloor}$ (If you have a grapher, try graphing the function for $-5 \le x \le 5$.)

28. $\lim_{\theta\to\infty} \dfrac{\cos\theta - 1}{\theta}$ (If you have a grapher, try graphing $f(x) = x(\cos(1/x) - 1)$ near the origin to "see" the limit at infinity.)

29. $\lim_{x\to\infty} \dfrac{x + \sin x + 2\sqrt{x}}{x + \sin x}$ **30.** $\lim_{x\to\infty} \dfrac{x^{2/3} + x^{-1}}{x^{2/3} + \cos^2 x}$

Continuous Extension

31. Can $f(x) = x(x^2 - 1)/|x^2 - 1|$ be extended to be continuous at $x = 1$ or -1? Give reasons for your answers. (Graph the function—you will find the graph interesting.)

32. Explain why the function $f(x) = \sin(1/x)$ has no continuous extension to $x = 0$.

T In Exercises 33–36, graph the function to see whether it appears to have a continuous extension to the given point a. If it does, use Trace and Zoom to find a good candidate for the extended function's value at a. If the function does not appear to have a continuous extension, can it be extended to be continuous from the right or left? If so, what do you think the extended function's value should be?

33. $f(x) = \dfrac{x - 1}{x - \sqrt[4]{x}}$, $a = 1$ **34.** $g(\theta) = \dfrac{5\cos\theta}{4\theta - 2\pi}$, $a = \pi/2$

35. $h(t) = (1 + |t|)^{1/t}$, $a = 0$ **36.** $k(x) = \dfrac{x}{1 - 2^{|x|}}$, $a = 0$

Roots

T 37. Let $f(x) = x^3 - x - 1$.

a. Show that f has a zero between -1 and 2.

b. Solve the equation $f(x) = 0$ graphically with an error of magnitude at most 10^{-8}.

c. It can be shown that the exact value of the solution in part (b) is

$$\left(\frac{1}{2} + \frac{\sqrt{69}}{18}\right)^{1/3} + \left(\frac{1}{2} - \frac{\sqrt{69}}{18}\right)^{1/3}$$

Evaluate this exact answer and compare it with the value you found in part (b).

T 38. Let $f(\theta) = \theta^3 - 2\theta + 2$.

a. Show that f has a zero between -2 and 0.

b. Solve the equation $f(\theta) = 0$ graphically with an error of magnitude at most 10^{-4}.

c. It can be shown that the exact value of the solution in part (b) is

$$\left(\sqrt{\frac{19}{27}} - 1\right)^{1/3} - \left(\sqrt{\frac{19}{27}} + 1\right)^{1/3}$$

Evaluate this exact answer and compare it with the value you found in part (b).

Chapter 2 Additional and Advanced Exercises

T **1. Assigning a value to 0^0** The rules of exponents (see Appendix 9) tell us that $a^0 = 1$ if a is any number different from zero. They also tell us that $0^n = 0$ if n is any positive number.

If we tried to extend these rules to include the case 0^0, we would get conflicting results. The first rule would say $0^0 = 1$, whereas the second would say $0^0 = 0$.

We are not dealing with a question of right or wrong here. Neither rule applies as it stands, so there is no contradiction. We could, in fact, define 0^0 to have any value we wanted as long as we could persuade others to agree.

What value would you like 0^0 to have? Here is an example that might help you to decide. (See Exercise 2 below for another example.)

a. Calculate x^x for $x = 0.1, 0.01, 0.001$, and so on as far as your calculator can go. Record the values you get. What pattern do you see?

b. Graph the function $y = x^x$ for $0 < x \le 1$. Even though the function is not defined for $x \le 0$, the graph will approach the y-axis from the right. Toward what y-value does it seem to be headed? Zoom in to further support your idea.

T **2. A reason you might want 0^0 to be something other than 0 or 1** As the number x increases through positive values, the numbers $1/x$ and $1/(\ln x)$ both approach zero. What happens to the number

$$f(x) = \left(\frac{1}{x}\right)^{1/(\ln x)}$$

as x increases? Here are two ways to find out.

a. Evaluate f for $x = 10, 100, 1000$, and so on as far as your calculator can reasonably go. What pattern do you see?

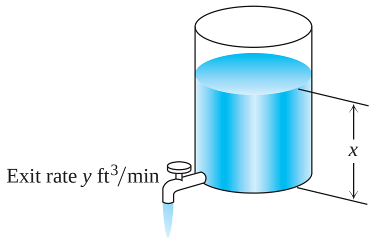
b. Graph f in a variety of graphing windows, including windows that contain the origin. What do you see? Trace the y-values along the graph. What do you find?

3. Lorentz contraction In relativity theory, the length of an object, say a rocket, appears to an observer to depend on the speed at which the object is traveling with respect to the observer. If the observer measures the rocket's length as L_0 at rest, then at speed υ the length will appear to be

$$L = L_0\sqrt{1 - \frac{\upsilon^2}{c^2}}.$$

This equation is the Lorentz contraction formula. Here, c is the speed of light in a vacuum, about 3×10^8 m/sec. What happens to L as υ increases? Find $\lim_{\upsilon \to c^-} L$. Why was the left-hand limit needed?

4. Controlling the flow from a draining tank Torricelli's law says that if you drain a tank like the one in the figure shown, the rate y at which water runs out is a constant times the square root of the water's depth x. The constant depends on the size and shape of the exit valve.



Suppose that $y = \sqrt{x}/2$ for a certain tank. You are trying to maintain a fairly constant exit rate by adding water to the tank with a hose from time to time. How deep must you keep the water if you want to maintain the exit rate

a. within 0.2 ft^3/min of the rate $y_0 = 1$ ft^3/min?

b. within 0.1 ft^3/min of the rate $y_0 = 1$ ft^3/min?

5. Thermal expansion in precise equipment As you may know, most metals expand when heated and contract when cooled. The dimensions of a piece of laboratory equipment are sometimes so critical that the shop where the equipment is made must be held at the same temperature as the laboratory where the equipment is to be used. A typical aluminum bar that is 10 cm wide at 70°F will be

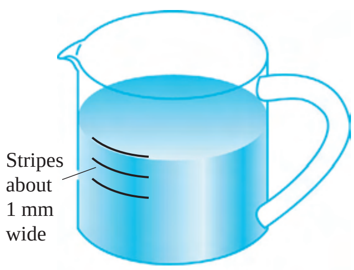
$$y = 10 + (t - 70) \times 10^{-4}$$

centimeters wide at a nearby temperature t. Suppose that you are using a bar like this in a gravity wave detector, where its width must stay within 0.0005 cm of the ideal 10 cm. How close to $t_0 = 70°$F must you maintain the temperature to ensure that this tolerance is not exceeded?

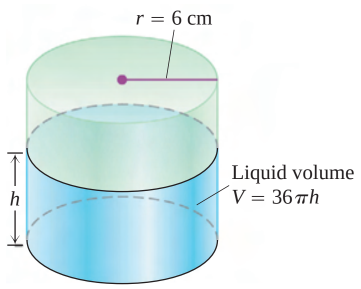
6. Stripes on a measuring cup The interior of a typical 1-L measuring cup is a right circular cylinder of radius 6 cm (see accompanying figure). The volume of water we put in the cup is therefore a function of the level h to which the cup is filled, the formula being

$$V = \pi 6^2 h = 36\pi h.$$

How closely must we measure h to measure out 1 L of water (1000 cm^3) with an error of no more than 1% (10 cm^3)?



(a)



(b)

A 1-L measuring cup (a), modeled as a right circular cylinder (b) of radius $r = 6$ cm

Precise Definition of Limit

In Exercises 7–10, use the formal definition of limit to prove that the function is continuous at x_0.

7. $f(x) = x^2 - 7, \quad x_0 = 1$ **8.** $g(x) = 1/(2x), \quad x_0 = 1/4$

9. $h(x) = \sqrt{2x - 3}, \quad x_0 = 2$ **10.** $F(x) = \sqrt{9 - x}, \quad x_0 = 5$

11. Uniqueness of limits Show that a function cannot have two different limits at the same point. That is, if $\lim_{x \to x_0} f(x) = L_1$ and $\lim_{x \to x_0} f(x) = L_2$, then $L_1 = L_2$.

12. Prove the limit Constant Multiple Rule:

$$\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x) \quad \text{for any constant } k.$$

13. One-sided limits If $\lim_{x \to 0^+} f(x) = A$ and $\lim_{x \to 0^-} f(x) = B$, find

a. $\lim_{x \to 0^+} f(x^3 - x)$ **b.** $\lim_{x \to 0^-} f(x^3 - x)$

c. $\lim_{x \to 0^+} f(x^2 - x^4)$ **d.** $\lim_{x \to 0^-} f(x^2 - x^4)$

14. Limits and continuity Which of the following statements are true, and which are false? If true, say why; if false, give a counterexample (that is, an example confirming the falsehood).

a. If $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} g(x)$ does not exist, then $\lim_{x \to a}(f(x) + g(x))$ does not exist.

b. If neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists, then $\lim_{x \to a} (f(x) + g(x))$ does not exist.

c. If f is continuous at x, then so is $|f|$.

d. If $|f|$ is continuous at a, then so is f.

In Exercises 15 and 16, use the formal definition of limit to prove that the function has a continuous extension to the given value of x.

15. $f(x) = \dfrac{x^2 - 1}{x + 1}, \quad x = -1$ **16.** $g(x) = \dfrac{x^2 - 2x - 3}{2x - 6}, \quad x = 3$

17. A function continuous at only one point Let

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

a. Show that f is continuous at $x = 0$.

b. Use the fact that every nonempty open interval of real numbers contains both rational and irrational numbers to show that f is not continuous at any nonzero value of x.

18. The Dirichlet ruler function If x is a rational number, then x can be written in a unique way as a quotient of integers m/n where $n > 0$ and m and n have no common factors greater than 1. (We say that such a fraction is in *lowest terms*. For example, 6/4 written in lowest terms is 3/2.) Let $f(x)$ be defined for all x in the interval [0, 1] by

$$f(x) = \begin{cases} 1/n, & \text{if } x = m/n \text{ is a rational number in lowest terms} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

For instance, $f(0) = f(1) = 1$, $f(1/2) = 1/2$, $f(1/3) = f(2/3) = 1/3$, $f(1/4) = f(3/4) = 1/4$, and so on.

a. Show that f is discontinuous at every rational number in [0, 1].

b. Show that f is continuous at every irrational number in [0, 1]. (*Hint*: If ϵ is a given positive number, show that there are only finitely many rational numbers r in [0, 1] such that $f(r) \ge \epsilon$.)

c. Sketch the graph of f. Why do you think f is called the "ruler function"?

19. Antipodal points Is there any reason to believe that there is always a pair of antipodal (diametrically opposite) points on Earth's equator where the temperatures are the same? Explain.

20. If $\lim_{x \to c} (f(x) + g(x)) = 3$ and $\lim_{x \to c} (f(x) - g(x)) = -1$, find $\lim_{x \to c} f(x)g(x)$.

21. Roots of a quadratic equation that is almost linear The equation $ax^2 + 2x - 1 = 0$, where a is a constant, has two roots if $a > -1$ and $a \ne 0$, one positive and one negative:

$$r_+(a) = \frac{-1 + \sqrt{1 + a}}{a}, \qquad r_-(a) = \frac{-1 - \sqrt{1 + a}}{a}.$$

a. What happens to $r_+(a)$ as $a \to 0$? As $a \to -1^+$?

b. What happens to $r_-(a)$ as $a \to 0$? As $a \to -1^+$?

c. Support your conclusions by graphing $r_+(a)$ and $r_-(a)$ as functions of a. Describe what you see.

d. For added support, graph $f(x) = ax^2 + 2x - 1$ simultaneously for $a = 1, 0.5, 0.2, 0.1$, and 0.05.

22. Root of an equation Show that the equation $x + 2\cos x = 0$ has at least one solution.

23. Bounded functions A real-valued function f is **bounded from above** on a set D if there exists a number N such that $f(x) \le N$ for all x in D. We call N, when it exists, an **upper bound** for f on D and say that f is bounded from above by N. In a similar manner, we say that f is **bounded from below** on D if there exists a number M such that $f(x) \ge M$ for all x in D. We call M, when it exists, a **lower bound** for f on D and say that f is bounded from below by M. We say that f is **bounded** on D if it is bounded from both above and below.

a. Show that f is bounded on D if and only if there exists a number B such that $|f(x)| \le B$ for all x in D.

b. Suppose that f is bounded from above by N. Show that if $\lim_{x \to x_0} f(x) = L$, then $L \le N$.

c. Suppose that f is bounded from below by M. Show that if $\lim_{x \to x_0} f(x) = L$, then $L \ge M$.

24. ***Max*** $\{a, b\}$ ***and min*** $\{a, b\}$

a. Show that the expression

$$\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$$

equals a if $a \geq b$ and equals b if $b \geq a$. In other words, $\max\{a, b\}$ gives the larger of the two numbers a and b.

b. Find a similar expression for $\min\{a, b\}$, the smaller of a and b.

Generalized Limits Involving $\dfrac{\sin\theta}{\theta}$

The formula $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ can be generalized. If $\lim_{x\to c} f(x) = 0$ and $f(x)$ is never zero in an open interval containing the point $x = c$, except possibly c itself, then

$$\lim_{x\to c}\frac{\sin f(x)}{f(x)} = 1.$$

Here are several examples.

a. $\displaystyle\lim_{x\to 0}\frac{\sin x^2}{x^2} = 1.$

b. $\displaystyle\lim_{x\to 0}\frac{\sin x^2}{x} = \lim_{x\to 0}\frac{\sin x^2}{x^2}\lim_{x\to 0}\frac{x^2}{x} = 1\cdot 0 = 0.$

c. $\displaystyle\lim_{x\to -1}\frac{\sin(x^2-x-2)}{x+1} = \lim_{x\to -1}\frac{\sin(x^2-x-2)}{(x^2-x-2)}\cdot$

$$\lim_{x\to -1}\frac{(x^2-x-2)}{x+1} = 1\cdot\lim_{x\to -1}\frac{(x+1)(x-2)}{x+1} = -3.$$

d. $\displaystyle\lim_{x\to 1}\frac{\sin\left(1-\sqrt{x}\right)}{x-1} = \lim_{x\to 1}\frac{\sin\left(1-\sqrt{x}\right)}{1-\sqrt{x}}\,\frac{1-\sqrt{x}}{x-1} =$

$$1\cdot\lim_{x\to 1}\frac{\left(1-\sqrt{x}\right)\left(1+\sqrt{x}\right)}{(x-1)\left(1+\sqrt{x}\right)} = \lim_{x\to 1}\frac{1-x}{(x-1)\left(1+\sqrt{x}\right)} = -\frac{1}{2}.$$

Find the limits in Exercises 25–30.

25. $\displaystyle\lim_{x\to 0}\frac{\sin(1-\cos x)}{x}$

26. $\displaystyle\lim_{x\to 0^+}\frac{\sin x}{\sin\sqrt{x}}$

27. $\displaystyle\lim_{x\to 0}\frac{\sin(\sin x)}{x}$

28. $\displaystyle\lim_{x\to 0}\frac{\sin(x^2+x)}{x}$

29. $\displaystyle\lim_{x\to 2}\frac{\sin(x^2-4)}{x-2}$

30. $\displaystyle\lim_{x\to 9}\frac{\sin\left(\sqrt{x}-3\right)}{x-9}$

Chapter 3 Practice Exercises

Derivatives of Functions

Find the derivatives of the functions in Exercises 1-40.

1. $y = x^5 - 0.125x^2 + 0.25x$

2. $y = 3 - 0.7x^3 + 0.3x^7$

3. $y = x^3 - 3(x^2 + \pi^2)$

4. $y = x^7 + \sqrt{7x} - \dfrac{1}{\pi + 1}$

5. $y = (x + 1)^2(x^2 + 2x)$

6. $y = (2x - 5)(4 - x)^{-1}$

7. $y = (\theta^2 + \sec\theta + 1)^3$

8. $y = \left(-1 - \dfrac{\csc\theta}{2} - \dfrac{\theta^2}{4}\right)^2$

9. $s = \dfrac{\sqrt{t}}{1 + \sqrt{t}}$

10. $s = \dfrac{1}{\sqrt{t} - 1}$

11. $y = 2\tan^2 x - \sec^2 x$

12. $y = \dfrac{1}{\sin^2 x} - \dfrac{2}{\sin x}$

13. $s = \cos^4(1 - 2t)$

14. $s = \cot^3\left(\dfrac{2}{t}\right)$

15. $s = (\sec t + \tan t)^5$

16. $s = \csc^5(1 - t + 3t^2)$

17. $r = \sqrt{2\theta \sin\theta}$

18. $r = 2\theta\sqrt{\cos\theta}$

19. $r = \sin\sqrt{2\theta}$

20. $r = \sin\left(\theta + \sqrt{\theta + 1}\right)$

21. $y = \dfrac{1}{2}x^2 \csc\dfrac{2}{x}$

22. $y = 2\sqrt{x}\sin\sqrt{x}$

23. $y = x^{-1/2}\sec(2x)^2$

24. $y = \sqrt{x}\csc(x + 1)^3$

25. $y = 5\cot x^2$

26. $y = x^2 \cot 5x$

27. $y = x^2 \sin^2(2x^2)$

28. $y = x^{-2}\sin^2(x^3)$

29. $s = \left(\dfrac{4t}{t + 1}\right)^{-2}$

30. $s = \dfrac{-1}{15(15t - 1)^3}$

31. $y = \left(\dfrac{\sqrt{x}}{1 + x}\right)^2$

32. $y = \left(\dfrac{2\sqrt{x}}{2\sqrt{x} + 1}\right)^2$

33. $y = \sqrt{\dfrac{x^2 + x}{x^2}}$

34. $y = 4x\sqrt{x + \sqrt{x}}$

35. $r = \left(\dfrac{\sin\theta}{\cos\theta - 1}\right)^2$

36. $r = \left(\dfrac{1 + \sin\theta}{1 - \cos\theta}\right)^2$

37. $y = (2x + 1)\sqrt{2x + 1}$

38. $y = 20(3x - 4)^{1/4}(3x - 4)^{-1/5}$

39. $y = \dfrac{3}{(5x^2 + \sin 2x)^{3/2}}$

40. $y = (3 + \cos^3 3x)^{-1/3}$

Implicit Differentiation

In Exercises 41–48, find dy/dx.

41. $xy + 2x + 3y = 1$

42. $x^2 + xy + y^2 - 5x = 2$

43. $x^3 + 4xy - 3y^{4/3} = 2x$

44. $5x^{4/5} + 10y^{6/5} = 15$

45. $\sqrt{xy} = 1$

46. $x^2y^2 = 1$

47. $y^2 = \dfrac{x}{x + 1}$

48. $y^2 = \sqrt{\dfrac{1 + x}{1 - x}}$

In Exercises 49 and 50, find dp/dq.

49. $p^3 + 4pq - 3q^2 = 2$

50. $q = (5p^2 + 2p)^{-3/2}$

In Exercises 51 and 52, find dr/ds.

51. $r\cos 2s + \sin^2 s = \pi$

52. $2rs - r - s + s^2 = -3$

53. Find d^2y/dx^2 by implicit differentiation:

a. $x^3 + y^3 = 1$ **b.** $y^2 = 1 - \dfrac{2}{x}$

54. a. By differentiating $x^2 - y^2 = 1$ implicitly, show that $dy/dx = x/y$.

b. Then show that $d^2y/dx^2 = -1/y^3$.

Numerical Values of Derivatives

55. Suppose that functions $f(x)$ and $g(x)$ and their first derivatives have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	-3	$1/2$
1	3	5	$1/2$	-4

Find the first derivatives of the following combinations at the given value of x.

a. $6f(x) - g(x), \quad x = 1$ **b.** $f(x)g^2(x), \quad x = 0$

c. $\dfrac{f(x)}{g(x) + 1}, \quad x = 1$ **d.** $f(g(x)), \quad x = 0$

e. $g(f(x)), \quad x = 0$ **f.** $(x + f(x))^{3/2}, \quad x = 1$

g. $f(x + g(x)), \quad x = 0$

56. Suppose that the function $f(x)$ and its first derivative have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$f'(x)$
0	9	-2
1	-3	$1/5$

Find the first derivatives of the following combinations at the given value of x.

a. $\sqrt{x}\,f(x), \quad x = 1$ **b.** $\sqrt{f(x)}, \quad x = 0$

c. $f(\sqrt{x}), \quad x = 1$ **d.** $f(1 - 5\tan x), \quad x = 0$

e. $\dfrac{f(x)}{2 + \cos x}, \quad x = 0$ **f.** $10\sin\left(\dfrac{\pi x}{2}\right)f^2(x), \quad x = 1$

57. Find the value of dy/dt at $t = 0$ if $y = 3\sin 2x$ and $x = t^2 + \pi$.

58. Find the value of ds/du at $u = 2$ if $s = t^2 + 5t$ and $t = (u^2 + 2u)^{1/3}$.

59. Find the value of dw/ds at $s = 0$ if $w = \sin\left(\sqrt{r} - 2\right)$ and $r = 8\sin(s + \pi/6)$.

60. Find the value of dr/dt at $t = 0$ if $r = (\theta^2 + 7)^{1/3}$ and $\theta^2 t + \theta = 1$.

61. If $y^3 + y = 2\cos x$, find the value of d^2y/dx^2 at the point (0, 1).

62. If $x^{1/3} + y^{1/3} = 4$, find d^2y/dx^2 at the point (8, 8).

Derivative Definition

In Exercises 63 and 64, find the derivative using the definition.

63. $f(t) = \dfrac{1}{2t + 1}$

64. $g(x) = 2x^2 + 1$

65. a. Graph the function

$$f(x) = \begin{cases} x^2, & -1 \le x < 0 \\ -x^2, & 0 \le x \le 1. \end{cases}$$

b. Is f continuous at $x = 0$?

c. Is f differentiable at $x = 0$?

Give reasons for your answers.

66. a. Graph the function

$$f(x) = \begin{cases} x, & -1 \le x < 0 \\ \tan x, & 0 \le x \le \pi/4. \end{cases}$$

b. Is f continuous at $x = 0$?

c. Is f differentiable at $x = 0$?

Give reasons for your answers.

67. a. Graph the function

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$$

b. Is f continuous at $x = 1$?

c. Is f differentiable at $x = 1$?

Give reasons for your answers.

68. For what value or values of the constant m, if any, is

$$f(x) = \begin{cases} \sin 2x, & x \le 0 \\ mx, & x > 0 \end{cases}$$

a. continuous at $x = 0$?

b. differentiable at $x = 0$?

Give reasons for your answers.

Slopes, Tangents, and Normals

69. Tangents with specified slope Are there any points on the curve $y = (x/2) + 1/(2x - 4)$ where the slope is $-3/2$? If so, find them.

70. Tangents with specified slope Are there any points on the curve $y = x - 1/(2x)$ where the slope is 3? If so, find them.

71. Horizontal tangents Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis.

72. Tangent intercepts Find the x- and y-intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.

73. Tangents perpendicular or parallel to lines Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is

a. perpendicular to the line $y = 1 - (x/24)$.

b. parallel to the line $y = \sqrt{2} - 12x$.

74. Intersecting tangents Show that the tangents to the curve $y = (\pi \sin x)/x$ at $x = \pi$ and $x = -\pi$ intersect at right angles.

75. Normals parallel to a line Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the normal is parallel to the line $y = -x/2$. Sketch the curve and normals together, labeling each with its equation.

76. Tangent and normal lines Find equations for the tangent and normal to the curve $y = 1 + \cos x$ at the point $(\pi/2, 1)$. Sketch the curve, tangent, and normal together, labeling each with its equation.

77. Tangent parabola The parabola $y = x^2 + C$ is to be tangent to the line $y = x$. Find C.

78. Slope of tangent Show that the tangent to the curve $y = x^3$ at any point (a, a^3) meets the curve again at a point where the slope is four times the slope at (a, a^3).

79. Tangent curve For what value of c is the curve $y = c/(x + 1)$ tangent to the line through the points $(0, 3)$ and $(5, -2)$?

80. Normal to a circle Show that the normal line at any point of the circle $x^2 + y^2 = a^2$ passes through the origin.

Tangents and Normals to Implicitly Defined Curves

In Exercises 81–86, find equations for the lines that are tangent and normal to the curve at the given point.

81. $x^2 + 2y^2 = 9, \quad (1, 2)$

82. $x^3 + y^2 = 2, \quad (1, 1)$

83. $xy + 2x - 5y = 2, \quad (3, 2)$

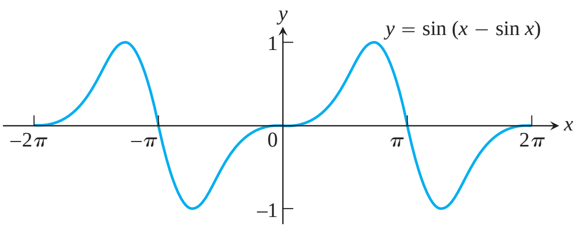
84. $(y - x)^2 = 2x + 4, \quad (6, 2)$

85. $x + \sqrt{xy} = 6, \quad (4, 1)$

86. $x^{3/2} + 2y^{3/2} = 17, \quad (1, 4)$

87. Find the slope of the curve $x^3y^3 + y^2 = x + y$ at the points $(1, 1)$ and $(1, -1)$.

88. The graph shown suggests that the curve $y = \sin(x - \sin x)$ might have horizontal tangents at the x-axis. Does it? Give reasons for your answer.



Tangents to Parametrized Curves

In Exercises 89 and 90, find an equation for the line in the xy-plane that is tangent to the curve at the point corresponding to the given value of t. Also, find the value of d^2y/dx^2 at this point.

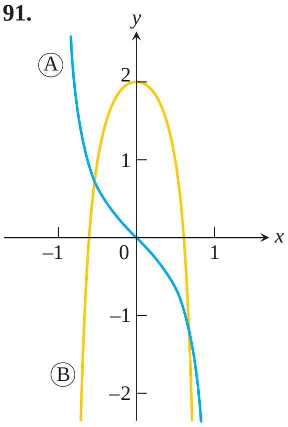
89. $x = (1/2)\tan t, \quad y = (1/2)\sec t, \quad t = \pi/3$

90. $x = 1 + 1/t^2, \quad y = 1 - 3/t, \quad t = 2$

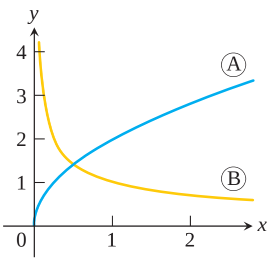
Analyzing Graphs

Each of the figures in Exercises 91 and 92 shows two graphs, the graph of a function $y = f(x)$ together with the graph of its derivative $f'(x)$. Which graph is which? How do you know?

91.



92.

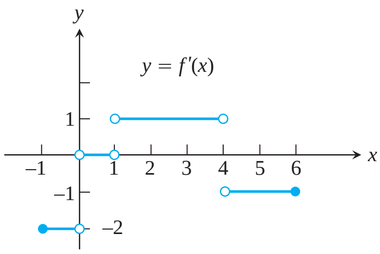


93. Use the following information to graph the function $y = f(x)$ for $-1 \leq x \leq 6$.

i. The graph of f is made of line segments joined end to end.

ii. The graph starts at the point $(-1, 2)$.

iii. The derivative of f, where defined, agrees with the step function shown here.



94. Repeat Exercise 93, supposing that the graph starts at $(-1, 0)$ instead of $(-1, 2)$.

Exercises 95 and 96 are about the graphs in Figure 3.53 (right-hand column). The graphs in part (a) show the numbers of rabbits and foxes in a small arctic population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on rabbits and, as the number of foxes increases, the rabbit population levels off and then drops. Figure 3.53b shows the graph of the derivative of the rabbit population. We made it by plotting slopes.

95. a. What is the value of the derivative of the rabbit population in Figure 3.53 when the number of rabbits is largest? Smallest?

b. What is the size of the rabbit population in Figure 3.53 when its derivative is largest? Smallest (negative value)?

96. In what units should the slopes of the rabbit and fox population curves be measured?

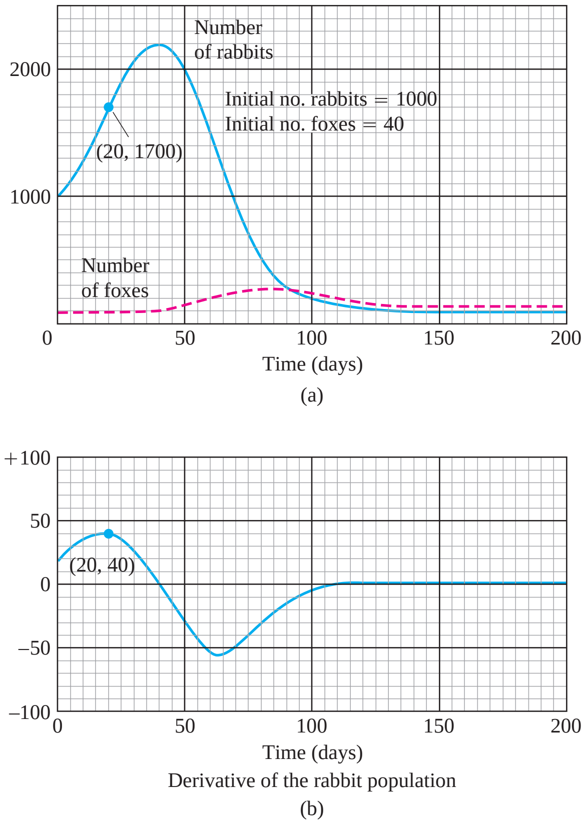


FIGURE 3.53 Rabbits and foxes in an arctic predator-prey food chain.

Trigonometric Limits

97. $\lim_{x \to 0} \dfrac{\sin x}{2x^2 - x}$

98. $\lim_{x \to 0} \dfrac{3x - \tan 7x}{2x}$

99. $\lim_{r \to 0} \dfrac{\sin r}{\tan 2r}$

100. $\lim_{\theta \to 0} \dfrac{\sin(\sin\theta)}{\theta}$

101. $\lim_{\theta \to (\pi/2)^-} \dfrac{4\tan^2\theta + \tan\theta + 1}{\tan^2\theta + 5}$

102. $\lim_{\theta \to 0^+} \dfrac{1 - 2\cot^2\theta}{5\cot^2\theta - 7\cot\theta - 8}$

103. $\lim_{x \to 0} \dfrac{x \sin x}{2 - 2\cos x}$

104. $\lim_{\theta \to 0} \dfrac{1 - \cos\theta}{\theta^2}$

Show how to extend the functions in Exercises 105 and 106 to be continuous at the origin.

105. $g(x) = \dfrac{\tan(\tan x)}{\tan x}$

106. $f(x) = \dfrac{\tan(\tan x)}{\sin(\sin x)}$

Related Rates

107. Right circular cylinder The total surface area S of a right circular cylinder is related to the base radius r and height h by the equation $S = 2\pi r^2 + 2\pi rh$.

a. How is dS/dt related to dr/dt if h is constant?

b. How is dS/dt related to dh/dt if r is constant?

c. How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?

d. How is dr/dt related to dh/dt if S is constant?

108. Right circular cone The lateral surface area S of a right circular cone is related to the base radius r and height h by the equation $S = \pi r\sqrt{r^2 + h^2}$.

a. How is dS/dt related to dr/dt if h is constant?

b. How is dS/dt related to dh/dt if r is constant?

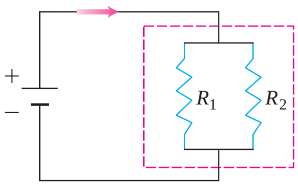
c. How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?

109. Circle's changing area The radius of a circle is changing at the rate of $-2/\pi$ m/sec. At what rate is the circle's area changing when $r = 10$ m?

110. Cube's changing edges The volume of a cube is increasing at the rate of 1200 cm^3/min at the instant its edges are 20 cm long. At what rate are the lengths of the edges changing at that instant?

111. Resistors connected in parallel If two resistors of R_1 and R_2 ohms are connected in parallel in an electric circuit to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$



If R_1 is decreasing at the rate of 1 ohm/sec and R_2 is increasing at the rate of 0.5 ohm/sec, at what rate is R changing when $R_1 = 75$ ohms and $R_2 = 50$ ohms?

112. Impedance in a series circuit The impedance Z (ohms) in a series circuit is related to the resistance R (ohms) and reactance X (ohms) by the equation $Z = \sqrt{R^2 + X^2}$. If R is increasing at 3 ohms/sec and X is decreasing at 2 ohms/sec, at what rate is Z changing when $R = 10$ ohms and $X = 20$ ohms?

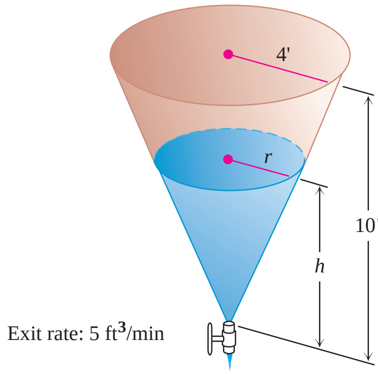
113. Speed of moving particle The coordinates of a particle moving in the metric xy-plane are differentiable functions of time t with $dx/dt = 10$ m/sec and $dy/dt = 5$ m/sec. How fast is the particle moving away from the origin as it passes through the point $(3, -4)$?

114. Motion of a particle A particle moves along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second. Find dx/dt when $x = 3$.

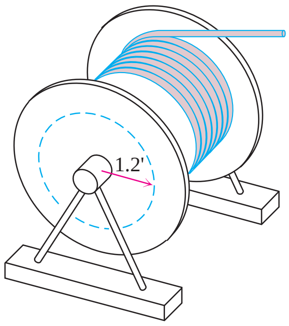
115. Draining a tank Water drains from the conical tank shown in the accompanying figure at the rate of 5 ft^3/min.

a. What is the relation between the variables h and r in the figure?

b. How fast is the water level dropping when $h = 6$ ft?



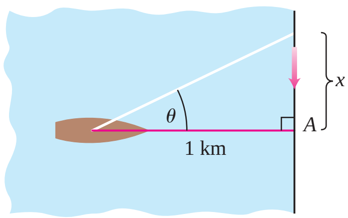
116. Rotating spool As television cable is pulled from a large spool to be strung from the telephone poles along a street, it unwinds from the spool in layers of constant radius (see accompanying figure). If the truck pulling the cable moves at a steady 6 ft/sec (a touch over 4 mph), use the equation $s = r\theta$ to find how fast (radians per second) the spool is turning when the layer of radius 1.2 ft is being unwound.



117. Moving searchlight beam The figure shows a boat 1 km offshore, sweeping the shore with a searchlight. The light turns at a constant rate, $d\theta/dt = -0.6$ rad/sec.

a. How fast is the light moving along the shore when it reaches point A?

b. How many revolutions per minute is 0.6 rad/sec?



118. Points moving on coordinate axes Points A and B move along the x- and y-axes, respectively, in such a way that the distance r (meters) along the perpendicular from the origin to the line AB remains constant. How fast is OA changing, and is it increasing, or decreasing, when $OB = 2r$ and B is moving toward O at the rate of $0.3r$ m/sec?

Linearization

119. Find the linearizations of

a. $\tan x$ at $x = -\pi/4$ **b.** $\sec x$ at $x = -\pi/4$.

Graph the curves and linearizations together.

120. We can obtain a useful linear approximation of the function $f(x) = 1/(1 + \tan x)$ at $x = 0$ by combining the approximations

$$\frac{1}{1+x} \approx 1 - x \quad \text{and} \quad \tan x \approx x$$

to get

$$\frac{1}{1+\tan x} \approx 1 - x.$$

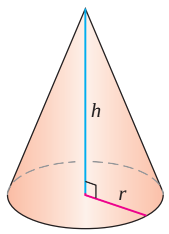
Show that this result is the standard linear approximation of $1/(1 + \tan x)$ at $x = 0$.

121. Find the linearization of $f(x) = \sqrt{1+x} + \sin x - 0.5$ at $x = 0$.

122. Find the linearization of $f(x) = 2/(1 - x) + \sqrt{1 + x} - 3.1$ at $x = 0$.

Differential Estimates of Change

123. Surface area of a cone Write a formula that estimates the change that occurs in the lateral surface area of a right circular cone when the height changes from h_0 to $h_0 + dh$ and the radius does not change.



$V = \frac{1}{3}\pi r^2 h$

$S = \pi r\sqrt{r^2 + h^2}$

(Lateral surface area)

124. Controlling error

a. How accurately should you measure the edge of a cube to be reasonably sure of calculating the cube's surface area with an error of no more than 2%?

b. Suppose that the edge is measured with the accuracy required in part (a). About how accurately can the cube's volume be calculated from the edge measurement? To find out, estimate the percentage error in the volume calculation that might result from using the edge measurement.

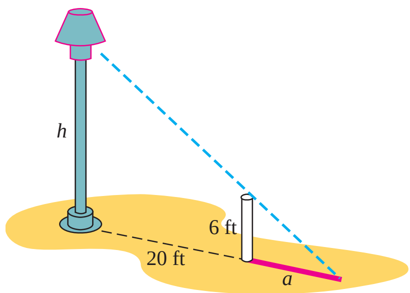
125. Compounding error The circumference of the equator of a sphere is measured as 10 cm with a possible error of 0.4 cm. This measurement is then used to calculate the radius. The radius is then used to calculate the surface area and volume of the sphere. Estimate the percentage errors in the calculated values of

a. the radius.

b. the surface area.

c. the volume.

126. Finding height To find the height of a lamppost (see accompanying figure), you stand a 6 ft pole 20 ft from the lamp and measure the length a of its shadow, finding it to be 15 ft, give or take an inch. Calculate the height of the lamppost using the value $a = 15$ and estimate the possible error in the result.



Chapter 4 Practice Exercises

Existence of Extreme Values

1. Does $f(x) = x^3 + 2x + \tan x$ have any local maximum or minimum values? Give reasons for your answer.
2. Does $g(x) = \csc x + 2 \cot x$ have any local maximum values? Give reasons for your answer.
3. Does $f(x) = (7 + x)(11 - 3x)^{1/3}$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of f.

4. Find values of a and b such that the function

$$f(x) = \frac{ax + b}{x^2 - 1}$$

has a local extreme value of 1 at $x = 3$. Is this extreme value a local maximum, or a local minimum? Give reasons for your answer.

5. The greatest integer function $f(x) = \lfloor x \rfloor$, defined for all values of x, assumes a local maximum value of 0 at each point of $[0, 1)$. Could any of these local maximum values also be local minimum values of f? Give reasons for your answer.

6. a. Give an example of a differentiable function f whose first derivative is zero at some point c even though f has neither a local maximum nor a local minimum at c.

b. How is this consistent with Theorem 2 in Section 4.1? Give reasons for your answer.

7. The function $y = 1/x$ does not take on either a maximum or a minimum on the interval $0 < x < 1$ even though the function is continuous on this interval. Does this contradict the Extreme Value Theorem for continuous functions? Why?

8. What are the maximum and minimum values of the function $y = |x|$ on the interval $-1 \le x < 1$? Notice that the interval is not closed. Is this consistent with the Extreme Value Theorem for continuous functions? Why?

T **9.** A graph that is large enough to show a function's global behavior may fail to reveal important local features. The graph of $f(x) = (x^8/8) - (x^6/2) - x^5 + 5x^3$ is a case in point.

a. Graph f over the interval $-2.5 \le x \le 2.5$. Where does the graph appear to have local extreme values or points of inflection?

b. Now factor $f'(x)$ and show that f has a local maximum at $x = \sqrt[3]{5} \approx 1.70998$ and local minima at $x = \pm\sqrt{3} \approx \pm 1.73205$.

c. Zoom in on the graph to find a viewing window that shows the presence of the extreme values at $x = \sqrt[3]{5}$ and $x = \sqrt{3}$.

The moral here is that without calculus the existence of two of the three extreme values would probably have gone unnoticed. On any normal graph of the function, the values would lie close enough together to fall within the dimensions of a single pixel on the screen.

(*Source*: *Uses of Technology in the Mathematics Curriculum*, by Benny Evans and Jerry Johnson, Oklahoma State University, published in 1990 under National Science Foundation Grant USE-8950044.)

T **10.** (*Continuation of Exercise 9.*)

a. Graph $f(x) = (x^8/8) - (2/5)x^5 - 5x - (5/x^2) + 11$ over the interval $-2 \le x \le 2$. Where does the graph appear to have local extreme values or points of inflection?

b. Show that f has a local maximum value at $x = \sqrt[7]{5} \approx 1.2585$ and a local minimum value at $x = \sqrt[3]{2} \approx 1.2599$.

c. Zoom in to find a viewing window that shows the presence of the extreme values at $x = \sqrt[7]{5}$ and $x = \sqrt[3]{2}$.

The Mean Value Theorem

11. a. Show that $g(t) = \sin^2 t - 3t$ decreases on every interval in its domain.

b. How many solutions does the equation $\sin^2 t - 3t = 5$ have? Give reasons for your answer.

12. a. Show that $y = \tan\theta$ increases on every interval in its domain.

b. If the conclusion in part (a) is really correct, how do you explain the fact that $\tan\pi = 0$ is less than $\tan(\pi/4) = 1$?

13. a. Show that the equation $x^4 + 2x^2 - 2 = 0$ has exactly one solution on $[0, 1]$.

T **b.** Find the solution to as many decimal places as you can.

14. a. Show that $f(x) = x/(x + 1)$ increases on every interval in its domain.

b. Show that $f(x) = x^3 + 2x$ has no local maximum or minimum values.

15. Water in a reservoir As a result of a heavy rain, the volume of water in a reservoir increased by 1400 acre-ft in 24 hours. Show that at some instant during that period the reservoir's volume was increasing at a rate in excess of 225,000 gal/min. (An acre-foot is 43,560 ft^3, the volume that would cover 1 acre to the depth of 1 ft. A cubic foot holds 7.48 gal.)

16. The formula $F(x) = 3x + C$ gives a different function for each value of C. All of these functions, however, have the same derivative with respect to x, namely $F'(x) = 3$. Are these the only differentiable functions whose derivative is 3? Could there be any others? Give reasons for your answers.

17. Show that

$$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{d}{dx}\left(-\frac{1}{x+1}\right)$$

even though

$$\frac{x}{x+1} \ne -\frac{1}{x+1}.$$

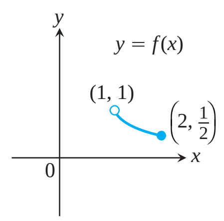
Doesn't this contradict Corollary 2 of the Mean Value Theorem? Give reasons for your answer.

18. Calculate the first derivatives of $f(x) = x^2/(x^2 + 1)$ and $g(x) = -1/(x^2 + 1)$. What can you conclude about the graphs of these functions?

Conclusions from Graphs

In Exercises 19 and 20, use the graph to answer the questions.

19. Identify any global extreme values of f and the values of x at which they occur.

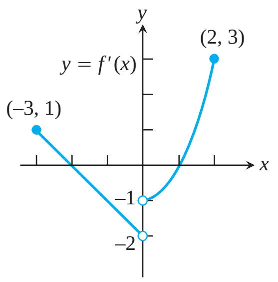


20. Estimate the intervals on which the function $y = f(x)$ is

a. increasing.

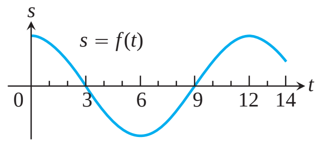
b. decreasing.

c. Use the given graph of f' to indicate where any local extreme values of the function occur, and whether each extreme is a relative maximum or minimum.



Each of the graphs in Exercises 21 and 22 is the graph of the position function $s = f(t)$ of a body moving on a coordinate line (t represents time). At approximately what times (if any) is each body's **(a)** velocity equal to zero? **(b)** Acceleration equal to zero? During approximately what time intervals does the body move **(c)** forward? **(d)** Backward?

21.



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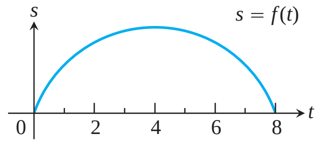


Graphs and Graphing

Graph the curves in Exercises 23–32.

23. $y = x^2 - (x^3/6)$ **24.** $y = x^3 - 3x^2 + 3$

25. $y = -x^3 + 6x^2 - 9x + 3$

26. $y = (1/8)(x^3 + 3x^2 - 9x - 27)$

27. $y = x^3(8 - x)$ **28.** $y = x^2(2x^2 - 9)$

29. $y = x - 3x^{2/3}$ **30.** $y = x^{1/3}(x - 4)$

31. $y = x\sqrt{3 - x}$ **32.** $y = x\sqrt{4 - x^2}$

Each of Exercises 33–38 gives the first derivative of a function $y = f(x)$. **(a)** At what points, if any, does the graph of f have a local maximum, local minimum, or inflection point? **(b)** Sketch the general shape of the graph.

33. $y' = 16 - x^2$ **34.** $y' = x^2 - x - 6$

35. $y' = 6x(x + 1)(x - 2)$ **36.** $y' = x^2(6 - 4x)$

37. $y' = x^4 - 2x^2$ **38.** $y' = 4x^2 - x^4$

In Exercises 39–42, graph each function. Then use the function's first derivative to explain what you see.

39. $y = x^{2/3} + (x - 1)^{1/3}$ **40.** $y = x^{2/3} + (x - 1)^{2/3}$

41. $y = x^{1/3} + (x - 1)^{1/3}$ **42.** $y = x^{2/3} - (x - 1)^{1/3}$

Sketch the graphs of the functions in Exercises 43–50.

43. $y = \dfrac{x + 1}{x - 3}$ **44.** $y = \dfrac{2x}{x + 5}$

45. $y = \dfrac{x^2 + 1}{x}$ **46.** $y = \dfrac{x^2 - x + 1}{x}$

47. $y = \dfrac{x^3 + 2}{2x}$ **48.** $y = \dfrac{x^4 - 1}{x^2}$

49. $y = \dfrac{x^2 - 4}{x^2 - 3}$ **50.** $y = \dfrac{x^2}{x^2 - 4}$

Applying l'Hôpital's Rule

Use l'Hôpital's Rule to find the limits in Exercises 51–62.

51. $\displaystyle\lim_{x \to 1} \frac{x^2 + 3x - 4}{x - 1}$ **52.** $\displaystyle\lim_{x \to 1} \frac{x^a - 1}{x^b - 1}$

53. $\displaystyle\lim_{x \to \pi} \frac{\tan x}{x}$ **54.** $\displaystyle\lim_{x \to 0} \frac{\tan x}{x + \sin x}$

55. $\displaystyle\lim_{x \to 0} \frac{\sin^2 x}{\tan(x^2)}$ **56.** $\displaystyle\lim_{x \to 0} \frac{\sin mx}{\sin nx}$

57. $\displaystyle\lim_{x \to \pi/2^-} \sec 7x \cos 3x$ **58.** $\displaystyle\lim_{x \to 0^+} \sqrt{x} \sec x$

59. $\displaystyle\lim_{x \to 0} (\csc x - \cot x)$ **60.** $\displaystyle\lim_{x \to 0} \left(\frac{1}{x^4} - \frac{1}{x^2}\right)$

61. $\displaystyle\lim_{x \to \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x}\right)$

62. $\displaystyle\lim_{x \to \infty} \left(\frac{x^3}{x^2 - 1} - \frac{x^3}{x^2 + 1}\right)$

Optimization

63. The sum of two nonnegative numbers is 36. Find the numbers if

a. the difference of their square roots is to be as large as possible.

b. the sum of their square roots is to be as large as possible.

64. The sum of two nonnegative numbers is 20. Find the numbers

a. if the product of one number and the square root of the other is to be as large as possible.

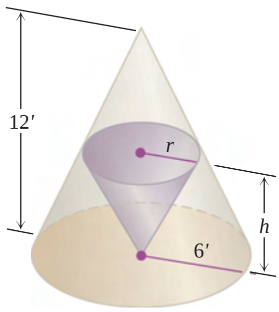
b. if one number plus the square root of the other is to be as large as possible.

65. An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

66. A customer has asked you to design an open-top rectangular stainless steel vat. It is to have a square base and a volume of 32 ft^3, to be welded from quarter-inch plate, and to weigh no more than necessary. What dimensions do you recommend?

67. Find the height and radius of the largest right circular cylinder that can be put in a sphere of radius $\sqrt{3}$.

68. The figure here shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base. What values of r and h will give the smaller cone the largest possible volume?



69. Manufacturing tires Your company can manufacture x hundred grade A tires and y hundred grade B tires a day, where $0 \le x \le 4$ and

$$y = \frac{40 - 10x}{5 - x}.$$

Your profit on a grade A tire is twice your profit on a grade B tire. What is the most profitable number of each kind to make?

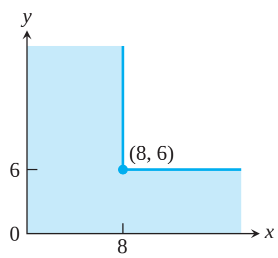
70. Particle motion The positions of two particles on the s-axis are $s_1 = \cos t$ and $s_2 = \cos(t + \pi/4)$.

a. What is the farthest apart the particles ever get?

b. When do the particles collide?

T **71. Open-top box** An open-top rectangular box is constructed from a 10-in.-by-16-in. piece of cardboard by cutting squares of equal side length from the corners and folding up the sides. Find analytically the dimensions of the box of largest volume and the maximum volume. Support your answers graphically.

72. The ladder problem What is the approximate length (in feet) of the longest ladder you can carry horizontally around the corner of the corridor shown here? Round your answer down to the nearest foot.



Newton's Method

73. Let $f(x) = 3x - x^3$. Show that the equation $f(x) = -4$ has a solution in the interval [2, 3] and use Newton's method to find it.

74. Let $f(x) = x^4 - x^3$. Show that the equation $f(x) = 75$ has a solution in the interval [3, 4] and use Newton's method to find it.

Finding Indefinite Integrals

Find the indefinite integrals (most general antiderivatives) in Exercises 75–90. Check your answers by differentiation.

75. $\int (x^3 + 5x - 7)\,dx$

76. $\int \left(8t^3 - \frac{t^2}{2} + t\right) dt$

77. $\int \left(3\sqrt{t} + \frac{4}{t^2}\right) dt$

78. $\int \left(\frac{1}{2\sqrt{t}} - \frac{3}{t^4}\right) dt$

79. $\int \frac{dr}{(r + 5)^2}$

80. $\int \frac{6\,dr}{\left(r - \sqrt{2}\right)^3}$

81. $\int 3\theta\sqrt{\theta^2 + 1}\,d\theta$

82. $\int \frac{\theta}{\sqrt{7 + \theta^2}}\,d\theta$

83. $\int x^3(1 + x^4)^{-1/4}\,dx$

84. $\int (2 - x)^{3/5}\,dx$

85. $\int \sec^2 \frac{s}{10}\,ds$

86. $\int \csc^2 \pi s\,ds$

87. $\int \csc \sqrt{2}\theta \cot \sqrt{2}\theta\,d\theta$

88. $\int \sec \frac{\theta}{3} \tan \frac{\theta}{3}\,d\theta$

89. $\int \sin^2 \frac{x}{4}\,dx$

90. $\int \cos^2 \frac{x}{2}\,dx \quad \left(\textit{Hint: } \cos^2\theta = \frac{1 + \cos 2\theta}{2}\right)$

Initial Value Problems

Solve the initial value problems in Exercises 91–94.

91. $\frac{dy}{dx} = \frac{x^2 + 1}{x^2}, \quad y(1) = -1$

92. $\frac{dy}{dx} = \left(x + \frac{1}{x}\right)^2, \quad y(1) = 1$

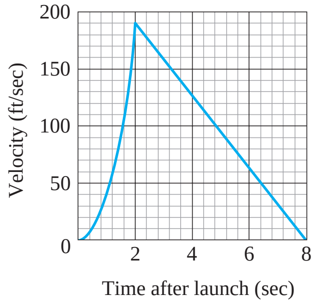
93. $\frac{d^2r}{dt^2} = 15\sqrt{t} + \frac{3}{\sqrt{t}}; \quad r'(1) = 8, \quad r(1) = 0$

94. $\frac{d^3r}{dt^3} = -\cos t; \quad r''(0) = r'(0) = 0, \quad r(0) = -1$

Chapter 5 Practice Exercises

Finite Sums and Estimates

1. The accompanying figure shows the graph of the velocity (ft/sec) of a model rocket for the first 8 sec after launch. The rocket accelerated straight up for the first 2 sec and then coasted to reach its maximum height at $t = 8$ sec.

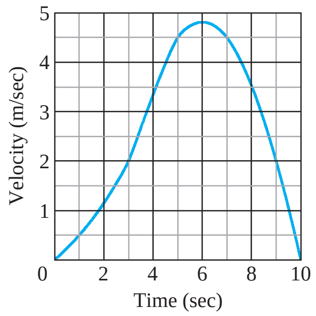


a. Assuming that the rocket was launched from ground level, about how high did it go? (This is the rocket in Section 3.3, Exercise 17, but you do not need to do Exercise 17 to do the exercise here.)

b. Sketch a graph of the rocket's height aboveground as a function of time for $0 \le t \le 8$.

2. a. The accompanying figure shows the velocity (m/sec) of a body moving along the s-axis during the time interval from $t = 0$ to $t = 10$ sec. About how far did the body travel during those 10 sec?

b. Sketch a graph of s as a function of t for $0 \le t \le 10$ assuming $s(0) = 0$.



3. Suppose that $\sum_{k=1}^{10} a_k = -2$ and $\sum_{k=1}^{10} b_k = 25$. Find the value of

a. $\sum_{k=1}^{10} \frac{a_k}{4}$ **b.** $\sum_{k=1}^{10} (b_k - 3a_k)$

c. $\sum_{k=1}^{10} (a_k + b_k - 1)$ **d.** $\sum_{k=1}^{10} \left(\frac{5}{2} - b_k\right)$

4. Suppose that $\sum_{k=1}^{20} a_k = 0$ and $\sum_{k=1}^{20} b_k = 7$. Find the values of

a. $\sum_{k=1}^{20} 3a_k$ **b.** $\sum_{k=1}^{20} (a_k + b_k)$

c. $\sum_{k=1}^{20} \left(\frac{1}{2} - \frac{2b_k}{7}\right)$ **d.** $\sum_{k=1}^{20} (a_k - 2)$

Definite Integrals

In Exercises 5–8, express each limit as a definite integral. Then evaluate the integral to find the value of the limit. In each case, P is a partition of the given interval and the numbers c_k are chosen from the subintervals of P.

5. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (2c_k - 1)^{-1/2} \, \Delta x_k$, where P is a partition of $[1, 5]$

6. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} c_k (c_k{}^2 - 1)^{1/3} \, \Delta x_k$, where P is a partition of $[1, 3]$

7. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} \left(\cos\left(\frac{c_k}{2}\right)\right) \Delta x_k$, where P is a partition of $[-\pi, 0]$

8. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (\sin c_k)(\cos c_k) \, \Delta x_k$, where P is a partition of $[0, \pi/2]$

9. If $\int_{-2}^{2} 3f(x)\,dx = 12$, $\int_{-2}^{5} f(x)\,dx = 6$, and $\int_{-2}^{5} g(x)\,dx = 2$, find the values of the following.

a. $\int_{-2}^{2} f(x)\,dx$ **b.** $\int_{2}^{5} f(x)\,dx$

c. $\int_{5}^{-2} g(x)\,dx$ **d.** $\int_{-2}^{5} (-\pi g(x))\,dx$

e. $\int_{-2}^{5} \left(\frac{f(x) + g(x)}{5}\right) dx$

10. If $\int_{0}^{2} f(x)\,dx = \pi$, $\int_{0}^{2} 7g(x)\,dx = 7$, and $\int_{0}^{1} g(x)\,dx = 2$, find the values of the following.

a. $\int_{0}^{2} g(x)\,dx$ **b.** $\int_{1}^{2} g(x)\,dx$

c. $\int_{2}^{0} f(x)\,dx$ **d.** $\int_{0}^{2} \sqrt{2}\, f(x)\,dx$

e. $\int_{0}^{2} (g(x) - 3f(x))\,dx$

Area

In Exercise 11–14, find the total area of the region between the graph of f and the x-axis.

11. $f(x) = x^2 - 4x + 3, \quad 0 \le x \le 3$

12. $f(x) = 1 - (x^2/4), \quad -2 \le x \le 3$

13. $f(x) = 5 - 5x^{2/3}, \quad -1 \le x \le 8$

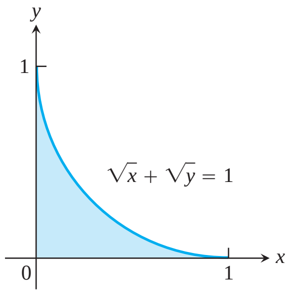
14. $f(x) = 1 - \sqrt{x}, \quad 0 \le x \le 4$

Find the areas of the regions enclosed by the curves and lines in Exercises 15–26.

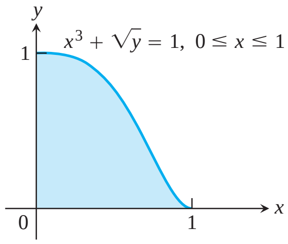
15. $y = x, \quad y = 1/x^2, \quad x = 2$

16. $y = x, \quad y = 1/\sqrt{x}, \quad x = 2$

17. $\sqrt{x} + \sqrt{y} = 1, \quad x = 0, \quad y = 0$



18. $x^3 + \sqrt{y} = 1, \quad x = 0, \quad y = 0, \quad \text{for} \quad 0 \le x \le 1$



19. $x = 2y^2, \quad x = 0, \quad y = 3$

20. $x = 4 - y^2, \quad x = 0$

21. $y^2 = 4x, \quad y = 4x - 2$

22. $y^2 = 4x + 4, \quad y = 4x - 16$

23. $y = \sin x, \quad y = x, \quad 0 \le x \le \pi/4$

24. $y = |\sin x|, \quad y = 1, \quad -\pi/2 \le x \le \pi/2$

25. $y = 2\sin x, \quad y = \sin 2x, \quad 0 \le x \le \pi$

26. $y = 8\cos x, \quad y = \sec^2 x, \quad -\pi/3 \le x \le \pi/3$

27. Find the area of the "triangular" region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$.

28. Find the area of the "triangular" region bounded on the left by $y = \sqrt{x}$, on the right by $y = 6 - x$, and below by $y = 1$.

29. Find the extreme values of $f(x) = x^3 - 3x^2$ and find the area of the region enclosed by the graph of f and the x-axis.

30. Find the area of the region cut from the first quadrant by the curve $x^{1/2} + y^{1/2} = a^{1/2}$.

31. Find the total area of the region enclosed by the curve $x = y^{2/3}$ and the lines $x = y$ and $y = -1$.

32. Find the total area of the region between the curves $y = \sin x$ and $y = \cos x$ for $0 \le x \le 3\pi/2$.

Initial Value Problems

33. Show that $y = x^2 + \int_1^x \frac{1}{t}\,dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = 2 - \frac{1}{x^2}; \quad y'(1) = 3, \quad y(1) = 1.$$

34. Show that $y = \int_0^x \left(1 + 2\sqrt{\sec t}\right) dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = \sqrt{\sec x}\tan x; \quad y'(0) = 3, \quad y(0) = 0.$$

Express the solutions of the initial value problems in Exercises 35 and 36 in terms of integrals.

35. $\dfrac{dy}{dx} = \dfrac{\sin x}{x}, \quad y(5) = -3$

36. $\dfrac{dy}{dx} = \sqrt{2 - \sin^2 x}, \quad y(-1) = 2$

Evaluating Indefinite Integrals

Evaluate the integrals in Exercises 37–44.

37. $\int 2(\cos x)^{-1/2} \sin x\,dx$

38. $\int (\tan x)^{-3/2} \sec^2 x\,dx$

39. $\int (2\theta + 1 + 2\cos(2\theta + 1))\,d\theta$

40. $\int \left(\dfrac{1}{\sqrt{2\theta - \pi}} + 2\sec^2(2\theta - \pi)\right) d\theta$

41. $\int \left(t - \dfrac{2}{t}\right)\left(t + \dfrac{2}{t}\right) dt$

42. $\int \dfrac{(t + 1)^2 - 1}{t^4}\,dt$

43. $\int \sqrt{t}\sin(2t^{3/2})\,dt$

44. $\int \sec\theta\tan\theta\sqrt{1 + \sec\theta}\,d\theta$

Evaluating Definite Integrals

Evaluate the integrals in Exercises 45–70.

45. $\int_{-1}^{1} (3x^2 - 4x + 7)\,dx$

46. $\int_0^1 (8s^3 - 12s^2 + 5)\,ds$

47. $\int_1^2 \dfrac{4}{v^2}\,dv$

48. $\int_1^{27} x^{-4/3}\,dx$

49. $\int_1^4 \dfrac{dt}{t\sqrt{t}}$

50. $\int_1^4 \dfrac{\left(1 + \sqrt{u}\right)^{1/2}}{\sqrt{u}}\,du$

51. $\int_0^1 \dfrac{36\,dx}{(2x + 1)^3}$

52. $\int_0^1 \dfrac{dr}{\sqrt[3]{(7 - 5r)^2}}$

53. $\int_{1/8}^{1} x^{-1/3}(1 - x^{2/3})^{3/2}\,dx$

54. $\int_0^{1/2} x^3(1 + 9x^4)^{-3/2}\,dx$

55. $\int_0^{\pi} \sin^2 5r\,dr$

56. $\int_0^{\pi/4} \cos^2\left(4t - \dfrac{\pi}{4}\right) dt$

57. $\displaystyle\int_0^{\pi/3} \sec^2\theta\, d\theta$

58. $\displaystyle\int_{\pi/4}^{3\pi/4} \csc^2 x\, dx$

59. $\displaystyle\int_{\pi}^{3\pi} \cot^2\frac{x}{6}\, dx$

60. $\displaystyle\int_0^{\pi} \tan^2\frac{\theta}{3}\, d\theta$

61. $\displaystyle\int_{-\pi/3}^{0} \sec x \tan x\, dx$

62. $\displaystyle\int_{\pi/4}^{3\pi/4} \csc z \cot z\, dz$

63. $\displaystyle\int_0^{\pi/2} 5(\sin x)^{3/2}\cos x\, dx$

64. $\displaystyle\int_{-1}^{1} 2x\sin(1 - x^2)\, dx$

65. $\displaystyle\int_{-\pi/2}^{\pi/2} 15\sin^4 3x\cos 3x\, dx$

66. $\displaystyle\int_0^{2\pi/3} \cos^{-4}\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) dx$

67. $\displaystyle\int_0^{\pi/2} \frac{3\sin x\cos x}{\sqrt{1 + 3\sin^2 x}}\, dx$

68. $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{(1 + 7\tan x)^{2/3}}\, dx$

69. $\displaystyle\int_0^{\pi/3} \frac{\tan\theta}{\sqrt{2\sec\theta}}\, d\theta$

70. $\displaystyle\int_{\pi^2/36}^{\pi^2/4} \frac{\cos\sqrt{t}}{\sqrt{t}\sin\sqrt{t}}\, dt$

Average Values

71. Find the average value of $f(x) = mx + b$

a. over $[-1, 1]$

b. over $[-k, k]$

72. Find the average value of

a. $y = \sqrt{3x}$ over $[0, 3]$

b. $y = \sqrt{ax}$ over $[0, a]$

73. Let f be a function that is differentiable on $[a, b]$. In Chapter 2 we defined the average rate of change of f over $[a, b]$ to be

$$\frac{f(b) - f(a)}{b - a}$$

and the instantaneous rate of change of f at x to be $f'(x)$. In this chapter we defined the average value of a function. For the new definition of average to be consistent with the old one, we should have

$$\frac{f(b) - f(a)}{b - a} = \text{average value of } f' \text{ on } [a, b].$$

Is this the case? Give reasons for your answer.

74. Is it true that the average value of an integrable function over an interval of length 2 is half the function's integral over the interval? Give reasons for your answer.

T **75.** Compute the average value of the temperature function

$$f(x) = 37\sin\left(\frac{2\pi}{365}(x - 101)\right) + 25$$

for a 365-day year. This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is 25.7°F, which is slightly higher than the average value of $f(x)$. Figure 3.33 shows why.

T **76. Specific heat of a gas** Specific heat C_v is the amount of heat required to raise the temperature of a given mass of gas with constant volume by 1°C, measured in units of cal/deg-mole (calories per degree gram molecule). The specific heat of oxygen depends on its temperature T and satisfies the formula

$$C_v = 8.27 + 10^{-5}(26T - 1.87T^2).$$

Find the average value of C_v for $20° \le T \le 675°$C and the temperature at which it is attained.

Differentiating Integrals

In Exercises 77–80, find dy/dx.

77. $y = \displaystyle\int_2^{x} \sqrt{2 + \cos^3 t}\, dt$

78. $y = \displaystyle\int_2^{7x^2} \sqrt{2 + \cos^3 t}\, dt$

79. $y = \displaystyle\int_x^{1} \frac{6}{3 + t^4}\, dt$

80. $y = \displaystyle\int_{\sec x}^{2} \frac{1}{t^2 + 1}\, dt$

Theory and Examples

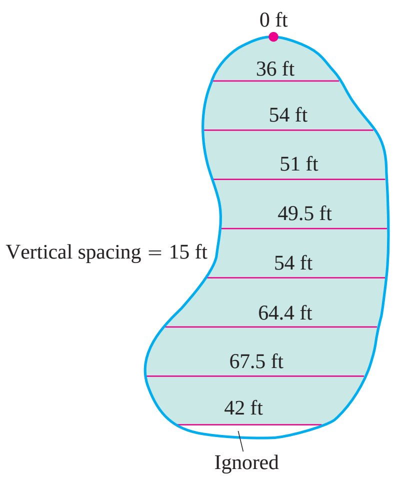
81. Is it true that every function $y = f(x)$ that is differentiable on $[a, b]$ is itself the derivative of some function on $[a, b]$? Give reasons for your answer.

82. Suppose that $F(x)$ is an antiderivative of $f(x) = \sqrt{1 + x^4}$. Express $\int_0^1 \sqrt{1 + x^4}\, dx$ in terms of F and give a reason for your answer.

83. Find dy/dx if $y = \int_x^1 \sqrt{1 + t^2}\, dt$. Explain the main steps in your calculation.

84. Find dy/dx if $y = \int_{\cos x}^0 (1/(1 - t^2))\, dt$. Explain the main steps in your calculation.

85. A new parking lot To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for \$10,000. The cost to clear the land will be \$0.10 a square foot, and the lot will cost \$2.00 a square foot to pave. Can the job be done for \$10,000? Use a lower sum estimate to see. (Answers may vary slightly, depending on the estimate used.)



86. Skydivers A and B are in a helicopter hovering at 6400 ft. Skydiver A jumps and descends for 4 sec before opening her parachute. The helicopter then climbs to 7000 ft and hovers there. Forty-five seconds after A leaves the aircraft, B jumps and descends for 13 sec before opening his parachute. Both skydivers descend at 16 ft/sec with parachutes open. Assume that the skydivers fall freely (no effective air resistance) before their parachutes open.

a. At what altitude does A's parachute open?

b. At what altitude does B's parachute open?

c. Which skydiver lands first?

Average Daily Inventory

Average value is used in economics to study such things as average daily inventory. If $I(t)$ is the number of radios, tires, shoes, or whatever product a firm has on hand on day t (we call I an **inventory function**), the average value of I over a time period $[0, T]$ is called the firm's average daily inventory for the period.

$$\textbf{Average daily inventory} = \text{av}(I) = \frac{1}{T}\int_0^T I(t)\,dt.$$

If h is the dollar cost of holding one item per day, the product $\text{av}(I) \cdot h$ is the **average daily holding cost** for the period.

87. As a wholesaler, Tracey Burr Distributors receives a shipment of 1200 cases of chocolate bars every 30 days. TBD sells the chocolate to retailers at a steady rate, and t days after a shipment arrives, its inventory of cases on hand is $I(t) = 1200 - 40t$, $0 \le t \le 30$. What is TBD's average daily inventory for the 30-day period? What is its average daily holding cost if the cost of holding one case is 3¢ a day?

88. Rich Wholesale Foods, a manufacturer of cookies, stores its cases of cookies in an air-conditioned warehouse for shipment every 14 days. Rich tries to keep 600 cases on reserve to meet occasional peaks in demand, so a typical 14-day inventory function is $I(t) = 600 + 600t$, $0 \le t \le 14$. The daily holding cost for each case is 4¢ per day. Find Rich's average daily inventory and average daily holding cost.

89. Solon Container receives 450 drums of plastic pellets every 30 days. The inventory function (drums on hand as a function of days) is $I(t) = 450 - t^2/2$. Find the average daily inventory. If the holding cost for one drum is 2¢ per day, find the average daily holding cost.

90. Mitchell Mailorder receives a shipment of 600 cases of athletic socks every 60 days. The number of cases on hand t days after the shipment arrives is $I(t) = 600 - 20\sqrt{15t}$. Find the average daily inventory. If the holding cost for one case is 1/2¢ per day, find the average daily holding cost.

Chapter 6 Practice Exercises

Volumes

Find the volumes of the solids in Exercises 1–16.

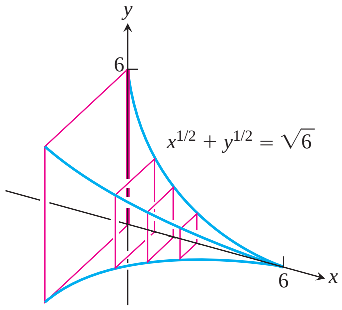
1. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = \sqrt{x}$.

2. The base of the solid is the region in the first quadrant between the line $y = x$ and the parabola $y = 2\sqrt{x}$. The cross-sections of

the solid perpendicular to the x-axis are equilateral triangles whose bases stretch from the line to the curve.

3. The solid lies between planes perpendicular to the x-axis at $x = \pi/4$ and $x = 5\pi/4$. The cross-sections between these planes are circular disks whose diameters run from the curve $y = 2\cos x$ to the curve $y = 2\sin x$.

4. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 6$. The cross-sections between these planes are squares whose bases run from the x-axis up to the curve $x^{1/2} + y^{1/2} = \sqrt{6}$.



5. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections of the solid perpendicular to the x-axis between these planes are circular disks whose diameters run from the curve $x^2 = 4y$ to the curve $y^2 = 4x$.

6. The base of the solid is the region bounded by the parabola $y^2 = 4x$ and the line $x = 1$ in the xy-plane. Each cross-section perpendicular to the x-axis is an equilateral triangle with one edge in the plane. (The triangles all lie on the same side of the plane.)

7. Find the volume of the solid generated by revolving the region bounded by the x-axis, the curve $y = 3x^4$, and the lines $x = 1$ and $x = -1$ about **(a)** the x-axis; **(b)** the y-axis; **(c)** the line $x = 1$; **(d)** the line $y = 3$.

8. Find the volume of the solid generated by revolving the "triangular" region bounded by the curve $y = 4/x^3$ and the lines $x = 1$ and $y = 1/2$ about **(a)** the x-axis; **(b)** the y-axis; **(c)** the line $x = 2$; **(d)** the line $y = 4$.

9. Find the volume of the solid generated by revolving the region bounded on the left by the parabola $x = y^2 + 1$ and on the right by the line $x = 5$ about **(a)** the x-axis; **(b)** the y-axis; **(c)** the line $x = 5$.

10. Find the volume of the solid generated by revolving the region bounded by the parabola $y^2 = 4x$ and the line $y = x$ about **(a)** the x-axis; **(b)** the y-axis; **(c)** the line $x = 4$; **(d)** the line $y = 4$.

11. Find the volume of the solid generated by revolving the "triangular" region bounded by the x-axis, the line $x = \pi/3$, and the curve $y = \tan x$ in the first quadrant about the x-axis.

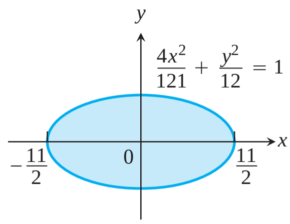
12. Find the volume of the solid generated by revolving the region bounded by the curve $y = \sin x$ and the lines $x = 0$, $x = \pi$, and $y = 2$ about the line $y = 2$.

13. Find the volume of the solid generated by revolving the region between the x-axis and the curve $y = x^2 - 2x$ about **(a)** the x-axis; **(b)** the line $y = -1$; **(c)** the line $x = 2$; **(d)** the line $y = 2$.

14. Find the volume of the solid generated by revolving about the x-axis the region bounded by $y = 2\tan x$, $y = 0$, $x = -\pi/4$, and $x = \pi/4$. (The region lies in the first and third quadrants and resembles a skewed bowtie.)

15. Volume of a solid sphere hole A round hole of radius $\sqrt{3}$ ft is bored through the center of a solid sphere of a radius 2 ft. Find the volume of material removed from the sphere.

16. Volume of a football The profile of a football resembles the ellipse shown here. Find the football's volume to the nearest cubic inch.



Lengths of Curves

Find the lengths of the curves in Exercises 17–23.

17. $y = x^{1/2} - (1/3)x^{3/2}, \quad 1 \le x \le 4$

18. $x = y^{2/3}, \quad 1 \le y \le 8$

19. $y = (5/12)x^{6/5} - (5/8)x^{4/5}, \quad 1 \le x \le 32$

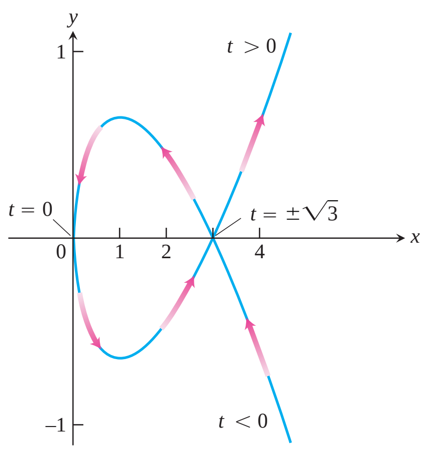
20. $x = (y^3/12) + (1/y), \quad 1 \le y \le 2$

21. $x = 5\cos t - \cos 5t, \quad y = 5\sin t - \sin 5t, \quad 0 \le t \le \pi/2$

22. $x = t^3 - 6t^2, \quad y = t^3 + 6t^2, \quad 0 \le t \le 1$

23. $x = 3\cos\theta, \quad y = 3\sin\theta, \quad 0 \le \theta \le \dfrac{3\pi}{2}$

24. Find the length of the enclosed loop $x = t^2$, $y = (t^3/3) - t$ shown here. The loop starts at $t = -\sqrt{3}$ and ends at $t = \sqrt{3}$.



Centroids and Centers of Mass

25. Find the centroid of a thin, flat plate covering the region enclosed by the parabolas $y = 2x^2$ and $y = 3 - x^2$.

26. Find the centroid of a thin, flat plate covering the region enclosed by the x-axis, the lines $x = 2$ and $x = -2$, and the parabola $y = x^2$.

27. Find the centroid of a thin, flat plate covering the "triangular" region in the first quadrant bounded by the y-axis, the parabola $y = x^2/4$, and the line $y = 4$.

28. Find the centroid of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$.

29. Find the center of mass of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$ if the density function is $\delta(y) = 1 + y$. (Use horizontal strips.)

30. **a.** Find the center of mass of a thin plate of constant density covering the region between the curve $y = 3/x^{3/2}$ and the x-axis from $x = 1$ to $x = 9$.

b. Find the plate's center of mass if, instead of being constant, the density is $\delta(x) = x$. (Use vertical strips.)

Areas of Surfaces of Revolution

In Exercises 31–36, find the areas of the surfaces generated by revolving the curves about the given axes.

31. $y = \sqrt{2x + 1}, \quad 0 \le x \le 3; \quad x\text{-axis}$

32. $y = x^3/3, \quad 0 \le x \le 1; \quad x\text{-axis}$

33. $x = \sqrt{4y - y^2}, \quad 1 \le y \le 2; \quad y\text{-axis}$

34. $x = \sqrt{y}, \quad 2 \le y \le 6; \quad y\text{-axis}$

35. $x = t^2/2, \quad y = 2t, \quad 0 \le t \le \sqrt{5}; \quad x\text{-axis}$

36. $x = t^2 + 1/(2t), \quad y = 4\sqrt{t}, \quad 1/\sqrt{2} \le t \le 1; \quad y\text{-axis}$

Work

37. Lifting equipment A rock climber is about to haul up 100 N (about 22.5 lb) of equipment that has been hanging beneath her on 40 m of rope that weighs 0.8 newton per meter. How much work will it take? (*Hint:* Solve for the rope and equipment separately, then add.)

38. Leaky tank truck You drove an 800-gal tank truck of water from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-ft elevation change in 50 min. Assuming that the water leaked out at a steady rate, how much work was spent in carrying water to the top? Do not count the work done in getting yourself and the truck there. Water weights 8 lb/U.S. gal.

39. Stretching a spring If a force of 20 lb is required to hold a spring 1 ft beyond its unstressed length, how much work does it take to stretch the spring this far? An additional foot?

40. Garage door spring A force of 200 N will stretch a garage door spring 0.8 m beyond its unstressed length. How far will a 300-N force stretch the spring? How much work does it take to stretch the spring this far from its unstressed length?

41. Pumping a reservoir A reservoir shaped like a right circular cone, point down, 20 ft across the top and 8 ft deep, is full of water. How much work does it take to pump the water to a level 6 ft above the top?

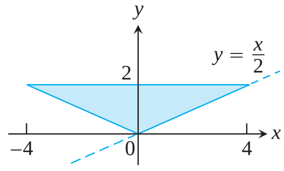
42. Pumping a reservoir (*Continuation of Exercise 41.*) The reservoir is filled to a depth of 5 ft, and the water is to be pumped to the same level as the top. How much work does it take?

43. Pumping a conical tank A right circular conical tank, point down, with top radius 5 ft and height 10 ft is filled with a liquid whose weight-density is 60 lb/ft^3. How much work does it take to pump the liquid to a point 2 ft above the tank? If the pump is driven by a motor rated at 275 ft-lb/sec (1/2 hp), how long will it take to empty the tank?

44. Pumping a cylindrical tank A storage tank is a right circular cylinder 20 ft long and 8 ft in diameter with its axis horizontal. If the tank is half full of olive oil weighing 57 lb/ft^3, find the work done in emptying it through a pipe that runs from the bottom of the tank to an outlet that is 6 ft above the top of the tank.

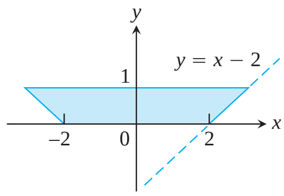
Fluid Force

45. Trough of water The vertical triangular plate shown here is the end plate of a trough full of water $(w = 62.4)$. What is the fluid force against the plate?



UNITS IN FEET

46. Trough of maple syrup The vertical trapezoid plate shown here is the end plate of a trough full of maple syrup weighing 75 lb/ft^3. What is the force exerted by the syrup against the end plate of the trough when the syrup is 10 in. deep?

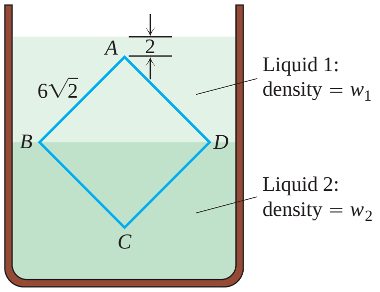


UNITS IN FEET

47. Force on a parabolic gate A flat vertical gate in the face of a dam is shaped like the parabolic region between the curve $y = 4x^2$ and the line $y = 4$, with measurements in feet. The top of the gate lies 5 ft below the surface of the water. Find the force exerted by the water against the gate $(w = 62.4)$.

T 48. You plan to store mercury ($w = 849$ lb/ft^3) in a vertical rectangular tank with a 1 ft square base side whose interior side wall can withstand a total fluid force of 40,000 lb. About how many cubic feet of mercury can you store in the tank at any one time?

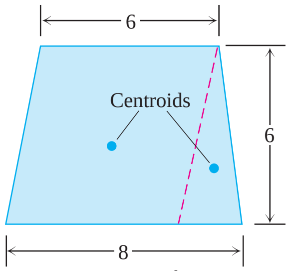
49. The container profiled in the accompanying figure is filled with two nonmixing liquids of weight-density w_1 and w_2. Find the fluid force on one side of the vertical square plate $ABCD$. The points B and D lie in the boundary layer and the square is $6\sqrt{2}$ ft on a side.



50. The isosceles trapezoidal plate shown here is submerged vertically in water ($w = 62.4$) with its upper edge 4 ft below the surface. Find the fluid force on one side of the plate in two different ways:

a. By evaluating an integral.

b. By dividing the plate into a parallelogram and an isosceles triangle, locating their centroids, and using the equation $F = w\bar{h}A$ from Section 6.7.



Chapter 7 Practice Exercises

Differentiation

In Exercises 1–24, find the derivative of y with respect to the appropriate variable.

1. $y = 10e^{-x/5}$
2. $y = \sqrt{2}e^{\sqrt{2}x}$
3. $y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}$
4. $y = x^2e^{-2/x}$
5. $y = \ln(\sin^2\theta)$
6. $y = \ln(\sec^2\theta)$
7. $y = \log_2(x^2/2)$
8. $y = \log_5(3x - 7)$
9. $y = 8^{-t}$
10. $y = 9^{2t}$
11. $y = 5x^{3.6}$
12. $y = \sqrt{2}x^{-\sqrt{2}}$
13. $y = (x + 2)^{x+2}$
14. $y = 2(\ln x)^{x/2}$
15. $y = \sin^{-1}\sqrt{1 - u^2}, \quad 0 < u < 1$
16. $y = \sin^{-1}\left(\frac{1}{\sqrt{v}}\right), \quad v > 1$
17. $y = \ln\cos^{-1}x$
18. $y = z\cos^{-1}z - \sqrt{1 - z^2}$
19. $y = t\tan^{-1}t - \frac{1}{2}\ln t$
20. $y = (1 + t^2)\cot^{-1}2t$
21. $y = z\sec^{-1}z - \sqrt{z^2 - 1}, \quad z > 1$
22. $y = 2\sqrt{x - 1}\sec^{-1}\sqrt{x}$
23. $y = \csc^{-1}(\sec\theta), \quad 0 < \theta < \pi/2$
24. $y = (1 + x^2)e^{\tan^{-1}x}$

Logarithmic Differentiation

In Exercises 25–30, use logarithmic differentiation to find the derivative of y with respect to the appropriate variable.

25. $y = \frac{2(x^2 + 1)}{\sqrt{\cos 2x}}$
26. $y = \sqrt[10]{\frac{3x + 4}{2x - 4}}$
27. $y = \left(\frac{(t + 1)(t - 1)}{(t - 2)(t + 3)}\right)^5, \quad t > 2$
28. $y = \frac{2u2^u}{\sqrt{u^2 + 1}}$
29. $y = (\sin\theta)^{\sqrt{\theta}}$
30. $y = (\ln x)^{1/(\ln x)}$

Integration

Evaluate the integrals in Exercises 31–78.

31. $\int e^x \sin(e^x)\,dx$
32. $\int e^t\cos(3e^t - 2)\,dt$

33. $\int e^x \sec^2(e^x - 7)\,dx$

34. $\int e^y \csc(e^y + 1)\cot(e^y + 1)\,dy$

35. $\int \sec^2(x)e^{\tan x}\,dx$

36. $\int \csc^2 x\, e^{\cot x}\,dx$

37. $\int_{-1}^{1} \frac{dx}{3x - 4}$

38. $\int_{1}^{e} \frac{\sqrt{\ln x}}{x}\,dx$

39. $\int_{0}^{\pi} \tan\frac{x}{3}\,dx$

40. $\int_{1/6}^{1/4} 2\cot \pi x\,dx$

41. $\int_{0}^{4} \frac{2t}{t^2 - 25}\,dt$

42. $\int_{-\pi/2}^{\pi/6} \frac{\cos t}{1 - \sin t}\,dt$

43. $\int \frac{\tan(\ln v)}{v}\,dv$

44. $\int \frac{dv}{v \ln v}$

45. $\int \frac{(\ln x)^{-3}}{x}\,dx$

46. $\int \frac{\ln(x - 5)}{x - 5}\,dx$

47. $\int \frac{1}{r}\csc^2(1 + \ln r)\,dr$

48. $\int \frac{\cos(1 - \ln v)}{v}\,dv$

49. $\int x3^{x^2}\,dx$

50. $\int 2^{\tan x}\sec^2 x\,dx$

51. $\int_{1}^{7} \frac{3}{x}\,dx$

52. $\int_{1}^{32} \frac{1}{5x}\,dx$

53. $\int_{1}^{4} \left(\frac{x}{8} + \frac{1}{2x}\right)dx$

54. $\int_{1}^{8} \left(\frac{2}{3x} - \frac{8}{x^2}\right)dx$

55. $\int_{-2}^{-1} e^{-(x+1)}\,dx$

56. $\int_{-\ln 2}^{0} e^{2w}\,dw$

57. $\int_{0}^{\ln 5} e^r(3e^r + 1)^{-3/2}\,dr$

58. $\int_{0}^{\ln 9} e^\theta(e^\theta - 1)^{1/2}\,d\theta$

59. $\int_{1}^{e} \frac{1}{x}(1 + 7\ln x)^{-1/3}\,dx$

60. $\int_{e}^{e^2} \frac{1}{x\sqrt{\ln x}}\,dx$

61. $\int_{1}^{3} \frac{(\ln(v + 1))^2}{v + 1}\,dv$

62. $\int_{2}^{4} (1 + \ln t)t \ln t\,dt$

63. $\int_{1}^{8} \frac{\log_4 \theta}{\theta}\,d\theta$

64. $\int_{1}^{e} \frac{8\ln 3 \log_3 \theta}{\theta}\,d\theta$

65. $\int_{-3/4}^{3/4} \frac{6\,dx}{\sqrt{9 - 4x^2}}$

66. $\int_{-1/5}^{1/5} \frac{6\,dx}{\sqrt{4 - 25x^2}}$

67. $\int_{-2}^{2} \frac{3\,dt}{4 + 3t^2}$

68. $\int_{\sqrt{3}}^{3} \frac{dt}{3 + t^2}$

69. $\int \frac{dy}{y\sqrt{4y^2 - 1}}$

70. $\int \frac{24\,dy}{y\sqrt{y^2 - 16}}$

71. $\int_{\sqrt{2}/3}^{2/3} \frac{dy}{|y|\sqrt{9y^2 - 1}}$

72. $\int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{dy}{|y|\sqrt{5y^2 - 3}}$

73. $\int \frac{dx}{\sqrt{-2x - x^2}}$

74. $\int \frac{dx}{\sqrt{-x^2 + 4x - 1}}$

75. $\int_{-2}^{-1} \frac{2\,dv}{v^2 + 4v + 5}$

76. $\int_{-1}^{1} \frac{3\,dv}{4v^2 + 4v + 4}$

77. $\int \frac{dt}{(t + 1)\sqrt{t^2 + 2t - 8}}$

78. $\int \frac{dt}{(3t + 1)\sqrt{9t^2 + 6t}}$

Solving Equations with Logarithmic or Exponential Terms

In Exercises 79–84, solve for y.

79. $3^y = 2^{y+1}$

80. $4^{-y} = 3^{y+2}$

81. $9e^{2y} = x^2$

82. $3^y = 3\ln x$

83. $\ln(y - 1) = x + \ln y$

84. $\ln(10\ln y) = \ln 5x$

Evaluating Limits

Find the limits in Exercises 85–96.

85. $\lim_{x\to 0} \frac{10^x - 1}{x}$

86. $\lim_{\theta\to 0} \frac{3^\theta - 1}{\theta}$

87. $\lim_{x\to 0} \frac{2^{\sin x} - 1}{e^x - 1}$

88. $\lim_{x\to 0} \frac{2^{-\sin x} - 1}{e^x - 1}$

89. $\lim_{x\to 0} \frac{5 - 5\cos x}{e^x - x - 1}$

90. $\lim_{x\to 0} \frac{4 - 4e^x}{xe^x}$

91. $\lim_{t\to 0^+} \frac{t - \ln(1 + 2t)}{t^2}$

92. $\lim_{x\to 4} \frac{\sin^2(\pi x)}{e^{x-4} + 3 - x}$

93. $\lim_{t\to 0^+} \left(\frac{e^t}{t} - \frac{1}{t}\right)$

94. $\lim_{y\to 0^+} e^{-1/y}\ln y$

95. $\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^x$

96. $\lim_{x\to 0^+} \left(1 + \frac{3}{x}\right)^x$

Comparing Growth Rates of Functions

97. Does f grow faster, slower, or at the same rate as g as $x \to \infty$? Give reasons for your answers.

   a. $f(x) = \log_2 x$, $g(x) = \log_3 x$
   b. $f(x) = x$, $g(x) = x + \frac{1}{x}$
   c. $f(x) = x/100$, $g(x) = xe^{-x}$
   d. $f(x) = x$, $g(x) = \tan^{-1} x$
   e. $f(x) = \csc^{-1} x$, $g(x) = 1/x$
   f. $f(x) = \sinh x$, $g(x) = e^x$

98. Does f grow faster, slower, or at the same rate as g as $x \to \infty$? Give reasons for your answers.

   a. $f(x) = 3^{-x}$, $g(x) = 2^{-x}$
   b. $f(x) = \ln 2x$, $g(x) = \ln x^2$
   c. $f(x) = 10x^3 + 2x^2$, $g(x) = e^x$
   d. $f(x) = \tan^{-1}(1/x)$, $g(x) = 1/x$
   e. $f(x) = \sin^{-1}(1/x)$, $g(x) = 1/x^2$
   f. $f(x) = \operatorname{sech} x$, $g(x) = e^{-x}$

99. True, or false? Give reasons for your answers.

a. $\frac{1}{x^2} + \frac{1}{x^4} = O\left(\frac{1}{x^2}\right)$ **b.** $\frac{1}{x^2} + \frac{1}{x^4} = O\left(\frac{1}{x^4}\right)$

c. $x = o(x + \ln x)$ **d.** $\ln(\ln x) = o(\ln x)$

e. $\tan^{-1} x = O(1)$ **f.** $\cosh x = O(e^x)$

100. True, or false? Give reasons for your answers.

a. $\frac{1}{x^4} = O\left(\frac{1}{x^2} + \frac{1}{x^4}\right)$ **b.** $\frac{1}{x^4} = o\left(\frac{1}{x^2} + \frac{1}{x^4}\right)$

c. $\ln x = o(x + 1)$ **d.** $\ln 2x = O(\ln x)$

e. $\sec^{-1} x = O(1)$ **f.** $\sinh x = O(e^x)$

Theory and Applications

101. The function $f(x) = e^x + x$, being differentiable and one-to-one, has a differentiable inverse $f^{-1}(x)$. Find the value of df^{-1}/dx at the point $f(\ln 2)$.

102. Find the inverse of the function $f(x) = 1 + (1/x), x \neq 0$. Then show that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$ and that

$$\left.\frac{df^{-1}}{dx}\right|_{f(x)} = \frac{1}{f'(x)}.$$

In Exercises 103 and 104, find the absolute maximum and minimum values of each function on the given interval.

103. $y = x \ln 2x - x, \quad \left[\frac{1}{2e}, \frac{e}{2}\right]$

104. $y = 10x(2 - \ln x), \quad (0, e^2]$

105. Area Find the area between the curve $y = 2(\ln x)/x$ and the x-axis from $x = 1$ to $x = e$.

106. Area

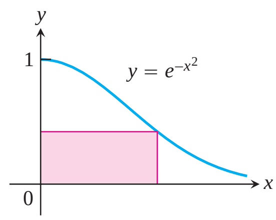
a. Show that the area between the curve $y = 1/x$ and the x-axis from $x = 10$ to $x = 20$ is the same as the area between the curve and the x-axis from $x = 1$ to $x = 2$.

b. Show that the area between the curve $y = 1/x$ and the x-axis from ka to kb is the same as the area between the curve and the x-axis from $x = a$ to $x = b$ $(0 < a < b, k > 0)$.

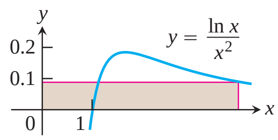
107. A particle is traveling upward and to the right along the curve $y = \ln x$. Its x-coordinate is increasing at the rate $(dx/dt) = \sqrt{x}$ m/sec. At what rate is the y-coordinate changing at the point $(e^2, 2)$?

108. A girl is sliding down a slide shaped like the curve $y = 9e^{-x/3}$. Her y-coordinate is changing at the rate $dy/dt = (-1/4)\sqrt{9 - y}$ ft/sec. At approximately what rate is her x-coordinate changing when she reaches the bottom of the slide at $x = 9$ ft? (Take e^3 to be 20 and round your answer to the nearest ft/sec.)

109. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = e^{-x^2}$. What dimensions give the rectangle its largest area, and what is that area?



110. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = (\ln x)/x^2$. What dimensions give the rectangle its largest area, and what is that area?



111. The functions $f(x) = \ln 5x$ and $g(x) = \ln 3x$ differ by a constant. What constant? Give reasons for your answer.

112. a. If $(\ln x)/x = (\ln 2)/2$, must $x = 2$?

b. If $(\ln x)/x = -2 \ln 2$, must $x = 1/2$?

Give reasons for your answers.

113. The quotient $(\log_4 x)/(\log_2 x)$ has a constant value. What value? Give reasons for your answer.

T **114.** $\log_x(2)$ **vs.** $\log_2(x)$ How does $f(x) = \log_x(2)$ compare with $g(x) = \log_2(x)$? Here is one way to find out.

a. Use the equation $\log_a b = (\ln b)/(\ln a)$ to express $f(x)$ and $g(x)$ in terms of natural logarithms.

b. Graph f and g together. Comment on the behavior of f in relation to the signs and values of g.

T **115.** Graph the following functions and use what you see to locate and estimate the extreme values, identify the coordinates of the inflection points, and identify the intervals on which the graphs are concave up and concave down. Then confirm your estimates by working with the functions' derivatives.

a. $y = (\ln x)/\sqrt{x}$ **b.** $y = e^{-x^2}$ **c.** $y = (1 + x)e^{-x}$

T **116.** Graph $f(x) = x \ln x$. Does the function appear to have an absolute minimum value? Confirm your answer with calculus.

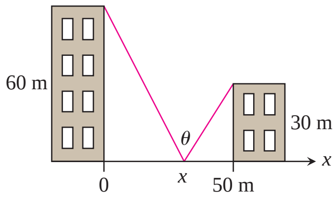
117. What is the age of a sample of charcoal in which 90% of the carbon-14 originally present has decayed?

118. Cooling a pie A deep-dish apple pie, whose internal temperature was 220°F when removed from the oven, was set out on a breezy 40°F porch to cool. Fifteen minutes later, the pie's internal temperature was 180°F. How long did it take the pie to cool from there to 70°F?

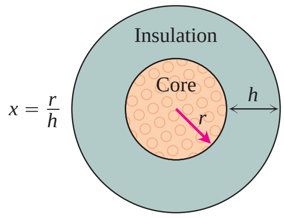
119. Locating a solar station You are under contract to build a solar station at ground level on the east–west line between the two buildings shown here. How far from the taller building should you place the station to maximize the number of hours it will be in the sun on a day when the sun passes directly overhead? Begin by observing that

$$\theta = \pi - \cot^{-1}\frac{x}{60} - \cot^{-1}\frac{50-x}{30}.$$

Then find the value of x that maximizes θ.



120. A round underwater transmission cable consists of a core of copper wires surrounded by nonconducting insulation. If x denotes the ratio of the radius of the core to the thickness of the insulation, it is known that the speed of the transmission signal is given by the equation $\upsilon = x^2 \ln(1/x)$. If the radius of the core is 1 cm, what insulation thickness h will allow the greatest transmission speed?



Chapter 8 Practice Exercises

Integration Using Substitutions

Evaluate the integrals in Exercises 1–82. To transform each integral into a recognizable basic form, it may be necessary to use one or more of the techniques of algebraic substitution, completing the square, separating fractions, long division, or trigonometric substitution.

1. $\int x\sqrt{4x^2 - 9}\, dx$
2. $\int 6x\sqrt{3x^2 + 5}\, dx$
3. $\int x(2x + 1)^{1/2}\, dx$
4. $\int x(1 - x)^{-1/2}\, dx$
5. $\int \frac{x\, dx}{\sqrt{8x^2 + 1}}$
6. $\int \frac{x\, dx}{\sqrt{9 - 4x^2}}$
7. $\int \frac{y\, dy}{25 + y^2}$
8. $\int \frac{y^3\, dy}{4 + y^4}$
9. $\int \frac{t^3\, dt}{\sqrt{9 - 4t^4}}$
10. $\int \frac{2t\, dt}{t^4 + 1}$
11. $\int z^{2/3}(z^{5/3} + 1)^{2/3}\, dz$
12. $\int z^{-1/5}(1 + z^{4/5})^{-1/2}\, dz$
13. $\int \frac{\sin 2\theta\, d\theta}{(1 - \cos 2\theta)^2}$
14. $\int \frac{\cos\theta\, d\theta}{(1 + \sin\theta)^{1/2}}$
15. $\int \frac{\sin t}{3 + 4\cos t}\, dt$
16. $\int \frac{\cos 2t}{1 + \sin 2t}\, dt$
17. $\int \sin 2x\, e^{\cos 2x}\, dx$
18. $\int \sec x \tan x\, e^{\sec x}\, dx$
19. $\int e^{\theta} \sin(e^{\theta}) \cos^2(e^{\theta})\, d\theta$
20. $\int e^{\theta} \sec^2(e^{\theta})\, d\theta$
21. $\int 2^{x-1}\, dx$
22. $\int 5^{x\sqrt{2}}\, dx$
23. $\int \frac{dv}{v \ln v}$
24. $\int \frac{dv}{v(2 + \ln v)}$
25. $\int \frac{dx}{(x^2 + 1)(2 + \tan^{-1} x)}$
26. $\int \frac{\sin^{-1} x}{\sqrt{1 - x^2}}\, dx$
27. $\int \frac{2\, dx}{\sqrt{1 - 4x^2}}$
28. $\int \frac{dx}{\sqrt{49 - x^2}}$
29. $\int \frac{dt}{\sqrt{16 - 9t^2}}$
30. $\int \frac{dt}{\sqrt{9 - 4t^2}}$
31. $\int \frac{dt}{9 + t^2}$
32. $\int \frac{dt}{1 + 25t^2}$
33. $\int \frac{4\, dx}{5x\sqrt{25x^2 - 16}}$
34. $\int \frac{6\, dx}{x\sqrt{4x^2 - 9}}$
35. $\int \frac{dx}{\sqrt{4x - x^2}}$
36. $\int \frac{dx}{\sqrt{4x - x^2 - 3}}$
37. $\int \frac{dy}{y^2 - 4y + 8}$
38. $\int \frac{dt}{t^2 + 4t + 5}$
39. $\int \frac{dx}{(x - 1)\sqrt{x^2 - 2x}}$
40. $\int \frac{dv}{(v + 1)\sqrt{v^2 + 2v}}$
41. $\int \sin^2 x\, dx$
42. $\int \cos^2 3x\, dx$

43. $\int \sin^3 \frac{\theta}{2}\, d\theta$

44. $\int \sin^3 \theta \cos^2 \theta \, d\theta$

45. $\int \tan^3 2t \, dt$

46. $\int 6 \sec^4 t \, dt$

47. $\int \frac{dx}{2 \sin x \cos x}$

48. $\int \frac{2\, dx}{\cos^2 x - \sin^2 x}$

49. $\int_{\pi/4}^{\pi/2} \sqrt{\csc^2 y - 1}\, dy$

50. $\int_{\pi/4}^{3\pi/4} \sqrt{\cot^2 t + 1}\, dt$

51. $\int_0^{\pi} \sqrt{1 - \cos^2 2x}\, dx$

52. $\int_0^{2\pi} \sqrt{1 - \sin^2 \frac{x}{2}}\, dx$

53. $\int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos 2t}\, dt$

54. $\int_{\pi}^{2\pi} \sqrt{1 + \cos 2t}\, dt$

55. $\int \frac{x^2}{x^2 + 4}\, dx$

56. $\int \frac{x^3}{9 + x^2}\, dx$

57. $\int \frac{4x^2 + 3}{2x - 1}\, dx$

58. $\int \frac{2x}{x - 4}\, dx$

59. $\int \frac{2y - 1}{y^2 + 4}\, dy$

60. $\int \frac{y + 4}{y^2 + 1}\, dy$

61. $\int \frac{t + 2}{\sqrt{4 - t^2}}\, dt$

62. $\int \frac{2t^2 + \sqrt{1 - t^2}}{t\sqrt{1 - t^2}}\, dt$

63. $\int \frac{\tan x \, dx}{\tan x + \sec x}$

64. $\int \frac{\cot x}{\cot x + \csc x}\, dx$

65. $\int \sec (5 - 3x)\, dx$

66. $\int x \csc (x^2 + 3)\, dx$

67. $\int \cot \left(\frac{x}{4}\right) dx$

68. $\int \tan (2x - 7)\, dx$

69. $\int x\sqrt{1 - x}\, dx$

70. $\int 3x\sqrt{2x + 1}\, dx$

71. $\int \sqrt{z^2 + 1}\, dz$

72. $\int (16 + z^2)^{-3/2}\, dz$

73. $\int \frac{dy}{\sqrt{25 + y^2}}$

74. $\int \frac{dy}{\sqrt{25 + 9y^2}}$

75. $\int \frac{dx}{x^2\sqrt{1 - x^2}}$

76. $\int \frac{x^3\, dx}{\sqrt{1 - x^2}}$

77. $\int \frac{x^2\, dx}{\sqrt{1 - x^2}}$

78. $\int \sqrt{4 - x^2}\, dx$

79. $\int \frac{dx}{\sqrt{x^2 - 9}}$

80. $\int \frac{12\, dx}{(x^2 - 1)^{3/2}}$

81. $\int \frac{\sqrt{w^2 - 1}}{w}\, dw$

82. $\int \frac{\sqrt{z^2 - 16}}{z}\, dz$

Integration by Parts

Evaluate the integrals in Exercises 83–90 using integration by parts.

83. $\int \ln (x + 1)\, dx$

84. $\int x^2 \ln x \, dx$

85. $\int \tan^{-1} 3x \, dx$

86. $\int \cos^{-1} \left(\frac{x}{2}\right) dx$

87. $\int (x + 1)^2 e^x \, dx$

88. $\int x^2 \sin (1 - x)\, dx$

89. $\int e^x \cos 2x \, dx$

90. $\int e^{-2x} \sin 3x \, dx$

Partial Fractions

Evaluate the integrals in Exercises 91–110. It may be necessary to use a substitution first.

91. $\int \frac{x\, dx}{x^2 - 3x + 2}$

92. $\int \frac{x\, dx}{x^2 + 4x + 3}$

93. $\int \frac{dx}{x(x + 1)^2}$

94. $\int \frac{x + 1}{x^2(x - 1)}\, dx$

95. $\int \frac{\sin \theta \, d\theta}{\cos^2 \theta + \cos \theta - 2}$

96. $\int \frac{\cos \theta \, d\theta}{\sin^2 \theta + \sin \theta - 6}$

97. $\int \frac{3x^2 + 4x + 4}{x^3 + x}\, dx$

98. $\int \frac{4x\, dx}{x^3 + 4x}$

99. $\int \frac{v + 3}{2v^3 - 8v}\, dv$

100. $\int \frac{(3v - 7)\, dv}{(v - 1)(v - 2)(v - 3)}$

101. $\int \frac{dt}{t^4 + 4t^2 + 3}$

102. $\int \frac{t\, dt}{t^4 - t^2 - 2}$

103. $\int \frac{x^3 + x^2}{x^2 + x - 2}\, dx$

104. $\int \frac{x^3 + 1}{x^3 - x}\, dx$

105. $\int \frac{x^3 + 4x^2}{x^2 + 4x + 3}\, dx$

106. $\int \frac{2x^3 + x^2 - 21x + 24}{x^2 + 2x - 8}\, dx$

107. $\int \frac{dx}{x\left(3\sqrt{x + 1}\right)}$

108. $\int \frac{dx}{x\left(1 + \sqrt[3]{x}\right)}$

109. $\int \frac{ds}{e^s - 1}$

110. $\int \frac{ds}{\sqrt{e^s + 1}}$

Trigonometric Substitutions

Evaluate the integrals in Exercises 111–114 **(a)** without using a trigonometric substitution, **(b)** using a trigonometric substitution.

111. $\int \frac{y\, dy}{\sqrt{16 - y^2}}$

112. $\int \frac{x\, dx}{\sqrt{4 + x^2}}$

113. $\int \frac{x\, dx}{4 - x^2}$

114. $\int \frac{t\, dt}{\sqrt{4t^2 - 1}}$

Quadratic Terms

Evaluate the integrals in Exercises 115–118.

115. $\int \frac{x\, dx}{9 - x^2}$

116. $\int \frac{dx}{x(9 - x^2)}$

117. $\int \frac{dx}{9 - x^2}$

118. $\int \frac{dx}{\sqrt{9 - x^2}}$

Trigonometric Integrals

Evaluate the integrals in Exercises 119–126.

119. $\int \sin^3 x \cos^4 x\, dx$

120. $\int \cos^5 x \sin^5 x\, dx$

121. $\int \tan^4 x \sec^2 x\, dx$

122. $\int \tan^3 x \sec^3 x\, dx$

123. $\int \sin 5\theta \cos 6\theta\, d\theta$

124. $\int \cos 3\theta \cos 3\theta\, d\theta$

125. $\int \sqrt{1 + \cos(t/2)}\, dt$

126. $\int e^t \sqrt{\tan^2 e^t + 1}\, dt$

Numerical Integration

127. According to the error-bound formula for Simpson's Rule, how many subintervals should you use to be sure of estimating the value of

$$\ln 3 = \int_1^3 \frac{1}{x}\, dx$$

by Simpson's Rule with an error of no more than 10^{-4} in absolute value? (Remember that for Simpson's Rule, the number of subintervals has to be even.)

128. A brief calculation shows that if $0 \le x \le 1$, then the second derivative of $f(x) = \sqrt{1 + x^4}$ lies between 0 and 8. Based on this, about how many subdivisions would you need to estimate the integral of f from 0 to 1 with an error no greater than 10^{-3} in absolute value using the Trapezoidal Rule?

129. A direct calculation shows that

$$\int_0^{\pi} 2 \sin^2 x\, dx = \pi.$$

How close do you come to this value by using the Trapezoidal Rule with $n = 6$? Simpson's Rule with $n = 6$? Try them and find out.

130. You are planning to use Simpson's Rule to estimate the value of the integral

$$\int_1^2 f(x)\, dx$$

with an error magnitude less than 10^{-5}. You have determined that $|f^{(4)}(x)| \le 3$ throughout the interval of integration. How many subintervals should you use to assure the required accuracy? (Remember that for Simpson's Rule the number has to be even.)

131. Mean temperature Compute the average value of the temperature function

$$f(x) = 37 \sin\left(\frac{2\pi}{365}(x - 101)\right) + 25$$

for a 365-day year. This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is 25.7°F, which is slightly higher than the average value of $f(x)$.

132. Heat capacity of a gas Heat capacity C_v is the amount of heat required to raise the temperature of a given mass of gas with constant volume by 1°C, measured in units of cal/deg-mol (calories per degree gram molecular weight). The heat capacity of oxygen depends on its temperature T and satisfies the formula

$$C_v = 8.27 + 10^{-5}(26T - 1.87T^2).$$

Find the average value of C_v for $20° \le T \le 675°$C and the temperature at which it is attained.

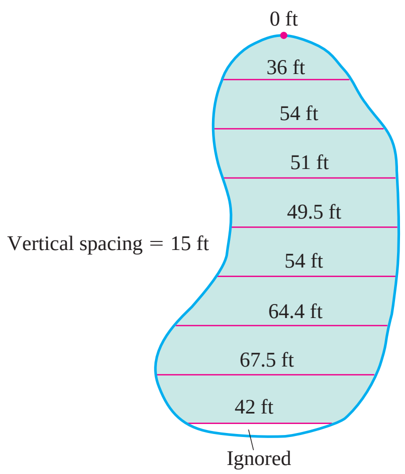
133. Fuel efficiency An automobile computer gives a digital readout of fuel consumption in gallons per hour. During a trip, a passenger recorded the fuel consumption every 5 min for a full hour of travel.

Time	Gal/h	Time	Gal/h
0	2.5	35	2.5
5	2.4	40	2.4
10	2.3	45	2.3
15	2.4	50	2.4
20	2.4	55	2.4
25	2.5	60	2.3
30	2.6		

a. Use the Trapezoidal Rule to approximate the total fuel consumption during the hour.

b. If the automobile covered 60 mi in the hour, what was its fuel efficiency (in miles per gallon) for that portion of the trip?

134. A new parking lot To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for $11,000. The cost to clear the land will be $0.10 a square foot, and the lot will cost $2.00 a square foot to pave. Use Simpson's Rule to find out if the job can be done for $11,000.



Improper Integrals

Evaluate the improper integrals in Exercises 135–144.

135. $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

136. $\int_0^1 \ln x \, dx$

137. $\int_{-1}^1 \frac{dy}{y^{2/3}}$

138. $\int_{-2}^0 \frac{d\theta}{(\theta+1)^{3/5}}$

139. $\int_3^\infty \frac{2\,du}{u^2-2u}$

140. $\int_1^\infty \frac{3v-1}{4v^3-v^2}\,dv$

141. $\int_0^\infty x^2e^{-x}\,dx$

142. $\int_{-\infty}^0 xe^{3x}\,dx$

143. $\int_{-\infty}^\infty \frac{dx}{4x^2+9}$

144. $\int_{-\infty}^\infty \frac{4dx}{x^2+16}$

Convergence or Divergence

Which of the improper integrals in Exercises 145–150 converge and which diverge?

145. $\int_6^\infty \frac{d\theta}{\sqrt{\theta^2+1}}$

146. $\int_0^\infty e^{-u}\cos u\,du$

147. $\int_1^\infty \frac{\ln z}{z}\,dz$

148. $\int_1^\infty \frac{e^{-t}}{\sqrt{t}}\,dt$

149. $\int_{-\infty}^\infty \frac{2\,dx}{e^x+e^{-x}}$

150. $\int_{-\infty}^\infty \frac{dx}{x^2(1+e^x)}$

Assorted Integrations

Evaluate the integrals in Exercises 151–218. The integrals are listed in random order.

151. $\int \frac{x\,dx}{1+\sqrt{x}}$

152. $\int \frac{x^3+2}{4-x^2}\,dx$

153. $\int \frac{dx}{x(x^2+1)^2}$

154. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

155. $\int \frac{dx}{\sqrt{-2x-x^2}}$

156. $\int \frac{(t-1)\,dt}{\sqrt{t^2-2t}}$

157. $\int \frac{du}{\sqrt{1+u^2}}$

158. $\int e^t\cos e^t\,dt$

159. $\int \frac{2-\cos x+\sin x}{\sin^2 x}\,dx$

160. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta$

161. $\int \frac{9\,dv}{81-v^4}$

162. $\int \frac{\cos x\,dx}{1+\sin^2 x}$

163. $\int \theta\cos(2\theta+1)\,d\theta$

164. $\int_2^\infty \frac{dx}{(x-1)^2}$

165. $\int \frac{x^3\,dx}{x^2-2x+1}$

166. $\int \frac{d\theta}{\sqrt{1+\sqrt{\theta}}}$

167. $\int \frac{2\sin\sqrt{x}\,dx}{\sqrt{x}\sec\sqrt{x}}$

168. $\int \frac{x^5\,dx}{x^4-16}$

169. $\int \frac{dy}{\sin y\cos y}$

170. $\int \frac{d\theta}{\theta^2-2\theta+4}$

171. $\int \frac{\tan x}{\cos^2 x}\,dx$

172. $\int \frac{dr}{(r+1)\sqrt{r^2+2r}}$

173. $\int \frac{(r+2)\,dr}{\sqrt{-r^2-4r}}$

174. $\int \frac{y\,dy}{4+y^4}$

175. $\int \frac{\sin 2\theta\,d\theta}{(1+\cos 2\theta)^2}$

176. $\int \frac{dx}{(x^2-1)^2}$

177. $\int_{\pi/4}^{\pi/2} \sqrt{1+\cos 4x}\,dx$

178. $\int (15)^{2x+1}\,dx$

179. $\int \frac{x\,dx}{\sqrt{2-x}}$

180. $\int \frac{\sqrt{1-v^2}}{v^2}\,dv$

181. $\int \frac{dy}{y^2-2y+2}$

182. $\int \ln\sqrt{x-1}\,dx$

183. $\int \theta^2\tan(\theta^3)\,d\theta$

184. $\int \frac{x\,dx}{\sqrt{8-2x^2-x^4}}$

185. $\int \frac{z+1}{z^2(z^2+4)}\,dz$

186. $\int x^3e^{(x^2)}\,dx$

187. $\int \frac{t\,dt}{\sqrt{9-4t^2}}$

188. $\int_0^{\pi/10} \sqrt{1+\cos 5\theta}\,d\theta$

189. $\int \frac{\cot\theta\,d\theta}{1+\sin^2\theta}$

190. $\int \frac{\tan^{-1} x}{x^2}\,dx$

191. $\int \frac{\tan\sqrt{y}}{2\sqrt{y}}\,dy$

192. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2}$

193. $\int \frac{\theta^2 d\theta}{4-\theta^2}$

194. $\int \frac{1-\cos 2x}{1+\cos 2x}\,dx$

195. $\int \frac{\cos(\sin^{-1} x)}{\sqrt{1-x^2}}\,dx$

196. $\int \frac{\cos x\,dx}{\sin^3 x-\sin x}$

197. $\int \sin\frac{x}{2}\cos\frac{x}{2}\,dx$

198. $\int \frac{x^2-x+2}{(x^2+2)^2}\,dx$

199. $\int \frac{e^t\,dt}{1+e^t}$

200. $\int \tan^3 t\,dt$

201. $\int_1^\infty \frac{\ln y}{y^3}\,dy$

202. $\int \frac{3+\sec^2 x+\sin x}{\tan x}\,dx$

203. $\int \frac{\cot v\,dv}{\ln\sin v}$

204. $\int \frac{dx}{(2x-1)\sqrt{x^2-x}}$

205. $\int e^{\ln\sqrt{x}}\,dx$

206. $\int e^\theta\sqrt{3+4e^\theta}\,d\theta$

207. $\int \frac{\sin 5t\,dt}{1+(\cos 5t)^2}$

208. $\int \frac{dv}{\sqrt{e^{2v}-1}}$

209. $\int (27)^{3\theta+1}\,d\theta$

210. $\int x^5\sin x\,dx$

211. $\int \frac{dr}{1+\sqrt{r}}$

212. $\int \frac{4x^3-20x}{x^4-10x^2+9}\,dx$

213. $\int \frac{8\,dy}{y^3(y+2)}$

214. $\int \frac{(t+1)\,dt}{(t^2+2t)^{2/3}}$

215. $\int \frac{8\,dm}{m\sqrt{49m^2-4}}$

216. $\displaystyle\int \frac{dt}{t(1 + \ln t)\sqrt{(\ln t)(2 + \ln t)}}$

217. $\displaystyle\int_0^1 3(x - 1)^2 \left(\int_0^x \sqrt{1 + (t - 1)^4}\, dt \right) dx$

218. $\displaystyle\int_2^\infty \frac{4v^3 + v - 1}{v^2(v - 1)(v^2 + 1)}\, dv$

219. Suppose for a certain function f it is known that

$$f'(x) = \frac{\cos x}{x}, \quad f(\pi\ 2) = a, \quad \text{and} \quad f(3\pi\ 2) = b.$$

Use integration by parts to evaluate

$$\int_{\pi/2}^{3\pi/2} f(x)\, dx.$$

220. Find a positive number a satisfying

$$\int_0^a \frac{dx}{1 + x^2} = \int_a^\infty \frac{dx}{1 + x^2}.$$

Chapter 11 Practice Exercises

Convergent or Divergent Sequences

Which of the sequences whose nth terms appear in Exercises 1–18 converge, and which diverge? Find the limit of each convergent sequence.

1. $a_n = 1 + \dfrac{(-1)^n}{n}$

2. $a_n = \dfrac{1 - (-1)^n}{\sqrt{n}}$

3. $a_n = \dfrac{1 - 2^n}{2^n}$

4. $a_n = 1 + (0.9)^n$

5. $a_n = \sin \dfrac{n\pi}{2}$

6. $a_n = \sin n\pi$

7. $a_n = \dfrac{\ln (n^2)}{n}$

8. $a_n = \dfrac{\ln (2n + 1)}{n}$

9. $a_n = \dfrac{n + \ln n}{n}$

10. $a_n = \dfrac{\ln (2n^3 + 1)}{n}$

11. $a_n = \left(\dfrac{n - 5}{n}\right)^n$

12. $a_n = \left(1 + \dfrac{1}{n}\right)^{-n}$

13. $a_n = \sqrt[n]{\dfrac{3^n}{n}}$

14. $a_n = \left(\dfrac{3}{n}\right)^{1/n}$

15. $a_n = n(2^{1/n} - 1)$

16. $a_n = \sqrt[n]{2n + 1}$

17. $a_n = \dfrac{(n + 1)!}{n!}$

18. $a_n = \dfrac{(-4)^n}{n!}$

Convergent Series

Find the sums of the series in Exercises 19–24.

19. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(2n - 3)(2n - 1)}$

20. $\displaystyle\sum_{n=2}^{\infty} \frac{-2}{n(n + 1)}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{9}{(3n - 1)(3n + 2)}$

22. $\displaystyle\sum_{n=3}^{\infty} \frac{-8}{(4n - 3)(4n + 1)}$

23. $\displaystyle\sum_{n=0}^{\infty} e^{-n}$

24. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n}$

Convergent or Divergent Series

Which of the series in Exercises 25–40 converge absolutely, which converge conditionally, and which diverge? Give reasons for your answers.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{-5}{n}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^3}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln (n + 1)}$

30. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^2}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

32. $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{\ln (\ln n)}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2 + 1}}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n^2}{n^3 + 1}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{n + 1}{n!}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n^2 + 1)}{2n^2 + n - 1}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n 3^n}{n^n}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n + 1)(n + 2)}}$

40. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2 - 1}}$

Power Series

In Exercises 41–50, **(a)** find the series' radius and interval of convergence. Then identify the values of x for which the series converges **(b)** absolutely and **(c)** conditionally.

41. $\displaystyle\sum_{n=1}^{\infty} \frac{(x + 4)^n}{n3^n}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{(x - 1)^{2n-2}}{(2n - 1)!}$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3x - 1)^n}{n^2}$

44. $\displaystyle\sum_{n=0}^{\infty} \frac{(n + 1)(2x + 1)^n}{(2n + 1)2^n}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

47. $\sum_{n=0}^{\infty} \frac{(n+1)x^{2n-1}}{3^n}$

48. $\sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{2n+1}}{2n+1}$

49. $\sum_{n=1}^{\infty} (\operatorname{csch} n)x^n$

50. $\sum_{n=1}^{\infty} (\coth n)x^n$

Maclaurin Series

Each of the series in Exercises 51–56 is the value of the Taylor series at $x = 0$ of a function $f(x)$ at a particular point. What function and what point? What is the sum of the series?

51. $1 - \frac{1}{4} + \frac{1}{16} - \cdots + (-1)^n \frac{1}{4^n} + \cdots$

52. $\frac{2}{3} - \frac{4}{18} + \frac{8}{81} - \cdots + (-1)^{n-1} \frac{2^n}{n3^n} + \cdots$

53. $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \cdots + (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} + \cdots$

54. $1 - \frac{\pi^2}{9 \cdot 2!} + \frac{\pi^4}{81 \cdot 4!} - \cdots + (-1)^n \frac{\pi^{2n}}{3^{2n}(2n)!} + \cdots$

55. $1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^n}{n!} + \cdots$

56. $\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \cdots$

$$+ (-1)^{n-1} \frac{1}{(2n-1)\left(\sqrt{3}\right)^{2n-1}} + \cdots$$

Find Taylor series at $x = 0$ for the functions in Exercises 57–64.

57. $\frac{1}{1-2x}$

58. $\frac{1}{1+x^3}$

59. $\sin \pi x$

60. $\sin \frac{2x}{3}$

61. $\cos (x^{5/2})$

62. $\cos \sqrt{5x}$

63. $e^{(\pi x/2)}$

64. e^{-x^2}

Taylor Series

In Exercises 65–68, find the first four nonzero terms of the Taylor series generated by f at $x = a$.

65. $f(x) = \sqrt{3 + x^2}$ at $x = -1$

66. $f(x) = 1/(1 - x)$ at $x = 2$

67. $f(x) = 1/(x + 1)$ at $x = 3$

68. $f(x) = 1/x$ at $x = a > 0$

Initial Value Problems

Use power series to solve the initial value problems in Exercises 69–76.

69. $y' + y = 0, \quad y(0) = -1$

70. $y' - y = 0, \quad y(0) = -3$

71. $y' + 2y = 0, \quad y(0) = 3$

72. $y' + y = 1, \quad y(0) = 0$

73. $y' - y = 3x, \quad y(0) = -1$

74. $y' + y = x, \quad y(0) = 0$

75. $y' - y = x, \quad y(0) = 1$

76. $y' - y = -x, \quad y(0) = 2$

Nonelementary Integrals

Use series to approximate the values of the integrals in Exercises 77–80 with an error of magnitude less than 10^{-8}. (The answer section gives the integrals' values rounded to 10 decimal places.)

77. $\int_0^{1/2} e^{-x^3}\, dx$

78. $\int_0^1 x \sin (x^3)\, dx$

79. $\int_0^{1/2} \frac{\tan^{-1} x}{x}\, dx$

80. $\int_0^{1/64} \frac{\tan^{-1} x}{\sqrt{x}}\, dx$

Indeterminate Forms

In Exercises 81–86:

a. Use power series to evaluate the limit.

T **b.** Then use a grapher to support your calculation.

81. $\lim_{x \to 0} \frac{7 \sin x}{e^{2x} - 1}$

82. $\lim_{\theta \to 0} \frac{e^\theta - e^{-\theta} - 2\theta}{\theta - \sin \theta}$

83. $\lim_{t \to 0} \left(\frac{1}{2 - 2\cos t} - \frac{1}{t^2} \right)$

84. $\lim_{h \to 0} \frac{(\sin h)/h - \cos h}{h^2}$

85. $\lim_{z \to 0} \frac{1 - \cos^2 z}{\ln (1 - z) + \sin z}$

86. $\lim_{y \to 0} \frac{y^2}{\cos y - \cosh y}$

87. Use a series representation of $\sin 3x$ to find values of r and s for which

$$\lim_{x \to 0} \left(\frac{\sin 3x}{x^3} + \frac{r}{x^2} + s \right) = 0.$$

88. **a.** Show that the approximation $\csc x \approx 1/x + x/6$ in Section 11.10, Example 9, leads to the approximation $\sin x \approx 6x/(6 + x^2)$.

T **b.** Compare the accuracies of the approximations $\sin x \approx x$ and $\sin x \approx 6x/(6 + x^2)$ by comparing the graphs of $f(x) = \sin x - x$ and $g(x) = \sin x - (6x/(6 + x^2))$. Describe what you find.

Theory and Examples

89. **a.** Show that the series

$$\sum_{n=1}^{\infty} \left(\sin \frac{1}{2n} - \sin \frac{1}{2n+1} \right)$$

converges.

T **b.** Estimate the magnitude of the error involved in using the sum of the sines through $n = 20$ to approximate the sum of the series. Is the approximation too large, or too small? Give reasons for your answer.

90. **a.** Show that the series $\sum_{n=1}^{\infty} \left(\tan \frac{1}{2n} - \tan \frac{1}{2n+1} \right)$ converges.

T **b.** Estimate the magnitude of the error in using the sum of the tangents through $-\tan (1/41)$ to approximate the sum of the series. Is the approximation too large, or too small? Give reasons for your answer.

91. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n - 1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} x^n.$$

92. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n + 1)}{4 \cdot 9 \cdot 14 \cdot \cdots \cdot (5n - 1)} (x - 1)^n.$$

93. Find a closed-form formula for the nth partial sum of the series $\sum_{n=2}^{\infty} \ln(1 - (1/n^2))$ and use it to determine the convergence or divergence of the series.

94. Evaluate $\sum_{k=2}^{\infty} (1/(k^2 - 1))$ by finding the limits as $n \to \infty$ of the series' nth partial sum.

95. a. Find the interval of convergence of the series

$$y = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots + \frac{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n - 2)}{(3n)!} x^{3n} + \cdots.$$

b. Show that the function defined by the series satisfies a differential equation of the form

$$\frac{d^2y}{dx^2} = x^a y + b$$

and find the values of the constants a and b.

96. a. Find the Maclaurin series for the function $x^2/(1 + x)$.

b. Does the series converge at $x = 1$? Explain.

97. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty} a_n b_n$? Give reasons for your answer.

98. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty} a_n b_n$? Give reasons for your answer.

99. Prove that the sequence $\{x_n\}$ and the series $\sum_{k=1}^{\infty} (x_{k+1} - x_k)$ both converge or both diverge.

100. Prove that $\sum_{n=1}^{\infty} (a_n/(1 + a_n))$ converges if $a_n > 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges.

101. (*Continuation of Section 4.7, Exercise 27.*) If you did Exercise 27 in Section 4.7, you saw that in practice Newton's method stopped too far from the root of $f(x) = (x - 1)^{40}$ to give a useful estimate of its value, $x = 1$. Prove that nevertheless, for any starting value $x_0 \neq 1$, the sequence $x_0, x_1, x_2, \ldots, x_n, \ldots$ of approximations generated by Newton's method really does converge to 1.

102. Suppose that $a_1, a_2, a_3, \ldots, a_n$ are positive numbers satisfying the following conditions:

i. $a_1 \geq a_2 \geq a_3 \geq \cdots;$

ii. the series $a_2 + a_4 + a_8 + a_{16} + \cdots$ diverges.

Show that the series

$$\frac{a_1}{1} + \frac{a_2}{2} + \frac{a_3}{3} + \cdots$$

diverges.

103. Use the result in Exercise 102 to show that

$$1 + \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

diverges.

104. Suppose you wish to obtain a quick estimate for the value of $\int_0^1 x^2 e^x \, dx$. There are several ways to do this.

a. Use the Trapezoidal Rule with $n = 2$ to estimate $\int_0^1 x^2 e^x \, dx$.

b. Write out the first three nonzero terms of the Taylor series at $x = 0$ for $x^2 e^x$ to obtain the fourth Taylor polynomial $P(x)$ for $x^2 e^x$. Use $\int_0^1 P(x) \, dx$ to obtain another estimate for $\int_0^1 x^2 e^x \, dx$.

c. The second derivative of $f(x) = x^2 e^x$ is positive for all $x > 0$. Explain why this enables you to conclude that the Trapezoidal Rule estimate obtained in part (a) is too large. (*Hint:* What does the second derivative tell you about the graph of a function? How does this relate to the trapezoidal approximation of the area under this graph?)

d. All the derivatives of $f(x) = x^2 e^x$ are positive for $x > 0$. Explain why this enables you to conclude that all Maclaurin polynomial approximations to $f(x)$ for x in [0, 1] will be too small. (*Hint:* $f(x) = P_n(x) + R_n(x)$.)

e. Use integration by parts to evaluate $\int_0^1 x^2 e^x \, dx$.

Fourier Series

Find the Fourier series for the functions in Exercises 105–108. Sketch each function.

105. $f(x) = \begin{cases} 0, & 0 \leq x \leq \pi \\ 1, & \pi < x \leq 2\pi \end{cases}$

106. $f(x) = \begin{cases} x, & 0 \leq x \leq \pi \\ 1, & \pi < x \leq 2\pi \end{cases}$

107. $f(x) = \begin{cases} \pi - x, & 0 \leq x \leq \pi \\ x - 2\pi, & \pi < x \leq 2\pi \end{cases}$

108. $f(x) = |\sin x|, \quad 0 \leq x \leq 2\pi$