LECTURE MATERIALS

FOR

LINEAR ALGEBRA

(AS PART OF THE COURSE ZCA 110)

PREPARED BY

YOON TIEM LEONG

SCHOOL OF PHYSICS

USM

Chapter 1 Matrices

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Give an example of a 3×2 matrix. Call it A.

2. Identify the element a_{13} and a_{23} in A defined in (1).

3. Give any example of a square matrix of order 3. Call it S.

4. List down all the diagonal elements in (3).

5. Calculate the trace of S as defined in 3.

6. Give an example of a pair of equal matrices.

7. Give an example of a zero matrices or order 2×4.

8. Consider a square matrix of order 3, $A = [a_{ij}]$, where $a_{ij}=1$ for all $i,j = 1,2,3$, and a square matrix of the same order, $B = [b_{ij}]$, where $b_{ij}=2$ if $i=j$, $b_{ij}=0$ if $i \neq j$. Calculate the matrix $C = A + B$.

9. What is the negative of B, with B defined in (8).

10. If there exist a matrix D such that $A + D = B$, determine D.

11. Given $X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, Y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, find, if possible, (i) the product *XY*, (ii) the product *YX*.

12. If *K* is a matrix of order 3 by 2, *M* a matrix of order 2 by 3, what is the order of the product (i) *KM*? (ii) *MK*?

13. Give an example of a pair of 3-square matrices *A*, *B* such that (i) $AB \neq BA$, (ii) $AB = BA$.

14. Give an example of a pair of 3-square matrices *A*, *B* such that (i) $AB = 0$ but $A \neq 0$ (ii) $AB = 0$ but $B \neq 0$, (iii) $AB = 0$ with $A=0$, $B=0$.

15. Give an example of a set of 3-square matrices *A*, *B*, *C* such that (i) $AB=AC$ with $B \neq C$, (ii) $AB=AC$ with $B=C$.

Chapter 2 Some types of matricesQ

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Give an example of a 3 × 3 (*i*) upper triangular matrix (*ii*) lower triangular matrix. (iii) Give an example of a 3 × 3 matrix which is both an upper and lower triangular matrix.

2. Give an example of a (*i*) 2 × 3 upper triangular matrix (*ii*) 2 × 3 lower triangular matrix.

3. (*i*) Write out explicitly I_4. (*ii*) What is $(I_4)^n$, with n a positive integer.

4. Given a $n \times n$ matrix A with $a_{ii}=\sqrt[n]{n}$, $i = 1, 2, \ldots, n$, where n a positive integer, and $a_{ji}=0$ for $i \neq j$. What is the name for this type of matrix?

5. Refer to (4), express A explicitly if n=3.

6. Consider the matrix A as defined in (4). Find A^n.

7. Given B an n-square diagonal matrix, with $b_{ii}= j$, where $j = 1, 2,\ldots, n$. Let $C = AB$. Let c_{ij} denote the (i, j) element of the matrix C. (*i*) What is c_{ij} if $i \neq j$? What the element c_{jj}?

8. Say A is a matrix of order $n \times n$. Give three simplest matrices you can think of that will commute with A?

9. Give an example of a matrix that anti-commute with I_n.

10. Give two simplest examples you can think of for an idempotent matrix.

11. (*i*) Given an $n \times n$ matrix $F = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. What is the period of F? (*ii*) Construct a 3-square matrix which has a period of 3. *Hint*: you may like to test out some trial matrix in which each row or column has only a single none zero element.

12. If A is nilpotent of order n, what is A^{n+1}? n positive integer.

13. Given an n-square matrix A is the inverse of matrix B, then we write $A = B^{-1}$. (*i*) Express the inverse of A in term of B. (*ii*) What is AB? (*ii*) What is BA?

14. Prove that $(AB)^{-1} = B^{-1}A^{-1}$.

15. Give an easiest example you can think of where the inverse of a matrix is also the matrix itself.

16. Let Q be a scalar n-square matrix such that $Q = kI_n$, where k an scalar. What is Q^{-1}?

17. Given the order of matrix A is $m \times n$, what is the order of its transpose, A'? Note that sometimes the transpose of matrix A is written as A^T instead of A'.

18. Consider a 2 by 2 matrix *A*. (*i*) what is the sufficient condition that *A* is also equal to its transpose (i.e. $A'=A$)?

19. Consider a 3 by 3 non-zero diagonal matrix *B*. (*i*) Is *B* symmetric? (*ii*) Could *B* ever be skew-symmetric?
[Ans: (*i*) YES, (ii), NEVER.]

20. Let *A* be an *n*-square matrix. (*i*) What is the symmetry of $A+A^{\mathrm{T}}$ (i.e. is it symmetric or skew-symmetric?). (*i*) What is the symmetry of A-A^{T}?
[(*i*) symmetric, (*ii*) skew-symmetric.]

21. Given the matrix, $C = \begin{pmatrix} w & y \\ x & z \end{pmatrix}$, (*i*) express *C* as a sum of a symmetric matrix and a skew-symmetric matrix.

22. (*i*) Express explicitly $\overline{C}$, the conjugate of the matrix *C* in (21). (*ii*) Express explicitly the conjugate of the matrix *kC*, where *k* a real scalar.

23. (*i*) The transpose of the conjugate of matrix *C* is written as … ? (*ii*) The conjugate of transpose of the matrix *C* is written as …? (*iii*) Are both of these equal each to other?

24. (*i*) What is A^*? (*ii*) If *A* is a matrix containing no complex element, is there any difference between A^* and A^{T}? (*iii*) In general, if *A* is a matrix containing complex element, will A^* equal A^{T} in general?

25. Is a real, symmetric matrix Hermitian?

26. (*i*) By making use of Theorem X in page 13 of Ayers 1982 impression, very quickly give an example of a 2 by 2 Hermitian matrix and an example of a 2 by 2 skew-Hermitian matrix.

27. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, C = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}$. (*i*)

What is the order of S = diag(A, B, C), the direct sum of A, B, C? (*ii*) What is the order of S^2?

Chapter 3 Determinant of a square matrixQ

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. See if you understand what permutation is.

   (*i*) How many inversions are there in the permutation 1234? (*ii*) 4321? (*iii*) state whether (*i*) is odd or even? (*iv*) state whether (*ii*) is odd or even. (*v*) Give an example of an odd permutation.

2. See if you can do this: $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -3 & -3 & -3 \end{vmatrix} =$

3. See if you can do this without expanding the determinant:

   (i) $\begin{vmatrix} 43 & 88 & 22.34 & 113.8 \\ 65 & 43 & 43 & 24.9 \\ 86 & 443 & 3 & 23 \\ 0 & 0 & 0 & 0 \end{vmatrix} =$

   (ii) $\begin{vmatrix} 1 & 0 & 5 \\ 2 & 0 & 6 \\ 4 & 0 & 8 \end{vmatrix} =$

4. Can you define the determinant of a matrix that is not square? How?

5. Given that you know $\begin{vmatrix} 1 & 4 & 1 \\ 2 & 2 & 3 \\ 0 & -3 & -3 \end{vmatrix}$ =21, find

$\begin{vmatrix} 1 & 2 & 0 \\ 4 & 2 & -3 \\ 1 & 3 & -3 \end{vmatrix}$ without expansion.

6. Given that $\begin{vmatrix} 1 & 2 & 1 \\ 8 & 4 & 4 \\ -3 & 2 & -3 \end{vmatrix} = 32$, find

$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ -3 & 2 & -3 \end{vmatrix}$ without expansion.

7. Given that $\begin{vmatrix} g & j & p \\ y & t & l \\ u & s & m \end{vmatrix} = 53.889$, find

$(i) \begin{vmatrix} y & t & l \\ g & j & p \\ u & s & m \end{vmatrix}$, $(ii) \begin{vmatrix} u & s & m \\ y & t & l \\ g & j & p \end{vmatrix}$

8. Given $|A| = \begin{vmatrix} l & o & v & e \\ s & u & k & a \\ l & i & k & e \\ a & i & 0 & 0 \end{vmatrix}$ =999, what is

$(i) \begin{vmatrix} o & v & l & e \\ u & k & s & a \\ i & k & l & e \\ i & 0 & a & 0 \end{vmatrix}$, $(ii) \begin{vmatrix} o & v & e & l \\ u & k & a & s \\ i & k & e & l \\ i & 0 & 0 & a \end{vmatrix}$

9. How do you convince yourself that

$(i) \begin{vmatrix} 0 & 0 & 0 \\ y & t & l \\ g & j & p \end{vmatrix} = \begin{vmatrix} u & s & m \\ 0 & 0 & 0 \\ g & j & p \end{vmatrix} = 0$?

$(ii)\ \begin{vmatrix} o & v & 0 & l \\ u & k & 0 & s \\ i & k & 0 & l \\ i & 0 & 0 & a \end{vmatrix} = \begin{vmatrix} o & v & e & l \\ u & k & a & s \\ i & k & e & l \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0?$

10. Given $\begin{vmatrix} 2 & t & l \\ 2 & j & p \\ 4 & s & m \end{vmatrix} = 10, \begin{vmatrix} 3 & t & l \\ 4 & j & p \\ 6 & s & m \end{vmatrix} = 9,$ what

is $\begin{vmatrix} 5 & t & l \\ 6 & j & p \\ 10 & s & m \end{vmatrix}$?

11. Say $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = x$.

(i) Consider the operation "add 2×(second row) to the first row". If this operation is applied to the determinant above, what do you get? Write down the expression explicitly. (ii) Without expansion, work out what is the value of the determinant in (i).

12. Consider the matrix $X = [x_{ij}] = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

(i) What is the first minor of x_{11}? Of x_{23}? Conventionally what symbol you will use to represent these quantities?

(ii) What is the cofactor of x_{11}? Of x_{23}? Conventionally what symbol you will use to represent these quantities?

(iii) What is the matrix of the minors x_{ij},

$[|M_{ij}|]$?

(iv) What is the matrix of the cofactors of x_{ij},

$[\alpha_{ij}]$?

13. Consider the matrix in (12). Calculate

(*i*) $\sum_{j}^{3} x_{1j}\alpha_{1j}$ (*ii*) $\sum_{i}^{3} x_{i3}\alpha_{i3}$

(*ii*) What is the quantity represented by the sum in (*i*) and (*iii*)? Do you get the same value for both (*i*) and (*ii*)?
(*iv*) So, what is the determinant of *X*?

14. If you were to manually find the determinant of

$$\begin{vmatrix} 44.1 & 0 & 52.3 \\ 33.6 & 1.98 & 7.1 \\ 4.2 & 0 & 1.6 \end{vmatrix}$$, along which column or row would you like to follow to calculate the determinant?

Chapter 4 Evaluation of determinantsQ

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

Consider the operation "add to row *j* by (*k* × row *i*)", *j* ≠*i*, where *k* is a non-zero scalar. We will conveniently represent the above operation by $R_j^i(k)$, and we have $A \xrightarrow{R_j^i(k)} A' = R_j^i(k)A$.

Sometimes this operation is symbolized by {*j*} → {*j*} + *k*{*i*}, meaning: "replace row *j* by (row *j* + *k* times row *i*)". In Ayer's notation, this operation is denoted by $H_{ij}(k)$. We will adopt the $R_j^i(k)$ notation for the rest of the course. There is also a similar operation that acts on the columns, denoted by $C_j^i(k)$.

1. Given $A=\begin{pmatrix}1&2&3\\4&5&9\\8&10&6\end{pmatrix}$, what is $A' = R_3^2(-2)A$?

2. (*i*) What is the determinant of *A* in (1)? (*ii*) That of *A*′? (*iii*) What is your conclusion?

3. If you understand the idea intended to be conveyed in (1), what is your strategy if you were asked to evaluate $\begin{vmatrix}1&66&54\\4&11&9\\8&10&43\end{vmatrix}$? Inspect the values of the elements then think of the best strategy possible.

4. How about evaluating $\begin{vmatrix}1&4&8\\7&11&10\\7&8&16\end{vmatrix}$?

5. Given $A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$. Carry out successive operations of $R_j^i(k)$ to transform *A* into a matrix which has as many zero elements as possible in row 3. Call the resultant matrix *B*. (*i*) What is your resultant matrix, *B*? (*ii*) Is *A* and *B* equivalent? (*iii*) Does *A* and *B* have the determinant? Justify your answer. (*iv*) Evaluate |*A*|.

Chapter 5 EquivalenceQ

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Consider an *n*-square non-zero matrix *A*. (*i*) What is the highest possible rank of matrix *A*? (*ii*) What is the smallest possible rank of *A*? (*iii*) What is the condition on *A* for it to assume the highest possible rank? (*vi*) What kind of matrix *A* is if it fulfils the condition in (*iii*)?

2. In general, for an *n*-square non-zero matrix *A*, its first minors* are (*i*) ___-square minors (*ii*) Can you recognize what the "*n*-square minors" of *A* is?
(*We will refer "first minors" simply as "minors" in the future unless specify otherwise.)

3. Consider the 3-square matrix $Y=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 8\end{pmatrix}$. (*i*) The minors of *Y* are ____-square minors. (*ii*) What is the "3-square minor" of *Y*? (*iii*) What is the rank of *Y*? (*iv*) What is the 4-square minor of *Y*?

4. You may refer to Chapter 3, designed question (12).
Consider the 3-square matrix $X=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$. (*i*) What is the value of the "3-square minor" of *X*? (*ii*) How many "2-square minors" are there in *X*? (*iii*) Is ALL of the "2-square minors" zero? (*iv*) What is the rank of *X*?

5. Consider the 3-square matrix *Z*=$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$. (*i*) What is the value of the "3-square minor" of *Z*? (*ii*) How many "2-square minors" are there in *Z*? (*iii*) Is ALL of the "2-square minors" zero? (*iv*) How many "1-square minors" are there in *Z*? (*v*) Is ALL of the "1-square minors" zero? (*vi*)What is the rank of *X*?

6. Consider matrix *X* as defined in (4). What is *X′*, the image of *X* when it is transformed under the operations

   (*i*) R_2^1 (*ii*) C_2^1 (*iii*) $R_2(3)$ (*iv*) $C_2(3)$

   (*v*) $R_2^1(3)$ (*vi*) $C_2^1(3)$

7. (*i*) What do you call the operations in (6)?
   (*ii*) What is the determinant of *X′* in each case in (6)?
   (*iii*) What is the order of *X′* in each case in (6)?
   (*iv*) What is the rank of *X′* in each case in (6)?

8. Given two examples of equivalent matrices to *X* as defined in (6).

9. Let *A*=$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find a single row elementary transformation (let's call it $E_{A\rightarrow B}$) that transforms *A* into a 3-square diagonal matrix, *B*? We will use the notation $B=E_{A\rightarrow B}A$.

10. Find a single row elementary transformation (let's call it $E_{B\rightarrow I_3}$) that transforms *B* in (9) into the unit matrix I_3?

    We will denote $I_3=E_{B\rightarrow I_3}A$.

11. If we define the operation $U = E_{B\to I_3}\, E_{A\to B}$, what will you get if U is operated on A? We will denote the resultant matrix as $V = UA$.

12. Let $A=\begin{pmatrix}1 & 3 & 2\\ 0 & 2 & 0\\ 5 & 0 & 1\end{pmatrix}$. Transforms A into a 3-square diagonal matrix, call it D, by successively applying row elementary operations. I can achieve a 3-square diagonal matrix in three steps (I call it $R_{stp} = R_{stp3}\, R_{stp2}\, R_{stp1}$, so that $R_{stp}A$ is diagonal matrix.) How many steps you need, and what is your that R_{stp}?

13. Reduce the resultant diagonal matrix in (12) into a unit matrix by applying a Sequence of row elementary transformations. Call this operation R_d. What is your R_d?

14. Now, I will call the operation that transforms A, as in (12), into a 3-square matrix identity matrix $E_{A\to I_3}$.

    With this notation, we have $E_{A\to I_3} A = I_3$`. Write down the form of $E_{A\to I_3}$ in terms of the operations R_{stp} and R_d as obtained in (12) and (13).

"Canonical matrix" as mentioned in page 40, Ayers, is a general form of matrix that fulfills the set of properties (*a*)-(*d*) stated in the same page. A special case of canonical matrices are matrices in Row Reduce Echelon form (RREF). These are matrices that have the following properties:

1. Rows of all zeros, if there are any, appear at the bottom of the matrix.

2. The first nonzero entry of a nonzero row is 1. This is called a leading 1.

3. For each nonzero row, the leading 1 appears to the right and below any leading 1's in preceding rows.

4. Any column in which a leading 1 appears has zeros in every other entry.

A matrix in RREF appears as a staircase pattern of leading 1's descending from the upper left corner of the matrix. The columns of the leading 1's are columns of an identity matrix. A matrix is in row echelon form (REF) if properties 1, 2, and 3 above are satisfied.

15. Now, see if you can demonstrate your understanding on what RREF is by answering the following questions: Given the following matrices, state whether they are in RREF, REF or neither.

$(i)\ \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (ii) \begin{pmatrix} 0 & 0 & 1 & -6 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$

$(iii) \begin{pmatrix} 1 & 5 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} (iv)\ \begin{pmatrix} 0 & 0 & 1 & -6 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$

$(v) \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 2 & 5 \end{pmatrix}.$

16. Convert the above non-RREF matrices in to RREF via elementary row transformations.

17. Consider $B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$. (*i*) Transform B into a row echelon form via elementary row transformations. (*ii*) What is the reduced row echelon form of B? (*iii*) Is the REF as obtained in (*ii*) the same as your friend's? (*iv*) Is the RREF as obtained in (*iii*) the same as your friend's? (*v*) What can you conclude from (*iv*) and (*v*)? After this exercise you should have learnt the trick of reducing any generic matrix into RREF form.

Chapter 6 The adjoint of a square matrixQ

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Given $A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. (*i*) Obtain the cofactor α_{ij} for all *i*, *j*. (*ii*) Form the matrix of the cofactors of *A*, and call it *X*. (*iii*) How is *X* be related to adj *A*? (*iv*) What is |*A*|? (You should have been very familiar with *A*, see DQ in Chapter 5.) (*v*) What is adj*A*·*A*? Try to relate the answer in (*v*) to theorem II, page 50, Ayers.

2. How would you convince yourself that indeed adj*A*·*A*= *A*·adj*A*=diag(|*A*, *A*, *A*, …,|*A*|)?

3. Say *A*, *B* are two matrices conformable for a product in the order of *AB*. What is |*A*||*B*|?

4. How would you convince yourself that $|A|\ |\text{adj}A| = |A|^n$? [*Hint*: use DQ (2).]

5. Consider this statement: If *X* a square matrix and singular, then |adj*X*|=0. Is this statement true?

6. Try to relate this question with what you have learnt in the secondary school. Given $G=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, (*i*) what is Adj *G*? You should be able to write down the answer by inspection. (*ii*) What is |*G*|? (*ii*) Let *H* = Adj *G* /|*G*|, work out what is *HG*. (*iii*) Based on the answer of (*ii*), what is the product *GH*? (*iv*) So, what can you conclude from the above exercise?

Chapter 7 The inverse of a matrixQ

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. For the following two questions, please refer to the designed questions (12), (13), (14), in Chapter 5. There, $E_{A\to I_3}A = I_3$. Now, what would you get when if $E_{A\to I_3}$ is operated on I_3 instead of on A? In other words, ask yourself, what is $E_{A\to I_3} I_3$?

2. So, by now, have you learnt how to find the inverse of a matrix? Find A^{-1}, where A is as defined in DQ (12), Chapter 5, $A=\begin{pmatrix}1 & 3 & 2\\ 0 & 2 & 0\\ 5 & 0 & 1\end{pmatrix}$.

3. Now carry out the $E_{A\to I_3}A=$ and $E_{A\to I_3} I_3$ operations in a "two-in-one" manner, i.e. if the augmented matrix form. First form the augmented matrix of the form $(A|I)=\left(\begin{array}{ccc|ccc}1 & 3 & 2 & 1 & 0 & 0\\ 0 & 2 & 0 & 0 & 1 & 0\\ 5 & 0 & 1 & 0 & 0 & 1\end{array}\right)$. Carrying out $E_{A\to I_3}$ on both sides to arrive at $(I|A^{-1})$.

4. In the designed questions (12), (13), (14) of Chapter 5, you are asked to find a sequence of elementary transformations that transform a generic matrix A into a unit matrix. Now, if A were our old friend $\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$, what would happen if you were to attempt to reduce it into an identity matrix via a sequence of elementary transformations? Explain.

5. Deduce A^{-1}, where A is as defined (2), using $A^{-1}= \text{adj}A/|A|$. Do you get the same answer as in (2) where the inverse is obtained using row reduced echelon form method?

Chapter 9 Linear dependence of vectors and formsQ

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Give examples of two distinct, non-zero 2-vectors in row vector form. Call them X_1 and X_2. (*ii*) What is their sum, $X_3 = X_1 + X_2$? Write it down explicitly. (*iii*) Sketch a picture representing these vectors in a 2-dimensional space. Label your drawing properly (including the vectors and the axes).

2. (*i*) Repeat (*i*) and (*ii*) of the above question but for 2-vectors in *column* form (i.e. giving examples, writing down their sum). (*ii*) If you were to sketch a picture representing these 2-dimensional column vectors in a 2-dimensional plane, as you did in (1)(*iii*), will the drawing be the same as in (1)(*iii*)?

3. (*i*) Give an example of a simplest possible 3-vector in column vector form, with all positive, non-zero components you can think of. Call it X. (*iii*) Sketch a picture representing this vector in a 3-dimensional space. Label your drawing properly (including the vectors and the axes).

4. (*i*) Give an example of a simplest possible 4-vector in column vector form, with all positive, non-zero components you can think of. Call it X. (*ii*) Can you possibly sketch a picture to represent this vector in the similar manner as you do for (2) and (3)? Explain.

5. What is the dimensionality of the vectors in (1), (2), (3) and (4)?

6. Consider the 3-vectors pair X_1= [1, 2, 3], X_2= [-1, -2, -3]. (*i*) Find any possible values of k_1 and k_2, with $\{k_1\ k_2\} \neq \{0, 0\}$, such that $k_1X_b + k_1X_2 = 0$. (*ii*) Are the vectors linearly independent?

7. Explain why is that a zero 3-vector is always linearly dependent with any 3-vector?

8. Consider $X_1=[a, b, c]$ and $X_2=s[a, b, c]$, where s a non-zero scalar. (*i*) Are these 3-vectors linearly independent? (*ii*) Explain why you say so.

9. Consider X_1=[1, 2, 3] and X_2=[4, 5, 6]. (*i*) Are they linearly independent? (*ii*) Explain why you say so.

10. Give a set of three distinct, non-zero 2-vectors, X_1, X_2, X_3 that are linearly independent.

11. (*i*) Consider 3 distinct, non-zero 2-vectors. These vectors must be (linearly dependent / linearly independent). (*ii*), Consider 2 distinct, non-zero 3-vectors. Furthermore, these vectors are not in the form of $X_1=sX_2$ (in other words, they are not parallel nor anti-parallel). These vectors must be (linearly dependent / linearly independent).

12. Refer (9.5) in page 69, Ayers. Given a set of m vectors, we want to know whether they are linearly independent or otherwise. What is the easiest way (or one of the easier ways) to determine the linear independence of such a set of vectors?

    Ans: Use row elementary operations to reduce the matrix A formed by these vectors to RREF. The number of non-zero row in the RREF of A is the rank of the matrix A, r. The rank, r, also tells us how many linearly independent vectors are there in the set of m vectors. If $r = m$, then the set of this m vectors is linearly independent. If $r < m$, then the set of m vectors is linearly dependent.

In such a case, there are exactly r vectors of the set which are linearly independent while each of the remaining m-r vectors can be expressed as a linear combination of these r vectors.

13. Consider the set S containing the following 4 3-vectors: $K_1=[1,1,1]^T$, $K_2=[1,3,5]^T$, $K_3=[1,5, 3]^T$, $K_4=[5, 3, 1]^T$; $S=\{K_1, K_1, K_3\ K_4\}$. (i) Form the matrix A whose rows are made up of the vectors K_i^T , i=1, 2, 3, 4. (ii) Reduce A into RREF. (iii) What is the rank of A? (iv) How many linearly independent vectors are there in the set S? (v) Are the vectors in set S linearly independent?

14. ***The following question could be understood better after you have gone through Chapter 10 on Linear Equations*.

    There is another way to prove the linearly independence of a set of vectors. Consider a set of three vectors K_1= $(1, 4, 7)^T$, K_2= $(2, 5, 8)^T$, K_3= $(3, 6, 9)^T$. Let's find out whether they are linearly independent or otherwise. If the set of these vectors is linearly independent, then the only solution to the homogeneous equation system

    $$x_1K_1+x_2K_2+ x_3K_3 =0$$

    is the trivial solution, i.e. $X=(x_1, x_2, x_3)^T=(0, 0, 0)^T$.
    (i) If we write the homogeneous equation system in the matrix form of KX = 0. What is the matrix K? (ii) Reduce K into RREF to determine rank(K). (iii) How many unknowns are there in the HE system? (iv) By comparing your answer in (ii) and (iii) what can you say about the solution X? (iv) Is the set of three vectors linearly independent?

Chapter_10_Linear EquationsQ

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Consider a system of 3 linear equations in 3 unknown, x_1, x_2, x_3.

$$x_1 + x_2 + x_3 = 0$$

$$2x_1 + 2x_2 + 2x_3 = 0$$

$$3x_1 + 3x_2 + 3x_3 = 0$$

(*i*) Give the most trivial solution to the system of equations above. (*ii*) Give a not-most trivial solution to the system of equations above. (*iii*) Is the system consistent? (*iv*) Explain why you say so in (*iii*). (*v*) How many solutions are there for the linear equation system?

2. Consider a system of 2 linear equations in 2 unknown, x_1, x_2: $x_1 + x_2 = 2$, $x_1 + x_2 = 1$

   (*i*) Try sketching the two equations given in the x_2-x_1 plane. (*ii*) Do these two graphs intersect at all? (*iii*) Can you find a solution to the linear equation system given? (*iv*) Is the system consistent? (*v*) Explain why you say so in (*iv*).

3. Express the linear equation system in (1) and (2) in (*i*) matrix form $AX=H$. (*ii*) augmented matrix form $[A\ H]$.

4. Consider the following system of equation:

$$x_1 - x_2 = 1 \qquad \text{(Eq. 1)}$$

$$x_1 + x_2 = 2 \qquad \text{(Eq. 2)}$$

(*i*) Express the equation system above into augmented matrix form.
(*ii*) Perform the following operation: Replace (Eq. 1) by (Eq.1) + (Eq. 2), and call the resultant equation (Eq 1’). At the mean time, leave (Eq. 2) untouched. Write down the resultant equation system. What are the corresponding row

elementary operations that have an equivalent effect on the augmented matrix?

(*iii*) Now, perform a follow-up operations: Replace (Eq. 2) by (Eq. 2) added with (Eq 1') multiplied by (-1/2). Write down the resultant equations. What are the corresponding row elementary operations that have an equivalent effect on the augmented matrix?

(*iv*) Now, multiply (Eq. 1') by a factor of 1/2. Write down the resultant equations. What are the corresponding row elementary operations that have an equivalent effect on the augmented matrix?

(*v*) Now, reduce the augmented matrix in (*i*) into RREF. Do you get the same augmented matrix as resulted in (*iv*)? (*vi*) Can you read off the solutions of x_1, x_2 from the resultant RREF matrix by inspection?

5. (*i*) Express the following linear equation system in augmented matrix form [*A H*].

   $x_1 + x_2 + x_3 = 1$; $2x_1 + x_2 + 3x_3 = 2$;

   $3x_1 + x_2 + x_3 = 0$.

   (*ii*) Reduce the augmented matrix [*A H*] into reduced row echelon form. (*iii*) Write out the equation system represented by the RREF augmented matrix as obtained in (*ii*). Read off the solutions by inspection.

6. Choose the correct answer: (*i*) REF is a (special / general) case of RREF. (*ii*) The procedure you used in (5), in which the augmented matrix is reduced into RREF, to solve for the solutions, is called (Gaussian /Gaussian-Jordan) elimination.

7. The equation system in

(*i*) (1) is a (homogeneous/non-homogeneous) system.
(*ii*) (2) is a (homogeneous/non-homogeneous) system.
(*iii*) (5) is a (homogeneous/non-homogeneous) system.

8. Refer to theorem III, page 77, Ayers. Answer the following properties of the equation system in (5):
   (*i*) The system in (5) is (homogeneous or non-homogeneous).
   (*ii*) The number of unknowns in (5) is _______.
   (*iii*) The number of equation in (5) is _______.
   (*iv*) The determinant of $|A|$ in (5) is (zero / non-zero).

(*v*) The solution as obtained in (5) is (unique / not-unique).

10. Given an arbitrary linear equation system of the form $AX=H$, there are two possibility on the existence of its solution, i.e. either the solution exists or __________. If the solution exists, it could be ___________or ____________, ___________ or _____________.

11. Give a simple example of a non-homogeneous system that (*i*) has a unique solution. What is the unique solution? (*ii*) has no solution (i.e. not consistent); (*iii*) has non-unique solutions.

12. Give an example of a 2 by 2 homogeneous system that (*i*) has a unique solution; (*ii*) has no solution (i.e. not consistent); (*iii*) has non-unique solutions.

13. Given a non-homogeneous equation $AX=H$, with $|A| \neq 0$, A an n-square coefficient matrix, X column vector of n variables, H non-zero column vector of n components.
(*i*) List down all the ways you can think of that can be employed to solve the linear equation systems. I can think of 5, how many can you think of?
(*ii*) As an exercise, solve the given equation system using all the methods you have listed. You should get the same answer with all these different methods.

14. Consider an equation system $AX=H$, which represent m equations in n unknown. What is the sufficient condition that this equation system is consistent? (*Hint: find the answer in page 76 of Ayers "fundamental theorems."*)

15. Consider a homogeneous equation system $AX=0$, which has n equations in n unknowns.
(*i*) Does the system is guaranteed consistent? Explain by referring to your answer in (14).
(*ii*) What can you say about the solution X if the rank of A, r, is equal the number of unknowns, n? (Give your answer in terms of its existence, uniqueness and triviality.)
(*iii*) What can you say about the solution X if the rank

of A, r, is less than the number of unknowns, n? (Give your answer in terms of its existence, uniqueness and triviality.)
(*iv*) Consider the statement: ALL homogeneous equation system is consistent. Is this statement true?

16. Consider the homogeneous equation system in (15), $AX=0$. (*i*) If A is not singular, A^{-1} exist. What do you get if you operate A^{-1} on $AX=0$ from the left-hand-side (LHS)? Try to figure out what will happen to the solution X. (*ii*) So, what is your conclusion?

17. Consider (16) again. (*i*) If A is singular, i.e. A^{-1} does not exist, can you claim that $AX=0$ has no solution? (*ii*) What's the difference in the solution of a homogeneous equation system of singular coefficient matrix A and one that is not? Give your answer in terms of its existence, uniqueness and triviality.

18. Consider the equation system

$x_1 - 2x_2 + 3x_3 = 4$; $x_1 + x_2 + 2x_3 = 5$. (*i*) Express the system in matrix form. (*ii*) Is the number of unknowns larger than the number of equation? (*iii*) So, how many solutions would you expect? (*iii*) Solve the equation system using Gaussian-Jordan elimination.

19. Refer to solved problems 2, page 79 of Ayers. We will learn how to 'count' the rank of a matrix in this DQ. Consider a homogeneous system of 2 linear equations in 3 unknown, x_1, x_2, x_3, $AX=H$, where

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 4 & 5 & 3 & 0 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad H = \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$$

(i) Transform the augmented matrix $[A\ H]$ into $[A_1 \mid H_1]$, the RREF form of $[A \mid H]$ using row elementary operations. Express $[A_1 \mid H_1]$ explicitly.

(ii) Is the rank of $[A_1 \mid H_1]$, $[A \mid H]$ equal?

(iii) Count the number of non-zero row in $[A_1 \mid H_1]$. The number of non-zero row equals the rank of the augmented matrix $[A_1 \mid H_1]$. This number is the same as the number of leading 1 in the RREF.

(Note that this simple fact is not mentioned explicitly in Ayers.) Hence, what is the rank of $[A \mid H]$?

(iv) From the expression of $[A_1 \mid H_1]$, what is the rank of the matrix A_1? What is the rank of matrix A?

(v) By referring to the fundamental theorem I, page 76, Ayers, is the system consistent?

20. Consider the homogeneous equation system of

$$x_1 + 2x_2 + 2x_4 = 0\,,\; x_2 + 3x_3 = 0\,.$$

(i) Express the system in the matrix form of $AX=H$.

(ii) This is an equation system with _____equations and _______ unknowns.

(iii) Determine the rank of A and $[A \mid 0]$.

(iv) Is this system consistent?

(v) By comparing the rank of A and the number of unknowns, can you determine whether the system will admit non-trivial solution? State explicitly whether the non-trivial solutions are expected. (*Hint: refer to theorem IV, page 78, Ayers.*)

(vi) From the answer to (*ii*), state your expectation whether the solutions will be unique or otherwise.

(vii) Based on your answer to (*v*) you know what the rank of the matrix A is. Hence, how many linearly independent solutions do you expect for the HE system? (*Hint: refer to theorem VI, page 78, Ayers.*)

21. Solve the HE system in the previous DQ. Your solution should agree with the number of linearly independent solutions is as given in (*vi*) in the same question.

22. You may like to refer to Example 5, page 79, Ayers. Consider a non homogeneous equation system:

$$x_1 - 2x_2 + 3x_3 = 4\,,\quad x_1 + x_2 + 2x_3 = 5\,.$$

(i) Express the system in the matrix form of $AX=H$.

(ii) This is an equation system with _____equations and _______ unknowns.

(iii) Determine the rank of $[A \mid H]$ and A.

(iv) Is this system consistent?

(v) From the answer to (*ii*), state your expectation whether the solutions will be unique or otherwise.

(vi) Based on your answer to (*iii*) you know what

the rank of the matrix *A* is. Hence, how many linearly independent solutions do you expect for the HE system? (*Hint: refer to theorem VI, page 78, Ayers.*)

(vii) Obtain the solution.

23. You may like to try out the trick you have learnt from the above DQ on the solved problem 1 in page 79, Ayers.

1. **Definition:** n-vector
An vector $\boldsymbol{a}$ with n-component is an n-tuple of real numbers, $\boldsymbol{a} = \{a_1, a_2, \ldots, a_n\}$. We call this an n-vector. a_i, $i=1, 2, \ldots, n$ are the components of $\boldsymbol{a}$. It has n components.

2. As an special example, for $n=3$, $\boldsymbol{a}=\{a_1, a_2, a_3\}$. $\boldsymbol{a}$ can be imagined as a point in 3-space, the 3-dimensional space we human resides in. For example, the 3-vector $\boldsymbol{a}=\{0,0,0\}$ represents a point with spatial coordinates $\{0 , 0, 0\}$.

3. Imagine the collection of all possible 3-vectors into a set V containing all points in the 3-space. We call the set of all 3-vectors, (or in other words, all points in the 3-D space), R^3. Each vector in R^3 is equivalent to a point in the 3-space.

4. Similarly, R^2 is the set of all 2-vectors. R^2 is the set of all points in 2-space.

5. R, the set of all real number, is the set of all ‘1-vector’ (‘1-vector’ is just the real scalar we all familiar with). The collection of all ‘points’ in the 1-space is equivalent to the set of all points in a 1-dimensional ‘real-number line’.

6. For the 2-vectors and 3-vectors, we know that we can add and do scalar multiplication on them according to well-defined rules of vector addition and scalar multiplication. As an illustration, consider this: Given two 3-vectors in R^3, $\boldsymbol{a}=\{a_1, a_2, a_3\}$ and $\boldsymbol{b}=\{b_1, b_2, b_3\}$, the vector addition $\boldsymbol{a} + \boldsymbol{b}$ is defined as a new 3-vector, $\boldsymbol{c} = \{a_1+b_1, a_2+b_2, a_3+b_3\}$. Similarly, the scalar multiplication between a scalar k and a vector $\boldsymbol{a}$ is defined as a new vector $\boldsymbol{d}=\{ka_1, ka_2, ka_3\}$.

7. **Definition:** Consider a set V containing some elements on which operations of **vector addition** and **scalar multiplication** are defined. The set V is called a **vector space** if the following ten properties are satisfied:

DEFINITION 7.5 Vector Space

Let V be a set of elements on which two operations called **vector addition** and **scalar multiplication** are defined. Then V is said to be a **vector space** if the following ten properties are satisfied.

Axioms for Vector Addition

(*i*) If $\mathbf{x}$ and $\mathbf{y}$ are in V, then $\mathbf{x}+\mathbf{y}$ is in V.
(*ii*) For all $\mathbf{x}, \mathbf{y}$ in V, $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$. (commutative law)
(*iii*) For all $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in V, $\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}$. (associative law)
(*iv*) There is a unique vector $\mathbf{0}$ in V such that $\mathbf{0}+\mathbf{x}=\mathbf{x}+\mathbf{0}=\mathbf{0}$. (zero vector)
(*v*) For each $\mathbf{x}$ in V, there exists a vector $-\mathbf{x}$ such that $\mathbf{x}+(-\mathbf{x})=(-\mathbf{x})+\mathbf{x}=\mathbf{0}$. (negative of a vector)

Axioms for Scalar Multiplication

(*vi*) If k is any scalar and $\mathbf{x}$ is in V, then $k\mathbf{x}$ is in V.
(*vii*) $k(\mathbf{x}+\mathbf{y})=k\mathbf{x}+k\mathbf{y}$
(*viii*) $(k_1+k_2)\mathbf{x}=k_1\mathbf{x}+k_2\mathbf{x}$ (distributive laws)
(*ix*) $k_1(k_2\mathbf{x})=(k_1k_2)\mathbf{x}$
(*x*) $1\mathbf{x}=\mathbf{x}$

8. Consider the 3-space, R^3. As mentioned, this a vector space. Can you justify this claim by referring to the definition as given?

Ans:

This is a vector space because (*i*) vector addition and scalar multiplication are well defined on all of the 3-vectors, the elements in R^3, (*ii*) all of the 3-vectors, the elements in R^3, fulfill the 10 axioms. In particular, all 3-vectors are closed under vector addition and closed under scalar multiplication.

9. Explain what do you understand by (*i*) 'closure under vector addition'. (*ii*) 'closure under scalar multiplication'.

10. Consider R^2. Is it also a vector space? How about the set of all real number, the 1-space, R? How do you convince yourself that they are indeed also vector space?

11. **Definition:** A set of vectors V_s from a vector space V is a **subspace** of V if V_s is closed under addition and scalar multiplication.

Example: The set containing only the element 0, $V_s=\{0\}$, is a subspace of the vector space R, since the $\{0\}$ is

(*i*) a element vector from R,

(*ii*) closed under scalar multiplication:
$k{\cdot}0 = 0 \in V_s$,

(*iii*) closed under vector addition:
$0 + 0 = 0 \in V_s$.

Note that the subspace $\{0\}$ has only a single element. The criteria of being closed under addition are fulfilled: "if x and y are element is V_s, then $x + y$ is also an element in V_s". Here, x=0, y=0, because there is no any other element in V_s other than 0. In other words, 'any element' in {0} (the *x*), when vectorially added to 'any element' in {0} (the *y*) will result in $x + y = 0$, an element of V_s.

12. Every vector space *V* has at least two subspaces. One of if is the zero subspace, {**0**}, which is illustrated above. Can you think of what's the other one?

13. **Definition:** Consider a set *S* containing vectors $\mathbf{x}_1$, $\mathbf{x}_2$, …$\mathbf{x}_m$ in a vector space *V*. (To help you visualize better, think of *V* as the vector space of R^3 that contains an infinite number of 3-vectors. Think of *S* as a set containing, say, *m*=3 vectors selected from R^3.) We form linear combinations of these *m* vectors in the form of $k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \ldots k_m\mathbf{x}_m$, where k_i are scalars. The set of all linear combinations of the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, …$\mathbf{x}_m$ is called the **span** of the vectors, and is written as **Span(*S*)**.

14. Span(*S*) is a subspace of *V*. Span(*S*) is said to be a subspace spanned by the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, …$\mathbf{x}_n$.

15. If every vector in the vector space *V* can be written as a linear combination of the vectors in *S*, then *S* is called a **spanning set** for *V*.

Example: Let *V* be the vector space containing all 3-vectors, R^3. Consider the set *S*={**i**, **j**, **k**} containing the three rectangular unit vectors. The set of all linear combination $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, where *a*, *b*, *c* are scalar, is the span of the vectors {**i**, **j**, **k**}, Span(*S*). Span(*S*) is a subspace in R^3 spanned by **i**, **j**, **k**.

16. We say 'the set *S*={**i**, **j**, **k**} is a spanning set for R^3'. Think of Span(*S*) in terms of the set of all possible linear combination in terms of **i**, **j**, **k**, $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$. Can you imagine what does Span(*S*) represent? *Hint*: Imagine the point at the tip of the 3-vector $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$. Imagine the pervasive cloud form by the tip of $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ when *a*, *b*, *c* vary continuously.

17. Can you think of any other spanning set for R^3?

Ans: e.g. {**r,** θ, ϕ}.

18. Is the set {**i**, **j**, **k**, **i**+**j**, **k**+**i, i + k**} also spanning set for R^3?

19. Is {**i**, **j**} a spanning set for R^3? Explain your answer.

20. Consider the set S containing the following 4 3-vectors: K_1=[1,1,1]^T, K_2=[1,3,5]^T, K_3=[1,5, 3]^T, K_4=[5, 3, 1]^T; S={K_1, K_1, K_3 K_4}. How would you prove that the S is the spanning set of R^3 (or in other words, S span R^3)?
Hint: To prove that the set of vectors in S span R^3, one needs to prove the existence of the solution
X=[x_1, x_2, x_3, x_4]^T for the non-homogeneous equation
system $A = x_1K_1 + x_2K_2 + x_3K_3 + x_4K_4$, given an
arbitrary 3-vector A=$(a, b, c)^T$ from R^3**.** If the solution X exists, then S spans R^3, otherwise it doesn't. The reasoning is: If the solution X exists, this means that any arbitrary vector A from R^3 can always be expressed as a unique linear combination in the form
of $x_1K_1 + x_2K_2 + x_3K_3 + x_4K_4$. Hence, by definition, if
the set of vectors in S is a spanning set of R^3.

21. In general, given a set of m n-vectors, K_i=(k_1, k_2,…, k_n)^T , i=1,2,…m, we can determines whether they span a vector space R^n, the vector space containing the set of all n-vector by looking for the existence proof of solution X to the non-homogeneous system. The procedure is as followed:

22. Let K=$\left(K_1, K_2,..., K_m\right)$, an n by m matrix,
X=$\left(x_1, x_2,..., x_m\right)^T$, an m by 1 column vector,
A=$\left(a_1, a_2,..., a_n\right)^T$, an arbitrary n-vector in R^n. Consider
the NH system $A = x_1K_1 + x_2K_2 + \cdots + x_mK_m = KX$. If
the solution for the NH systems does not exist, i.e. rank[K] $\neq$ rank [K | A], then the set of vectors K_i does not span R^n. Otherwise, they do.

23. In (20), we see that the set $\{K_1, K_2, K_3, K_4\}$ comprises of 4 3-vector spans R^3. Can we span R^3 with less than 4 3-vectors (e.g., say, 3 or even 2 3-vectors)? In general, for a vector space *V* containing elements made up of *n*-vectors, we want to know what is the smallest number of linearly independent *n*-vector that spans the vector space *V*.

24. In fact, out of the four 3-vectors in the set *S* in (20), only three are linearly independent (refer DQ 13, Chapter 9), namely K_1, K_2, K_3, whereas K_4 can be expressed as a linear combination of the other three vectors.

    (*i*) Prove the linearly independence of the vector set K_1, K_2, K_3. (*Hint:* Refer to DQ 12, 13, 14 in Chapter 9.)

    (*ii*) Prove, using the procedure mentioned in (22) above, that this set of vectors K_1, K_2, K_3 spans R^3.

25. Now, we ask: can any of the 2 vectors (which are necessarily linearly independent) form the set $\{K_1, K_2, K_3\}$ span R^3? The answer can be proven to be negative. (Prove this). So, it appears that the minimum number of linearly independent 3-vectors to span R^3 is 3, not 2.

26. **Definition:** The minimum number of linearly independent vectors that is required to span a vector space is called the **dimension** of the vector space. In the above example, the dimension of the vector space R^3 is 3 since the minimum number of linearly independent vectors in R^3 is 3.

27. **Definition:** Consider a vector space *V* with dimension *r*. A set of *r* linearly independent vectors in *V* is called the **basis** (or basis set) of the vector space. It happens that given any set of *r* vectors, which are linearly independent, from *V*, they (*i*) will form a basis set for *V*, and (*ii*) any vector in *V* can be expressed as a unique linear combination in this set of *r* vectors.

28. Let's consider the vector space R^3. We know that the dimension of it is r=3.

(i) If I simply pick any three vectors in R^3, say $X_1 = (a, b, c)$, $X_2 = (d, e, f)$, $X_3 = (g, h, i)$, in general, will the set $\{X_1, X_2, X_3\}$ form a basis for R^3?

(ii) Is the basis set of R^3 unique?

(iii) How many basis set can R^3 possibly has?

29. Consider the set of three vectors in R^3, $S=\{E_1, E_2, E_3\}$, where $E_1= [1, 0, 0]^T$, $E_2= [0, 1, 0]^T$, $E_3= [0, 0, 1]^T$.

(i) Are the vectors in S linearly independent (you should be able to answer this simply question by visual inspection)?

(ii) Do the vectors in the set S form a basis set for R^3?

(iii) Do the vectors in the set S span R^3?

(iv) Can every vector in R^3 be expressed as linear combination of E_1, E_2, E_3?

(v) What's the name of these E-vectors? (*Hint: see page 88 of Ayers).* (Note: we will refer this basis set by the name 'the E-basis').

30. You may like to refer to Ayers page 88. Say I have an arbitrary vector in R^3, $X=(a, b, c)^T$.

(i) Write X as a linear combination of the unit vectors, E_i, defined in (30).

(ii) What are the components (or referred to as 'coordinates') of X relative to the E-basis? Write these components in the form of a column vector and call it 'the component vector of X relative to the E-basis', denoted by X_E.

31. In the previous question, we have an arbitrary vector in R^3, X. Let's say that the vector X when expressed in the E-basis is represented by the component vectors $X_E=(1, 2, 3)^T$. Normally, a vector is by default expressed in the E-basis. In general, other than the E-basis, we can also represent a vector in other basis set. To illustrate this point, let's consider another basis set $Z = \{Z_1, Z_2, Z_3\}$ ('the Z-basis'), where $Z_1= [2, -1, 3]^T$, $Z_2= [1, 2, -1]^T$, $Z_3= [1, -1, -1]^T$. What is the component vector of X relative to the Z-basis, X_Z? [*Hint:* In order to obtain X_Z, you need

to express X as a linear combination of $\{Z_1, Z_2, Z_3\}$: $X_E = a_1Z_1 + a_2Z_2 + a_3Z_3$. Then the component vector of X in the Z-basis is simply $X_Z = (a_1, a_2, a_3)^T$.]

32. Refer to Example 5, page 88 Ayers. Now, see if you can do things another way round: If the component vector of X is given in the Z representation, i.e. $X_Z = (1,2,3)^T$ is known. What is component vector of X in the E-basis? In other words, what is X_E? *Hint*: Follow the procedure as described in (32), then try to find a similar relation that relates X_E to X_Z in the form of

$$X_E = [\text{some matrix}] \cdot X_Z$$

Linear transformation and the change of basis

Abstract

This short note supplements the Linear algebra part of ZCA 110. In particular it discusses in understandable language (i) the idea of linear transformation involving different bases, as discussed in Chapter 12, Ayers, and (i) the idea of bases and coordinates, page 88-89, Ayers.

1 Going from one basis to another

Consider a generic $n-$vector, X, in $V_n(R)$. The vector X can be represented in different basis (as a simile: think of the appearance of an actor viewed through different coloured glasses by different autdience. Despite it is the same actor, to different audience the actor appears differently.) As an illustration, we will discuss how the vector be represented in two different basis. Let's agree to call these two generic basis the $W-$basis and the $Z-$basis.

The W-basis consists of a set of n $n-$vector, namely $\{W_1, W_2, \cdots, W_n\}$, where each of the W_i, $i = 1, 2, \cdots n$ is an n by 1 column vector, $W_i = (w_1, w_2, \cdots, w_n)^T$. Similarly, the $Z-$basis consists of the set $\{Z_1, Z_2, \cdots, Z_n\}$, $Z_i = (z_1, z_2, \cdots, z_n)^T$, $i = 1, 2, \cdots n$. The connection between the two bases can be worked out via the following consideration:

In the $W-$basis, the vector X presents itself as a linear combination in terms of W_i's, i.e.

$$X = a_1 W_1 + a_2 W_2 + \cdots a_n W_n. \tag{1}$$

a_i are scalars called the components of X in the $W-$basis. By definition, the components vector of X in the $W-$ basis is the column vector that contains all of the components, or coordinate, of vector X in the $W-$basis. It is denoted by $X_W = (a_1, a_2, \cdots, a_n)^T$. Let us arrange all of the basis vector W_i column-by-colum into the matrix $W = (W_1, W_2, \cdots, W_n)$, where W is a $n \times n$ matrix. Now, Eq. (1) can be compactly written in the form of

$$X = a_1 W_1 + a_2 W_2 + \cdots a_n W_n = (W_1, W_2, W_3, \cdots, W_n)(a_1, a_2, \cdots, a_n)^T = W \cdot X_W. \tag{2}$$

X_W is the coordinate vector of X relative to the W-basis.

Similarly, if the vector X were to be represented in the $Z-$basis,

$$X = b_1 Z_1 + b_2 Z_2 + \cdots b_n Z_n = (Z_1, Z_2, Z_3, \cdots, Z_n)(b_1, b_2, \cdots, b_n)^T = Z \cdot X_Z. \tag{3}$$

$X_Z = (b_1, b_2, \cdots, b_n)^T$ is the coordinate vector of X relative to the W-basis.

The vector X is the same vector irrespective of its basis representation, hence

$$X \quad = \quad W \cdot X_W = Z \cdot X_Z \tag{4}$$

Eq. (4) relates the coordinate vector of vector X represented in the Z-basis to that in the W-basis. The coodinate vector in one basis can be determined if the coordinate vector in the other is known, and vice versa.

For example, if we know X_W, we can determine X_Z by making use of Eq. (4): We form the matrix

$$P = Z^{-1} \cdot W, \tag{5}$$

operate it to X_W from the left to obtain

$$X_Z = P \cdot X_W = (Z^{-1} \cdot W) \cdot X_W. \tag{6}$$

Conversely, we can obtain X_W if X_Z is known via

$$X_W = P^{-1} X_Z. \tag{7}$$

2 Linear Transformation

In simple language, a transformation is an operation that operates onto a vector to make it into another vector. Normally the operation is realised via matrix multiplication. Say X is a vector to be transformed into anoter vector, call it Y. To implement the transformation, we will operate a matrix A onto X to make it into Y. Symbolically, $X \stackrel{A}{\rightarrow} Y$; operationally, $Y = AX$. The transformation matrix A contains the information (instruction) of how the vector is to be transformed (e.g. to rotate X about the origin by +90 degree, to reflect the vector X about the origin, etc.). Y, the resultant vector under the transformation, is called the image of X under transformation A.

A transformation can be carried out in any basis. Consider a vector X is tranformed into vector Y. Such a transformation can be represented in both the W-basis and the Z-basis. In each basis the transformation takes on different forms. Say A is the transformation matrix in the W-basis representation, whereas B is the correspoinding transformation in the Z-basis representation. The linear transformations in both bases are given by:

W-basis	Z-basis
$W = (W_1, W_2, \cdots, W_n)$	$Z = (Z_1, Z_2, \cdots, Z_n)$
$X \stackrel{A}{\rightarrow} Y$	$X \stackrel{B}{\rightarrow} Y$
In component form	In component form
$X_W \stackrel{A}{\rightarrow} Y_W$	$X_Z \stackrel{B}{\rightarrow} Y_Z$
In matrix form	In matrix form
$Y_W = AX_W$	$Y_Z = BX_Z$

Question: What is the definition of 'linear transformation'? (*Hint:* Refer to Ayers, page 94.)

Now, we shall prove that: If

$$\begin{aligned} X_W &\stackrel{A}{\rightarrow} Y_W \text{ in the } W\text{-basis} \\ X_Z &\stackrel{B}{\rightarrow} Y_Z \text{ in the } Z\text{-basis}, \end{aligned}$$

then the two transformation A, B are similar, i.e.,

$$B = Q^{-1}AQ,$$

where $Q = P^{-1} = (Z^{-1}W)^{-1} = W^{-1}Z$.

Question: What does it mean, mathematically, when it is said that matrix A and matrix B are 'similar'?

The proof is as followed: We begin with

$$Y_Z = BX_Z. \tag{8}$$

In Eq. (8), the LHS, i.e. Y_Z is related to Y_W via $Y_Z = PY_W$ [see Eq. (6)], whereas X_Z in the RHS is related to X_W via $X_Z = PX_W$ [see Eq. (7)]. Hence, Eq. (8) can be written as

$$\begin{aligned} & PY_W = B(PX_W) \\ \Rightarrow \quad & Y_W = (P^{-1}BP)X_W. \end{aligned} \tag{9}$$

Eq. (9) is just the tranformation of X_W into Y_W by A (in the W-basis), i.e.

$$Y_W = (P^{-1}BP)X_W \equiv AX_W. \tag{10}$$

Hence, we can identify

$$A = P^{-1}BP,$$

or

$$B = PAP^{-1} = Q^{-1}AQ,$$

where P is given by Eq. (5).

Questions: (*i*) Attempt example 2, page 96, Ayers, yourself. (*ii*) Attempt Solved Problem 1, in the same page, yourself. Try to make yourself proud by solving these problems without reading the solutions.

ZCA 110
Linear Algebra
Chapter 12
Basis and coordinates

Please refer to Chapter 11, page 88-89, Ayres.

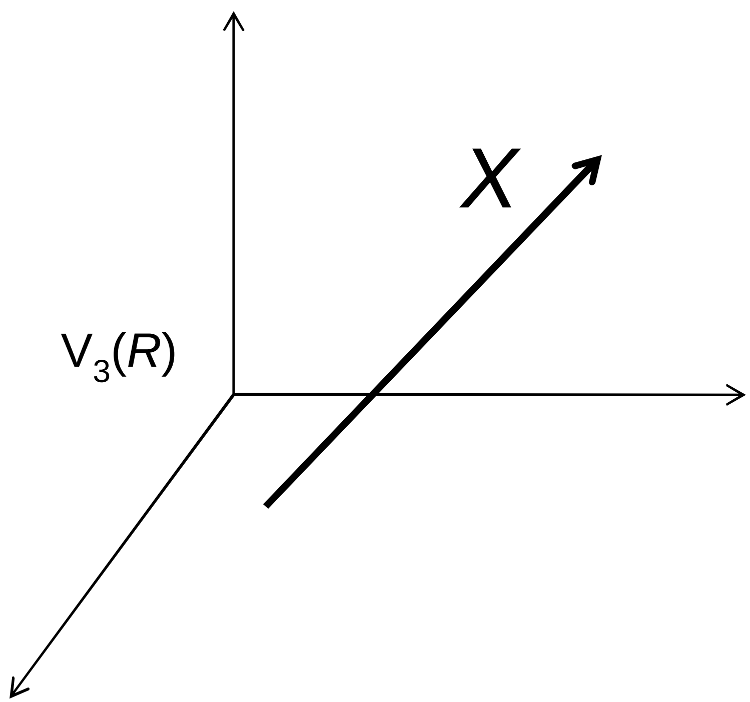
We will discuss how to relate a 3-vector in the vector space $V_3(R)$ in one basis to another basis.

- The description presented in these slides can be trivially generalised to general case of a *n*-vector living in a real $V_n(R)$ space.

A 3-vector living in $V_3(R)$

- Consider a 3-vector X in the real 3-space $V_3(R)$
- This is a mathematical object quantity with three component, expressed in the form of a column vector

$$X = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$



Basis

- Any vector in $V_3(R)$ can be expressed in any basis
- Consider an arbitrary basis, which shall name it as the *W*-basis
- The *W*-basis is comprised of three basis vector
- $\{W_i\} = \{W_1, W_2, W_3\}$, where each W_i is a 3-component column vector

W-basis

$$\{W_i\} = \{W_1, W_2, W_3\}$$

$$W_1 = \begin{pmatrix} w_{11} \\ w_{21} \\ w_{31} \end{pmatrix}, W_2 = \begin{pmatrix} w_{12} \\ w_{22} \\ w_{32} \end{pmatrix}, W_3 = \begin{pmatrix} w_{13} \\ w_{23} \\ w_{33} \end{pmatrix}$$

Vector *X* represented in *W*-basis

- In the W-basis an arbitrary vector *X* is represented as

$$X = a_1 W_1 + a_2 W_2 + a_3 W_3$$

- a_1, a_2, a_3 are the components of the vector *X* in the *W*-basis

- We put all the components a_i in a column vector called

"the coordinate vector of *X* relative to the W-basis"

$$X_W = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Vector X represented in Z-basis

- Likewise, the same vector X can also be represented in other basis, say, a Z-basis

$$X = b_1 Z_1 + b_2 Z_2 + b_3 Z_3$$

- b_1, b_2, b_3 are the components of the vector X in the Z-basis

- We put all the components b_i in a column vector called

"the coordinate vector of X relative to the Z-basis"

$$X_Z = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

What are the questions we want to answer?

- Given the knowledge of a vector in a specific basis, we want to know what is the representation of the same vector in another basis.

What are the questions we want to answer? (cont.)

- For example, if we know X_Z, and the bases $\{Z_i\}$, we want to know what is X_W.

- Usually, the bases in questions, $\{Z_i\}$ and $\{W_i\}$ are known.

How to relate X_Z to X_w (and vice versa)

$$X = \underbrace{b_1 Z_1 + b_2 Z_2 + b_3 Z_3}_{\text{representation in } Z\text{-basis}} = \underbrace{a_1 W_1 + a_2 W_2 + a_3 W_3}_{\text{representation in } W\text{-basis}}$$

$$= \underbrace{\begin{pmatrix} Z_1 & Z_2 & Z_3 \end{pmatrix}}_{z} \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{X_Z} = \underbrace{\begin{pmatrix} W_1 & W_2 & W_3 \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}}_{X_W}$$

X_Z is related to X_w via the "matrices of the bases", W and Z

$$X = ZX_Z = WX_W$$

$$\underbrace{Z}_{\text{a 3 by 3 matrix}} = \begin{pmatrix} Z_1 & Z_2 & Z_3 \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix};$$

$$\underbrace{W}_{\text{a 3 by 3 matrix}} = \begin{pmatrix} W_1 & W_2 & W_3 \end{pmatrix} = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}$$

$$X_Z = Z^{-1}WX_W \qquad X_W = W^{-1}ZX_Z$$

Example
(Example 5, Ayres pg 88)

If $Z_1=(2,-1,3)^T$, $Z_2=(1,2,-1)^T$, $Z_3=(1,-1,-1)^T$ is a basis of $V_3(R)$

W is the E-basis (elemental basis), $W_1=(1,0,0)^T$, $W_2=(0,1,0)^T$, $W_3=(0,0,1)^T$

If $X_Z=(1,2,3)^T$, then the component vector of X in the W-basis (E-basis), X_W is

$$X = ZX_Z = WX_W$$

$$\Rightarrow X_W = W^{-1}ZX_Z = IZX_Z = ZX_Z = \begin{pmatrix} Z_1 & Z_2 & Z_3 \end{pmatrix} X_Z$$

$$= \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 3 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix}$$

Conclusion for the previous example

The vector *X* in the *Z*-basis representation is given by $X_Z=(1,2,3)^T$

Whereas in the *W*-basis (which is actually the *E*-basis) representation the vector *X* is represented as $X_W=X_E=(7,0,2)^T$

Linear transformation

(Please refer to Chapter 12, Ayres)

Linear transformation

- Roughly speaking, linear transformation is an operation that maps a vector *X* into its image *Y* via a transformation matrix, *A*

$$X \xrightarrow{A} Y$$

- Operationally,

$$Y = AX$$

Y is called "the image of *X* under transformation *A*"

Transformation expressed in different bases

- As discussed in earlier slides, a vector must specified with respect to a basis.
- So is the transformation relation

In W-basis,

$$Y_W = A_W X_W$$

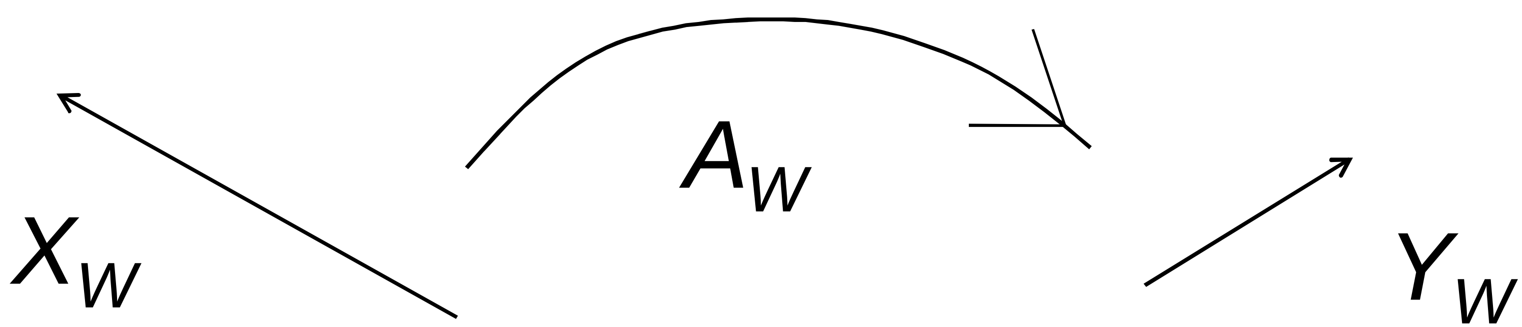
In Z-basis,

$$Y_Z = A_Z X_Z$$

Transformation expressed in *W*-basis

$$Y_W = A_W X_W$$

- X_W is the component vector of X in W-basis
- Y_W is the component vector of Y in W-basis
- X_W is the "original" vector, and Y_W is the image of X_W under the transformation A_W



A note of caution

Note that we are actually talking about the *same* operation A on X **but** NOT two different operations

A_W and A_Z are the same operation expressed in different bases

Both A_W and A_Z are non-identical matrices, but their effect on X is the same: both map X to Y in their respective basis.

A_W and A_Z are related via a similarity transformation

$$A_Z = Q^{-1} A_W Q$$

$$A_W = Q A_Z Q^{-1}$$

$$\text{where} \quad Q = W^{-1} Z, \quad Q^{-1} = Z^{-1} W$$

Proof of $A_W = Q A_Z Q^{-1}, Q \equiv W^{-1} Z$

$$Y_Z = A_Z X_Z$$

$$\text{LHS}: Y_Z = \left(Z^{-1} W\right) Y_W; \qquad \text{RHS}: A_Z X_Z = A_Z \left[\left(Z^{-1} W\right) X_W\right]$$

$$\text{LHS} = \text{RHS}:$$

$$Z^{-1} W Y_W = A_Z Z^{-1} W X_W$$

$$Y_W = \left(W^{-1} Z A_Z Z^{-1} W\right) X_W$$

Compare this to $Y_W = A_W X_W$,

$$A_W \equiv \left(W^{-1} Z\right) A_Z \left(Z^{-1} W\right) = Q A_Z Q^{-1} \Leftrightarrow A_Z = Q^{-1} A_W Q,$$

$$Q \equiv W^{-1} Z$$

How to remember the similarity transformation

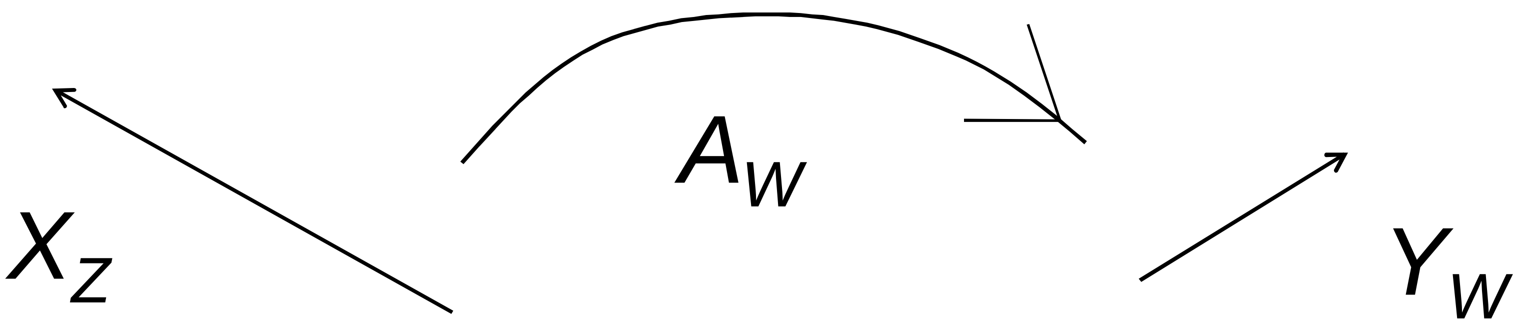
$$A_W = QA_Z Q^{-1}, Q \equiv W^{-1} Z$$

$$A_W = W^{-1} \underbrace{Z A_Z Z^{-1}} W \xrightarrow{X \leftrightarrow Z} A_Z = Z^{-1} \underbrace{W A_W W^{-1}} Z$$

Transformation expressed in Z-basis

$$Y_Z = A_Z X_Z$$

- X_Z is the component vector of X in Z-basis
- Y_Z is the component vector of Y in Z-basis
- X_Z is the “original” vector, and Y_Z is the image of X_Z under the transformation A_Z



What are the questions we want to answer?

- Usually in a given question, the bases $\{Z_i\}$, $\{W_i\}$ are known quantities.
- We are given (*i*) a transformation matrix A in a given basis, say, the Z-basis (i.e., A_Z is know), and (*ii*) an 'original vector' X in a basis, say X_Z.
- The question usually involves the calculation for

(1) Y_Z

(2) X_W

(3) A_W

(4) Y_W .

Ayres, Chapter 12, example 2, page 96

Given $A_Z = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 3 & -1 & -1 \end{pmatrix}$ relative to the Z-basis (which is the E-basis)

$$Z_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, Z_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, Z_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let the W – basis be $W_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, W_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, W_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

(a) Given the vector $X_Z = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ in the Z-basis, find the coordinate

of its image relative to the W-basis

Solutions

$$X_Z = \begin{pmatrix} 3 & 0 & 2 \end{pmatrix}^T$$

$$A_Z = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 3 & -1 & -1 \end{pmatrix}, Z_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, Z_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, Z_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad W_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, W_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, W_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

(a) Find Y_W :

$$Y_Z = A_Z X_Z = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$$

$$Y_W = W^{-1} Z Y_Z = W^{-1} Y_Z$$

$$W = \begin{pmatrix} W_1 & W_2 & W_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{pmatrix} \Rightarrow W^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{pmatrix}^{-1} = \frac{1}{9} \begin{pmatrix} 3 & 3 & 0 \\ 1 & 2 & 3 \\ 5 & -1 & -3 \end{pmatrix}$$

$$Y_W = W^{-1} Y_Z = \frac{1}{9} \begin{pmatrix} 3 & 3 & 0 \\ 1 & 2 & 3 \\ 5 & -1 & -3 \end{pmatrix} \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 14/3 \\ 20/9 \\ 19/9 \end{pmatrix}$$

(b) Find the linear transformation $Y_W = A_W X_W$ corresponding to $Y_Z = A_Z X_Z$.

Solution:

$$A_W = W^{-1} Z A_Z Z^{-1} W = W^{-1} A_Z W = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 3 & 3 & 0 \\ 1 & 2 & 3 \\ 5 & -1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 36 & 21 & -15 \\ 21 & 10 & -11 \\ -3 & 23 & -1 \end{pmatrix}$$

Hence the tranformation relative to the W-basis is

$$Y_W = \frac{1}{9} \begin{pmatrix} 36 & 21 & -15 \\ 21 & 10 & -11 \\ -3 & 23 & -1 \end{pmatrix} X_W.$$

(b) Use the result in (c) to find the image Y_W of $X_W = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$.

Solution:

$$Y_W = A_W X_W = \frac{1}{9}\begin{pmatrix} 36 & 21 & -15 \\ 21 & 10 & -11 \\ -3 & 23 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 7 \end{pmatrix}$$