

# **Calculus Lecture Notes for ZCA 110**

based on Thomas' Calculus, 11th Edition

by George B. Thomas, Maurice D. Weir, Joel Hass, Frank R. Giordano, Addison Wesley,  
11th edition

prepared by

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School of Physics

Universiti Sains Malaysia

August 2015

## **COURSE DETAILS**

Course name:	Calculus and Linear Algebra
Course code:	ZCA 110
Credit hours:	4 (i.e. 4 lectures per week for 14 weeks, plus tutorial sessions)

## **LECTURERS**

- Four separate classes for ZCA 110 (Groups: A, B, C, and D) handled concurrently by four lecturers:

A group: Dr. Norhaslinda Mohamed Tahrin (NMT)  
B group: Dr. Ramzun Maizan Ramli (RMR)  
C group: Dr. Yoon Tiem Leong (YTL)  
D group: Prof. Fauziah Sulaiman (FS)

## **COURSE DESCRIPTIONS**

- A core course offered by School of Physics
- Course conducted in English, but the students can answer the final exam either in Bahasa Malaysia or English

Duration: 7<sup>th</sup> September 2015 – 18<sup>th</sup> December 2015

Semester Break: 9<sup>th</sup> – 15<sup>th</sup> November 2015

Public Holidays: Wed 16<sup>th</sup> September (Hari Malaysia)  
Thur 24<sup>th</sup> September (Aidiladha)  
Wed 14<sup>th</sup> October (Maal Hijrah)

Meeting times:	Mon 10.00 – 10.50 am Wed 9.00 – 9.50 am Thurs 12.00 – 12.50 pm Fri 10.00 – 10.50 am
Pre-requisite:	None, BUT will assume that students are familiar with basic mathematics at STPM or Matrikulasi level (i.e. arithmetic of addition, subtraction, division and multiplication; basic algebra, geometry, trigonometry, simple differentiation, and integration)
E-learn:	For updates, announcements, assignments, etc.

## **CONTENTS**

**Preliminaries:** Sets, real numbers, rational and complex numbers (read the Appendix section of Thomas' calculus)

The course consists of two parts:

### **A. Calculus (~weeks 1 – 12)**

- Functions
- Limits and continuity
- Differentiation and its applications
- Integration and its applications
- Transcendental functions
- Sequences and series

### **B. Linear algebra (~weeks 12 – 14)**

- Matrix algebra, types of matrices
- Determinants, minors, cofactors
- Solving system of linear equations
- Vector spaces: subspaces, basis and dimension, linear transformations

## **OBJECTIVES**

### ***Calculus***

1. Differentiation: learn the different rules of differentiation, and its applications
2. Integration: learn the different techniques of integration, and its applications
3. To learn about sequence and series (basic concepts), including the calculus of transcendental functions

### ***Linear algebra***

1. To learn about matrix algebra and its types
2. To solve system of linear equations using matrix
3. To learn about vector spaces: subspaces, basis and dimension, and linear transformation

## **COURSE EXPECTATIONS**

After completing this course, students should be:

- Well-versed in the so-called foundation mathematics that will be needed for numerous applications in physics
- Well-prepared for more advanced mathematics courses as well (e.g. ZCT 112/3, ZCT 210/4, ZCT 219/4, etc.)

## **CONSULTATION HOURS**

Consult your respective group lecturers for details.



## **ASSESSMENT**

COMPONENTS	DESCRIPTION	WEIGHTAGE
Course work	Two (2) tests – 20% (10% each) Assignments – 20%	40%
Final examination	Will cover all topics	60%
Attendance	<ul style="list-style-type: none"><li>• will be recorded</li><li>• students missing tests without valid reasons/M.C. will get zero</li><li>• students with attendance less than 70% will be barred from sitting for the final examination</li></ul>	
Total		100%

## **TESTS**

	Dates	Time	Venue
<i>Test 1 (Calculus Part 1)</i>	6 <sup>th</sup> November 2015	10.00 – 11.00 am	E41*
<i>Test 2 (Calculus Part 2, linear algebra)</i>	11 <sup>th</sup> December 2015	10.00 – 11.00 am	E41*

\* Basement of PHS II (Adjacent to Eureka building)

Note: All students (A, B, C, D groups) will sit for the same tests and final examination. Topics covered will be announced later.

## **ASSIGNMENTS and TUTORIALS**

- About ten (10) assignments to be completed by students throughout the course duration
- Students are required to submit them to her/his respective tutors, and will be graded
- Assignments received after the respective due date will not be graded (which means that you will get zero for that particular assignment)
- Tutorial sessions – each session is to be held during one of the usual lecture hours. Details of which will be announced later by your respective group lecturers.

## **REFERENCES**

### *Main textbooks*

- (1) **Thomas' Calculus Early Transcendentals**, 11<sup>th</sup> edition, G.B. Thomas, as revised by MD Weir, J Hass and F.R. Giordano, Pearson international edition, 2008
- (2) **Schaum's Outline of Theory and Problems of Matrices**, SI (Metric) Edition, Frank Ayres, McGraw-Hill, 1974

### *Additional references*

1. S.L. Salas, E. Hille, and G.J. Etgen, *Calculus*, John Wiley & Sons, New York, 9th Edition, 2003, John Wiley & Sons.
2. Edwards and Penny, *Calculus*, 6th Edition, 2002, Prentice Hall.
3. Gerald L. Bradley and Karl J. Smith, *Calculus*, 2nd Edition, 1999, Prentice Hall.
4. Seymour Lipschutz and Marc Lipson, *Schaum's Outlines, Linear Algebra*, 3rd Edition, 2001, McGraw-Hill.
5. *Introductory Linear Algebra with Application* by Bernard Kolman and David R. Hill, 7<sup>th</sup> Edition, 2001, Prentice Hall.

**Lecture schedule – tentative**

WEEK	DATE	TOPICS TO BE COVERED
1	7 – 13 September 2015	PART I: CALCULUS Chapter 1 Functions Chapter 2 Limits and continuity
2	14 – 20 September 2015	Chapter 2 Limits and continuity Chapter 3 Differentiation
3	21 – 27 September 2015	Chapter 3 Differentiation
4	28 – 4 October 2015	Chapter 4 Applications of derivatives Chapter 5 Integration
5	5 – 11 October 2015	Chapter 5 Integration Chapter 6 Applications of integrals
6	12 – 18 October 2015	Chapter 7 Transcendental functions
7	19 – 25 October 2015	Chapter 8 Techniques of integration
8	26 – 1 November 2015	Chapter 8 Techniques of integration
9	2 – 8 November 2015	Chapter 8 Techniques of integration Chapter 9 Sequences and series  <b>Test 1</b>
	9 – 15 November 2015	Semester break
10	16 – 22 November 2015	Chapter 9 Sequences and series

11	23 – 29 November 2015	Chapter 9 Sequences and series
12	30 – 6 December 2015	PART II: LINEAR ALGEBRA Chapter 1 Matrices
13	7 – 13 December 2015	Chapter 1 Matrices Chapter 2 Vector spaces  <b>Test 2</b>
14	14 – 20 December 2015	Chapter 2 Vector spaces

# Chapter 1

## Preliminaries

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## Function

- $y = f(x)$
- $f$  represents function (a rule that tell us how to calculate the value of  $y$  from the variable  $x$ )
- $x$  : independent variable (input of  $f$ )
- $y$  : dependent variable (the corresponding output value of  $f$  at  $x$ )

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# 1.3

## Functions and Their Graphs

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### DEFINITION Function

A **function** from a set  $D$  to a set  $Y$  is a rule that assigns a *unique* (single) element  $f(x) \in Y$  to each element  $x \in D$ .

### Definition Domain of the function

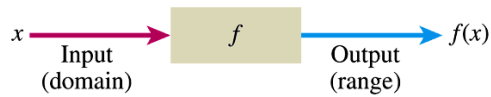
The set of  $D$  of all possible input values

### Definition Range of the function

The set of all values of  $f(x)$  as  $x$  varies throughout  $D$

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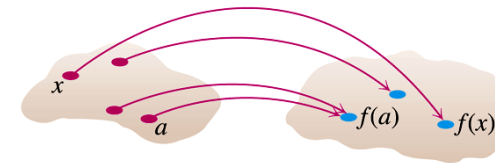
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**FIGURE 1.22** A diagram showing a function as a kind of machine.

## Natural Domain

- When a function  $y = f(x)$  is defined and the domain is not stated explicitly, the domain is assumed to be the largest set of real  $x$ -values for the formula gives real  $y$ -values.
- e.g. compare “ $y = x^2$ ” c.f. “ $y = x^2, x \geq 0$ ”
- Domain may be open, closed, half open, finite, infinite.



$D =$  domain set       $Y =$  set containing the range

**FIGURE 1.23** A function from a set  $D$  to a set  $Y$  assigns a unique element of  $Y$  to each element in  $D$ .

## Verify the domains and ranges of these functions

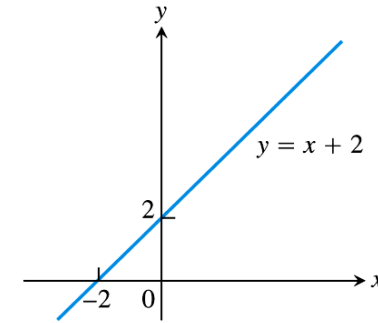
Function	Domain ( $x$ )	Range ( $y$ )
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

# Graphs of functions

- Graphs provide another way to visualise a function
- In set notation, a graph is
$$\{(x, f(x)) \mid x \in D\}$$
- The graph of a function is a useful picture of its behaviour.

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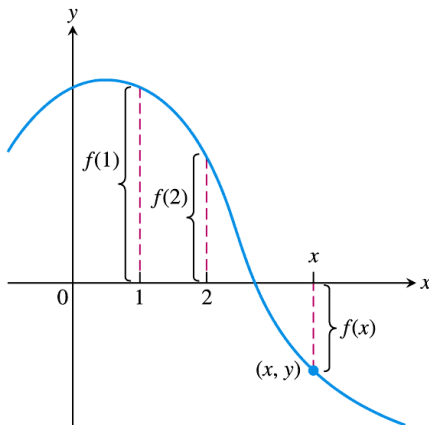
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**FIGURE 1.24** The graph of  $f(x) = x + 2$  is the set of points  $(x, y)$  for which  $y$  has the value  $x + 2$ .

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**FIGURE 1.25** If  $(x, y)$  lies on the graph of  $f$ , then the value  $y = f(x)$  is the height of the graph above the point  $x$  (or below  $x$  if  $f(x)$  is negative).

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## Example 2 Sketching a graph

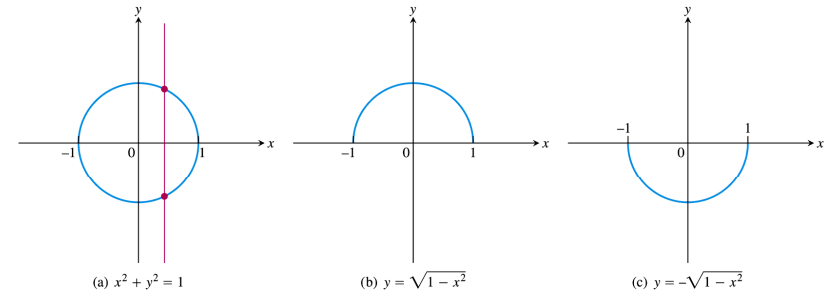
- Graph the function  $y = x^2$  over the interval  $[-2, 2]$

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## The vertical line test

- Since a function must be single valued over its domain, no vertical line can intersect the graph of a function more than once.
- If  $a$  is a point in the domain of a function  $f$ , the vertical line  $x=a$  can intersect the graph of  $f$  in a single point  $(a, f(a))$ .



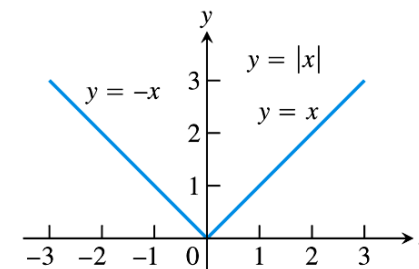
**FIGURE 1.28** (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of a function  $f(x) = \sqrt{1-x^2}$ . (c) The lower semicircle is the graph of a function  $g(x) = -\sqrt{1-x^2}$ .

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## Piecewise-defined functions

- The absolute value function

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



**FIGURE 1.29** The absolute value function has domain  $(-\infty, \infty)$  and range  $[0, \infty)$ .

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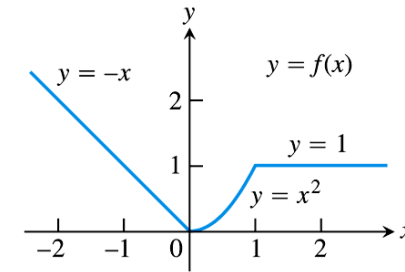
## Graphing piecewise-defined functions

- Note: this is *just one function* with a domain covering all real number

$$f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

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**FIGURE 1.30** To graph the function  $y = f(x)$  shown here, we apply different formulas to different parts of its domain (Example 5).

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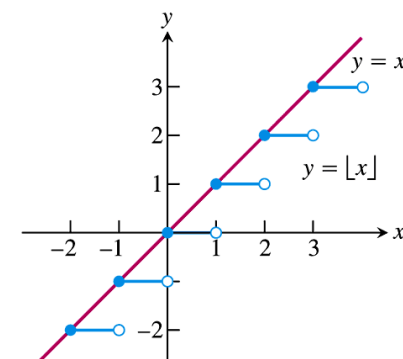
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## The greatest integer function

- Also called integer floor function
- $f = [x]$ , defined as greatest integer less than or equal to  $x$ .
- e.g.
- $[2.4] = 2$
- $[2] = 2$
- $[-2] = -2$ , etc.

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**FIGURE 1.31** The graph of the greatest integer function  $y = [x]$  lies on or below the line  $y = x$ , so it provides an integer floor for  $x$  (Example 6).

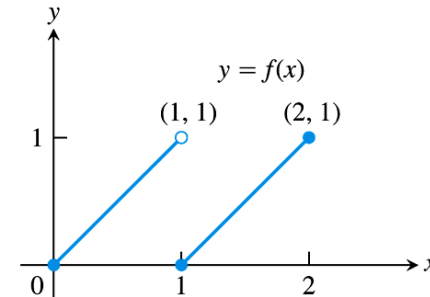
Note: the graph is the blue colour lines, not the one in red

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## Writing formulas for piecewise-defined functions

- Write a formula for the function  $y=f(x)$  in Figure 1.33



**FIGURE 1.33** The segment on the left contains  $(0, 0)$  but not  $(1, 1)$ . The segment on the right contains both of its endpoints (Example 8).

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## 1.4

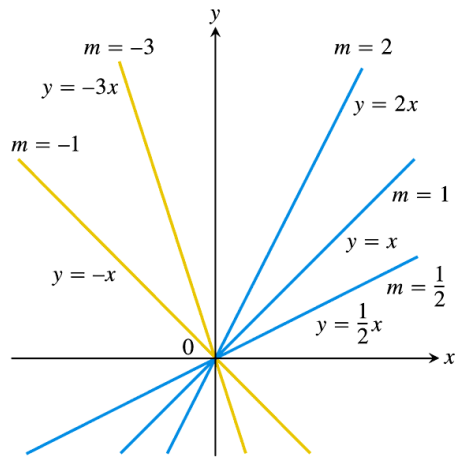
### Identifying Functions; Mathematical Models

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## Linear functions

- Linear function takes the form of
  - $y=mx + b$
  - $m, b$  constants
  - $m$  slope of the graph
  - $b$  intersection with the  $y$ -axis
- The linear function reduces to a constant function  $f = c$  when  $m = 0$ ,

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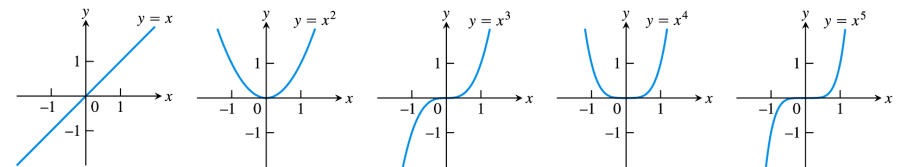
**FIGURE 1.34** The collection of lines  $y = mx$  has slope  $m$  and all lines pass through the origin.

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## Power functions

- $f(x) = x^a$
- a constant
- Case (a):  $a = n$ , a positive integer

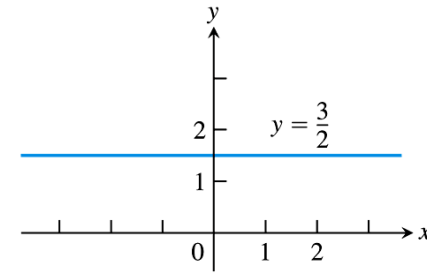


**FIGURE 1.36** Graphs of  $f(x) = x^n$ ,  $n = 1, 2, 3, 4, 5$  defined for  $-\infty < x < \infty$ .

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**FIGURE 1.35** A constant function has slope  $m = 0$ .

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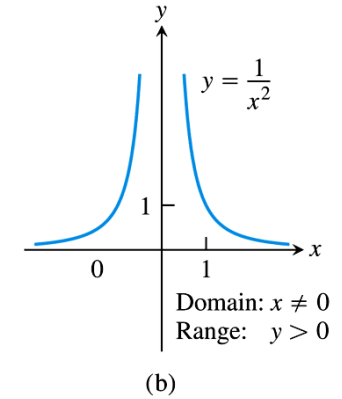
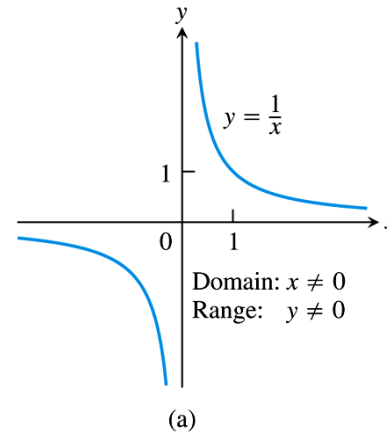
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## Power functions

- Case (b):
- $a = -1$  (hyperbola)
- or  $a = -2$



**FIGURE 1.37** Graphs of the power functions  $f(x) = x^a$  for part (a)  $a = -1$  and for part (b)  $a = -2$ .

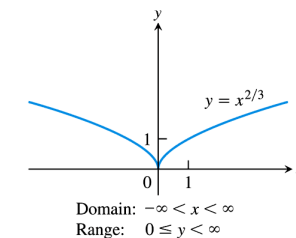
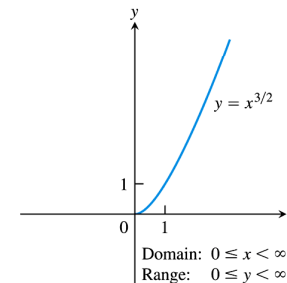
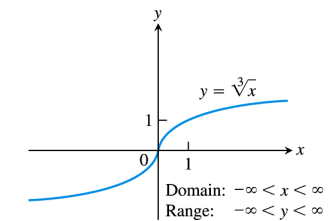
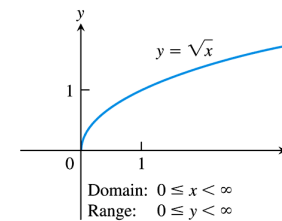
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## Power functions

- Case (c):
- $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2},$  and  $\frac{2}{3}$
- $f(x) = x^{1/2} = \sqrt{x}$  (square root), domain =  $[0 \leq x < \infty)$
- $g(x) = x^{1/3} = \sqrt[3]{x}$  (cube root), domain =  $(-\infty < x < \infty)$
- $p(x) = x^{2/3} = (x^{1/3})^2$ , domain = ?
- $q(x) = x^{3/2} = (x^3)^{1/2}$  domain = ?



**FIGURE 1.38** Graphs of the power functions  $f(x) = x^a$  for  $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2},$  and  $\frac{2}{3}$ .

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# Polynomials

- $p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0$
- $n$  nonnegative integer (1,2,3...)
- $a$ 's coefficients (real constants)
- If  $a_n \neq 0$ ,  $n$  is called the degree of the polynomial

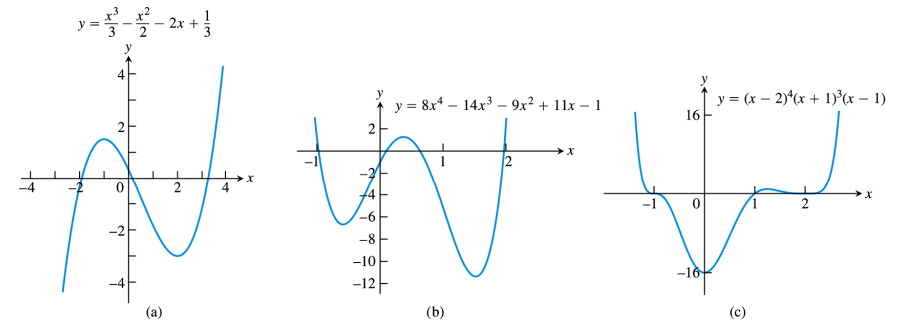


FIGURE 1.39 Graphs of three polynomial functions.

# Rational functions

- A rational function is a quotient of two polynomials:
- $f(x) = p(x) / q(x)$
- $p, q$  are polynomials.
- Domain of  $f(x)$  is the set of all real number  $x$  for which  $q(x) \neq 0$ .

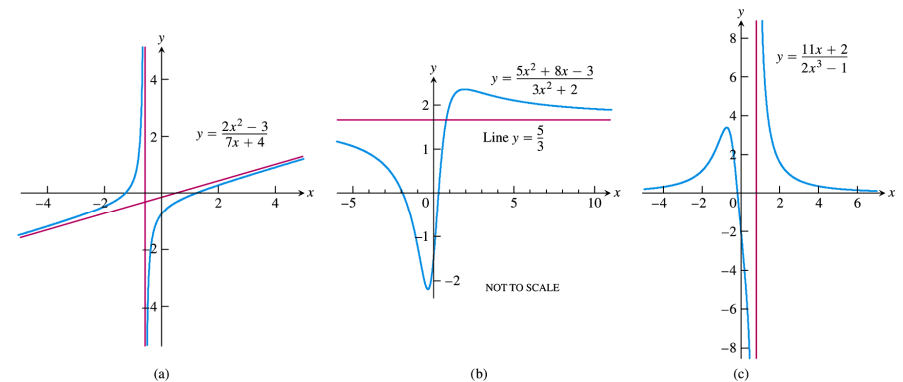


FIGURE 1.40 Graphs of three rational functions.

# Algebraic functions

- Functions constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots)

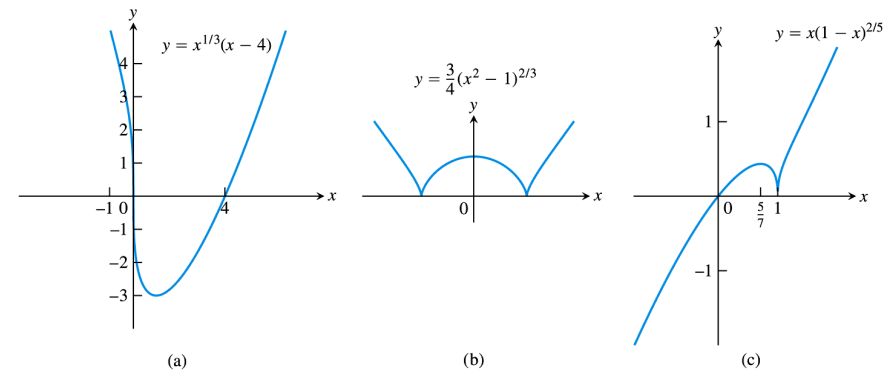


FIGURE 1.41 Graphs of three algebraic functions.

# Trigonometric functions

- More details in later chapter

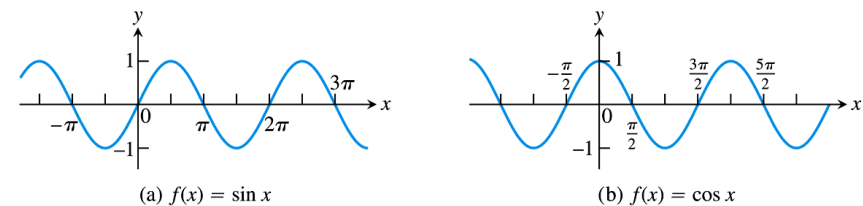


FIGURE 1.42 Graphs of the sine and cosine functions.

# Exponential functions

- $f(x) = a^x$
- Where  $a > 0$  and  $a \neq 0$ .  $a$  is called the 'base'.
- Domain  $(-\infty, \infty)$
- Range  $(0, \infty)$
- Hence,  $f(x) > 0$
- More in later chapter

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# Logarithmic functions

- $f(x) = \log_a x$
- $a$  is the base
- $a \neq 1, a > 0$
- Domain  $(0, \infty)$
- Range  $(-\infty, \infty)$
- They are the *inverse functions* of the exponential functions (more in later chapter)

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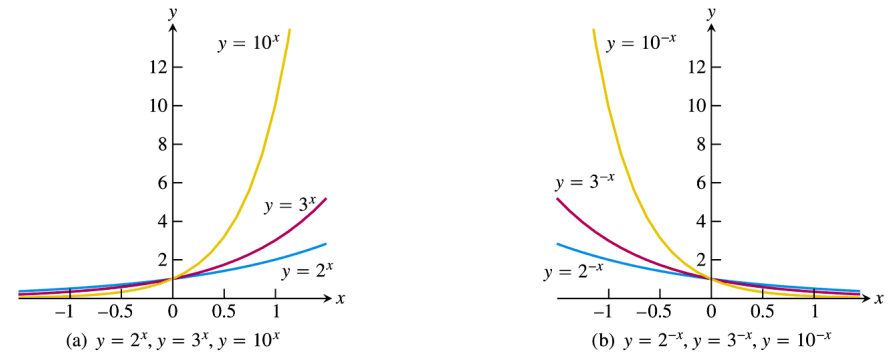


FIGURE 1.43 Graphs of exponential functions.

Note: graphs in (a) are reflections of the corresponding curves in (b) about the y-axis. This amounts to the symmetry operation of  $x \leftrightarrow -x$ .

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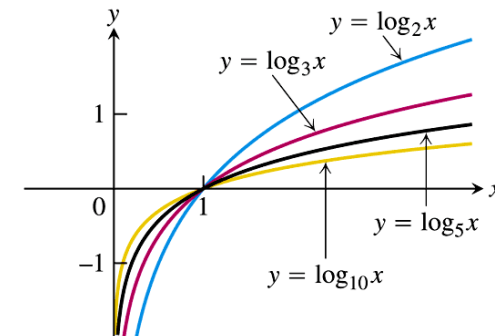


FIGURE 1.44 Graphs of four logarithmic functions.

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## Transcendental functions

- Functions that are not algebraic
- Include: trigonometric, inverse trigonometric, exponential, logarithmic, hyperbolic and many other functions

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## Example 1

- Recognizing Functions
- (a)  $f(x) = 1 + x - \frac{1}{2}x^5$
- (b)  $g(x) = 7^x$
- (c)  $h(z) = z^7$
- (d)  $y(t) = \sin(t - \pi/4)$

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## Increasing versus decreasing functions

- A function is said to be increasing if it rises as you move from left to right
- A function is said to be decreasing if it falls as you move from left to right

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Function	Where increasing	Where decreasing
$y = x^2$	$0 \leq x < \infty$	$-\infty < x \leq 0$
$y = x^3$	$-\infty < x < \infty$	Nowhere
$y = 1/x$	Nowhere	$-\infty < x < 0$ and $0 < x < \infty$
$y = 1/x^2$	$-\infty < x < 0$	$0 < x < \infty$
$y = \sqrt{x}$	$0 \leq x < \infty$	Nowhere
$y = x^{2/3}$	$0 \leq x < \infty$	$-\infty < x \leq 0$

$$y=x^2, y=x^3; y=1/x, y=1/x^2; y=x^{1/2}, y=x^{2/3}$$

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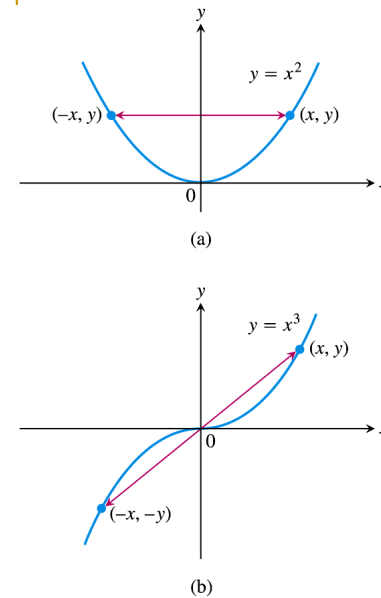
### DEFINITIONS Even Function, Odd Function

A function  $y = f(x)$  is an

**even function of  $x$**  if  $f(-x) = f(x)$ ,

**odd function of  $x$**  if  $f(-x) = -f(x)$ ,

for every  $x$  in the function's domain.



**FIGURE 1.46** In part (a) the graph of  $y = x^2$  (an even function) is symmetric about the  $y$ -axis. The graph of  $y = x^3$  (an odd function) in part (b) is symmetric about the origin.

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## Recognising even and odd functions

- $f(x) = x^2$  Even function as  $(-x)^2 = x^2$  for all  $x$ , symmetric about the all  $x$ , symmetric about the  $y$ -axis.
- $f(x) = x^2 + 1$  Even function as  $(-x)^2 + 1 = x^2 + 1$  for all  $x$ , symmetric about the all  $x$ , symmetric about the  $y$ -axis.

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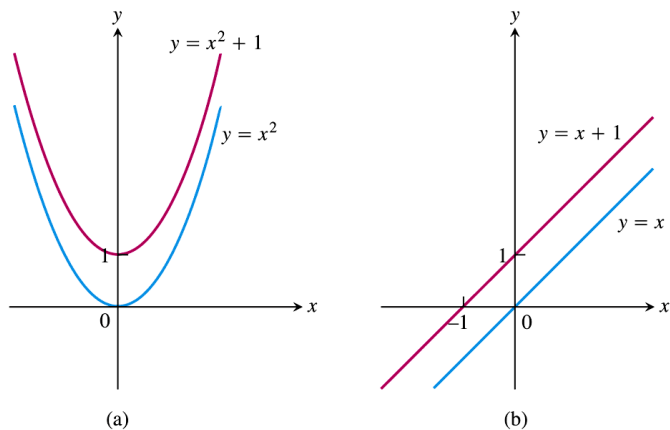
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## Recognising even and odd functions

- $f(x) = x$ . Odd function as  $(-x) = -x$  for all  $x$ , symmetric about origin.
- $f(x) = x+1$ . Odd function ?

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**FIGURE 1.47** (a) When we add the constant term 1 to the function  $y = x^2$ , the resulting function  $y = x^2 + 1$  is still even and its graph is still symmetric about the  $y$ -axis. (b) When we add the constant term 1 to the function  $y = x$ , the resulting function  $y = x + 1$  is no longer odd. The symmetry about the origin is lost (Example 2).

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## 1.5

### Combining Functions; Shifting and Scaling Graphs

54

## Sums, differences, products and quotients

- $f, g$  are functions
- For  $x \in D(f) \cap D(g)$ , we can define the functions of
- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$ ,
- $(cf)(x) = cf(x)$ ,  $c$  a real number
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ ,  $g(x) \neq 0$

55

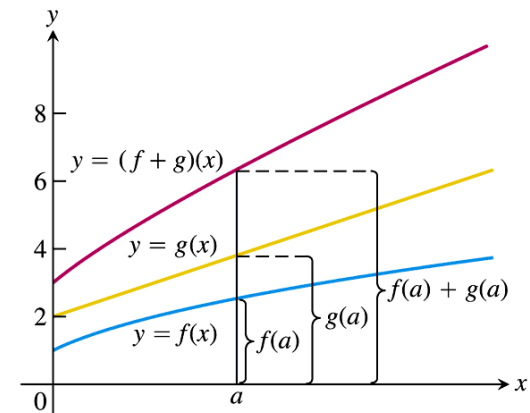
## Example 1

- $f(x) = \sqrt{x}$ ,  $g(x) = \sqrt{1-x}$ ,
- The domain common to both  $f, g$  is
- $D(f) \cap D(g) = [0, 1]$  (work it out)

56

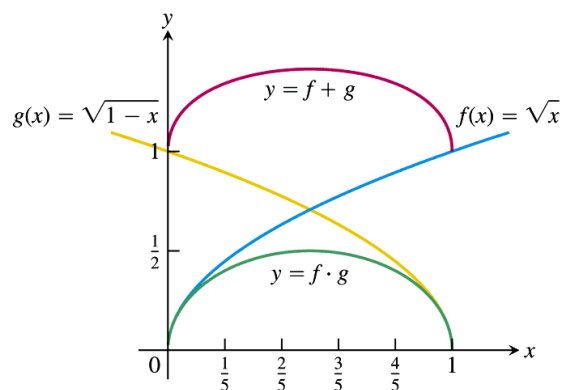
Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1-x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1-x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1-x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1-x)}$	$[0, 1]$
$f/g$	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0, 1)$ ( $x = 1$ excluded)
$g/f$	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0, 1]$ ( $x = 0$ excluded)

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**FIGURE 1.50** Graphical addition of two functions.

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**FIGURE 1.51** The domain of the function  $f + g$  is the intersection of the domains of  $f$  and  $g$ , the interval  $[0, 1]$  on the  $x$ -axis where these domains overlap. This interval is also the domain of the function  $f \cdot g$  (Example 1).

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## Composite functions

- Another way of combining functions

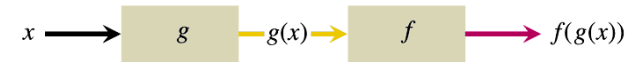
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**DEFINITION**    **Composition of Functions**

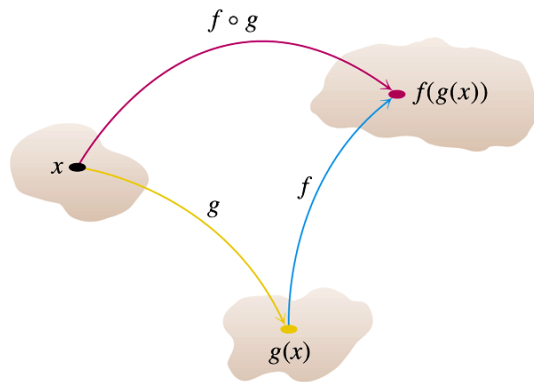
If  $f$  and  $g$  are functions, the **composite** function  $f \circ g$  (“ $f$  composed with  $g$ ”) is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of  $f \circ g$  consists of the numbers  $x$  in the domain of  $g$  for which  $g(x)$  lies in the domain of  $f$ .



**FIGURE 1.52** Two functions can be composed at  $x$  whenever the value of one function at  $x$  lies in the domain of the other. The composite is denoted by  $f \circ g$ .



**FIGURE 1.53** Arrow diagram for  $f \circ g$ .

## Example 2

- Viewing a function as a composite
- $y(x) = \sqrt{1 - x^2}$  is a composite of
- $g(x) = 1 - x^2$  and  $f(x) = \sqrt{x}$
- i.e.  $y(x) = f[g(x)] = \sqrt{1 - x^2}$
- Domain of the composite function is  $|x| \leq 1$ , or  $[-1, 1]$
- Is  $f[g(x)] = g[f(x)]$ ?

## Example 3

- Read it yourself
- Make sure that you know how to work out the domains and ranges of each composite functions listed

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## Shifting a graph of a function

### Shift Formulas

#### Vertical Shifts

$y = f(x) + k$  Shifts the graph of  $f$  *up*  $k$  units if  $k > 0$   
Shifts it *down*  $|k|$  units if  $k < 0$

#### Horizontal Shifts

$y = f(x + h)$  Shifts the graph of  $f$  *left*  $h$  units if  $h > 0$   
Shifts it *right*  $|h|$  units if  $h < 0$

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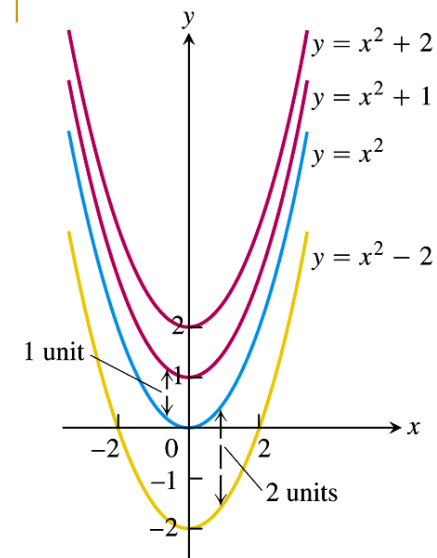
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## Example 4

- (a)  $y = x^2$ ,  $y = x^2 + 1$
- (b)  $y = x^2$ ,  $y = x^2 - 2$

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**FIGURE 1.54** To shift the graph of  $f(x) = x^2$  up (or down), we add positive (or negative) constants to the formula for  $f$  (Example 4a and b).

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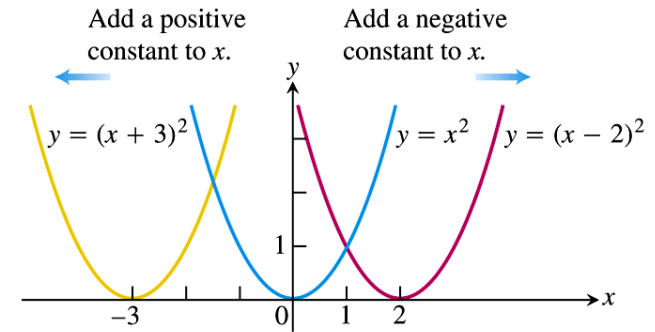
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## Example 4

- (c)  $y = x^2$ ,  $y = (x + 3)^2$ ,  $y = (x - 3)^2$

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**FIGURE 1.55** To shift the graph of  $y = x^2$  to the left, we add a positive constant to  $x$ . To shift the graph to the right, we add a negative constant to  $x$  (Example 4c).

70

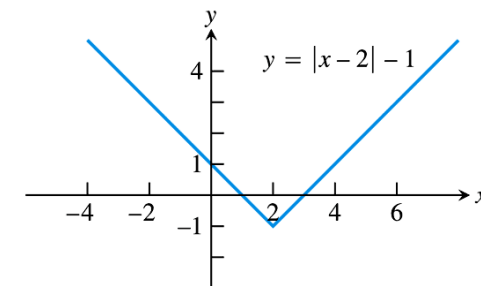
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## Example 4

- (d)  $y = |x|$ ,  $y = |x - 2| - 1$

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**FIGURE 1.56** Shifting the graph of  $y = |x|$  2 units to the right and 1 unit down (Example 4d).

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## Scaling and reflecting a graph of a function

- To scale a graph of a function is to stretch or compress it, vertically or horizontally.
- This is done by multiplying a constant  $c$  to the function or the independent variable

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## Example 5(a)

- Vertical stretching and compression of the graph  $y = \sqrt{x}$  by a factor of 3

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### Vertical and Horizontal Scaling and Reflecting Formulas

For  $c > 1$ ,

$y = cf(x)$  Stretches the graph of  $f$  vertically by a factor of  $c$ .

$y = \frac{1}{c}f(x)$  Compresses the graph of  $f$  vertically by a factor of  $c$ .

$y = f(cx)$  Compresses the graph of  $f$  horizontally by a factor of  $c$ .

$y = f(x/c)$  Stretches the graph of  $f$  horizontally by a factor of  $c$ .

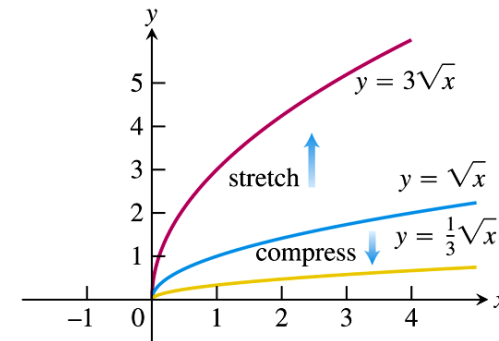
For  $c = -1$ ,

$y = -f(x)$  Reflects the graph of  $f$  across the  $x$ -axis.

$y = f(-x)$  Reflects the graph of  $f$  across the  $y$ -axis.

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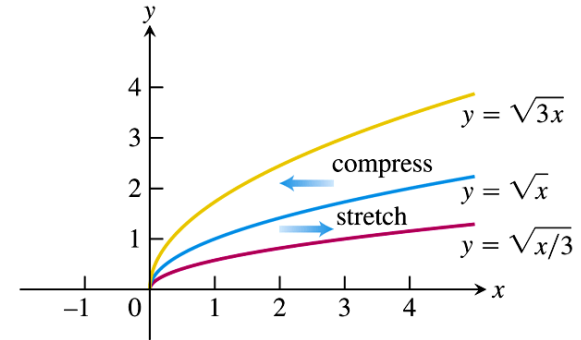
**FIGURE 1.57** Vertically stretching and compressing the graph  $y = \sqrt{x}$  by a factor of 3 (Example 5a).

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## Example 5(b)

- Horizontal stretching and compression of the graph  $y = \sqrt{x}$  by a factor of 3



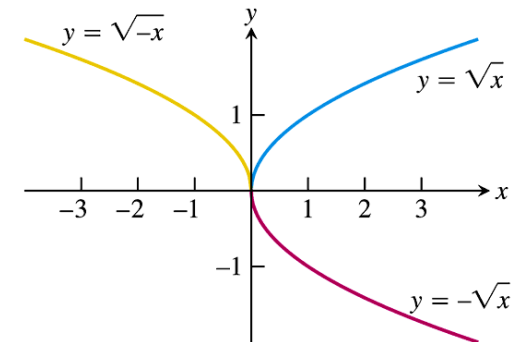
**FIGURE 1.58** Horizontally stretching and compressing the graph  $y = \sqrt{x}$  by a factor of 3 (Example 5b).

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## Example 5(c)

- Reflection across the x- and y- axes
- $c = -1$



**FIGURE 1.59** Reflections of the graph  $y = \sqrt{x}$  across the coordinate axes (Example 5c).

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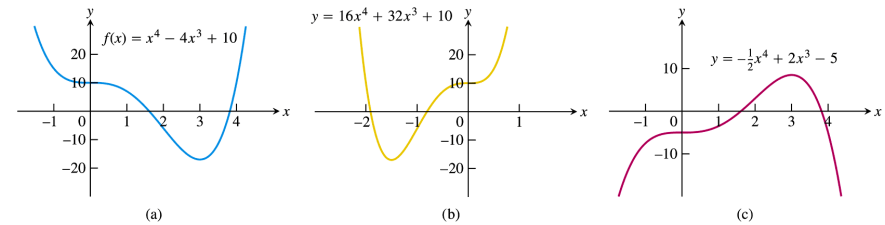
80



## EXAMPLE 6

### Combining Scalings and Reflections

- Given the function  $f(x) = x^4 - 4x^3 + 10$  (Figure 1.60a), find formulas to
- (a) compress the graph horizontally by a factor of 2 followed by a reflection across the  $y$ -axis (Figure 1.60b).
- (b) compress the graph vertically by a factor of 2 followed by a reflection across the  $x$ -axis (Figure 1.60c).



**FIGURE 1.60** (a) The original graph of  $f$ . (b) The horizontal compression of  $y = f(x)$  in part (a) by a factor of 2, followed by a reflection across the  $y$ -axis. (c) The vertical compression of  $y = f(x)$  in part (a) by a factor of 2, followed by a reflection across the  $x$ -axis (Example 6).

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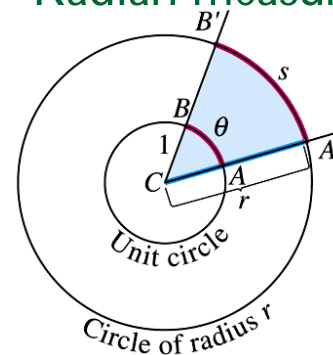
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## 1.6

### Trigonometric Functions

### Radian measure



**FIGURE 1.63** The radian measure of angle  $ACB$  is the length  $\theta$  of arc  $AB$  on the unit circle centered at  $C$ . The value of  $\theta$  can be found from any other circle, however, as the ratio  $s/r$ . Thus  $s = r\theta$  is the length of arc on a circle of radius  $r$  when  $\theta$  is measured in radians.

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### Conversion Formulas

$$1 \text{ degree} = \frac{\pi}{180} (\approx 0.02) \text{ radians}$$

Degrees to radians: multiply by  $\frac{\pi}{180}$

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57) \text{ degrees}$$

Radians to degrees: multiply by  $\frac{180}{\pi}$

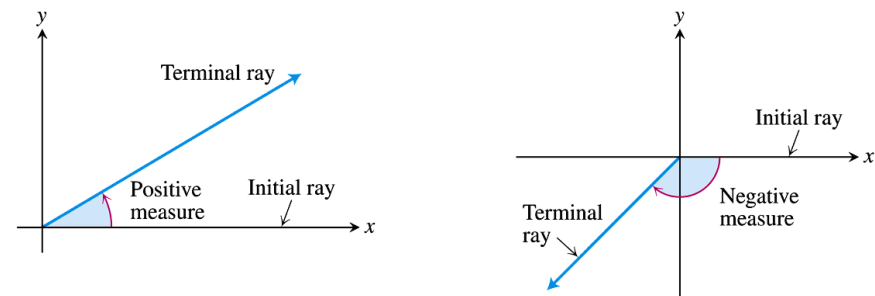


FIGURE 1.65 Angles in standard position in the  $xy$ -plane.

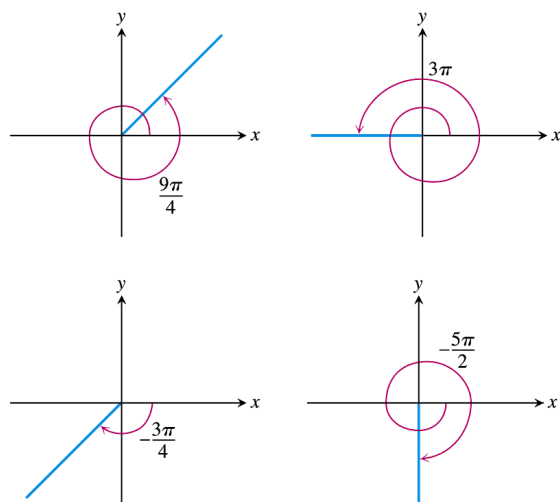
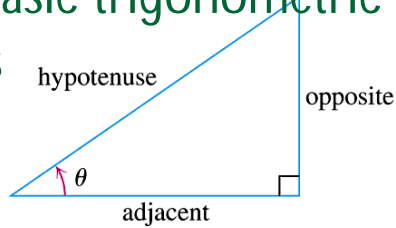


FIGURE 1.66 Nonzero radian measures can be positive or negative and can go beyond  $2\pi$ .

## Angle convention

- Be noted that angle will be expressed in terms of radian unless otherwise specified.
- Get used to the change of the unit

# The six basic trigonometric functions

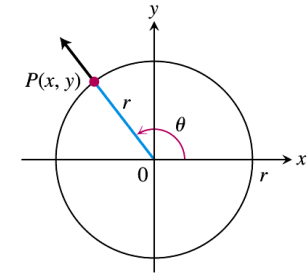


$$\begin{aligned} \sin \theta &= \frac{\text{opp}}{\text{hyp}} & \csc \theta &= \frac{\text{hyp}}{\text{opp}} \\ \cos \theta &= \frac{\text{adj}}{\text{hyp}} & \sec \theta &= \frac{\text{hyp}}{\text{adj}} \\ \tan \theta &= \frac{\text{opp}}{\text{adj}} & \cot \theta &= \frac{\text{adj}}{\text{opp}} \end{aligned}$$

**FIGURE 1.67** Trigonometric ratios of an acute angle.

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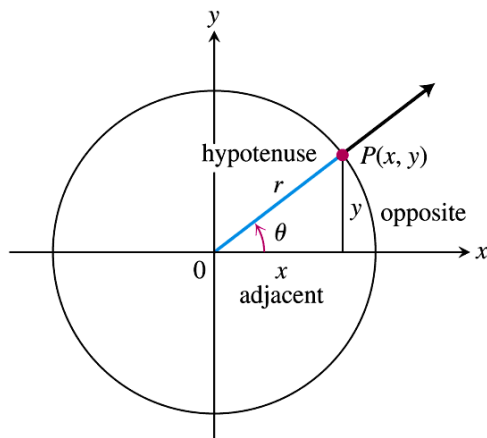
# Generalised definition of the six trigo functions



- Define the trigo functions in terms of the coordinates of the point  $P(x,y)$  on a circle of radius  $r$
- **sine:**  $\sin \theta = y/r$
- **cosine:**  $\cos \theta = x/r$
- **tangent:**  $\tan \theta = y/x$
- **cosecant:**  $\csc \theta = r/y$
- **secant:**  $\sec \theta = r/x$
- **cotangent:**  $\cot \theta = x/y$

**FIGURE 1.68** The trigonometric functions of a general angle  $\theta$  are defined in terms of  $x$ ,  $y$ , and  $r$ .

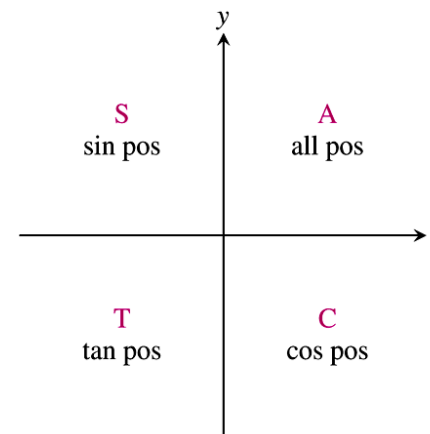
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**FIGURE 1.69** The new and old definitions agree for acute angles.

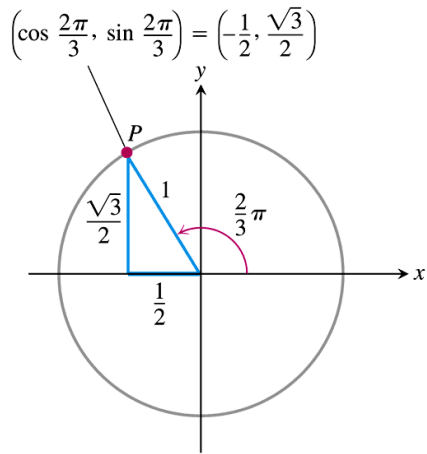
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# Mnemonic to remember when the basic trigo functions are positive or negative



**FIGURE 1.70** The CAST rule, remembered by the statement “All Students Take Calculus,” tells which trigonometric functions are positive in each quadrant.

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**FIGURE 1.71** The triangle for calculating the sine and cosine of  $2\pi/3$  radians. The side lengths come from the geometry of right triangles.

**TABLE 1.4** Values of  $\sin \theta$ ,  $\cos \theta$ , and  $\tan \theta$  for selected values of  $\theta$

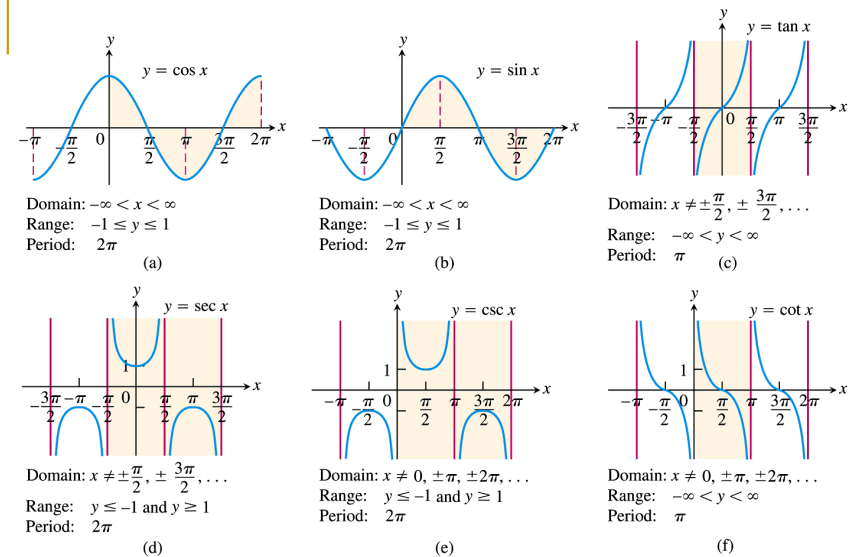
Degrees	-180	-135	-90	-45	0	30	45	60	90	120	135	150	180	270	360
$\theta$ (radians)	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$\sin \theta$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\cos \theta$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	0	1
$\tan \theta$	0	1		-1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0		0

## Periodicity and graphs of the trigonometric functions

Trigonometric functions are also periodic.

### DEFINITION Periodic Function

A function  $f(x)$  is **periodic** if there is a positive number  $p$  such that  $f(x + p) = f(x)$  for every value of  $x$ . The smallest such value of  $p$  is the **period** of  $f$ .



**FIGURE 1.73** Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure. The shading for each trigonometric function indicates its periodicity.

## Parity of the trigo functions

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

The parity is easily deduced from the graphs.

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Dividing identity (1) by  $\cos^2 \theta$  and  $\sin^2 \theta$  in turn gives the next two identities

$$\begin{aligned} 1 + \tan^2 \theta &= \sec^2 \theta. \\ 1 + \cot^2 \theta &= \csc^2 \theta. \end{aligned}$$

### Addition Formulas

$$\begin{aligned} \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \sin(A + B) &= \sin A \cos B + \cos A \sin B \end{aligned} \quad (2)$$

There are also similar formulas for  $\cos(A-B)$  and  $\sin(A-B)$ . Do you know how to deduce them?

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## Identities

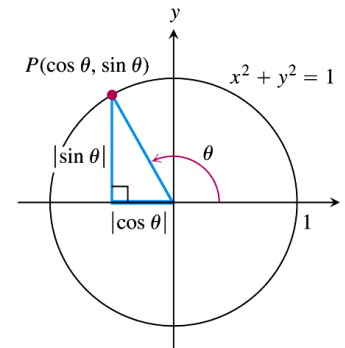


FIGURE 1.74 The reference triangle for a general angle  $\theta$ .

Applying Pythagorean theorem to the right triangle leads to the identity

$$\cos^2 \theta + \sin^2 \theta = 1. \quad (1)$$

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### Double-Angle Formulas

$$\begin{aligned} \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ \sin 2\theta &= 2 \sin \theta \cos \theta \end{aligned} \quad (3)$$

Identity (3) is derived by setting  $A = B$  in (2)

### Half-Angle Formulas

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad (4)$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad (5)$$

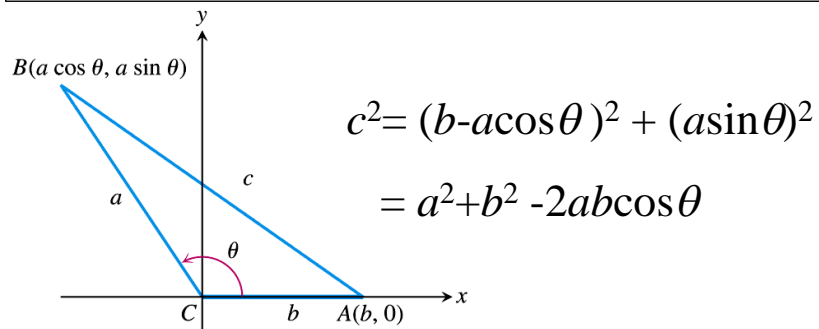
Identities (4,5) are derived by combining (1) and (3(i))

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# Law of cosines

$$c^2 = a^2 + b^2 - 2ab \cos \theta. \quad (6)$$



**FIGURE 1.75** The square of the distance between  $A$  and  $B$  gives the law of cosines.

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# Chapter 2

## Limits and Continuity

1

# 2.1

## Rates of Change and Limits

2

# Average Rates of change and Secant Lines

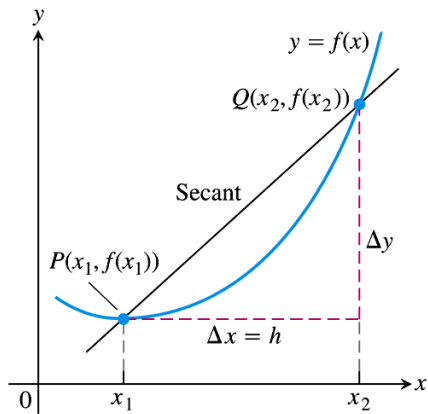
- Given an arbitrary function  $y=f(x)$ , we calculate the average rate of change of  $y$  with respect to  $x$  over the interval  $[x_1, x_2]$  by dividing the change in the value of  $y$ ,  $\Delta y$ , by the length  $\Delta x$

### DEFINITION Average Rate of Change over an Interval

The **average rate of change** of  $y = f(x)$  with respect to  $x$  over the interval  $[x_1, x_2]$  is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0.$$

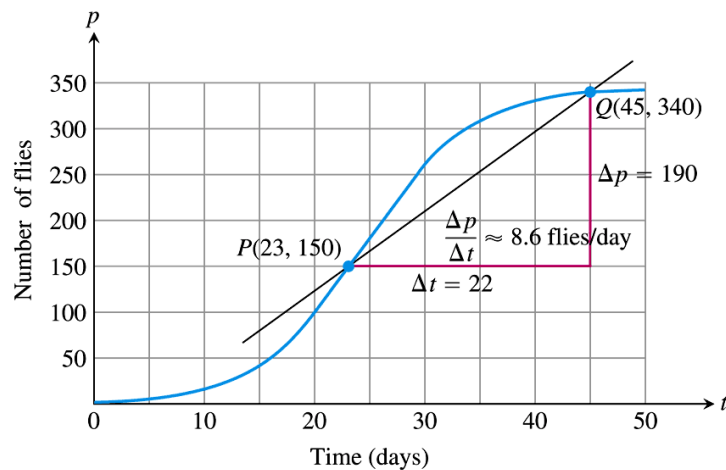
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**FIGURE 2.1** A secant to the graph  $y = f(x)$ . Its slope is  $\Delta y / \Delta x$ , the average rate of change of  $f$  over the interval  $[x_1, x_2]$ .

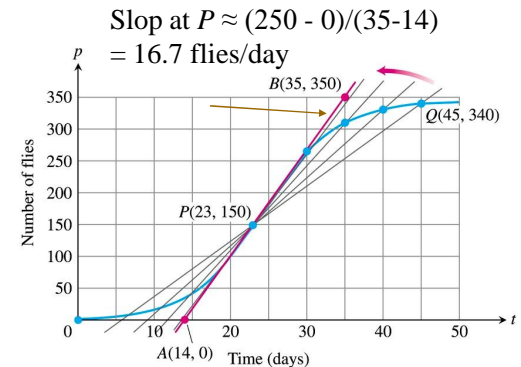
## Example 4

- Figure 2.2 shows how a population of fruit flies grew in a 50-day experiment.
- (a) Find the average growth rate from day 23 to day 45.
- (b) How fast was the number of the flies growing on day 23?



**FIGURE 2.2** Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope  $\Delta p / \Delta t$  of the secant line.

$Q$	Slope of $PQ = \Delta p / \Delta t$ (flies/day)
(45, 340)	$\frac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\frac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\frac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\frac{265 - 150}{30 - 23} \approx 16.4$



**FIGURE 2.3** The positions and slopes of four secants through the point  $P$  on the fruit fly graph (Example 4).

The grow rate at day 23 is calculated by examining the average rates of change over increasingly short time intervals starting at day 23. Geometrically, this is equivalent to evaluating the slopes of secants from  $P$  to  $Q$  with  $Q$  approaching  $P$ .

# Limits of function values

- Informal definition of limit:
- Let  $f$  be a function defined on an open interval about  $x_0$ , except possibly at  $x_0$  itself.
- If  $f$  gets arbitrarily close to  $L$  for all  $x$  sufficiently close to  $x_0$ , we say that  $f$  approaches the limit  $L$  as  $x$  approaches  $x_0$

$$\lim_{x \rightarrow x_0} f(x) = L$$

- “Arbitrarily close” is not yet defined here (hence the definition is informal).

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# Example 5

- How does the function behave near  $x=1$ ?

$$f(x) = \frac{x^2 - 1}{x - 1}$$

- Solution:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x+1 \quad \text{for } x \neq 1$$

9

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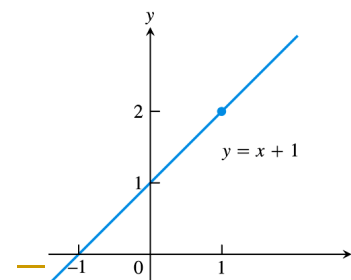
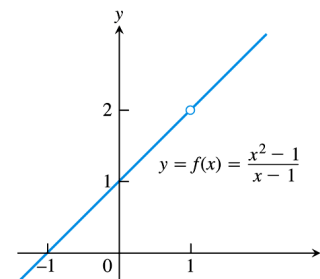
**TABLE 2.2** The closer  $x$  gets to 1, the closer  $f(x) = (x^2 - 1)/(x - 1)$  seems to get to 2

Values of $x$ below and above 1	$f(x) = \frac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

We say that  $f(x)$  approaches the limit 2 as  $x$  approaches 1,  $\lim_{x \rightarrow 1} f(x) = 2$  or  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$

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**FIGURE 2.4** The graph of  $f$  is identical with the line  $y = x + 1$  except at  $x = 1$ , where  $f$  is not defined (Example 5).

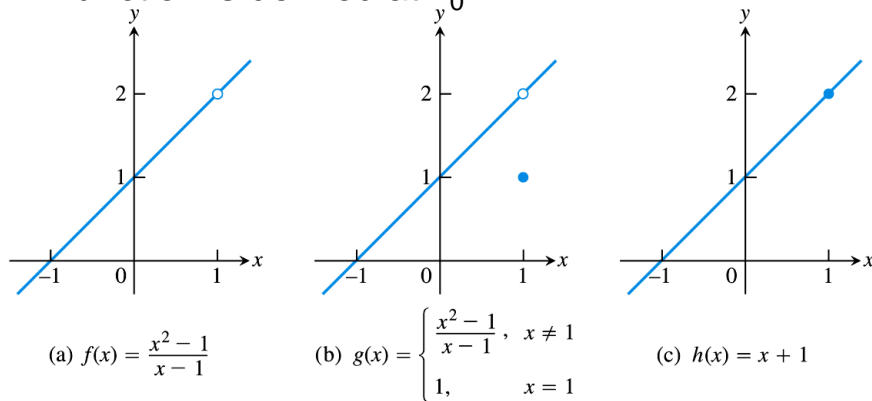
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## Example 6

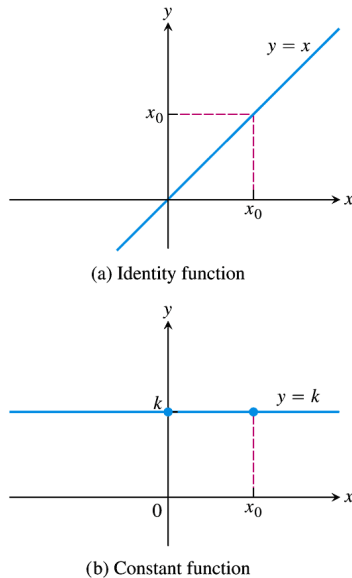
- The limit value does not depend on how the function is defined at  $x_0$ .



**FIGURE 2.5** The limits of  $f(x)$ ,  $g(x)$ , and  $h(x)$  all equal 2 as  $x$  approaches 1. However, only  $h(x)$  has the same function value as its limit at  $x = 1$  (Example 6).

## Example 7

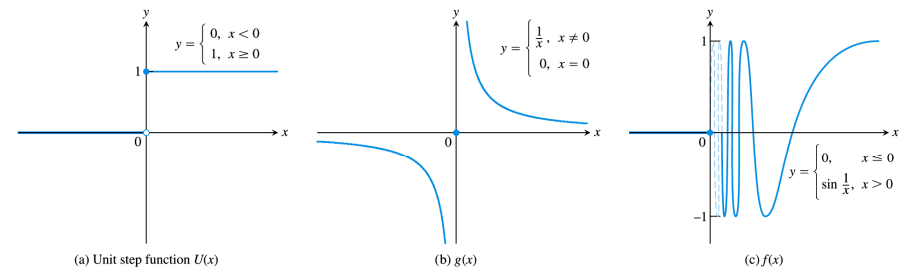
- In some special cases  $\lim_{x \rightarrow x_0} f(x)$  can be evaluated by calculating  $f(x_0)$ . For example, constant function, rational function and identity function for which  $x = x_0$  is defined
- (a)  $\lim_{x \rightarrow 2} (4) = 4$  (constant function)
- (b)  $\lim_{x \rightarrow -13} (4) = 4$  (constant function)
- (c)  $\lim_{x \rightarrow 3} x = 3$  (identity function)
- (d)  $\lim_{x \rightarrow 2} (5x - 3) = 10 - 3 = 7$  (polynomial function of degree 1)
- (e)  $\lim_{x \rightarrow -2} (3x + 4)/(x + 5) = (-6 + 4)/(-2 + 5) = -2/3$  (rational function)



**FIGURE 2.6** The functions in Example 8.

## Example 9

- A function may fail to have a limit exist at a point in its domain.



**FIGURE 2.7** None of these functions has a limit as  $x$  approaches 0 (Example 9).

Jump

Grow to  
infinities

Oscillate

## 2.2

### Calculating limits using the limits laws

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## The limit laws

- Theorem 1 tells how to calculate limits of functions that are arithmetic combinations of functions whose limit are already known.

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#### THEOREM 1 Limit Laws

If  $L$ ,  $M$ ,  $c$  and  $k$  are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:*  $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule:*  $\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule:*  $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:*  $\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:*  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule:* If  $r$  and  $s$  are integers with no common factor and  $s \neq 0$ , then

$$\lim_{x \rightarrow c} (f(x))^{r/s} = L^{r/s}$$

provided that  $L^{r/s}$  is a real number. (If  $s$  is even, we assume that  $L > 0$ .)

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

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## Example 1 Using the limit laws

- (a)  $\lim_{x \rightarrow c} (x^3 + 4x^2 - 3)$   
 $= \lim_{x \rightarrow c} x^3 + \lim_{x \rightarrow c} 4x^2 - \lim_{x \rightarrow c} 3$   
(sum and difference rule)  
 $= c^3 + 4c^2 - 3$   
(product and multiple rules)

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## Example 1

$$\begin{aligned} \blacksquare \text{ (b) } \lim_{x \rightarrow c} (x^4 + x^2 - 1)/(x^2 + 5) \\ = \lim_{x \rightarrow c} (x^4 + x^2 - 1) / \lim_{x \rightarrow c} (x^2 + 5) \end{aligned}$$

$$\begin{aligned} &= (\lim_{x \rightarrow c} x^4 + \lim_{x \rightarrow c} x^2 - \lim_{x \rightarrow c} 1) / (\lim_{x \rightarrow c} x^2 + \lim_{x \rightarrow c} 5) \\ &= (c^4 + c^2 - 1) / (c^2 + 5) \end{aligned}$$

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## Example 1

$$\begin{aligned} \blacksquare \text{ (c) } \lim_{x \rightarrow -2} \sqrt[1/2]{4x^2 - 3} &= \sqrt[1/2]{\lim_{x \rightarrow -2} (4x^2 - 3)} \\ &\text{Power rule with } r/s = 1/2 \end{aligned}$$

$$\begin{aligned} &= \sqrt[1/2]{[\lim_{x \rightarrow -2} 4x^2 - \lim_{x \rightarrow -2} 3]} \\ &= \sqrt[1/2]{[4(-2)^2 - 3]} = \sqrt{13} \end{aligned}$$

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### THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ , then

$$\lim_{x \rightarrow c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

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### THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If  $P(x)$  and  $Q(x)$  are polynomials and  $Q(c) \neq 0$ , then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

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## Example 2

- Limit of a rational function

$$\lim_{x \rightarrow -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

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## Eliminating zero denominators algebraically

### Identifying Common Factors

It can be shown that if  $Q(x)$  is a polynomial and  $Q(c) = 0$ , then  $(x - c)$  is a factor of  $Q(x)$ . Thus, if the numerator and denominator of a rational function of  $x$  are both zero at  $x = c$ , they have  $(x - c)$  as a common factor.

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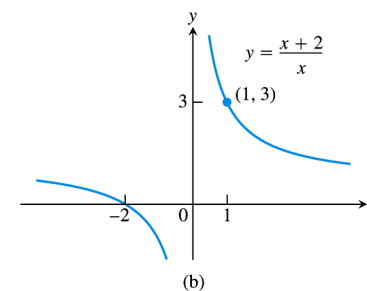
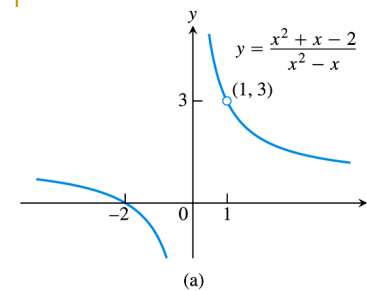
## Example 3 Canceling a common factor

- Evaluate  $\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}$
- Solution: We can't substitute  $x=1$  since  $f(x=1)$  is not defined. Since  $x \neq 1$ , we can cancel the common factor of  $x-1$ :

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \rightarrow 1} \frac{(x-1)(x+2)}{x(x-1)} = \lim_{x \rightarrow 1} \frac{(x+2)}{x} = 3$$

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**FIGURE 2.8** The graph of  $f(x) = (x^2 + x - 2)/(x^2 - x)$  in part (a) is the same as the graph of  $g(x) = (x + 2)/x$  in part (b) except at  $x = 1$ , where  $f$  is undefined. The functions have the same limit as  $x \rightarrow 1$  (Example 3).

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# The Sandwich theorem

## THEOREM 4 The Sandwich Theorem

Suppose that  $g(x) \leq f(x) \leq h(x)$  for all  $x$  in some open interval containing  $c$ , except possibly at  $x = c$  itself. Suppose also that

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L.$$

Then  $\lim_{x \rightarrow c} f(x) = L$ .

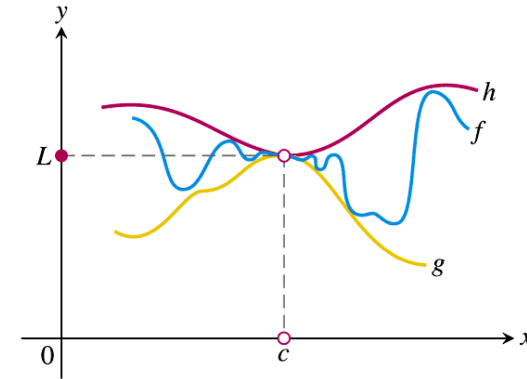


FIGURE 2.9 The graph of  $f$  is sandwiched between the graphs of  $g$  and  $h$ .

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# Example 6

- (a)
- The function  $y = \sin \theta$  is sandwiched between  $y = |\theta|$  and  $y = -|\theta|$  for all values of  $\theta$ . Since  $\lim_{\theta \rightarrow 0} (-|\theta|) = \lim_{\theta \rightarrow 0} (|\theta|) = 0$ , we have  $\lim_{\theta \rightarrow 0} \sin \theta = 0$ .
- (b)
- From the definition of  $\cos \theta$ ,  $0 \leq 1 - \cos \theta \leq |\theta|$  for all  $\theta$ , and we have the limit  $\lim_{\theta \rightarrow 0} \cos \theta = 1$

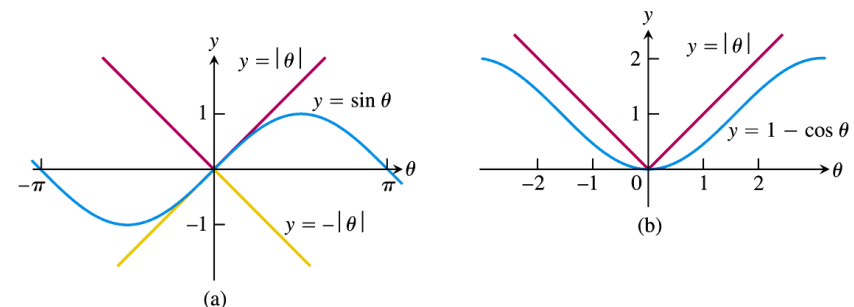


FIGURE 2.11 The Sandwich Theorem confirms that (a)  $\lim_{\theta \rightarrow 0} \sin \theta = 0$  and (b)  $\lim_{\theta \rightarrow 0} (1 - \cos \theta) = 0$  (Example 6).

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## Example 6(c)

- For any function  $f(x)$ , if  $\lim_{x \rightarrow 0} (|f(x)|) = 0$ , then  $\lim_{x \rightarrow 0} f(x) = 0$  due to the sandwich theorem.
- Proof:
- $-|f(x)| \leq f(x) \leq |f(x)|$
- Since  $\lim_{x \rightarrow 0} (|f(x)|) = \lim_{x \rightarrow 0} (-|f(x)|) = 0$
- $\Rightarrow \lim_{x \rightarrow 0} f(x) = 0$

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## Example 1 A linear function

- Consider the linear function  $y = 2x - 1$  near  $x_0 = 4$ . Intuitively it is close to 7 when  $x$  is close to 4, so  $\lim_{x \rightarrow 4} (2x-1) = 7$ . How close does  $x$  have to be so that  $y = 2x - 1$  differs from 7 by less than 2 units?

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## 2.3

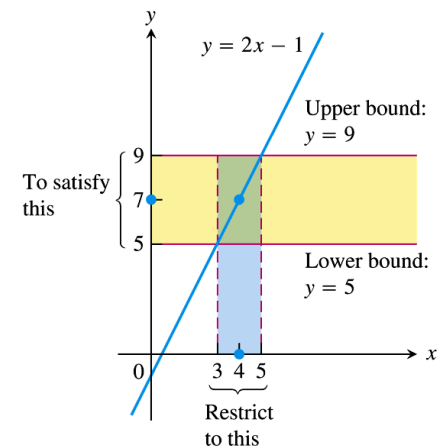
### The Precise Definition of a Limit

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## Solution

- For what value of  $x$  is  $|y-7| < 2$ ?
- First, find  $|y-7| < 2$  in terms of  $x$ :  
 $|y-7| < 2 \equiv |2x-8| < 2$   
 $\equiv -2 < 2x-8 < 2$   
 $\equiv 3 < x < 5$   
 $\equiv -1 < x - 4 < 1$   
Keeping  $x$  within 1 unit of  $x_0 = 4$  will keep  $y$  within 2 units of  $y_0 = 7$ .



**FIGURE 2.12** Keeping  $x$  within 1 unit of  $x_0 = 4$  will keep  $y$  within 2 units of  $y_0 = 7$  (Example 1).

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# Definition of limit

## DEFINITION Limit of a Function

Let  $f(x)$  be defined on an open interval about  $x_0$ , except possibly at  $x_0$  itself. We say that the **limit of  $f(x)$  as  $x$  approaches  $x_0$  is the number  $L$** , and write

$$\lim_{x \rightarrow x_0} f(x) = L,$$

if, for every number  $\epsilon > 0$ , there exists a corresponding number  $\delta > 0$  such that for all  $x$ ,

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

# Definition of limit

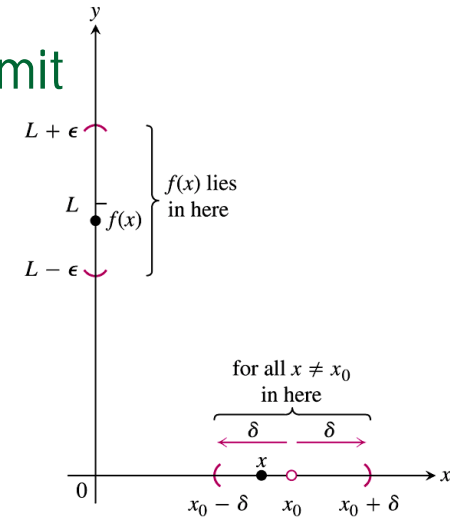


FIGURE 2.14 The relation of  $\delta$  and  $\epsilon$  in the definition of limit.

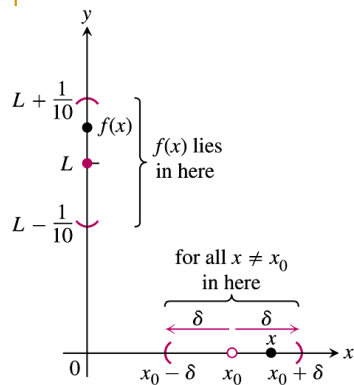
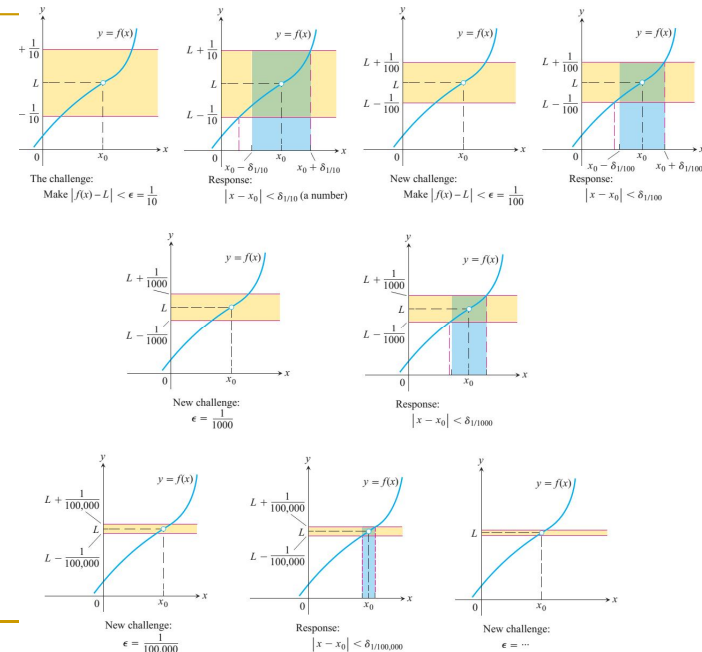


FIGURE 2.13 How should we define  $\delta > 0$  so that keeping  $x$  within the interval  $(x_0 - \delta, x_0 + \delta)$  will keep  $f(x)$  within the interval  $(L - \frac{1}{10}, L + \frac{1}{10})$ ?

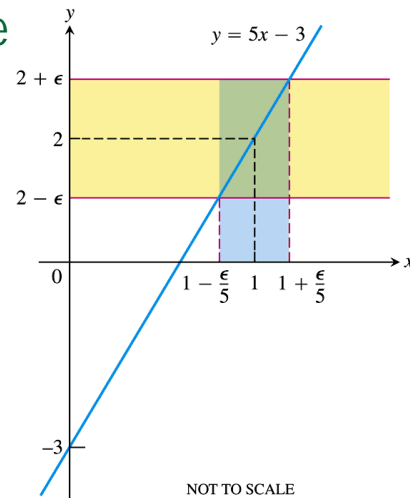
- The problem of proving  $L$  as the limit of  $f(x)$  as  $x$  approaches  $x_0$  is a problem of proving the existence of  $\delta$ , such that whenever
- $x_0 - \delta < x < x_0 + \delta$ ,
- $L + \epsilon < f(x) < L - \epsilon$  for any arbitrarily small value of  $\epsilon$ .
- As an example in Figure 2.13, given  $\epsilon = 1/10$ , can we find a corresponding value of  $\delta$ ?
- How about if  $\epsilon = 1/100$ ?  $\epsilon = 1/1234$ ?
- If for any arbitrarily small value of  $\epsilon$  we can always find a corresponding value of  $\delta$ , then we have successfully proven that  $L$  is the limit of  $f$  as  $x$  approaches  $x_0$



## Example 2 Testing the definition

- Show that

$$\lim_{x \rightarrow 1} (5x - 3) = 2$$

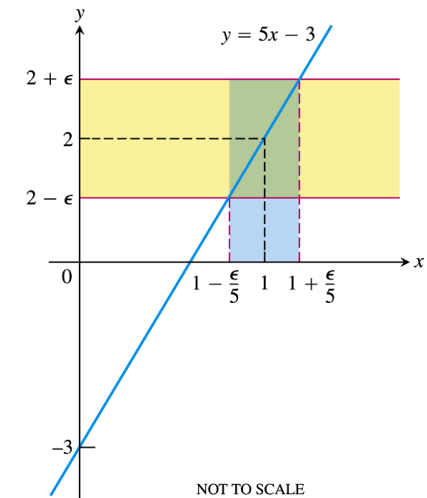


**FIGURE 2.15** If  $f(x) = 5x - 3$ , then  $0 < |x - 1| < \epsilon/5$  guarantees that  $|f(x) - 2| < \epsilon$  (Example 2).

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## Solution

- Set  $x_0 = 1$ ,  $f(x) = 5x - 3$ ,  $L = 2$ .
- For any given  $\epsilon$ , we have to find a suitable  $\delta > 0$  so that whenever  $0 < |x - 1| < \delta$ ,  $x \neq 1$ ,  $f(x)$  is within a distance  $\epsilon$  from  $L = 2$ , i.e.  $|f(x) - 2| < \epsilon$ .



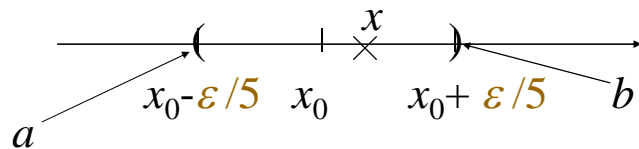
**FIGURE 2.15** If  $f(x) = 5x - 3$ , then  $0 < |x - 1| < \epsilon/5$  guarantees that  $|f(x) - 2| < \epsilon$  (Example 2).

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- First, obtain an open interval  $(a, b)$  in which  $|f(x) - 2| < \epsilon \equiv |5x - 5| < \epsilon \equiv -\epsilon/5 < x - 1 < \epsilon/5 \equiv -\epsilon/5 < x - x_0 < \epsilon/5$

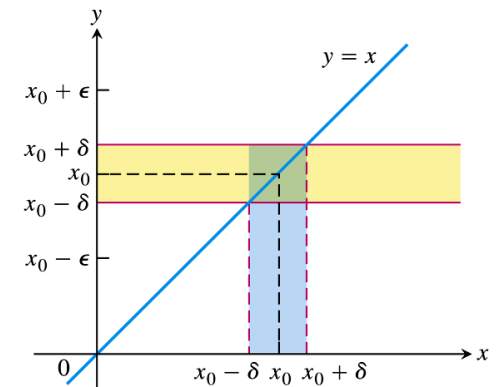


- choose  $\delta < \epsilon/5$ . This choice will guarantee that  $|f(x) - L| < \epsilon$  whenever  $x_0 - \delta < x < x_0 + \delta$ . We have shown that for any value of  $\epsilon$  given, we can always find a corresponding value of  $\delta$  that meets the "challenge" posed by an ever diminishing  $\epsilon$ . This is a proof of existence. Thus we have proven that the limit for  $f(x) = 5x - 3$  is  $L = 2$  when  $x \rightarrow x_0 = 1$ .

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## Example 3(a)

- Limits of the identity functions
- Prove  $\lim_{x \rightarrow x_0} x = x_0$



**FIGURE 2.16** For the function  $f(x) = x$ , we find that  $0 < |x - x_0| < \delta$  will guarantee  $|f(x) - x_0| < \epsilon$  whenever  $\delta \leq \epsilon$  (Example 3a).

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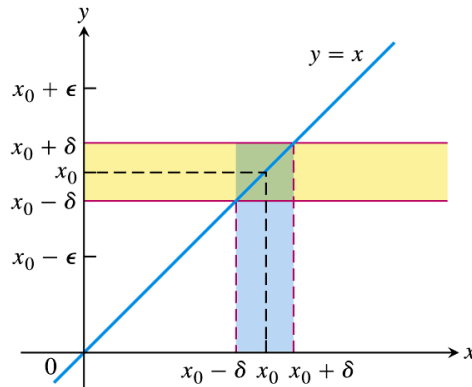
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## Solution

- Let  $\varepsilon > 0$ . We must find  $\delta > 0$  such that for all  $x$ ,  $0 < |x - x_0| < \delta$  implies  $|f(x) - x_0| < \varepsilon$ , here,  $f(x) = x$ , the identity function.
- Choose  $\delta < \varepsilon$  will do the job.
- The proof of the existence of  $\delta$  proves

$$\lim_{x \rightarrow x_0} x = x_0$$



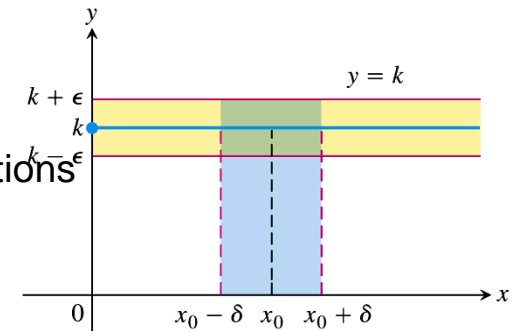
**FIGURE 2.16** For the function  $f(x) = x$ , we find that  $0 < |x - x_0| < \delta$  will guarantee  $|f(x) - x_0| < \varepsilon$  whenever  $\delta \leq \varepsilon$  (Example 3a).

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## Example 3(b)

- Limits constant functions
- Prove

$$\lim_{x \rightarrow x_0} k = k \quad (k \text{ constant})$$



**FIGURE 2.17** For the function  $f(x) = k$ , we find that  $|f(x) - k| < \varepsilon$  for any positive  $\delta$  (Example 3b).

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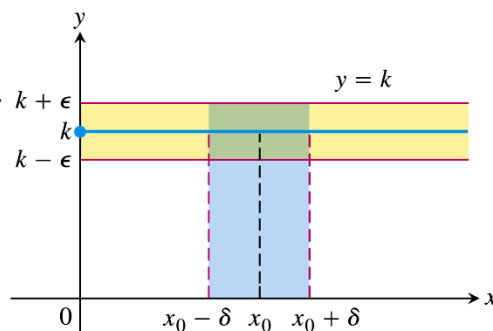
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## Solution

- Let  $\varepsilon > 0$ . We must find  $\delta > 0$  such that for all  $x$ ,  $0 < |x - x_0| < \delta$  implies  $|f(x) - k| < \varepsilon$ , here,  $f(x) = k$ , the constant function.
- Choose any  $\delta$  will do the job.
- The proof of the existence of  $\delta$  proves

$$\lim_{x \rightarrow x_0} k = k$$



**FIGURE 2.17** For the function  $f(x) = k$ , we find that  $|f(x) - k| < \varepsilon$  for any positive  $\delta$  (Example 3b).

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## Finding delta algebraically for given epsilons

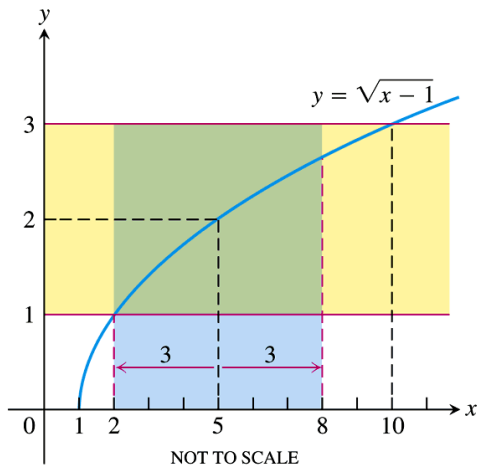
- Example 4: Finding delta algebraically
- For the limit  $\lim_{x \rightarrow 5} \sqrt{x-1} = 2$  find a  $\delta > 0$  that works for  $\varepsilon = 1$ . That is, find a  $\delta > 0$  such that for all  $x$ ,

$$0 < |x - 5| < \delta \Rightarrow 0 < |\sqrt{x-1} - 2| < 1$$

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**FIGURE 2.19** The function and intervals in Example 4.

## Solution

- $\delta$  is found by working backward:

### How to Find Algebraically a $\delta$ for a Given $f$ , $L$ , $x_0$ , and $\epsilon > 0$

The process of finding a  $\delta > 0$  such that for all  $x$

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

1. Solve the inequality  $|f(x) - L| < \epsilon$  to find an open interval  $(a, b)$  containing  $x_0$  on which the inequality holds for all  $x \neq x_0$ .
2. Find a value of  $\delta > 0$  that places the open interval  $(x_0 - \delta, x_0 + \delta)$  centered at  $x_0$  inside the interval  $(a, b)$ . The inequality  $|f(x) - L| < \epsilon$  will hold for all  $x \neq x_0$  in this  $\delta$ -interval.

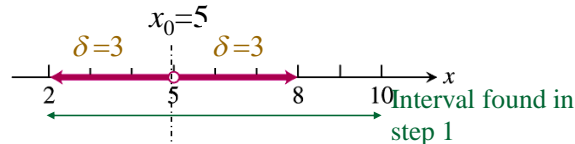
## Solution

- Step one: Solve the inequality  $|f(x) - L| < \epsilon$

$$0 < |\sqrt{x-1} - 2| < 1 \Rightarrow 2 < x < 10$$

- Step two: Find a value of  $\delta > 0$  that places the open interval  $(x_0 - \delta, x_0 + \delta)$  centered at  $x_0$  inside the open interval found in step one. Hence, we choose  $\delta = 3$  or a smaller number

By doing so, the inequality  $0 < |x - 5| < \delta$  will automatically place  $x$  between 2 and 10 to make  $0 < |f(x) - 2| < 1$



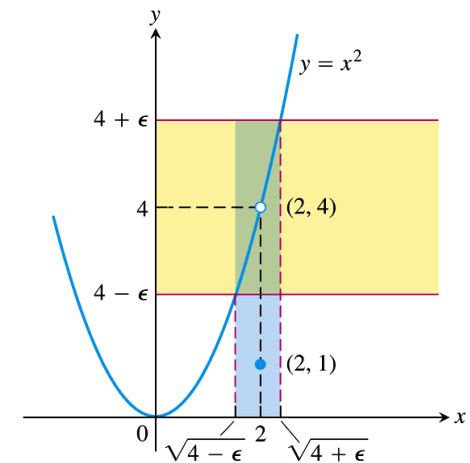
**FIGURE 2.18** An open interval of radius 3 about  $x_0 = 5$  will lie inside the open interval  $(2, 10)$ .

## Example 5

- Prove that

$$\lim_{x \rightarrow 2} f(x) = 4 \text{ if}$$

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$$



**FIGURE 2.20** An interval containing  $x = 2$  so that the function in Example 5 satisfies  $|f(x) - 4| < \epsilon$ .

# Solution

- Step one: Solve the inequality  $|f(x)-L|<\epsilon$ :

$$0 < |x^2 - 2| < \epsilon \Rightarrow \sqrt{4-\epsilon} < x < \sqrt{4+\epsilon}, x \neq 2$$

- Step two: Choose
- $\delta < \min [2-\sqrt{4-\epsilon}, \sqrt{4+\epsilon} - 2]$

For all  $x$  that obey

- $0 < |x - 2| < \delta$
- $\Rightarrow |f(x)-4| < \epsilon$
- This completes the proof.

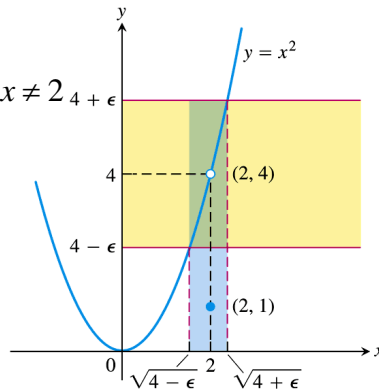


FIGURE 2.20 An interval containing  $x = 2$  so that the function in Example 5 satisfies  $|f(x) - 4| < \epsilon$ .

# 2.4

## One-Sided Limits and Limits at Infinity

Two sided limit does not exist for  $y$ ;

But

$y$  does has two one-sided limits

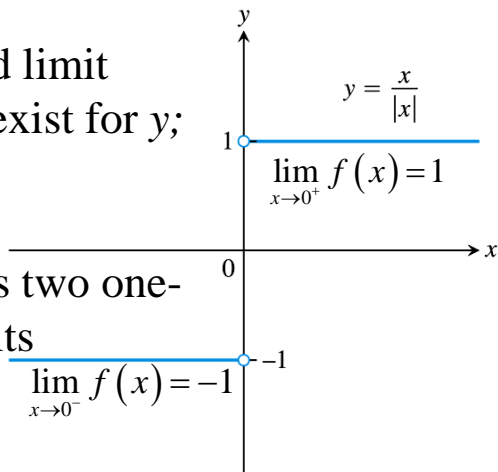


FIGURE 2.21 Different right-hand and left-hand limits at the origin.

# One-sided limits

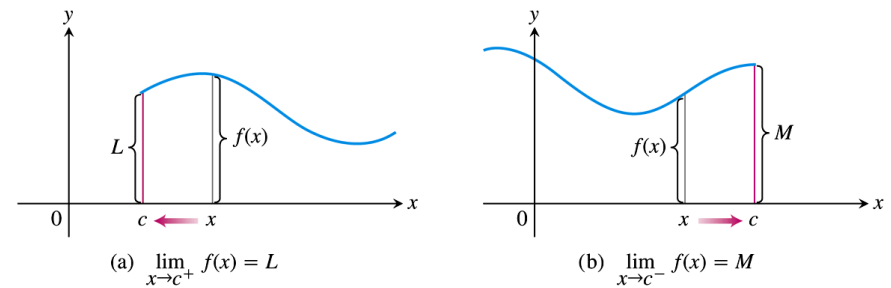
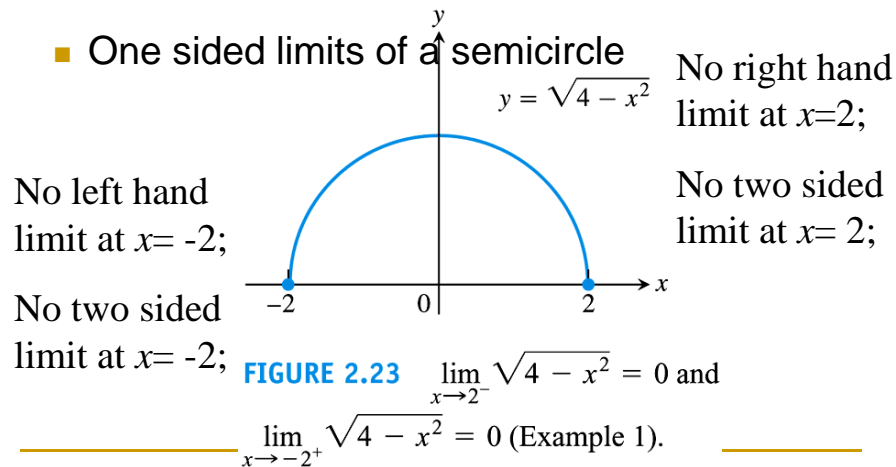


FIGURE 2.22 (a) Right-hand limit as  $x$  approaches  $c$ . (b) Left-hand limit as  $x$  approaches  $c$ .

Right-hand limit

Left-hand limit

## Example 1



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### THEOREM 6

A function  $f(x)$  has a limit as  $x$  approaches  $c$  if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

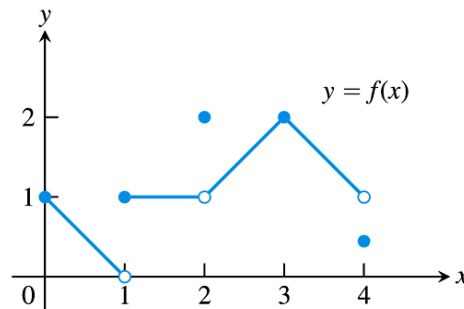
$$\lim_{x \rightarrow c} f(x) = L \iff \lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L.$$

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## Example 2

- Limits of the function graphed in Figure 2.24
- Can you write down all the limits at  $x=0$ ,  $x=1$ ,  $x=2$ ,  $x=3$ ,  $x=4$ ?
- What is the limit at other values of  $x$ ?



**FIGURE 2.24** Graph of the function in Example 2.

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## Precise definition of one-sided limits

### DEFINITIONS Right-Hand, Left-Hand Limits

We say that  $f(x)$  has **right-hand limit**  $L$  at  $x_0$ , and write

$$\lim_{x \rightarrow x_0^+} f(x) = L \quad (\text{See Figure 2.25})$$

if for every number  $\epsilon > 0$  there exists a corresponding number  $\delta > 0$  such that for all  $x$

$$x_0 < x < x_0 + \delta \implies |f(x) - L| < \epsilon.$$

We say that  $f$  has **left-hand limit**  $L$  at  $x_0$ , and write

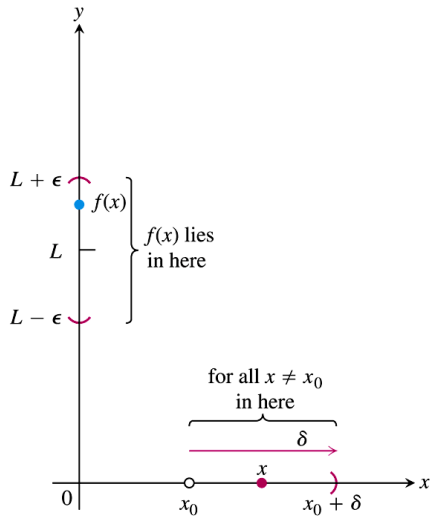
$$\lim_{x \rightarrow x_0^-} f(x) = L \quad (\text{See Figure 2.26})$$

if for every number  $\epsilon > 0$  there exists a corresponding number  $\delta > 0$  such that for all  $x$

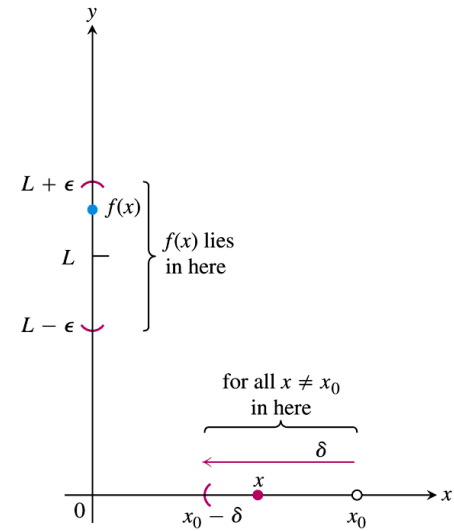
$$x_0 - \delta < x < x_0 \implies |f(x) - L| < \epsilon.$$

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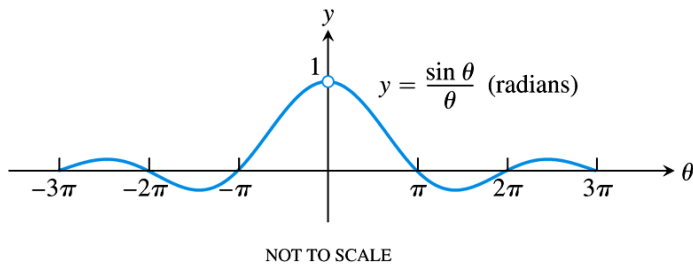


**FIGURE 2.25** Intervals associated with the definition of right-hand limit.



**FIGURE 2.26** Intervals associated with the definition of left-hand limit.

## Limits involving $(\sin \theta) / \theta$



**FIGURE 2.29** The graph of  $f(\theta) = (\sin \theta) / \theta$ .

### THEOREM 7

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad (\theta \text{ in radians}) \quad (1)$$

## Proof

$$\text{Area } \triangle OAP = \frac{1}{2} \sin \theta$$

$$\text{Area sector } OAP = \frac{\theta}{2}$$

$$\text{Area } \triangle OAT = \frac{1}{2} \tan \theta$$

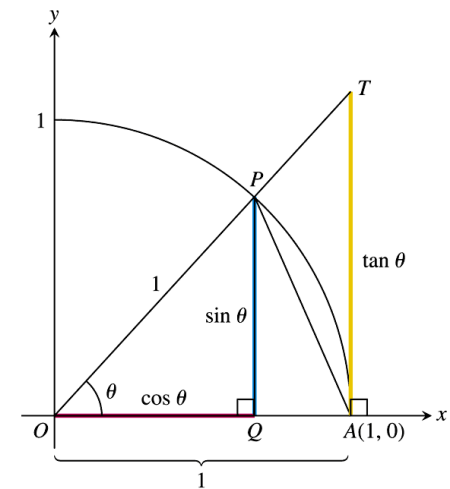
$$\frac{1}{2} \sin \theta < \frac{\theta}{2} < \frac{1}{2} \tan \theta$$

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}$$

$$1 > \frac{\sin \theta}{\theta} > \cos \theta$$

Taking limit  $\theta \rightarrow 0^{\pm}$ ,

$$\lim_{\theta \rightarrow 0^{\pm}} \frac{\sin \theta}{\theta} = 1 = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$$



**FIGURE 2.30** The figure for the proof of Theorem 7.  $TA/OA = \tan \theta$ , but  $OA = 1$ , so  $TA = \tan \theta$ .

## Example 5(a)

- Using theorem 7, show that

$$\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$$

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## Example 5(b)

- Using theorem 7, show that

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \frac{2}{5}$$

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## Finite limits as $x \rightarrow \pm\infty$

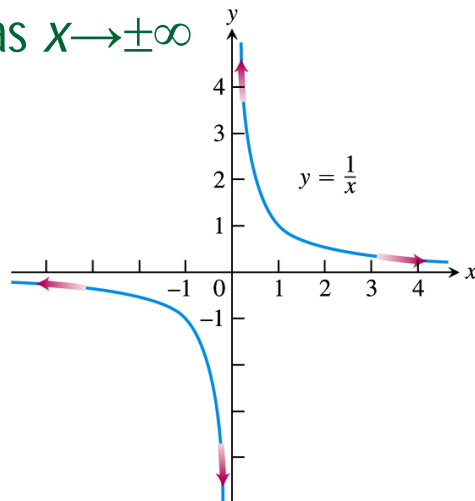


FIGURE 2.31 The graph of  $y = 1/x$ .

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## Precise definition

### DEFINITIONS Limit as $x$ approaches $\infty$ or $-\infty$

- We say that  $f(x)$  has the **limit  $L$  as  $x$  approaches infinity** and write

$$\lim_{x \rightarrow \infty} f(x) = L$$

if, for every number  $\epsilon > 0$ , there exists a corresponding number  $M$  such that for all  $x$

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

- We say that  $f(x)$  has the **limit  $L$  as  $x$  approaches minus infinity** and write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

if, for every number  $\epsilon > 0$ , there exists a corresponding number  $N$  such that for all  $x$

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

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## Example 6

- Limit at infinity for

$$f(x) = \frac{1}{x}$$

- (a) Show that

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

- (b) Show that

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

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### THEOREM 8 Limit Laws as $x \rightarrow \pm \infty$

If  $L$ ,  $M$ , and  $k$ , are real numbers and

$$\lim_{x \rightarrow \pm \infty} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow \pm \infty} g(x) = M, \quad \text{then}$$

- Sum Rule:**  $\lim_{x \rightarrow \pm \infty} (f(x) + g(x)) = L + M$
- Difference Rule:**  $\lim_{x \rightarrow \pm \infty} (f(x) - g(x)) = L - M$
- Product Rule:**  $\lim_{x \rightarrow \pm \infty} (f(x) \cdot g(x)) = L \cdot M$
- Constant Multiple Rule:**  $\lim_{x \rightarrow \pm \infty} (k \cdot f(x)) = k \cdot L$
- Quotient Rule:**  $\lim_{x \rightarrow \pm \infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
- Power Rule:** If  $r$  and  $s$  are integers with no common factors,  $s \neq 0$ , then

$$\lim_{x \rightarrow \pm \infty} (f(x))^{r/s} = L^{r/s}$$

provided that  $L^{r/s}$  is a real number. (If  $s$  is even, we assume that  $L > 0$ .)

70

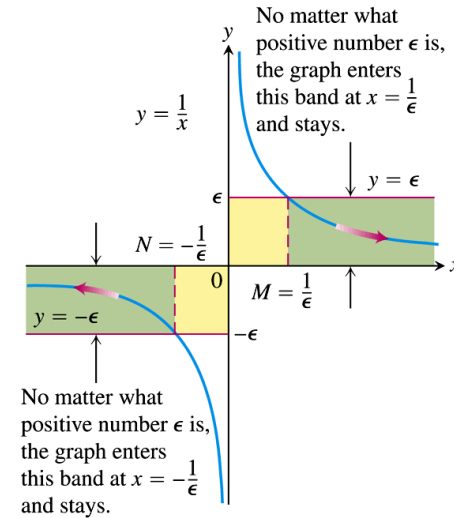


FIGURE 2.32 The geometry behind the argument in Example 6.

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## Example 7(a)

- Using Theorem 8

$$\lim_{x \rightarrow \infty} \left( 5 + \frac{1}{x} \right) = \lim_{x \rightarrow \infty} 5 + \lim_{x \rightarrow \infty} \frac{1}{x} = 5 + 0 = 5$$

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## Example 7(b)

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\pi\sqrt{3}}{x^2} &= \pi\sqrt{3} \lim_{x \rightarrow \infty} \frac{1}{x^2} \\ &= \pi\sqrt{3} \lim_{x \rightarrow \infty} \frac{1}{x} \cdot \lim_{x \rightarrow \infty} \frac{1}{x} \\ &= \pi\sqrt{3} \cdot 0 \cdot 0 = 0\end{aligned}$$

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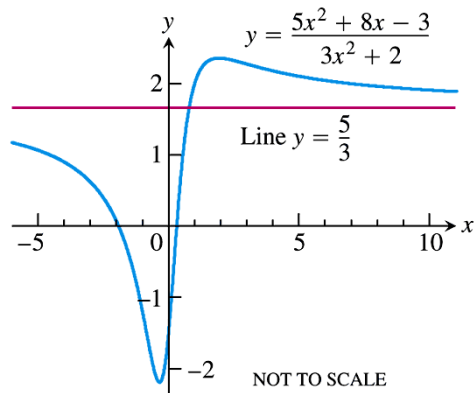
## Limits at infinity of rational functions

### ■ Example 8

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} &= \lim_{x \rightarrow \infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)} = \\ &= \frac{5 + \lim_{x \rightarrow \infty} (8/x) - \lim_{x \rightarrow \infty} (3/x^2)}{3 + \lim_{x \rightarrow \infty} (2/x^2)} = \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3}\end{aligned}$$

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**FIGURE 2.33** The graph of the function in Example 8. The graph approaches the line  $y = 5/3$  as  $|x|$  increases.

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## Example 9

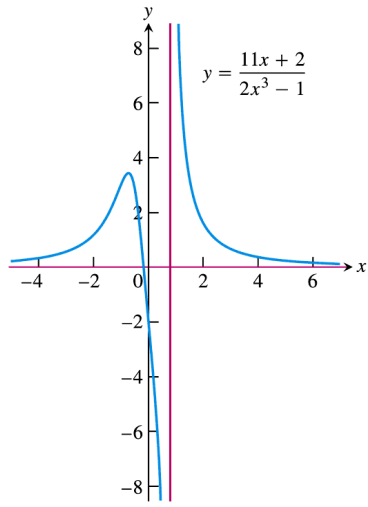
- Degree of numerator less than degree of denominator

$$\lim_{x \rightarrow \infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x \rightarrow \infty} \dots = 0$$

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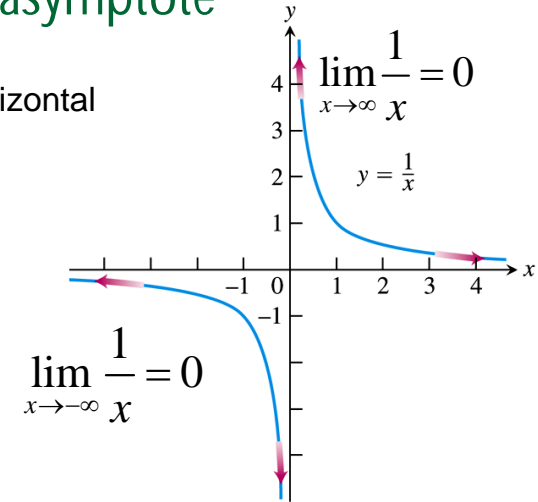


**FIGURE 2.34** The graph of the function in Example 9. The graph approaches the  $x$ -axis as  $|x|$  increases.

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## Horizontal asymptote

- $x$ -axis is a horizontal asymptote



**FIGURE 2.31** The graph of  $y = 1/x$ .

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### DEFINITION Horizontal Asymptote

A line  $y = b$  is a **horizontal asymptote** of the graph of a function  $y = f(x)$  if either

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b.$$

**Figure 2.33** has the line  $y=5/3$  as a horizontal asymptote on both the right and left because

$$\lim_{x \rightarrow \infty} f(x) = \frac{5}{3} \quad \lim_{x \rightarrow -\infty} f(x) = \frac{5}{3}$$

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## Oblique asymptote

- Happen when the degree of the numerator polynomial is one greater than the degree of the denominator
- By long division, recast  $f(x)$  into a linear function plus a remainder. The remainder shall  $\rightarrow 0$  as  $x \rightarrow \pm\infty$ . The linear function is the asymptote of the graph.

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# Example 12

- Find the oblique asymptote

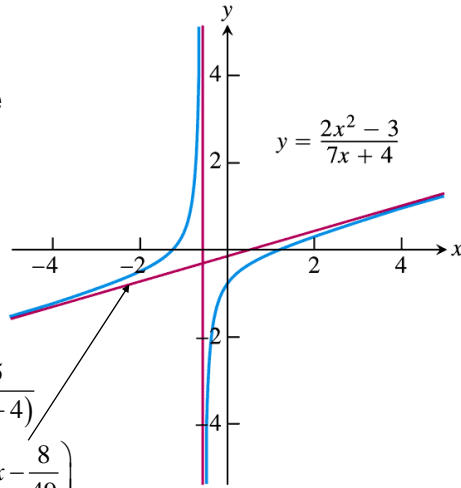
$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

- Solution

$$f(x) = \frac{2x^2 - 3}{7x + 4} = \overbrace{\left(\frac{2}{7}x - \frac{8}{49}\right)}^{\text{linear function}} + \frac{-115}{49(7x + 4)}$$

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \left(\frac{2}{7}x - \frac{8}{49}\right) + \lim_{x \rightarrow \pm\infty} \frac{-115}{49(7x + 4)}$$

$$= \lim_{x \rightarrow \pm\infty} \left(\frac{2}{7}x - \frac{8}{49}\right) + 0 = \lim_{x \rightarrow \pm\infty} \left(\frac{2}{7}x - \frac{8}{49}\right)$$

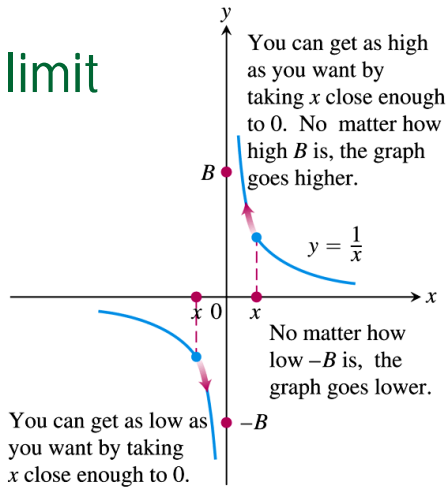


**FIGURE 2.36** The function in Example 12 has an oblique asymptote.

# 2.5

## Infinite Limits and Vertical Asymptotes

# Infinite limit

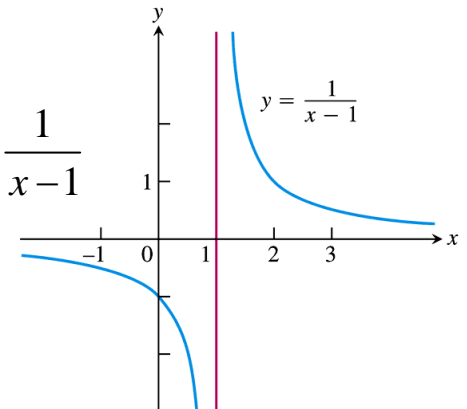


**FIGURE 2.37** One-sided infinite limits:

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

# Example 1

- Find  $\lim_{x \rightarrow 1^+} \frac{1}{x-1}$  and  $\lim_{x \rightarrow 1^-} \frac{1}{x-1}$



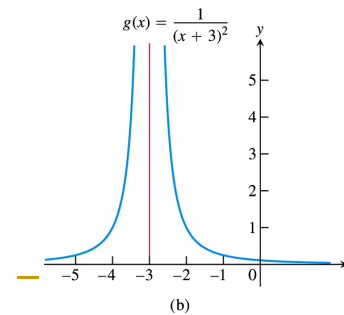
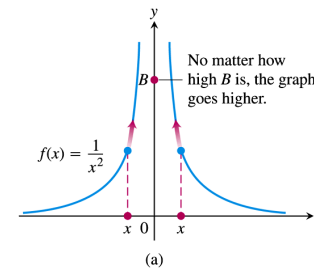
**FIGURE 2.38** Near  $x = 1$ , the function  $y = 1/(x - 1)$  behaves the way the function  $y = 1/x$  behaves near  $x = 0$ . Its graph is the graph of  $y = 1/x$  shifted 1 unit to the right (Example 1).

## Example 2 Two-sided infinite limit

- Discuss the behavior of

$$(a) \quad f(x) = \frac{1}{x^2} \text{ near } x = 0$$

$$(b) \quad g(x) = \frac{1}{(x+3)^2} \text{ near } x = -3$$



**FIGURE 2.39** The graphs of the functions in Example 2. (a)  $f(x)$  approaches infinity as  $x \rightarrow 0$ . (b)  $g(x)$  approaches infinity as  $x \rightarrow -3$ .

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## Example 3

- Rational functions can behave in various ways near zeros of their denominators

$$(a) \lim_{x \rightarrow 2} \frac{(x-2)^2}{x^2-4} = \lim_{x \rightarrow 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{(x-2)}{(x+2)} = 0$$

$$(b) \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{4}$$

$$(c) \lim_{x \rightarrow 2^+} \frac{x-3}{x^2-4} = \lim_{x \rightarrow 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty \quad (\text{note: } x > 2)$$

$$(d) \lim_{x \rightarrow 2^-} \frac{x-3}{x^2-4} = \lim_{x \rightarrow 2^-} \frac{x-3}{(x-2)(x+2)} = +\infty \quad (\text{note: } x < 2)$$

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## Example 3

$$(e) \lim_{x \rightarrow 2} \frac{x-3}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-3}{(x-2)(x+2)} \quad \text{limit does not exist}$$

$$(f) \lim_{x \rightarrow 2} \frac{2-x}{(x-2)^3} = -\lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x-2)^2} = -\lim_{x \rightarrow 2} \frac{1}{(x-2)^2} = -\infty$$

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# Precise definition of infinite limits

## DEFINITIONS Infinity, Negative Infinity as Limits

1. We say that  $f(x)$  **approaches infinity** as  $x$  approaches  $x_0$ , and write

$$\lim_{x \rightarrow x_0} f(x) = \infty,$$

if for every positive real number  $B$  there exists a corresponding  $\delta > 0$  such that for all  $x$

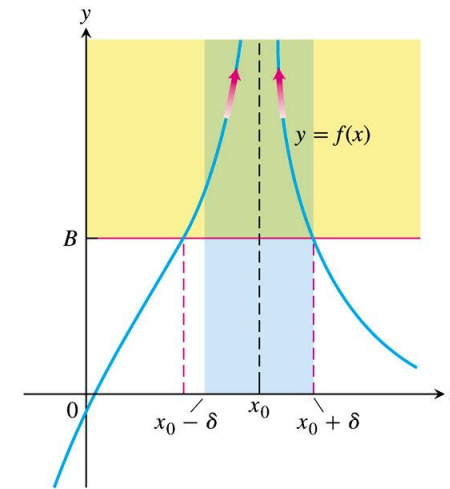
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > B.$$

2. We say that  $f(x)$  **approaches negative infinity** as  $x$  approaches  $x_0$ , and write

$$\lim_{x \rightarrow x_0} f(x) = -\infty,$$

if for every negative real number  $-B$  there exists a corresponding  $\delta > 0$  such that for all  $x$

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) < -B.$$

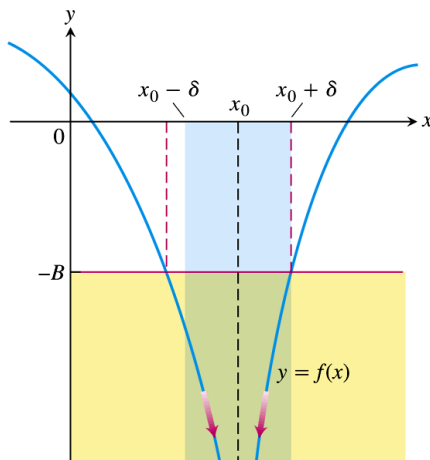


**FIGURE 2.40** For  $x_0 - \delta < x < x_0 + \delta$ , the graph of  $f(x)$  lies above the line  $y = B$ .

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**FIGURE 2.41** For  $x_0 - \delta < x < x_0 + \delta$ , the graph of  $f(x)$  lies below the line  $y = -B$ .

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## Example 4

■ Using definition of infinit limit

■ Prove that

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

Given  $B > 0$ , we want to find  $\delta > 0$  such that

$$0 < |x - 0| < \delta \quad \text{implies} \quad \frac{1}{x^2} > B$$

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## Example 4

Now

$$\frac{1}{x^2} > B \text{ if and only if } x^2 < 1/B \equiv |x| < 1/\sqrt{B}$$

By choosing  $\delta = 1/\sqrt{B}$

(or any smaller positive number), we see that

$$|x| < \delta \text{ implies } \frac{1}{x^2} > \frac{1}{\delta^2} \geq B$$

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## Vertical asymptotes

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

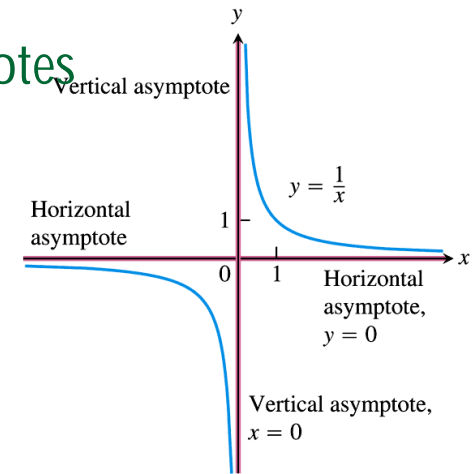


FIGURE 2.42 The coordinate axes are asymptotes of both branches of the hyperbola  $y = 1/x$ .

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## Example 5 Looking for asymptote

- Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x+3}{x+2}$$

- Solution:

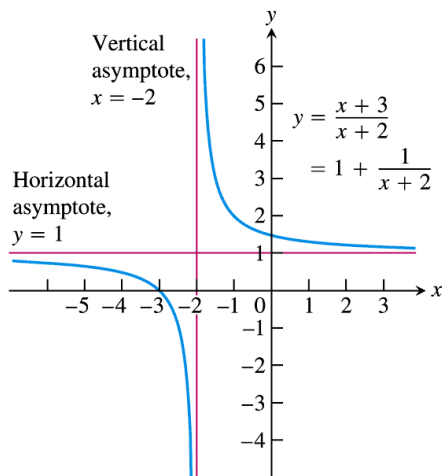
$$y = 1 + \frac{1}{x+2}$$

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**FIGURE 2.43** The lines  $y = 1$  and  $x = -2$  are asymptotes of the curve  $y = (x + 3)/(x + 2)$  (Example 5).

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## Asymptotes need not be two-sided

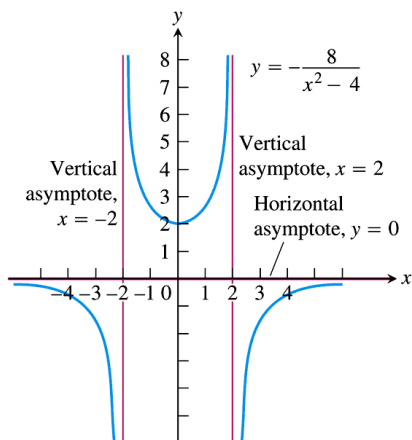
### ■ Example 6

$$f(x) = -\frac{8}{x^2 - 2}$$

### ■ Solution:

$$f(x) = -\frac{8}{x^2 - 2} = -\frac{8}{(x - 2)(x + 2)}$$

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**FIGURE 2.44** Graph of  $y = -8/(x^2 - 4)$ . Notice that the curve approaches the  $x$ -axis from only one side. Asymptotes do not have to be two-sided (Example 6).

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## Example 8

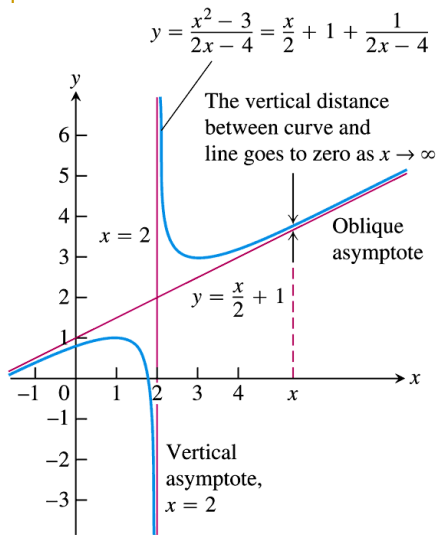
- A rational function with degree of numerator greater than degree of denominator

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

- Solution:

$$f(x) = \frac{x^2 - 3}{2x - 4} = \frac{\text{linear}}{2} + 1 + \frac{\text{remainder}}{2x - 4}$$

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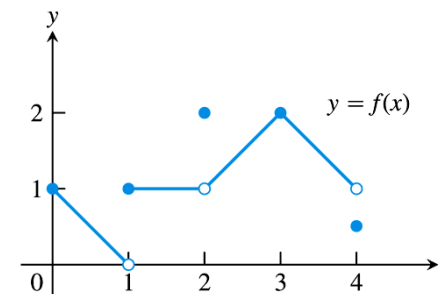
**FIGURE 2.47** The graph of  $f(x) = (x^2 - 3)/(2x - 4)$  has a vertical asymptote and an oblique asymptote (Example 8).

## 2.6

### Continuity

### Continuity at a point

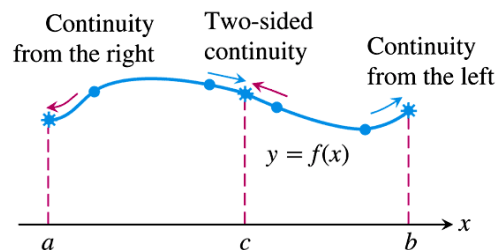
- Example 1
- Find the points at which the function  $f$  in Figure 2.50 is continuous and the points at which  $f$  is discontinuous.



**FIGURE 2.50** The function is continuous on  $[0, 4]$  except at  $x = 1$ ,  $x = 2$ , and  $x = 4$  (Example 1).

- $f$  continuous:
  - At  $x = 0$
  - At  $x = 3$
  - At  $0 < c < 4, c \neq 1, 2$
- $f$  discontinuous:
  - At  $x = 1$
  - At  $x = 2$
  - At  $x = 4$
  - $0 > c, c > 4$
  - Why?

- To define the continuity at any point in a function's domain, we need to define continuity at an interior point and continuity at an endpoint



**FIGURE 2.51** Continuity at points  $a$ ,  $b$ , and  $c$ .

#### DEFINITION Continuous at a Point

*Interior point:* A function  $y = f(x)$  is **continuous at an interior point  $c$**  of its domain if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

*Endpoint:* A function  $y = f(x)$  is **continuous at a left endpoint  $a$**  or is **continuous at a right endpoint  $b$**  of its domain if

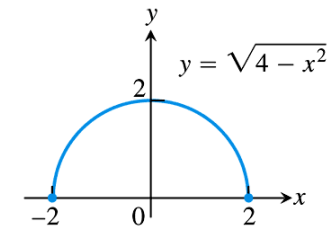
$$\lim_{x \rightarrow a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \rightarrow b^-} f(x) = f(b), \quad \text{respectively.}$$



## Example 2

- A function continuous throughout its domain

$$f(x) = \sqrt{4 - x^2}$$



**FIGURE 2.52** A function that is continuous at every domain point (Example 2).

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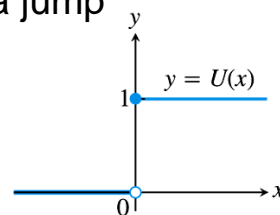
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## Example 3

- The unit step function has a jump discontinuity



**FIGURE 2.53** A function that is right-continuous, but not left-continuous, at the origin. It has a jump discontinuity there (Example 3).

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## Summarize continuity at a point in the form of a test

### Continuity Test

A function  $f(x)$  is continuous at  $x = c$  if and only if it meets the following three conditions.

1.  $f(c)$  exists ( $c$  lies in the domain of  $f$ )
2.  $\lim_{x \rightarrow c} f(x)$  exists ( $f$  has a limit as  $x \rightarrow c$ )
3.  $\lim_{x \rightarrow c} f(x) = f(c)$  (the limit equals the function value)

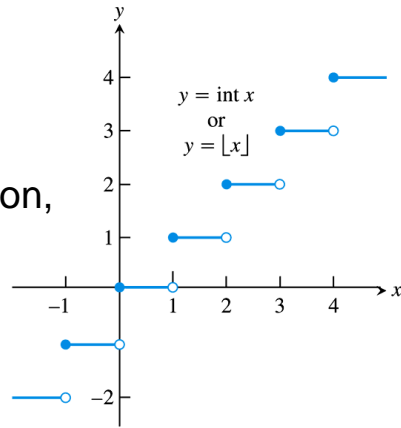
For one-sided continuity and continuity at an endpoint, the limits in part 2 and part 3 of the test should be replaced by the appropriate **one-sided limits**.

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## Example 4

- The greatest integer function,  $y = \text{int } x$
- $y = \text{int } x$
- The function is not continuous at the integer points since limit does not exist there (left and right limits not agree)



**FIGURE 2.54** The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

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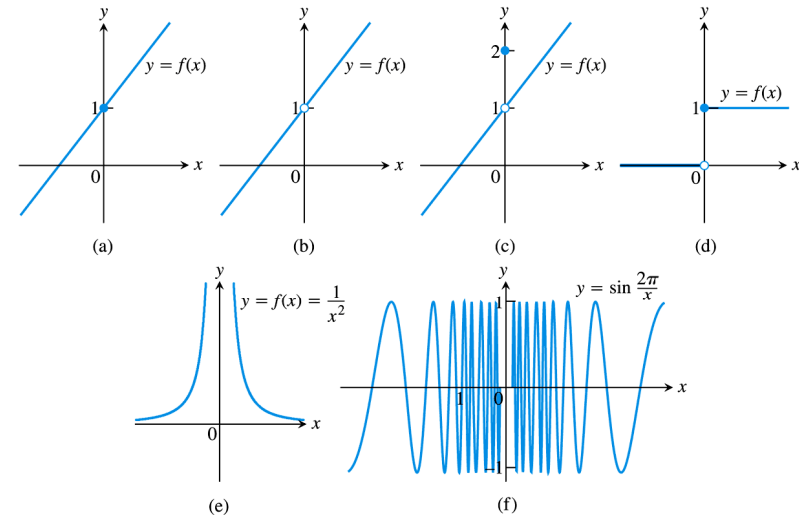
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## Discontinuity types

- (b), (c) removable discontinuity
- (d) jump discontinuity
- (e) infinite discontinuity
- (f) oscillating discontinuity

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**FIGURE 2.55** The function in (a) is continuous at  $x = 0$ ; the functions in (b) through (f) are not.

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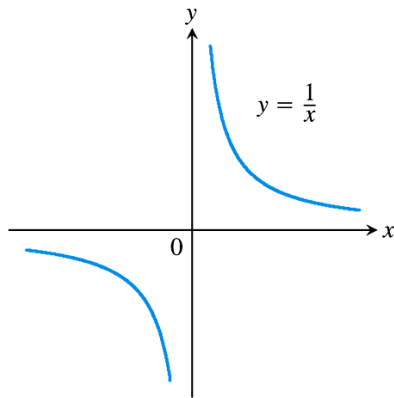
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## Continuous functions

- A function is continuous on an interval if and only if it is continuous at every point of the interval.
- Example: Figure 2.56
- $1/x$  not continuous on  $[-1, 1]$  but continuous over  $(-\infty, 0) \cup (0, \infty)$

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**FIGURE 2.56** The function  $y = 1/x$  is continuous at every value of  $x$  except  $x = 0$ . It has a point of discontinuity at  $x = 0$  (Example 5).

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## Example 5

- Identifying continuous function
- (a)  $f(x) = 1/x$
- (b)  $f(x) = x$
- Ask: is  $1/x$  continuous over its domain?

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### THEOREM 9 Properties of Continuous Functions

If the functions  $f$  and  $g$  are continuous at  $x = c$ , then the following combinations are continuous at  $x = c$ .

1. *Sums:*  $f + g$
2. *Differences:*  $f - g$
3. *Products:*  $f \cdot g$
4. *Constant multiples:*  $k \cdot f$ , for any number  $k$
5. *Quotients:*  $f/g$  provided  $g(c) \neq 0$
6. *Powers:*  $f^{r/s}$ , provided it is defined on an open interval containing  $c$ , where  $r$  and  $s$  are integers

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## Example 6

- Polynomial and rational functions are continuous
- (a) Every polynomial is continuous by
  - (i)  $\lim_{x \rightarrow c} P(x) = P(c)$
  - (ii) Theorem 9
- (b) If  $P(x)$  and  $Q(x)$  are polynomial, the rational function  $P(x)/Q(x)$  is continuous whenever it is defined.

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## Example 7

- Continuity of the absolute function
- $f(x) = |x|$  is everywhere continuous
  
- Continuity of the sinus and cosinus function
- $f(x) = \cos x$  and  $\sin x$  is everywhere continuous

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## Composites

- All composites of continuous functions are continuous

### THEOREM 10 Composite of Continuous Functions

If  $f$  is continuous at  $c$  and  $g$  is continuous at  $f(c)$ , then the composite  $g \circ f$  is continuous at  $c$ .

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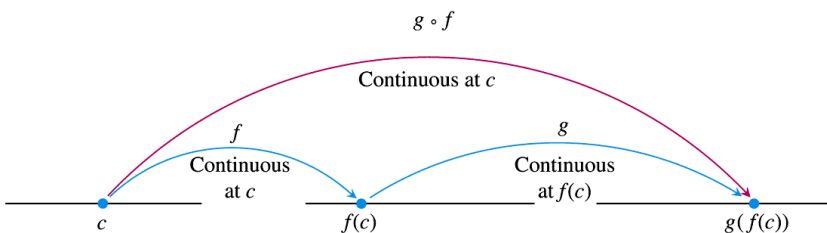


FIGURE 2.57 Composites of continuous functions are continuous.

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## Example 8

- Applying Theorems 9 and 10
- Show that the following functions are continuous everywhere on their respective domains.

$$(a) y = \sqrt{x^2 - 2x - 5} \quad (b) y = \frac{x^{2/3}}{1 + x^4}$$

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(a)  $y = \sqrt{x^2 - 2x - 5}$

(a) The square root function is continuous on  $[0, \infty)$  because it is a rational power of the continuous identity function  $f(x) = x$  (Part 6, Theorem 9). The given function is then the composite of the polynomial  $f(x) = x^2 - 2x - 5$  with the square root function  $g(t) = \sqrt{t}$ .

$$y(x) = f \circ g;$$

$$f(t) = \sqrt{t} = t^{1/2};$$

$$g(x) = x^2 + 2x - 5$$

$g(x)$  is continuous in all  $x$  since it is a polynomial, according to Example 6.

$f(t)$  is continuous in all  $t$  due to Part 6 in Theorem 9.

Hence,  $f[g(x)]$  is continuous, according to Theorem 10.

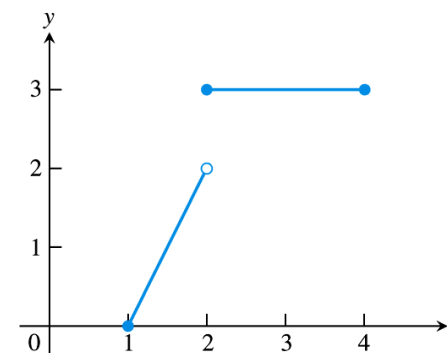
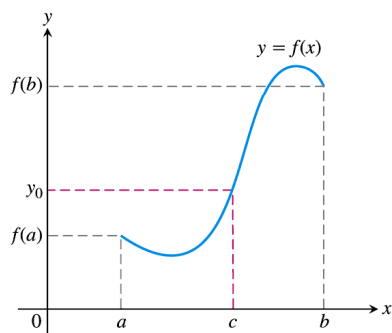
(b)  $y = \frac{x^{2/3}}{1 + x^4}$

The numerator is a rational power of the identity function; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.

This is the application of theorem 9.

**THEOREM 11 The Intermediate Value Theorem for Continuous Functions**

A function  $y = f(x)$  that is continuous on a closed interval  $[a, b]$  takes on every value between  $f(a)$  and  $f(b)$ . In other words, if  $y_0$  is any value between  $f(a)$  and  $f(b)$ , then  $y_0 = f(c)$  for some  $c$  in  $[a, b]$ .



**FIGURE 2.61** The function  $f(x) = \begin{cases} 2x - 2, & 1 \leq x < 2 \\ 3, & 2 \leq x \leq 4 \end{cases}$  does not take on all values between  $f(1) = 0$  and  $f(4) = 3$ ; it misses all the values between 2 and 3.

## Consequence of root finding

- A solution of the equation  $f(x)=0$  is called a root.
- For example,  $f(x)=x^2+x-6$ , the roots are  $x=2$ ,  $x=-3$  since  $f(-3)=f(2)=0$ .
- Say  $f$  is continuous over some interval.
- Say  $a, b$  (with  $a < b$ ) are in the domain of  $f$ , such that  $f(a)$  and  $f(b)$  have opposite signs.
- This means either  $f(a) < 0 < f(b)$  or  $f(b) < 0 < f(a)$
- Then, as a consequence of theorem 11, there must exist at least a point  $c$  between  $a$  and  $b$ , i.e.  $a < c < b$  such that  $f(c)=0$ .  $x=c$  is the root.

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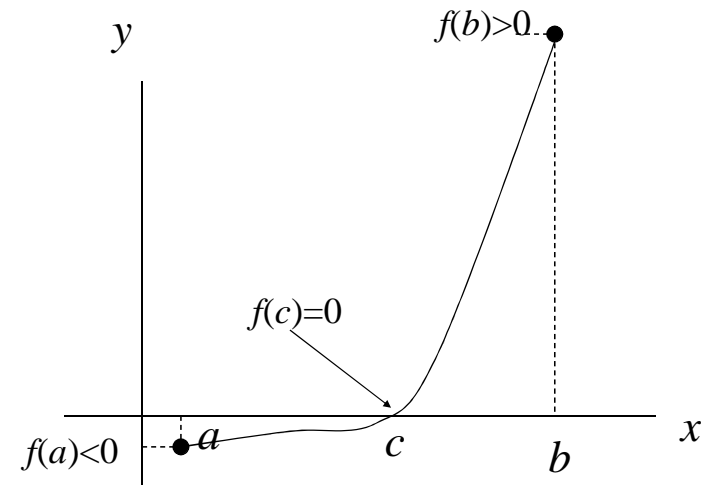
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## Example

- Consider the function  $f(x) = x - \cos x$
- Prove that there is at least one root for  $f(x)$  in the interval  $[0, \pi/2]$ .
- **Solution**
- $f(x)$  is continuous on  $(-\infty, \infty)$ .
- Say  $a = 0$ ,  $b = \pi/2$ .
- $f(x=0) = -1$ ;  $f(x = \pi/2) = \pi/2$
- $f(a)$  and  $f(b)$  have opposite signs
- Then, as a consequence of theorem 11, there must exist at least a point  $c$  between  $a$  and  $b$ , i.e.  $a=0 < c < b= \pi/2$  such that  $f(c)=0$ .  $x=c$  is the root.

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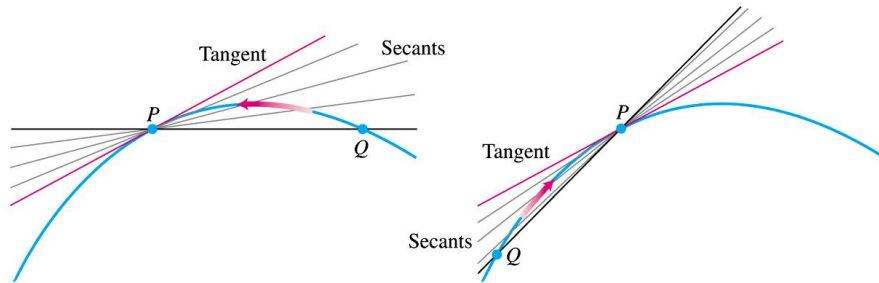
## 2.7

### Tangents and Derivatives

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# What is a tangent to a curve?



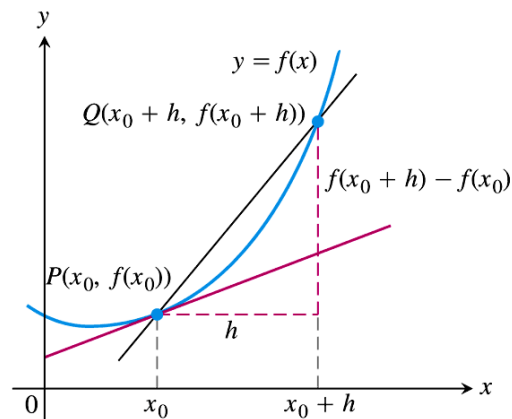
**FIGURE 2.65** The dynamic approach to tangency. The tangent to the curve at  $P$  is the line through  $P$  whose slope is the limit of the secant slopes as  $Q \rightarrow P$  from either side.

## DEFINITIONS Slope, Tangent Line

The **slope of the curve**  $y = f(x)$  at the point  $P(x_0, f(x_0))$  is the number

$$m = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad (\text{provided the limit exists}).$$

The **tangent line** to the curve at  $P$  is the line through  $P$  with this slope.



**FIGURE 2.67** The slope of the tangent line at  $P$  is  $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$ .

# Example 1: Tangent to a parabola

- Find the slope of the parabola  $y=x^2$  at the point  $P(2,4)$ . Write an equation for the tangent to the parabola at this point.

### Finding the Tangent to the Curve $y = f(x)$ at $(x_0, y_0)$

1. Calculate  $f(x_0)$  and  $f(x_0 + h)$ .
2. Calculate the slope

$$m = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

3. If the limit exists, find the tangent line as

$$y = y_0 + m(x - x_0).$$

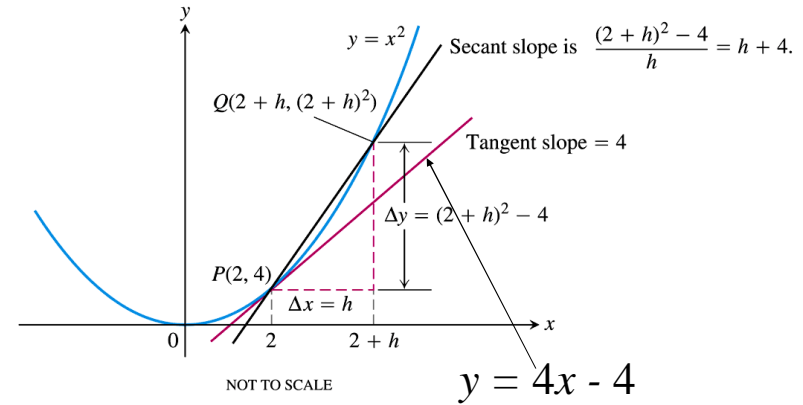


FIGURE 2.66 Finding the slope of the parabola  $y = x^2$  at the point  $P(2, 4)$  (Example 1).

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## Example 3

- Slope and tangent to  $y=1/x$ ,  $x \neq 0$
- (a) Find the slope of  $y=1/x$  at  $x = a \neq 0$
- (b) Where does the slope equal  $-1/4$ ?
- (c) What happens to the tangent of the curve at the point  $(a, 1/a)$  as  $a$  changes?

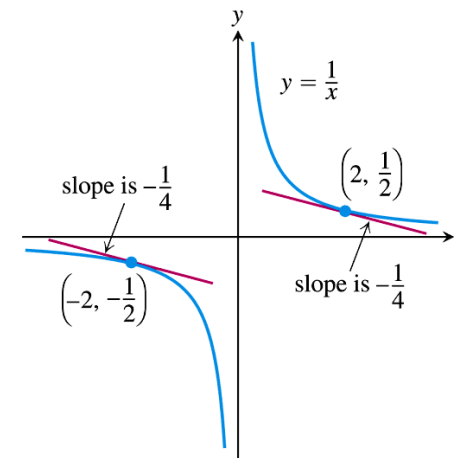


FIGURE 2.68 The two tangent lines to  $y = 1/x$  having slope  $-1/4$  (Example 3).

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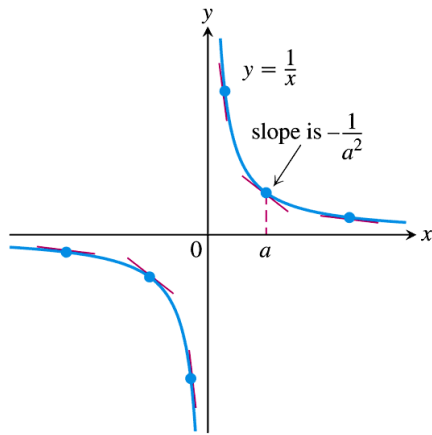
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**FIGURE 2.69** The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away.

**difference quotient of  $f$  at  $x_0$  with increment  $h$ .**

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

- If the limit  $h \rightarrow 0$  of the quotient exists, it is called

**the derivative of  $f$  at  $x_0$ .**

## Chapter 3

### Differentiation

## 3.1

### The Derivative as a Function

### DEFINITION Derivative Function

The **derivative** of the function  $f(x)$  with respect to the variable  $x$  is the function  $f'$  whose value at  $x$  is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

- The limit

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

- when it existed, is called the Derivative of  $f$  at  $x_0$ .
- View derivative as a function derived from  $f$

3

If write  $z = x + h$ , then  $h = z - x$

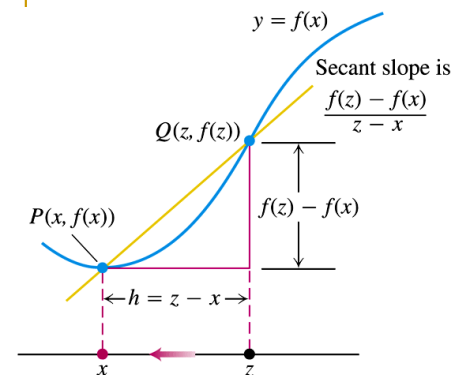
### Alternative Formula for the Derivative

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}.$$

5

- If  $f'$  exists at  $x$ ,  $f$  is said to be differentiable (has a derivative) at  $x$
- If  $f'$  exists at every point in the domain of  $f$ ,  $f$  is said to be differentiable.

4



Derivative of  $f$  at  $x$  is

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} \end{aligned}$$

**FIGURE 3.1** The way we write the difference quotient for the derivative of a function  $f$  depends on how we label the points involved.

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## Calculating derivatives from the definition

- Differentiation: an operation performed on a function  $y = f(x)$
- $d/dx$  operates on  $f(x)$
- Write as  $\frac{d}{dx} f(x)$
- $f'$  is taken as a shorthand notation for  $\frac{d}{dx} f(x)$

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## Example 2: Derivative of the square root function

- (a) Find the derivative of  $y = \sqrt{x}$  for  $x > 0$
- (b) Find the tangent line to the curve  $y = \sqrt{x}$  at  $x = 4$

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## Example 1: Applying the definition

- Differentiate

$$f(x) = \frac{x}{x-1}$$

- Solution:

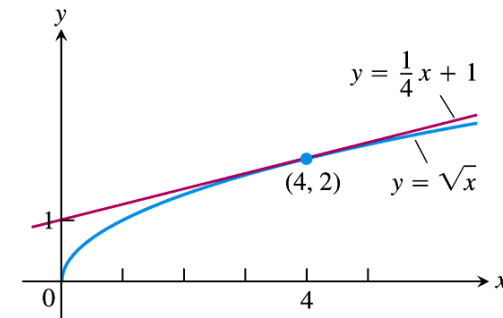
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\left(\frac{x+h}{x+h-1}\right) - \left(\frac{x}{x-1}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}$$

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**FIGURE 3.2** The curve  $y = \sqrt{x}$  and its tangent at  $(4, 2)$ . The tangent's slope is found by evaluating the derivative at  $x = 4$  (Example 2).

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## Notations

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

$$f'(a) = \left. \frac{dy}{dx} \right|_{x=a} = \left. \frac{df}{dx} \right|_{x=a} = \left. \frac{d}{dx} f(x) \right|_{x=a}$$

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## Differentiable on an interval; One sided derivatives

- A function  $y = f(x)$  is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval.
- It is differentiable on a closed interval  $[a, b]$  if it is differentiable on the interior  $(a, b)$  and if the limits

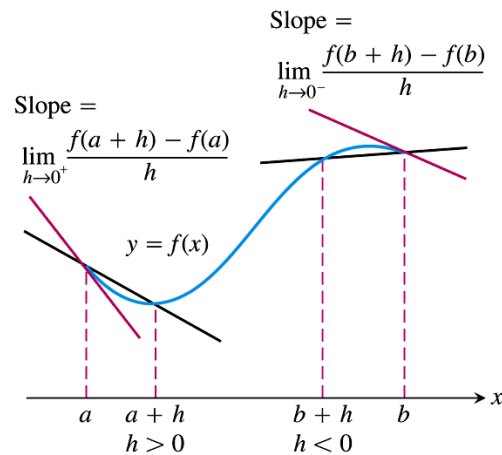
$$\lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h}$$

$$\lim_{h \rightarrow 0^-} \frac{f(b+h) - f(b)}{h}$$

exist at the endpoints

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**FIGURE 3.5** Derivatives at endpoints are one-sided limits.

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- A function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

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## Example 5

- $y = |x|$  is not differentiable at  $x = 0$ .
- Solution:
- For  $x > 0$ ,
$$\frac{d|x|}{dx} = \frac{d}{dx}(x) = 1$$
- For  $x < 0$ ,
$$\frac{d|x|}{dx} = \frac{d}{dx}(-x) = -1$$
- At  $x = 0$ , the right hand derivative and left hand derivative differ there. Hence  $f(x)$  not differentiable at  $x = 0$  but else where.

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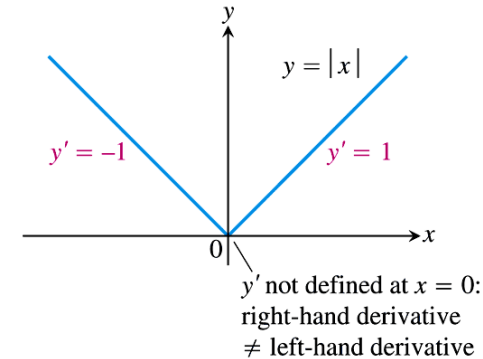
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## Example 6

- $y = \sqrt{x}$  is not differentiable at  $x = 0$
- The graph has a vertical tangent at  $x = 0$

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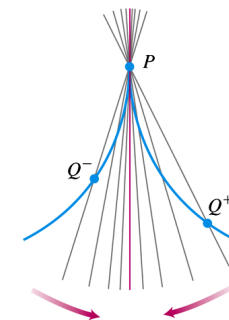
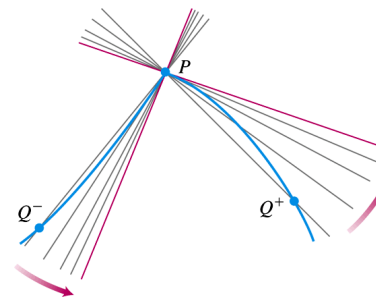
**FIGURE 3.6** The function  $y = |x|$  is not differentiable at the origin where the graph has a “corner.”

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## When Does a function not have a derivative at a point?

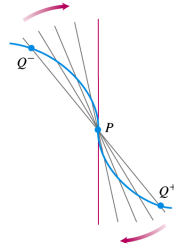
1. a *corner*, where the one-sided derivatives differ.
2. a *cusp*, where the slope of  $PQ$  approaches  $\infty$  from one side and  $-\infty$  from the other.



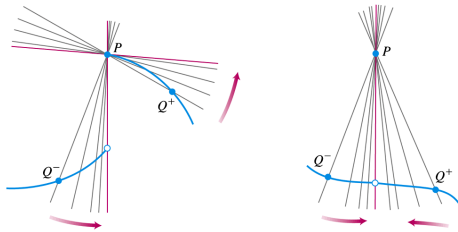
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3. a *vertical tangent*, where the slope of  $PQ$  approaches  $\infty$  from both sides or approaches  $-\infty$  from both sides (here,  $-\infty$ ).



4. a *discontinuity*.



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## Example

- $y = |x|$  is continuous everywhere, including  $x = 0$ , but it is not differentiable there.

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## Differentiable functions are continuous

### THEOREM 1 Differentiability Implies Continuity

If  $f$  has a derivative at  $x = c$ , then  $f$  is continuous at  $x = c$ .

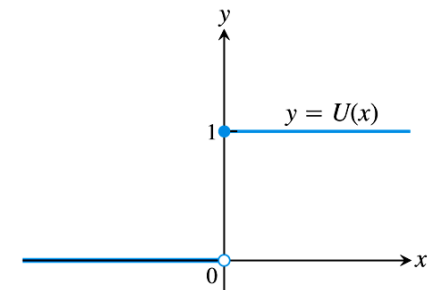
The converse is false: continuity does not necessarily implies differentiability

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## The equivalent form of Theorem 1

- If  $f$  is not continuous at  $x = c$ , then  $f$  is not differentiable at  $x = c$ .
- Example: the step function is discontinuous at  $x = 0$ , hence not differentiable at  $x = 0$ .



**FIGURE 3.7** The unit step function does not have the Intermediate Value Property and cannot be the derivative of a function on the real line.

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## The intermediate value property of derivatives

### THEOREM 2 Darboux's Theorem

If  $a$  and  $b$  are any two points in an interval on which  $f$  is differentiable, then  $f'$  takes on every value between  $f'(a)$  and  $f'(b)$ .

- See section 4.4

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## Powers, multiples, sums and differences

### RULE 1 Derivative of a Constant Function

If  $f$  has the constant value  $f(x) = c$ , then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

25

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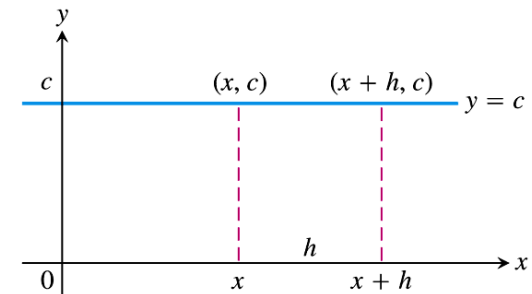
## 3.2

### Differentiation Rules

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### Example 1



**FIGURE 3.8** The rule  $(d/dx)(c) = 0$  is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

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### RULE 2 Power Rule for Positive Integers

If  $n$  is a positive integer, then

$$\frac{d}{dx} x^n = nx^{n-1}.$$

### RULE 3 Constant Multiple Rule

If  $u$  is a differentiable function of  $x$ , and  $c$  is a constant, then

$$\frac{d}{dx} (cu) = c \frac{du}{dx}.$$

In particular, if  $u = x^n$ ,  $\frac{d}{dx} (cx^n) = cx^{n-1}$

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## Example 3

$$\frac{d}{dx} (3x^2) = 3 \cdot 2x^{2-1} = 6x$$

$$\frac{d}{dx} (x^2) = 2x^{2-1} = 2x$$

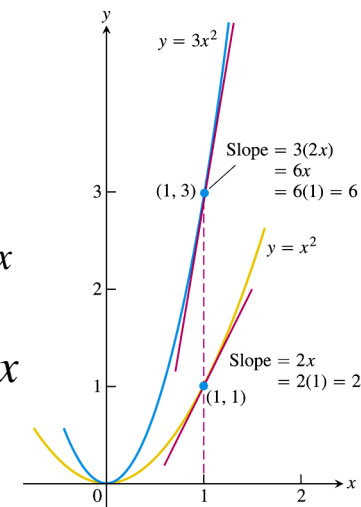


FIGURE 3.9 The graphs of  $y = x^2$  and  $y = 3x^2$ . Tripling the  $y$ -coordinates triples the slope (Example 3).

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## Example 5

$$y = x^3 + \frac{4}{3}x^2 - 5x + 1$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} (x^3) + \frac{d}{dx} \left( \frac{4}{3}x^2 \right) - \frac{d}{dx} (5x) + \frac{d}{dx} (1) \\ &= 3x^2 + \frac{8}{3}x - 5 \end{aligned}$$

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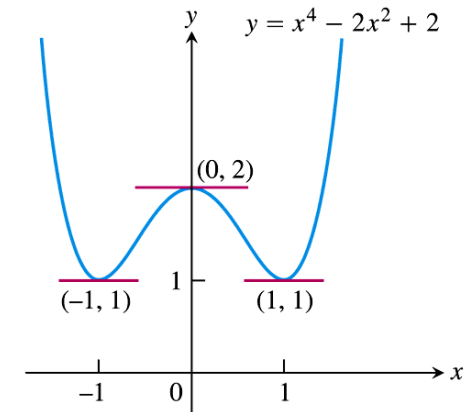
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## Example 6

- Does the curve  $y = x^4 - 2x^2 + 2$  have any horizontal tangents? If so, where?



**FIGURE 3.10** The curve  $y = x^4 - 2x^2 + 2$  and its horizontal tangents (Example 6).

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## Products and quotients

- Note that

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(x \cdot x) \neq \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1$$

### RULE 5 Derivative Product Rule

If  $u$  and  $v$  are differentiable at  $x$ , then so is their product  $uv$ , and

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}.$$

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## Example 7

- Find the derivative of

$$y = \frac{1}{x} \left( x^2 + \frac{1}{x} \right)$$

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## Example 8: Derivative from numerical values

- Let  $y = uv$ . Find  $y'(2)$  if  $u(2) = 3$ ,  $u'(2) = -4$ ,  $v(2) = 1$ ,  $v'(2) = 2$

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## Example 9

- Find the derivative of

$$y = (x^2 + 1)(x^3 + 3)$$

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## Negative integer powers of $x$

- The power rule for negative integers is the same as the rule for positive integers

### **RULE 6** Derivative Quotient Rule

If  $u$  and  $v$  are differentiable at  $x$  and if  $v(x) \neq 0$ , then the quotient  $u/v$  is differentiable at  $x$ , and

$$\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$

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### **RULE 7** Power Rule for Negative Integers

If  $n$  is a negative integer and  $x \neq 0$ , then

$$\frac{d}{dx} (x^n) = nx^{n-1}.$$

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## Example 11

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-1-1} = -x^{-2}$$

$$\frac{d}{dx}\left(\frac{4}{x^3}\right) = \frac{d}{dx}(4x^{-3}) = 4 \cdot (-3)x^{-3-1} = 12x^{-4}$$

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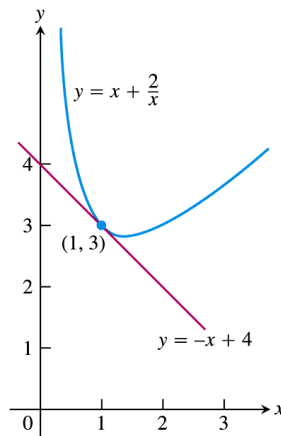
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## Example 12: Tangent to a curve

- Find the tangent to the curve  $y = x + \frac{2}{x}$  at the point  $(1, 3)$

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**FIGURE 3.11** The tangent to the curve  $y = x + (2/x)$  at  $(1, 3)$  in Example 12. The curve has a third-quadrant portion not shown here. We see how to graph functions like this one in Chapter 4.

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## Example 13

- Find the derivative of  $y = \frac{(x-1)(x^2-2x)}{x^4}$

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## Second- and higher-order derivative

- Second derivative

$$\begin{aligned} f''(x) &= \frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} (y') \\ &= y'' = D^2(f)(x) = D_x^2 f(x) \end{aligned}$$

- $n$ th derivative

$$y^{(n)} = \frac{d}{dx} y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y$$

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## 3.3

### The Derivative as a Rate of Change

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## Example 14

$$y = x^3 - 3x^2 + 2$$

$$y' = 3x^2 - 6x$$

$$y'' = 6x - 6$$

$$y''' = 6$$

$$y^{(4)} = 0$$

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## Instantaneous Rates of Change

### DEFINITION Instantaneous Rate of Change

The **instantaneous rate of change** of  $f$  with respect to  $x$  at  $x_0$  is the derivative

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

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## Example 1: How a circle's area changes with its diameter

- $A = \pi D^2/4$
- How fast does the area change with respect to the diameter when the diameter is 10 m?

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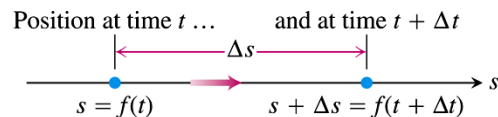
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## Motion along a line

- Position  $s = f(t)$
- Displacement,  $\Delta s = f(t + \Delta t) - f(t)$
- Average velocity
- $v_{av} = \Delta s / \Delta t = [f(t + \Delta t) - f(t)] / \Delta t$
- The instantaneous velocity is the limit of  $v_{av}$  when  $\Delta t \rightarrow 0$

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**FIGURE 3.12** The positions of a body moving along a coordinate line at time  $t$  and shortly later at time  $t + \Delta t$ .

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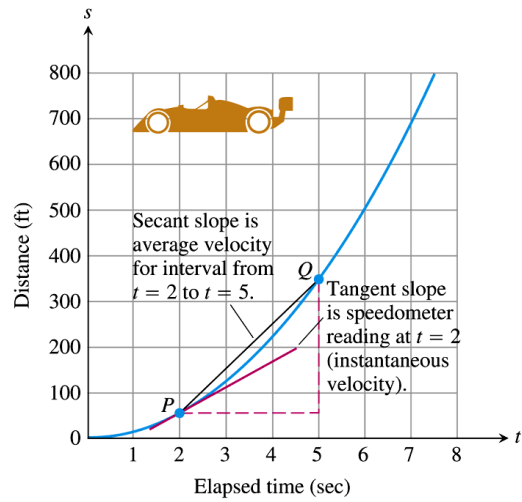
### DEFINITION Velocity

**Velocity (instantaneous velocity)** is the derivative of position with respect to time. If a body's position at time  $t$  is  $s = f(t)$ , then the body's velocity at time  $t$  is

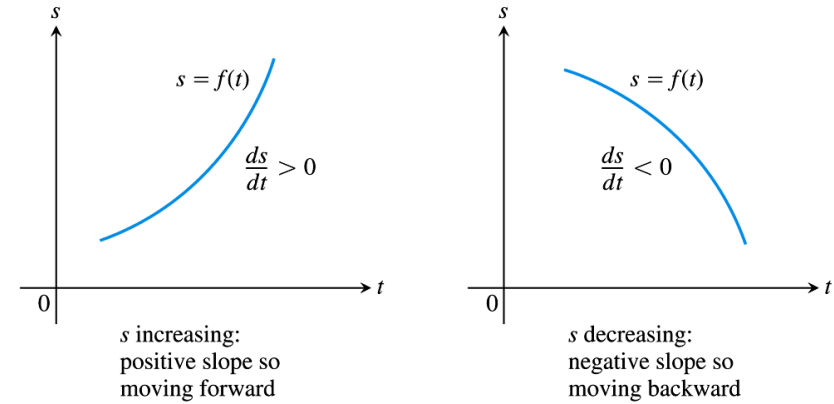
$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

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**FIGURE 3.13** The time-to-distance graph for Example 2. The slope of the tangent line at  $P$  is the instantaneous velocity at  $t = 2$  sec.



**FIGURE 3.14** For motion  $s = f(t)$  along a straight line,  $v = ds/dt$  is positive when  $s$  increases and negative when  $s$  decreases.

## Example 3

- Horizontal motion

### DEFINITION Speed

**Speed** is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left| \frac{ds}{dt} \right|$$

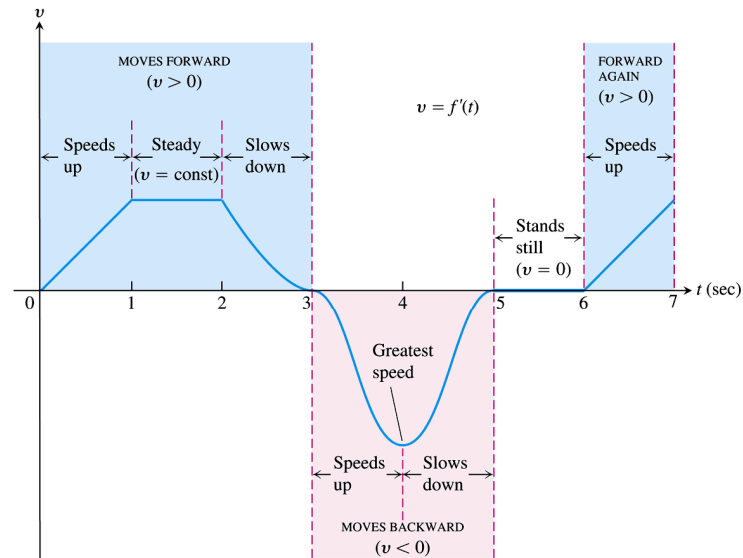


FIGURE 3.15 The velocity graph for Example 3.

**DEFINITIONS Acceleration, Jerk**

**Acceleration** is the derivative of velocity with respect to time. If a body's position at time  $t$  is  $s = f(t)$ , then the body's acceleration at time  $t$  is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

**Jerk** is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

## Example 4

- Modeling free fall  $s = \frac{1}{2}gt^2$
- Consider the free fall of a heavy ball released from rest at  $t = 0$  sec.
- (a) How many meters does the ball fall in the first 2 sec?
- (b) What is the velocity, speed and acceleration then?

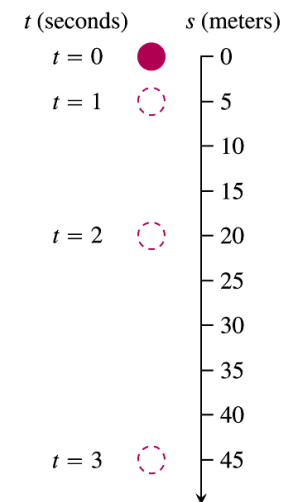


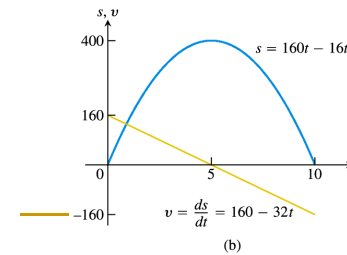
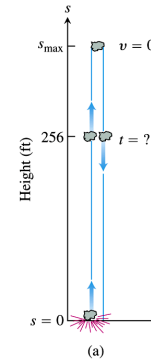
FIGURE 3.16 A ball bearing falling from rest (Example 4).

## Modeling vertical motion

- A dynamite blast blows a heavy rock straight up with a launch velocity of 160 m/sec. It reaches a height of  $s = 160t - 16t^2$  ft after  $t$  sec.
- (a) How high does the rock go?
- (b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?
- (c) What is the acceleration of the rock at any time  $t$  during its flight?
- (d) When does the rock hit the ground again?

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**FIGURE 3.17** (a) The rock in Example 5. (b) The graphs of  $s$  and  $v$  as functions of time;  $s$  is largest when  $v = ds/dt = 0$ . The graph of  $s$  is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

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## 3.4

### Derivatives of Trigonometric Functions

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## Derivative of the sine function

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

$$\frac{d}{dx} \sin x = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} = \dots$$

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## Derivative of the cosine function

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx} \cos x = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} = \dots$$

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## Example 2

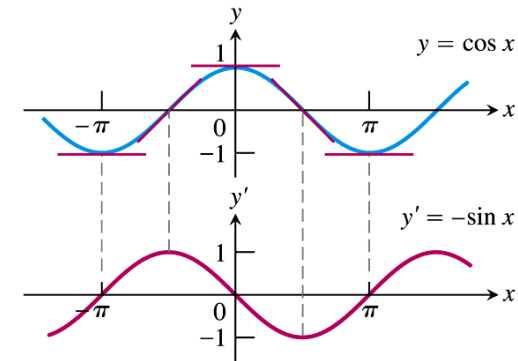
$$(a) y = 5x + \cos x$$

$$(b) y = \sin x \cos x$$

$$(c) y = \frac{\cos x}{1 - \sin x}$$

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**FIGURE 3.23** The curve  $y' = -\sin x$  as the graph of the slopes of the tangents to the curve  $y = \cos x$ .

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## Derivative of the other basic trigonometric functions

### Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

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## Example 5

- Find  $d(\tan x)/dx$

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## Example 6

- Find  $y''$  if  $y = \sec x$

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## Example 7: Finding a trigonometric limit

- Trigonometric functions are differentiable, hence are continuous throughout their domains.
- So we can calculate limits of algebraic combinations and composites of trigonometric functions by direct substitution.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{2 + \sec x}}{\cos(\pi - \tan x)} &= \frac{\sqrt{2 + \sec 0}}{\cos(\pi - \tan 0)} \\ &= \frac{\sqrt{2+1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}\end{aligned}$$

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- Note that you can only evaluate the limit of the form

$$\lim_{x \rightarrow x_0} \frac{P(x)}{Q(x)}$$

by direct substitution, i.e.,

$$\lim_{x \rightarrow x_0} \frac{P(x)}{Q(x)} = \frac{P(x_0)}{Q(x_0)}$$

only when  $P(x)$  and  $Q(x)$  are both continuous at

$x_0$

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## 3.5

### The Chain Rule and Parametric Equations

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## Derivative of a composite function

- Example 1: Relating derivatives
- $y = (3/2)x = (1/2)(3x)$
- $y = y[u(x)]$
- $y(u) = u/2; u(x) = 3x$
- $dy/dx = 3/2;$
- $dy/du = 1/2; du/dx = 3;$
- $dy/dx = (dy/du) \cdot (du/dx)$  (Not an accident)

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## Differentiating composite functions

- Example:
- $y = f(u) = \sin u$
- $u = g(x) = x^2 - 4$
- How to differentiate  $F(x) = f \circ g = f[g(x)]$ ?
- Use chain rule

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## Example 2

$$y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$$

$$y = u^2; u = 3x^2 + 1$$

$$\frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 6$$

$$= 2(3x^2 + 1) \cdot 6x = 36x^3 + 12x$$

c.f.

$$\frac{dy}{dx} = \frac{d}{dx}(9x^4 + 6x^2 + 1) = 36x^3 + 12x$$

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### THEOREM 3 The Chain Rule

If  $f(u)$  is differentiable at the point  $u = g(x)$  and  $g(x)$  is differentiable at  $x$ , then the composite function  $(f \circ g)(x) = f(g(x))$  is differentiable at  $x$ , and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if  $y = f(u)$  and  $u = g(x)$ , then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where  $dy/du$  is evaluated at  $u = g(x)$ .

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## Example 3

- Applying the chain rule
- $x(t) = \cos(t^2 + 1)$ . Find  $dx/dt$ .
- Solution:
- $x(u) = \cos(u)$ ;  $u(t) = t^2 + 1$ ;
- $dx/dt = (dx/du) \cdot (du/dt) = \dots$

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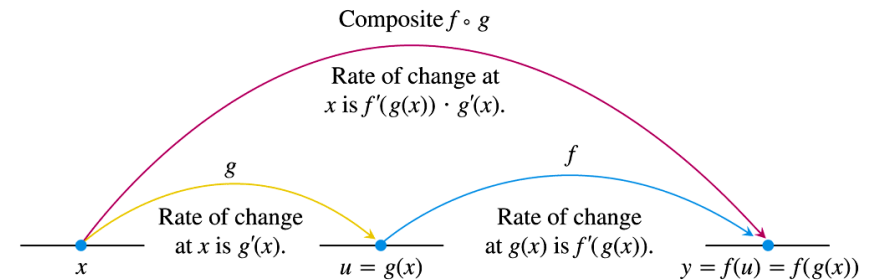


FIGURE 3.27 Rates of change multiply: The derivative of  $f \circ g$  at  $x$  is the derivative of  $f$  at  $g(x)$  times the derivative of  $g$  at  $x$ .

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## Alternative form of chain rule

- If  $y = f[g(x)]$ , then
- $dy/dx = f'[g(x)] \cdot g'(x)$
- Think of  $f$  as 'outside function',  $g$  as 'inside-function', then
- $dy/dx =$  differentiate the outside function and evaluate it at the inside function let alone; then multiply by the derivative of the inside function.

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## Example 4

- Differentiating from the outside in:

$$y = \sin(x^2 + x) = f(u) = f[g(x)]$$

$$\underbrace{f(u) = \sin u}_{\text{outside function}}; \quad \underbrace{g(x) = x^2 + x}_{\text{inside function}}$$

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x)$$

$$\frac{dy}{dx} = \cos \underbrace{(x^2 + x)}_{\substack{\text{inside function} \\ \text{left alone}}} \cdot \underbrace{(2x + 1)}_{\substack{\text{derivative of} \\ \text{the inside function}}}$$

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## Example 5

- A three-link 'chain'
- Find the derivative of  $g(t) = \tan(5 - \sin 2t)$

$$g'(t) = \frac{d}{dt} (\tan(5 - \sin 2t))$$

$$= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt} (5 - \sin 2t)$$

$$= \sec^2(5 - \sin 2t) \cdot \left( 0 - \cos 2t \cdot \frac{d}{dt} (2t) \right)$$

$$= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2$$

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## Example 6

- Applying the power chain rule

$$(a) \frac{d}{dx} (5x^3 - x^4)^7$$

$$(b) \frac{d}{dx} \left( \frac{1}{3x-2} \right) = \frac{d}{dx} (3x-2)^{-1}$$

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## Example 7

- (a) Find the slope of tangent to the curve  $y = \sin^5 x$  at the point where  $x = \pi/3$
- (b) Show that the slope of every line tangent to the curve  $y = 1/(1-2x)^3$  is positive

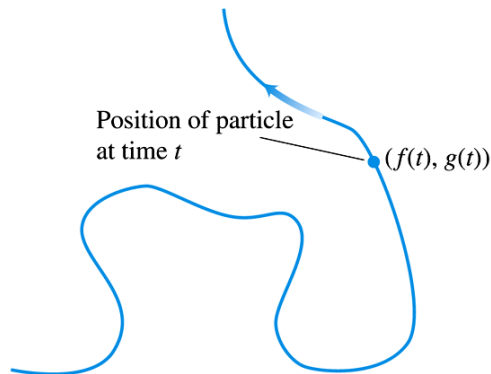
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# Parametric equations

- A way of expressing both the coordinates of a point on a curve,  $(x, y)$  as a function of a third variable,  $t$ .
- The path or locus traced by a point particle on a curve is then well described by a set of two equations:
- $x = f(t)$ ,  $y = g(t)$

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**FIGURE 3.29** The path traced by a particle moving in the  $xy$ -plane is not always the graph of a function of  $x$  or a function of  $y$ .

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## DEFINITION Parametric Curve

If  $x$  and  $y$  are given as functions

$$x = f(t), \quad y = g(t)$$

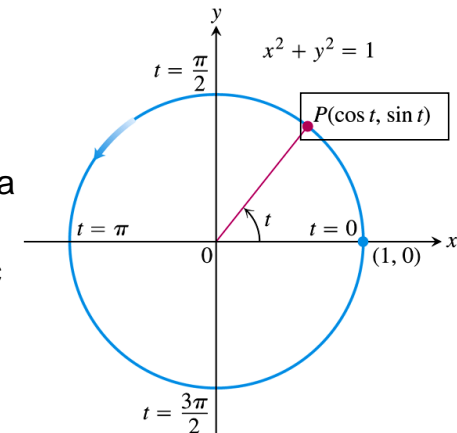
over an interval of  $t$ -values, then the set of points  $(x, y) = (f(t), g(t))$  defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

The variable  $t$  is a parameter for the curve

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## Example 9

- Moving counterclockwise on a circle
- Graph the parametric curves
- $x = \cos t$ ,  $y = \sin t$ ,  $0 \leq t \leq 2\pi$

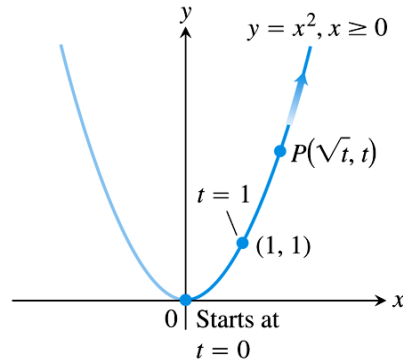


**FIGURE 3.30** The equations  $x = \cos t$  and  $y = \sin t$  describe motion on the circle  $x^2 + y^2 = 1$ . The arrow shows the direction of increasing  $t$  (Example 9).

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## Example 10

- Moving along a parabola
- $x = \sqrt{t}$ ,  $y = t$ ,  $0 \leq t$
- Determine the relation between  $x$  and  $y$  by eliminating  $t$ .
- $y = t = (x)^2 = x^2$
- The path traced out by  $P$  (the locus) is only half the parabola,  $x \geq 0$



**FIGURE 3.31** The equations  $x = \sqrt{t}$  and  $y = t$  and the interval  $t \geq 0$  describe the motion of a particle that traces the right-hand half of the parabola  $y = x^2$  (Example 10).

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### Parametric Formula for $dy/dx$

If all three derivatives exist and  $dx/dt \neq 0$ ,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad (2)$$

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## Slopes of parametrized curves

- A parametrized curve  $x = f(t)$ ,  $y = g(t)$  is differentiable at  $t$  if  $f$  and  $g$  are differentiable at  $t$ .
- At a point on a differentiable parametrized curve where  $y$  is also a differentiable function of  $x$ , i.e.  $y = y(x) = y[x(t)]$ ,
- chain rule relates  $dx/dt$ ,  $dy/dt$ ,  $dy/dx$  via

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

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## Example 12

- Differentiating with a parameter
- If  $x = 2t + 3$  and  $y = t^2 - 1$ , find the value of  $dy/dx$  at  $t = 6$ .

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### Parametric Formula for $d^2y/dx^2$

If the equations  $x = f(t)$ ,  $y = g(t)$  define  $y$  as a twice-differentiable function of  $x$ , then at any point where  $dx/dt \neq 0$ ,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \quad (3)$$

(3) is just the parametric formula (2) by

$$y \rightarrow y' = dy/dx$$

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## 3.6

### Implicit Differentiation

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## Example 14 Finding $d^2y/dx^2$ for a parametrised curve

- Find  $d^2y/dx^2$  as a function of  $t$  if  $x = t - t^2$ ,  
 $y = t - t^3$ .

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### Implicit Differentiation

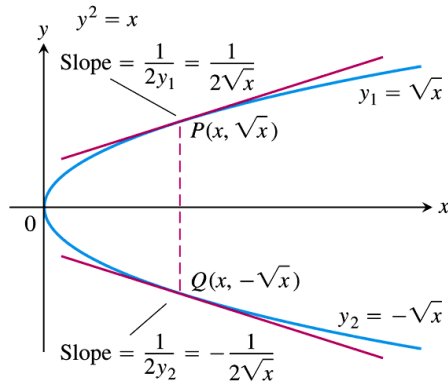
1. Differentiate both sides of the equation with respect to  $x$ , treating  $y$  as a differentiable function of  $x$ .
2. Collect the terms with  $dy/dx$  on one side of the equation.
3. Solve for  $dy/dx$ .

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## Example 1: Differentiating implicitly

- Find  $dy/dx$  if  $y^2 = x$



**FIGURE 3.37** The equation  $y^2 - x = 0$ , or  $y^2 = x$  as it is usually written, defines two differentiable functions of  $x$  on the interval  $x \geq 0$ . Example 1 shows how to find the derivatives of these functions without solving the equation  $y^2 = x$  for  $y$ .

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## Example 2

- Slope of a circle at a point
- Find the slope of circle  $x^2 + y^2 = 25$  at  $(3, -4)$

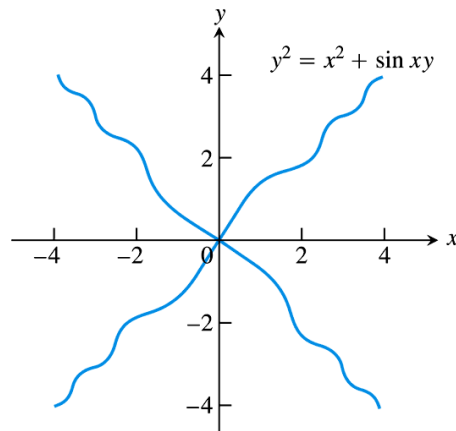
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## Example 3

- Differentiating implicitly
- Find  $dy/dx$  if  $y^2 = x^2 + \sin xy$



**FIGURE 3.39** The graph of  $y^2 = x^2 + \sin xy$  in Example 3. The example shows how to find slopes on this implicitly defined curve.

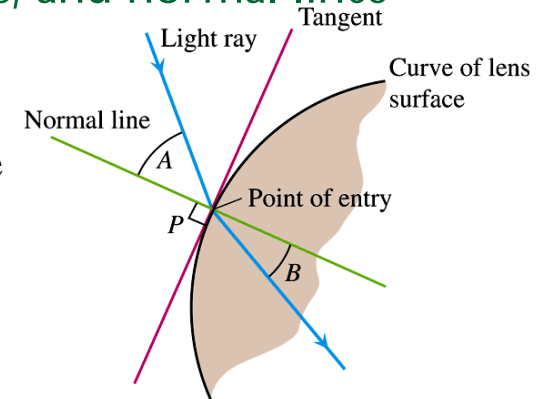
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## Lenses, tangents, and normal lines

If slope of tangent is  $m_t$ , the slope of normal,  $m_n$ , is given by the relation

$$m_n m_t = -1, \text{ or}$$

$$m_n = -1/m_t$$



**FIGURE 3.40** The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

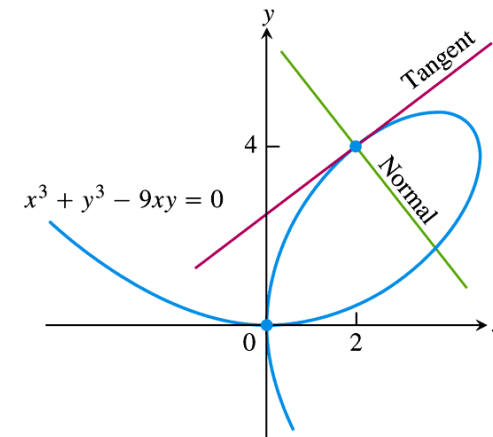
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## Example 4

- Tangent and normal to the folium of Descartes
- Show that the point (2,4) lies on the curve  $x^2 + y^3 - 9xy = 0$ . Then find the tangent and normal to the curve there.



**FIGURE 3.41** Example 4 shows how to find equations for the tangent and normal to the folium of Descartes at (2, 4).

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## Derivative of higher order

- Example 5
- Finding a second derivative implicitly
- Find  $d^2y/dx^2$  if  $2x^3 - 3y^2 = 8$ .

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## Rational powers of differentiable functions

### **THEOREM 4** Power Rule for Rational Powers

If  $p/q$  is a rational number, then  $x^{p/q}$  is differentiable at every interior point of the domain of  $x^{(p/q)-1}$ , and

$$\frac{d}{dx} x^{p/q} = \frac{p}{q} x^{(p/q)-1}.$$

Theorem 4 is proved based on  $d/dx(x^n) = nx^{n-1}$  (where  $n$  is an integer) using implicit differentiation

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- Theorem 4 provide a extension of the power chain rule to rational power:

$$\frac{d}{dx} u^{p/q} = \frac{p}{q} u^{(p/q)-1} \frac{du}{dx}$$

- $u \neq 0$  if  $(p/q) < 1$ ,  $(p/q)$  rational number,  $u$  a differential function of  $x$

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## Proof of Theorem 4

- Let  $p$  and  $q$  be integers with  $q > 0$  and

$$y = x^{p/q} \equiv y^q = x^p$$

- Explicitly differentiating both sides with respect to  $x$ ...

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## Example 6

- Using the rational power rule
  - (a)  $d/dx (x^{1/2}) = 1/2x^{-1/2}$  for  $x > 0$
  - (b)  $d/dx (x^{2/3}) = 2/3 x^{-1/3}$  for  $x \neq 0$
  - (c)  $d/dx (x^{-4/3}) = -4/3 x^{-7/3}$  for  $x \neq 0$

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## Example 7

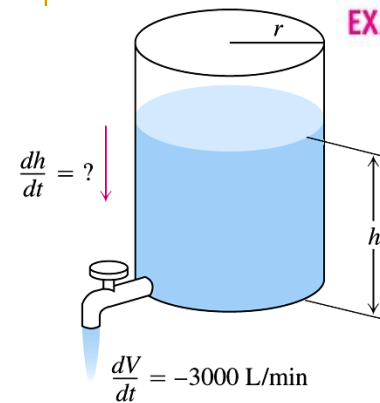
- Using the rational power and chain rules
  - (a) Differentiate  $(1-x^2)^{1/4}$
  - (b) Differentiate  $(\cos x)^{-1/5}$

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# 3.7

## Related Rates



### EXAMPLE 1 Pumping Out a Tank

How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L / min?

FIGURE 3.42 The rate of change of fluid volume in a cylindrical tank is related to the rate of change of fluid level in the tank (Example 1).

## Solution

Geometrically, Volume  $V$ , is a function of height  $h$ ,  $V=V(h)$   
Height,  $h$ , is a function of time,  $h=h(t)$ .  $r$ , radius, is fixed.

Combining both,  $V=V[r(t)]$

By chain rule, the derivative of  $V$  with respect to  $t$  is

$$\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} \longrightarrow \frac{dh}{dt} = \frac{dV}{dt} / \frac{dV}{dh}$$

We are asked to find  $\frac{dh}{dt}$ . given  $\frac{dV}{dt} = -3000 \text{ L/min}$

## Solution

$$V = \pi r^2 h \Rightarrow \frac{dV}{dh} = \pi r^2 \quad \frac{dV}{dt} = -3000 \text{ L/min}$$

$$\frac{dh}{dt} = \frac{dV}{dt} / \frac{dV}{dh} \longrightarrow \frac{dh}{dt} = \frac{-3000 \text{ L}}{\text{min}} \cdot \frac{1}{\pi r^2}$$

In this example, conversion of unit must be taken care of properly

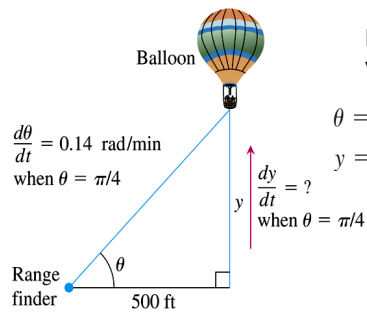
$$1 \text{ m}^3 = 1000 \text{ L}$$

$$\text{If } r = 1 \text{ m, } \frac{dh}{dt} = \frac{-3000(10^{-3} \text{ m}^3)}{\text{min}} \cdot \frac{1}{\pi (1 \text{ m})^2} = -\frac{3 \text{ m}}{\pi \text{ min}}$$

$$\text{If } r = 10 \text{ m, } \frac{dh}{dt} = \frac{-3000(10^{-3} \text{ m}^3)}{\text{min}} \cdot \frac{1}{\pi (10 \text{ m})^2} = -\frac{3 \text{ m}}{100\pi \text{ min}}$$

## EXAMPLE 2 A Rising Balloon

A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is  $\pi/4$ , the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?



Draw the scenario and label the relevant variables (and name them)

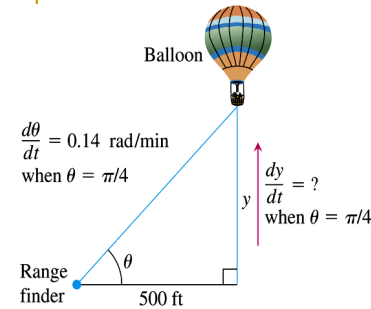
$\theta$  = the angle in radians the range finder makes with the ground.

$y$  = the height in feet of the balloon.

**FIGURE 3.43** The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

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## EXAMPLE 2 A Rising Balloon



Geometrically,  $y$  is a function of angle  $\theta$ .

$$y = x \tan \theta$$

$\theta$  is a function of time,  $\theta = \theta(t)$ .

$x$ , the horizontal distance, is fixed.

Combining both,  $y = y[\theta(t)]$

By chain rule, the derivative of  $y$  with respect to  $t$  is

$$\frac{dy}{dt} = \frac{dy}{d\theta} \frac{d\theta}{dt}$$

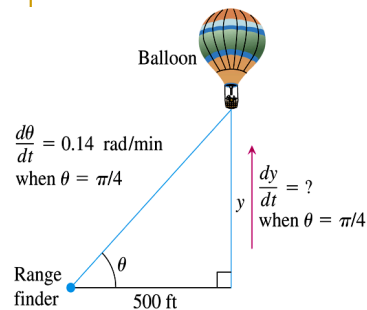
**FIGURE 3.43** The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

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## EXAMPLE 2 A Rising Balloon



$$y = x \tan \theta \quad \frac{dy}{d\theta} = x \sec^2 \theta$$

$$\frac{dy}{dt} = x \sec^2 \theta \frac{d\theta}{dt}$$

$$\text{Given } \frac{d\theta}{dt} = 0.14 \text{ (rad/min)}$$

at  $\theta = \pi/4$ ,

$$\frac{dy}{dt} = \frac{500 \text{ ft} \cdot \sec^2 \frac{\pi}{4}}{(0.14 \text{ rad/min})}$$

$$= 500 \text{ ft} \cdot (\sqrt{2})^2 \cdot (0.14 \text{ rad/min}) = 140 \text{ ft/min}$$

113 Note: radian is dimensionless (hence unit-less)

# 3.8

## Linearization and differentials

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## Linearization

- Say you have a very complicated function,  $f(x) = \sin(\cot^2 x)$ , and you want to calculate the value of  $f(x)$  at  $x = \pi/2 + \delta$ , where  $\delta$  is a very tiny number. The value sought can be estimated within some accuracy using linearization.

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## Refer to graph Figure 3.47.

- The point-slope equation of the tangent line passing through the point  $(a, f(a))$  on a differentiable function  $f$  at  $x=a$  is
- $y = mx + c$ , where  $c$  is  $c = f(a) - f'(a)a$
- Hence the tangent line is the graph of the linear function
- $$L(x) = f'(a)x + f(a) - af'(a)$$
$$= f(a) + f'(a)(x - a)$$

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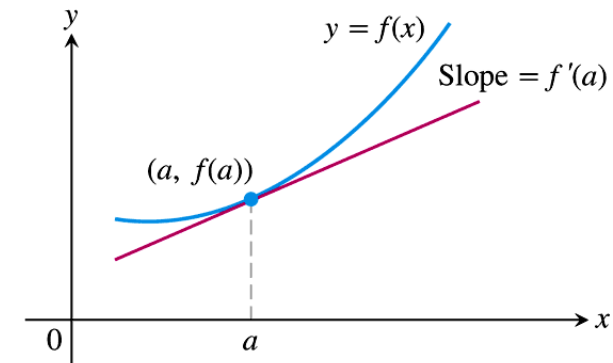
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## Definitions

- The tangent line  $L(x) = f(a) + f'(a)(x - a)$  gives a good approximation to  $f(x)$  as long as  $x$  is not too far away from  $x=a$ .
- Or in other words, we say that  $L(x)$  is the linearization of  $f$  at  $a$ .
- The approximation  $f(x) \approx L(x)$  of  $f$  by  $L$  is the standard linear approximation of  $f$  at  $a$ .
- The point  $x = a$  is the center of the approximation.

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**FIGURE 3.47** The tangent to the curve  $y = f(x)$  at  $x = a$  is the line  $L(x) = f(a) + f'(a)(x - a)$ .

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## Example 1 Finding Linearization

- Find the linearization of

$$f(x) = \sqrt{1+x}$$

- at  $x = 0$ .

$$f(x) = \sqrt{1+x}$$

$$f'(x) = \frac{1}{2(1+x)^{1/2}}$$

The linearization of  $f(x)$  at  $x = a$  is

$$f(x) \approx f(a) + f'(a)(x-a) = (1+a)^{1/2} + \frac{1}{2(1+a)^{1/2}}(x-a)$$

$a = 0$ ,

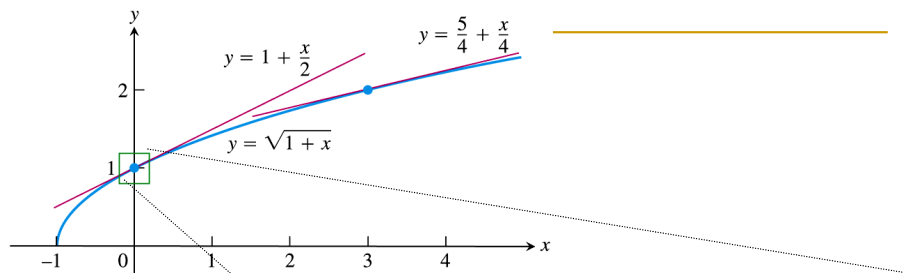
$$f'(0) = \frac{1}{2}; f(0) = 1;$$

The linearization of  $f(x)$  at  $x = a = 0$  is  $L(x) = 1 + x/2$

We write  $f(x) \approx L(x) = 1 + x/2$

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**FIGURE 3.48** The graph of  $y = \sqrt{1+x}$  and its linearizations at  $x = 0$  and  $x = 3$ . Figure 3.49 shows a magnified view of the small window about 1 on the y-axis.

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## Accuracy of the linearized approximation

- We find that the approximation of  $f(x)$  by  $L(x)$  gets worsened as  $|x - a|$  increases (or in other words,  $x$  gets further away from  $a$ ).

Approximation	True value	True value - approximation
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$< 10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$< 10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$< 10^{-5}$

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## What $\frac{dy}{dx}$ is not

- Note that the derivative notation  $\frac{dy}{dx}$  is not a ratio
- i.e. the derivative of the function  $y = y(x)$  with respect to  $x$ , is not to be understood as the ratio of two values, namely,  $dy$  and  $dx$ .
- $dy/dx$  here denotes the a new quantity derived from  $y$  when the operation  $D = d/dx$  is performed on the function  $y$ ,  
 $(d/dx)[y] = D [y]$

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## Differential

- Definition:
- Let  $y = f(x)$  be a differentiable function. The differential  $dy$  is
  - $dy = f'(x)dx$
- $dy$  is an dependent variable, i.e., the value of  $dy$  depends on  $f'(x)$  and  $dx$  where  $dx$  is viewed as an independent variable.
- Once  $f'(x)$  and  $dx$  is fixed, then the value of differential  $dy$  can be calculated.

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## Example 4 Finding the differential $dy$

- (a) Find  $dy$  if  $y=x^5 + 37x$ .
- (b) Find the value of  $dy$  when  $x=1$  and  $dx = 0.2$

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## $dy \div dx = f'(x)$

- Referring to the definition of the differentials  $dy$  and  $dx$ , if we take the ratio of  $dy$  and  $dx$ , i.e.  $dy \div dx$ , we get
$$dy \div dx = f'(x) dx / dx = f'(x) \equiv \frac{dy}{dx}$$
- In other words, the ratio of the differential  $dy$  and  $dx$  is equal to the derivative by definition.

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## Differential of $f$ , $df$

- We sometimes use the notation  $df$  in place of  $dy$ , so that

$$dy = f'(x) dx$$

is now written in terms of

- $df = f'(x) dx$
- $df$  is called the differential of  $f$

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## Example of differential of $f$

- If  $y = f(x) = 3x^2 - 6$ , then the differential of  $f$  is

$$df = f'(x) dx = 6x dx$$

- Note that in the above expression, if we take the ratio  $df / dx$ , we obtain

- $df / dx = f'(x) = 6x$

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## The differential form of a function

- For every differentiable function  $y=f(x)$ , we can obtain its derivative,  $\frac{dy}{dx}$

- Corresponds to every derivative  $\frac{dy}{dx}$  there is a differential  $df$  such that

$$df = \left( \frac{dy}{dx} \right) \cdot dx$$

In addition, if  $f = u + v$ , then  $df = \left( \frac{du}{dx} \right) \cdot dx + \left( \frac{dv}{dx} \right) \cdot dx$

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## Example 5

- If  $y = f(x) = \tan 2x$ , the derivative is

$$\frac{dy}{dx} = 2 \sec^2 2x$$

- Correspond to the derivative, the differential of the function,  $df$ , is given by the product of the derivative  $dy/dx$  and the independent differential  $dx$ :

$$df = d(\tan 2x) = \left( \frac{dy}{dx} \right) \cdot dx = (2 \sec^2 2x) \cdot dx$$

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## Example 5

- If  $y = f(x) = \tan 2x$ , then the differential form of the function,

$$y = f(x) = \left(\frac{x}{x+1}\right)$$

$$\begin{aligned} df &= d\left(\frac{x}{x+1}\right) = \left(\frac{dy}{dx}\right) \cdot dx = \frac{(x+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x+1)}{(x+1)^2} \cdot dx \\ &= \frac{(x+1)\left[\frac{d}{dx}(x) \cdot dx\right] - x\left[\frac{d}{dx}(x+1) \cdot dx\right]}{(x+1)^2} = \frac{(x+1)[dx] - x[d(x+1)]}{(x+1)^2} \\ &= \frac{(x+1)dx - xd(x+1)}{(x+1)^2} = \frac{(x+1)dx - x\left[\frac{d}{dx}(x+1) \cdot dx\right]}{(x+1)^2} = \frac{(x+1)dx - x[1 \cdot dx]}{(x+1)^2} = \frac{dx}{(x+1)^2} \end{aligned}$$

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## The derivative $dy/dx$ is not $dy$ divided by $dx$

- Due to the definition of the differentials  $dy$ ,  $dx$  that their ratio,  $dy/dx$  equals to the derivative of the differentiable function  $y = f(x)$ , i.e.  $\frac{dy}{dx} = \frac{d}{dx}(y) \equiv f'(x)$
- we can then move the differential  $dy$  or  $dx$  around, such as  $dy = f'(x)dx$
- When we do so, we need to be reminded that  $dy$  and  $dx$  are differentials, a pair of variables, instead of thinking that the derivative  $\frac{d}{dx}y$  is made up of a numerator " $dy$ " and a denominator " $dx$ " that are separable

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## $dy$ , $df$ : any difference?

- Sometimes for a given function,  $y = f(x)$ , the notation  $dy$  is used in place of the notation  $df$ .
- Operationally speaking, it does not matter whether one uses  $dy$  or  $df$ .

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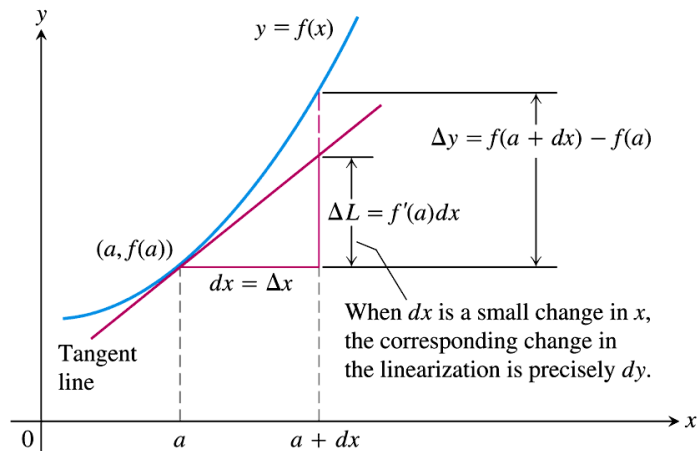
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## Estimation with differential

- Referring to figure 3.51, geometrically, one can see that if  $x$ , originally at  $x=a$ , changes by  $dx$  (where  $dx$  is an independent variable, the differential of  $x$ ),  $f(a)$  will change by
$$\Delta y = f(a+dx) - f(a)$$
- $\Delta y$  can be approximated by the change of the linearization of  $f$  at  $x=a$ ,  $L(x)=f(a)+f'(a)(x-a)$ ,
$$\Delta y \approx \Delta L = L(a+dx)-L(a) = f'(a)dx = df(a)$$

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**FIGURE 3.51** Geometrically, the differential  $dy$  is the change  $\Delta L$  in the linearization of  $f$  when  $x = a$  changes by an amount  $dx = \Delta x$ .

## $\Delta y \approx dy$ allows estimation of $f(a+dx)$

- In other words,  $\Delta y$  centered around  $x=a$  is approximated by  $df(a)$  ( $\equiv dy$ , where the differential is evaluated at  $x=a$ ):

$$\Delta y \approx dy$$

- or equivalently,

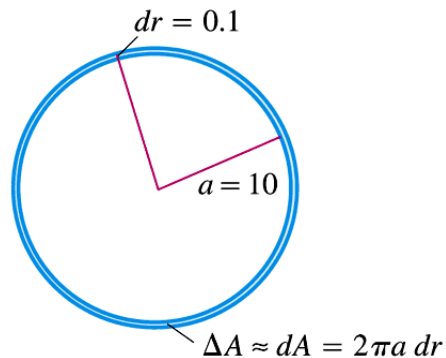
$$\Delta y = f(a+dx) - f(a) \approx dy = f'(a)dx$$

- This also allows us to estimate the value of  $f(a+dx)$  if  $f'(a)$ ,  $f(a)$  are known, and  $dx$  is not too large, via

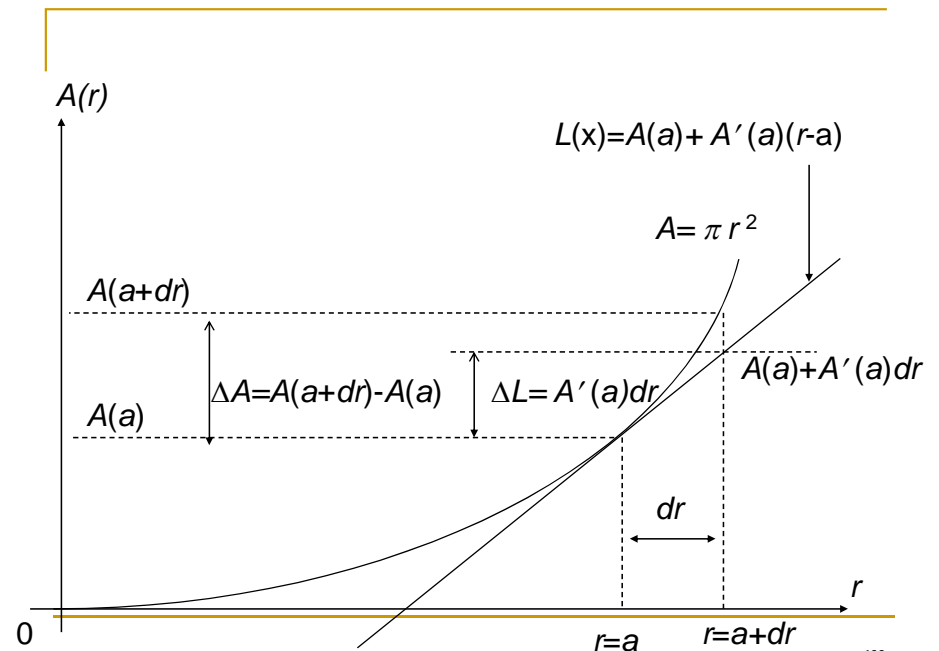
$$f(a+dx) \approx f(a) + f'(a)dx$$

## Example 6

- Figure 3.52
- The radius  $r$  of a circle increases from  $a=10$  m to 10.1 m. Use  $dA$  to estimate the increase in circle's area  $A$ . Estimate the area of the enlarged circle and compare your estimate to your true value.



**FIGURE 3.52** When  $dr$  is small compared with  $a$ , as it is when  $dr = 0.1$  and  $a = 10$ , the differential  $dA = 2\pi a dr$  gives a way to estimate the area of the circle with radius  $r = a + dr$  (Example 6).



## Solution to example 6

- Let  $a = 10$  m,  $a+dr = 10.1$  m  $\Rightarrow dr = 0.1$  m
- $A(r) = \pi r^2 \Rightarrow A(a) = \pi(10 \text{ m})^2 = 100\pi \text{ cm}^2$
- $\Delta A \approx A'(a)dr = 2\pi(a)dr = 2\pi(10 \text{ m})(0.1 \text{ m}) = 2\pi \text{ m}^2$ .
- $A(a+dr) = A(a) + \Delta A \approx A(a) + A'(a)dr = 102\pi \text{ m}^2$  (this is an estimation)
- c.f the true area is  $\pi(a+dr)^2 = \pi(10.1)^2 = 102.01\pi \text{ m}^2$

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## 4.1

### Extreme Values of Functions

2

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## Chapter 4

### Applications of Derivatives

1

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#### DEFINITIONS Absolute Maximum, Absolute Minimum

Let  $f$  be a function with domain  $D$ . Then  $f$  has an **absolute maximum** value on  $D$  at a point  $c$  if

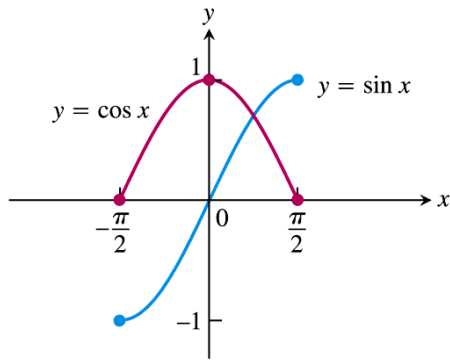
$$f(x) \leq f(c) \quad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on  $D$  at  $c$  if

$$f(x) \geq f(c) \quad \text{for all } x \text{ in } D.$$

3

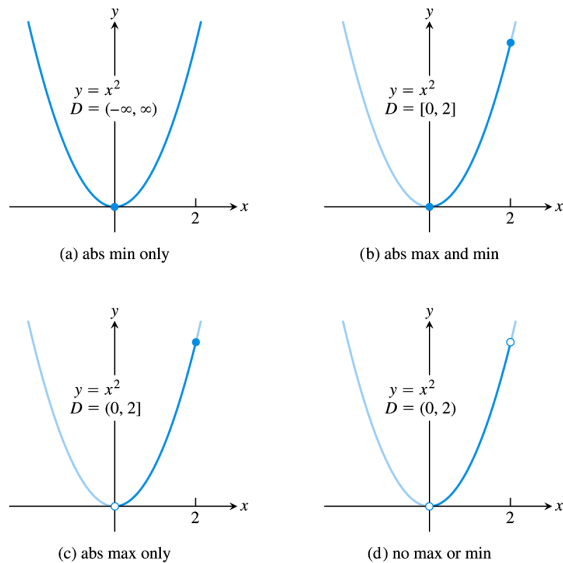
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**FIGURE 4.1** Absolute extrema for the sine and cosine functions on  $[-\pi/2, \pi/2]$ . These values can depend on the domain of a function.

## Example 1

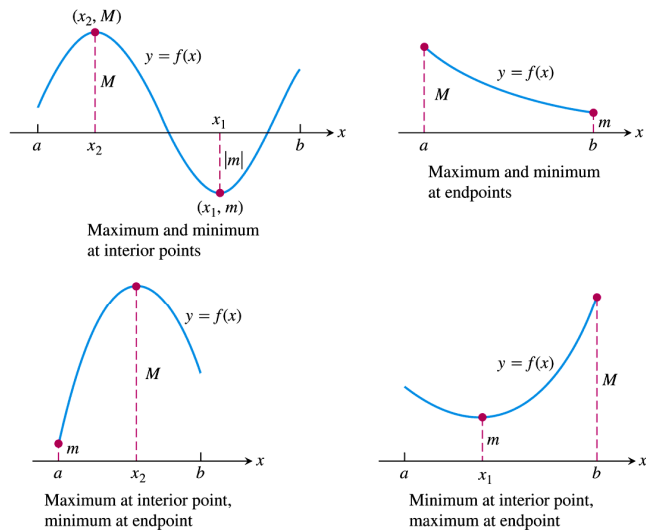
- Exploring absolute extrema
- The absolute extrema of the following functions on their domains can be seen in Figure 4.2



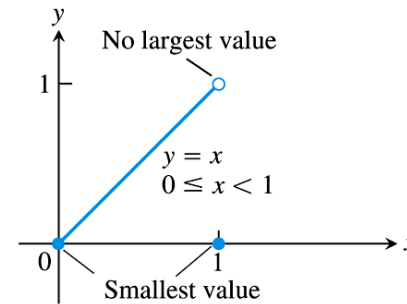
**FIGURE 4.2** Graphs for Example 1.

### **THEOREM 1** The Extreme Value Theorem

If  $f$  is continuous on a closed interval  $[a, b]$ , then  $f$  attains both an absolute maximum value  $M$  and an absolute minimum value  $m$  in  $[a, b]$ . That is, there are numbers  $x_1$  and  $x_2$  in  $[a, b]$  with  $f(x_1) = m$ ,  $f(x_2) = M$ , and  $m \leq f(x) \leq M$  for every other  $x$  in  $[a, b]$  (Figure 4.3).



**FIGURE 4.3** Some possibilities for a continuous function's maximum and minimum on a closed interval  $[a, b]$ .

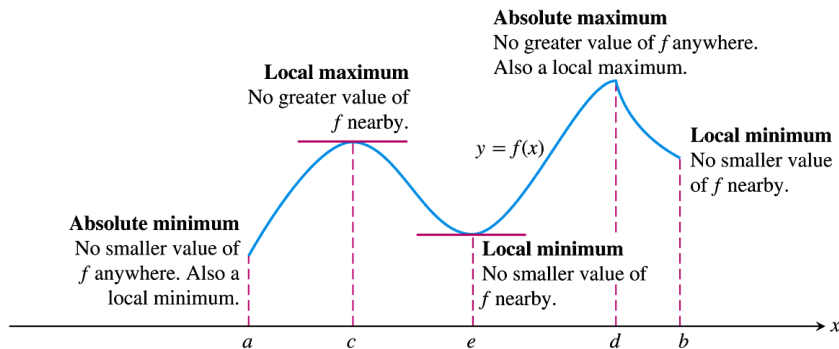


**FIGURE 4.4** Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of  $[0, 1]$  except  $x = 1$ , yet its graph over  $[0, 1]$  does not have a highest point.

## Local (relative) extreme values



**FIGURE 4.5** How to classify maxima and minima.

### DEFINITIONS Local Maximum, Local Minimum

A function  $f$  has a **local maximum** value at an interior point  $c$  of its domain if

$$f(x) \leq f(c) \quad \text{for all } x \text{ in some open interval containing } c.$$

A function  $f$  has a **local minimum** value at an interior point  $c$  of its domain if

$$f(x) \geq f(c) \quad \text{for all } x \text{ in some open interval containing } c.$$

## Finding Extrema...with a not-always-effective method.

### THEOREM 2 The First Derivative Theorem for Local Extreme Values

If  $f$  has a local maximum or minimum value at an interior point  $c$  of its domain, and if  $f'$  is defined at  $c$ , then

$$f'(c) = 0.$$

Be careful not to misinterpret theorem 2 because its converse is false. A differentiable function may have a critical point at  $x = c$  without having a local extreme value there. E.g. at point  $x = 0$  of function  $y = x^3$ .

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### DEFINITION Critical Point

An interior point of the domain of a function  $f$  where  $f'$  is zero or undefined is a **critical point** of  $f$ .

How to find the absolute extrema of a continuous function  $f$  on a finite closed interval

1. Evaluate  $f$  at all critical point and endpoints
2. Take the largest and smallest of these values.

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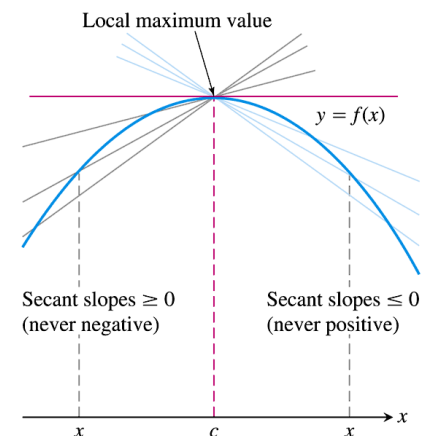


FIGURE 4.6 A curve with a local maximum value. The slope at  $c$ , simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

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## Example 2: Finding absolute extrema

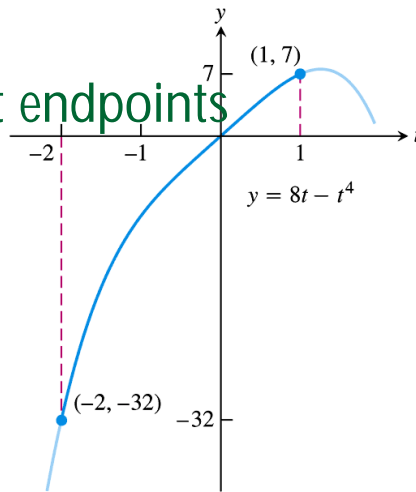
- Find the absolute maximum and minimum of  $f(x) = x^2$  on  $[-2, 1]$ .

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### Example 3: Absolute extrema at endpoints

- Find the absolute extrema values of  $g(t) = 8t - t^4$  on  $[-2, 1]$ .



**FIGURE 4.7** The extreme values of  $g(t) = 8t - t^4$  on  $[-2, 1]$  (Example 3).

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### Example 4: Finding absolute extrema on a closed interval

- Find the absolute maximum and minimum values of  $f(x) = x^{2/3}$  on the interval  $[-2, 3]$ .

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

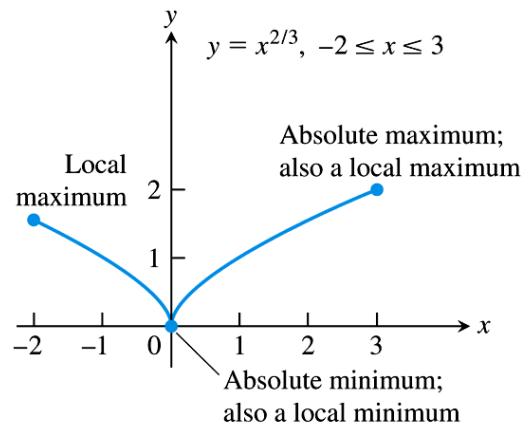
has no zeros but is undefined at the interior point  $x = 0$

The point  $(0, f(0))$  is a critical point by definition

17

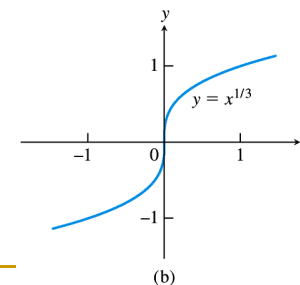
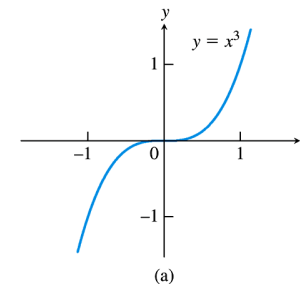
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**FIGURE 4.8** The extreme values of  $f(x) = x^{2/3}$  on  $[-2, 3]$  occur at  $x = 0$  and  $x = 3$  (Example 4).

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- Not every critical point or endpoints signals the presence of an extreme value.

**FIGURE 4.9** Critical points without extreme values. (a)  $y' = 3x^2$  is 0 at  $x = 0$ , but  $y = x^3$  has no extremum there. (b)  $y' = (1/3)x^{-2/3}$  is undefined at  $x = 0$ , but  $y = x^{1/3}$  has no extremum there.

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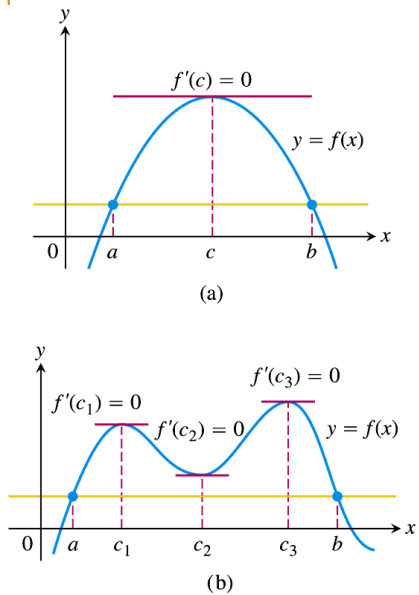
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# 4.2

## The Mean Value Theorem



**FIGURE 4.10** Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

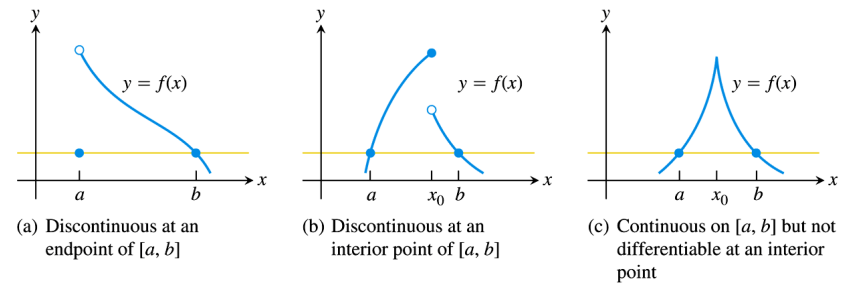
### THEOREM 3 Rolle's Theorem

Suppose that  $y = f(x)$  is continuous at every point of the closed interval  $[a, b]$  and differentiable at every point of its interior  $(a, b)$ . If

$$f(a) = f(b),$$

then there is at least one number  $c$  in  $(a, b)$  at which

$$f'(c) = 0.$$

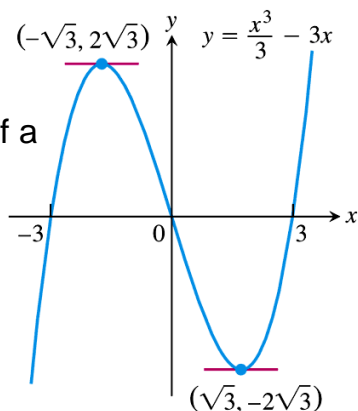


**FIGURE 4.11** There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.

## Example 1

- Horizontal tangents of a cubic polynomial

$$f(x) = \frac{x^3}{3} - 3x$$



**FIGURE 4.12** As predicted by Rolle's Theorem, this curve has horizontal tangents between the points where it crosses the  $x$ -axis (Example 1).

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## Example 2 Solution of an equation $f(x)=0$

- Show that the equation

$$x^3 + 3x + 1 = 0$$

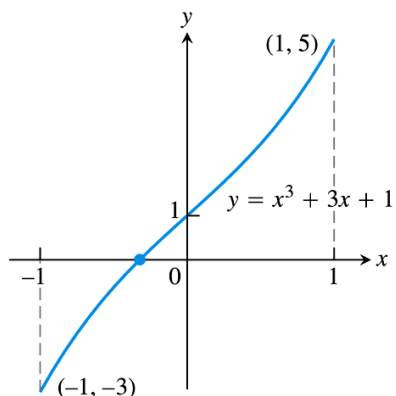
has exactly one real solution.

### Solution

- Apply Intermediate value theorem to show that there exist at least one root
- Apply Rolle's theorem to prove the uniqueness of the root.

$$f'(x) = 3x^2 + 3 \text{ is never zero}$$

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**FIGURE 4.13** The only real zero of the polynomial  $y = x^3 + 3x + 1$  is the one shown here where the curve crosses the  $x$ -axis between  $-1$  and  $0$  (Example 2).

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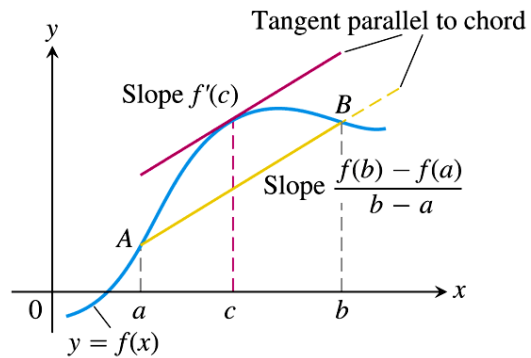
## The mean value theorem

### THEOREM 4 The Mean Value Theorem

Suppose  $y = f(x)$  is continuous on a closed interval  $[a, b]$  and differentiable on the interval's interior  $(a, b)$ . Then there is at least one point  $c$  in  $(a, b)$  at which

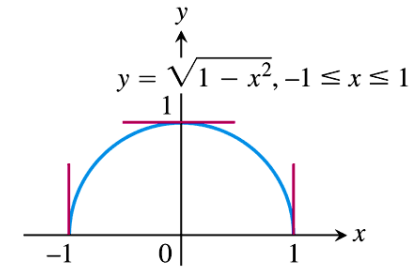
$$\frac{f(b) - f(a)}{b - a} = f'(c). \quad (1)$$

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**FIGURE 4.14** Geometrically, the Mean Value Theorem says that somewhere between  $A$  and  $B$  the curve has at least one tangent parallel to chord  $AB$ .

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**FIGURE 4.17** The function  $f(x) = \sqrt{1 - x^2}$  satisfies the hypotheses (and conclusion) of the Mean Value Theorem on  $[-1, 1]$  even though  $f$  is not differentiable at  $-1$  and  $1$ .

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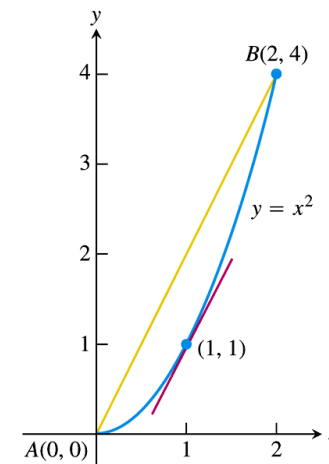
### Example 3

- The function  $f(x) = x^2$  is continuous for  $0 \leq x \leq 2$  and differentiable for  $0 < x < 2$ .

At some point  $c$  in the interval  $0 < x < 2$  the derivative  $f'(x) = 2x$  must have the value  $(4 - 0)/(2 - 0) = 2$ .

In this case,  $f'(c) = 2c = 2$ .

That is, at  $x = c = 1$ ,  $f'(c)$  is the slope of the chord  $AB$  (see Figure 4.18)



**FIGURE 4.18** As we find in Example 3,  $c = 1$  is where the tangent is parallel to the chord.

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## Mathematical consequences

### COROLLARY 1 Functions with Zero Derivatives Are Constant

If  $f'(x) = 0$  at each point  $x$  of an open interval  $(a, b)$ , then  $f(x) = C$  for all  $x \in (a, b)$ , where  $C$  is a constant.

### COROLLARY 2 Functions with the Same Derivative Differ by a Constant

If  $f'(x) = g'(x)$  at each point  $x$  in an open interval  $(a, b)$ , then there exists a constant  $C$  such that  $f(x) = g(x) + C$  for all  $x \in (a, b)$ . That is,  $f - g$  is a constant on  $(a, b)$ .

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## Proof of Corollary 2

- At each point  $x \in (a, b)$  the derivative of the difference between function  $h = f - g$  is
$$h'(x) = f'(x) - g'(x) = 0 \text{ (because } f'(x) = g'(x)\text{)}$$
- Thus  $h(x) = C$  on  $(a, b)$  by Corollary 1. That is  $f(x) - g(x) = C$  on  $(a, b)$ , so
$$f(x) = C + g(x).$$

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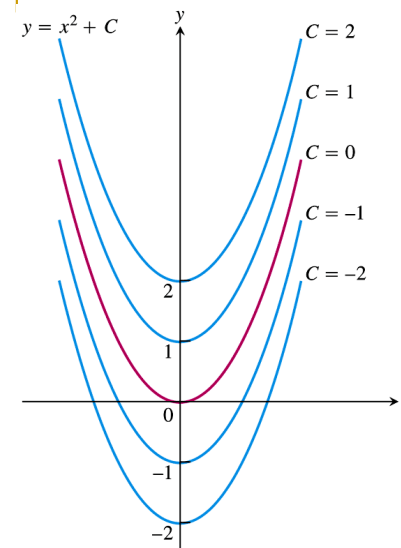
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## Corollary 1 can be proven using the Mean Value Theorem

- Say  $x_1, x_2 \in (a, b)$  such that  $x_1 < x_2$
- By the MVT on  $[x_1, x_2]$  there exist some point  $c$  between  $x_1$  and  $x_2$  such that
$$f'(c) = [f(x_2) - f(x_1)] / (x_2 - x_1)$$
- Since  $f'(c) = 0$  for all  $c$  lying in  $(a, b)$ ,
$$f(x_2) - f(x_1) = 0, \text{ hence } f(x_2) = f(x_1) \text{ for } x_1, x_2 \in (a, b).$$
- This is equivalent to  $f(x) = \text{a constant for } x \in (a, b)$ .

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**FIGURE 4.20** From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives on an interval can differ only by a vertical shift there. The graphs of the functions with derivative  $2x$  are the parabolas  $y = x^2 + C$ , shown here for selected values of  $C$ .

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## Example 5

- Find the function  $f(x)$  whose derivative is  $\sin x$  and whose graph passes through the point  $(0, 2)$ .

**Solution** Since  $f(x)$  has the same derivative as  $g(x) = -\cos x$ , we know that  $f(x) = -\cos x + C$  for some constant  $C$ . The value of  $C$  can be determined from the condition that  $f(0) = 2$  (the graph of  $f$  passes through  $(0, 2)$ ):

$$f(0) = -\cos(0) + C = 2, \quad \text{so} \quad C = 3.$$

The function is  $f(x) = -\cos x + 3$ . ■

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## Increasing functions and decreasing functions

### DEFINITIONS Increasing, Decreasing Function

Let  $f$  be a function defined on an interval  $I$  and let  $x_1$  and  $x_2$  be any two points in  $I$ .

- If  $f(x_1) < f(x_2)$  whenever  $x_1 < x_2$ , then  $f$  is said to be **increasing** on  $I$ .
- If  $f(x_2) < f(x_1)$  whenever  $x_1 < x_2$ , then  $f$  is said to be **decreasing** on  $I$ .

A function that is increasing or decreasing on  $I$  is called **monotonic** on  $I$ .

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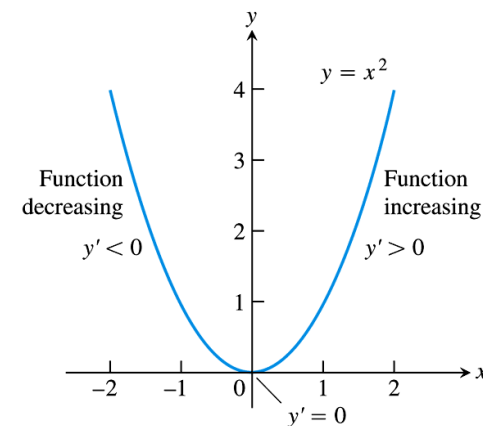
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## 4.3

### Monotonic Functions and The First Derivative Test

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**FIGURE 4.21** The function  $f(x) = x^2$  is monotonic on the intervals  $(-\infty, 0]$  and  $[0, \infty)$ , but it is not monotonic on  $(-\infty, \infty)$ .

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**COROLLARY 3 First Derivative Test for Monotonic Functions**

Suppose that  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ .

If  $f'(x) > 0$  at each point  $x \in (a, b)$ , then  $f$  is increasing on  $[a, b]$ .

If  $f'(x) < 0$  at each point  $x \in (a, b)$ , then  $f$  is decreasing on  $[a, b]$ .

Mean value theorem is used to prove Corollary 3

# Example 1

- Using the first derivative test for monotonic functions  $f(x) = x^3 - 12x - 5$
- Find the critical point of  $f(x) = x^3 - 12x - 5$  and identify the intervals on which  $f$  is increasing and decreasing.

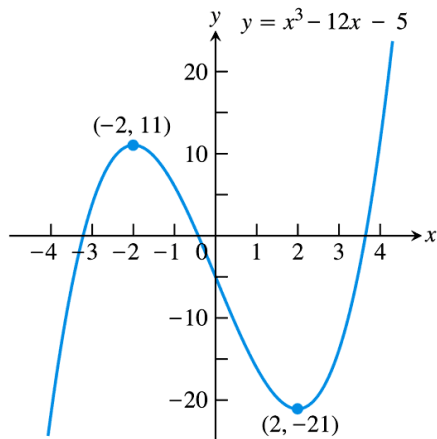
**Solution**

$$f'(x) = 3(x+2)(x-2)$$

$$f' > 0 \text{ for } -\infty < x < -2$$

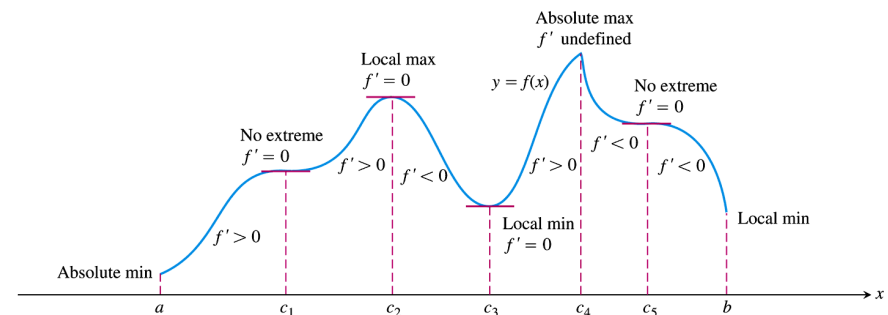
$$f' < 0 \text{ for } -2 < x < 2$$

$$f' > 0 \text{ for } 2 < x < \infty$$



**FIGURE 4.22** The function  $f(x) = x^3 - 12x - 5$  is monotonic on three separate intervals (Example 1).

# First derivative test for local extrema



**FIGURE 4.23** A function's first derivative tells how the graph rises and falls.

## Example 2: Using the first derivative test for local extrema

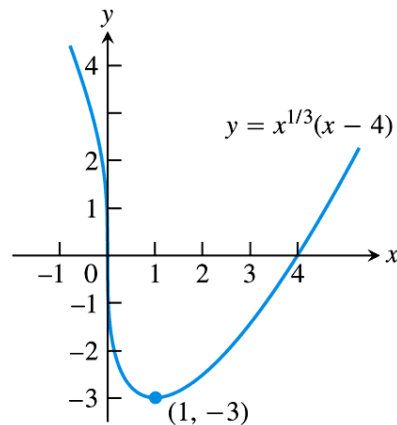
### First Derivative Test for Local Extrema

Suppose that  $c$  is a critical point of a continuous function  $f$ , and that  $f$  is differentiable at every point in some interval containing  $c$  except possibly at  $c$  itself. Moving across  $c$  from left to right,

1. if  $f'$  changes from negative to positive at  $c$ , then  $f$  has a local minimum at  $c$ ;
2. if  $f'$  changes from positive to negative at  $c$ , then  $f$  has a local maximum at  $c$ ;
3. if  $f'$  does not change sign at  $c$  (that is,  $f'$  is positive on both sides of  $c$  or negative on both sides), then  $f$  has no local extremum at  $c$ .

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**FIGURE 4.24** The function  $f(x) = x^{1/3}(x - 4)$  decreases when  $x < 1$  and increases when  $x > 1$  (Example 2).

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- Find the critical point of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}$$

- Identify the intervals on which  $f$  is increasing and decreasing. Find the function's local and absolute extreme values.

$$f' = \frac{4(x-1)}{3x^{2/3}}; f' - \text{ve for } x < 1;$$

$$f' - \text{ve for } 0 < x < 1; f' + \text{ve for } x > 1$$

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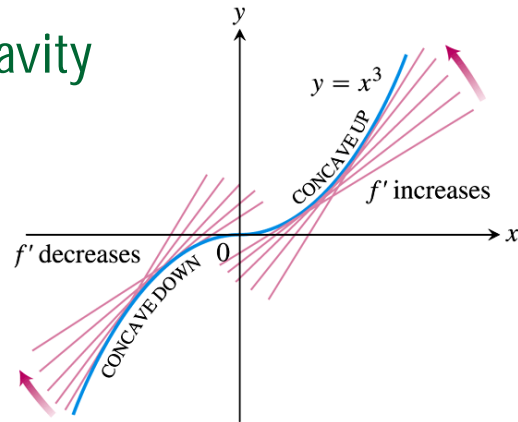
## 4.4

### Concavity and Curve Sketching

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# Concavity



**FIGURE 4.25** The graph of  $f(x) = x^3$  is concave down on  $(-\infty, 0)$  and concave up on  $(0, \infty)$  (Example 1a).

go back

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## The Second Derivative Test for Concavity

Let  $y = f(x)$  be twice-differentiable on an interval  $I$ .

1. If  $f'' > 0$  on  $I$ , the graph of  $f$  over  $I$  is concave up.
2. If  $f'' < 0$  on  $I$ , the graph of  $f$  over  $I$  is concave down.

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## DEFINITION Concave Up, Concave Down

The graph of a differentiable function  $y = f(x)$  is

- (a) **concave up** on an open interval  $I$  if  $f'$  is increasing on  $I$
- (b) **concave down** on an open interval  $I$  if  $f'$  is decreasing on  $I$ .

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## Example 1(a): Applying the concavity test

- Check the concavity of the curve  $y = x^3$
- Solution:  $y'' = 6x$
- $y'' < 0$  for  $x < 0$ ;  $y'' > 0$  for  $x > 0$ ;

Link to Figure 4.25

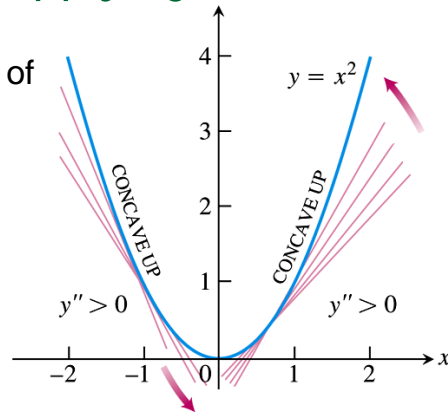
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## Example 1(b): Applying the concavity test

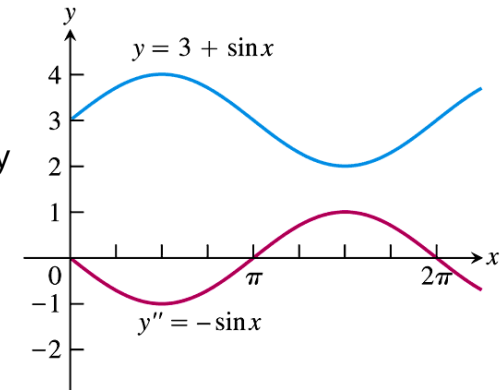
- Check the concavity of the curve  $y = x^2$
- Solution:  $y'' = 2 > 0$



**FIGURE 4.26** The graph of  $f(x) = x^2$  is concave up on every interval (Example 1b).

## Example 2

- Determining concavity
- Determine the concavity of  $y = 3 + \sin x$  on  $[0, 2\pi]$ .



**FIGURE 4.27** Using the graph of  $y''$  to determine the concavity of  $y$  (Example 2).

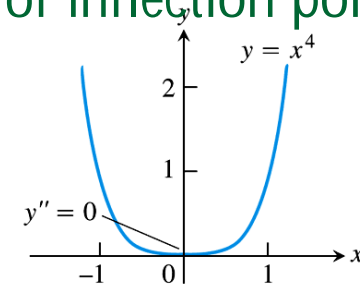
## Point of inflection

### DEFINITION Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

## Example 3: $y'' = 0$ not necessarily means existence of inflection point

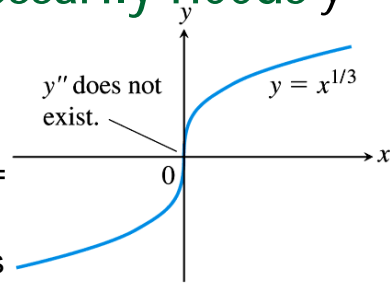
- An inflection point may not exist where  $y'' = 0$
- The curve  $y = x^4$  has no inflection point at  $x=0$ . Even though  $y'' = 12x^2$  is zero there, it does not change sign.



**FIGURE 4.28** The graph of  $y = x^4$  has no inflection point at the origin, even though  $y'' = 0$  there (Example 3).

## Example 4: Existence of inflection does not necessarily needs $y'' = 0$ means

- An inflection point may occur where  $y'' = 0$  does not exist
- The curve  $y = x^{1/3}$  has a point of inflection at  $x=0$  but  $y''$  does not exist there.
- $y'' = -(2/9)x^{5/3}$



**FIGURE 4.29** A point where  $y''$  fails to exist can be a point of inflection (Example 4).

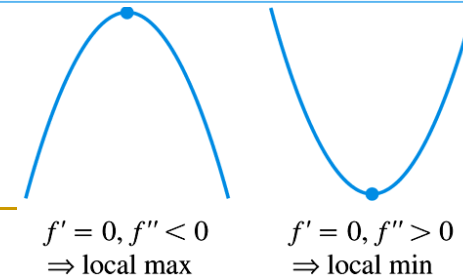
56

## Second derivative test for local extrema

### THEOREM 5 Second Derivative Test for Local Extrema

Suppose  $f''$  is continuous on an open interval that contains  $x = c$ .

- If  $f'(c) = 0$  and  $f''(c) < 0$ , then  $f$  has a local maximum at  $x = c$ .
- If  $f'(c) = 0$  and  $f''(c) > 0$ , then  $f$  has a local minimum at  $x = c$ .
- If  $f'(c) = 0$  and  $f''(c) = 0$ , then the test fails. The function  $f$  may have a local maximum, a local minimum, or neither.



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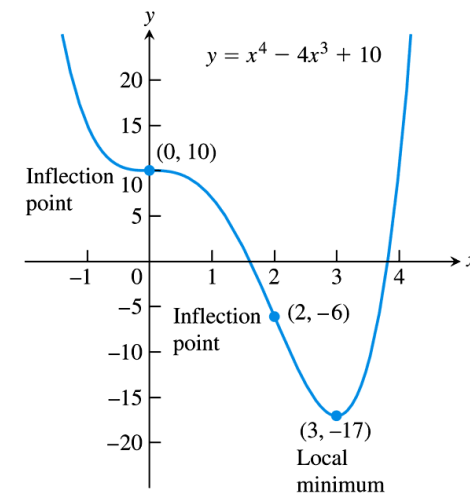
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## Example 6: Using $f'$ and $f''$ to graph $f$

- Sketch a graph of the function  $f(x) = x^4 - 4x^3 + 10$  using the following steps.
  - Identify where the extrema of  $f$  occur
  - Find the intervals on which  $f$  is increasing and the intervals on which  $f$  is decreasing
  - Find where the graph of  $f$  is concave up and where it is concave down.
  - Identify the slanted/vertical/horizontal asymptotes, if there is any
  - Sketch the general shape of the graph for  $f$ .
  - Plot the specific points. Then sketch the graph.

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**FIGURE 4.30** The graph of  $f(x) = x^4 - 4x^3 + 10$  (Example 6).

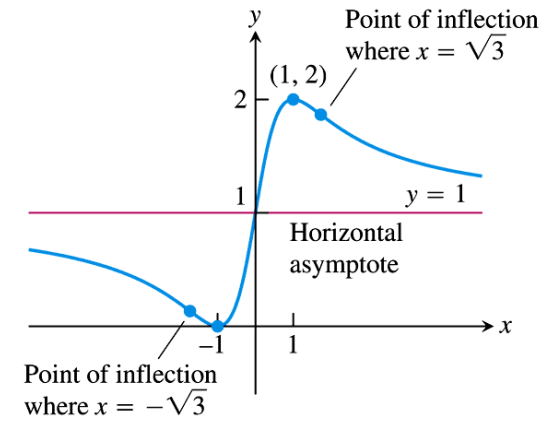
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## Example

- Using the graphing strategy
- Sketch the graph of
- $f(x) = (x + 1)^2 / (x^2 + 1)$ .



**FIGURE 4.31** The graph of  $y = \frac{(x + 1)^2}{1 + x^2}$  (Example 7).

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## Learning about functions from derivatives

<p><math>y = f(x)</math> Differentiable <math>\Rightarrow</math> smooth, connected; graph may rise and fall</p>	<p><math>y = f(x)</math> <math>y' &gt; 0 \Rightarrow</math> rises from left to right; may be wavy</p>	<p><math>y = f(x)</math> <math>y' &lt; 0 \Rightarrow</math> falls from left to right; may be wavy</p>
<p>or</p> <p><math>y'' &gt; 0 \Rightarrow</math> concave up throughout; no waves; graph may rise or fall</p>	<p>or</p> <p><math>y'' &lt; 0 \Rightarrow</math> concave down throughout; no waves; graph may rise or fall</p>	<p><math>y''</math> changes sign Inflection point</p>
<p>or</p> <p><math>y'</math> changes sign <math>\Rightarrow</math> graph has local maximum or local minimum</p>	<p><math>y' = 0</math> and <math>y'' &lt; 0</math> at a point; graph has local maximum</p>	<p><math>y' = 0</math> and <math>y'' &gt; 0</math> at a point; graph has local minimum</p>

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## 4.5

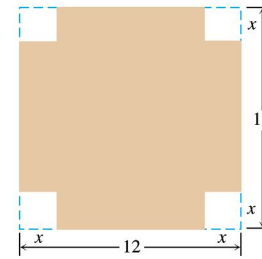
### Applied Optimization Problems

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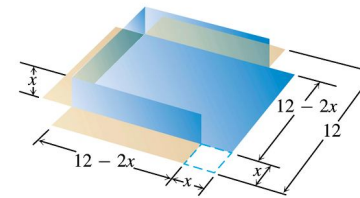
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## Example 1

- An open-top box is to be cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?



(a)



(b)

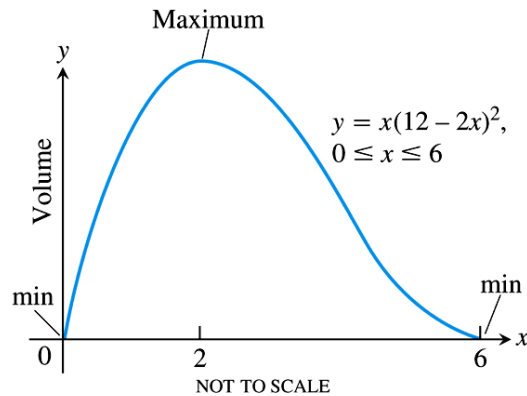
**FIGURE 4.32** An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume (Example 1)?

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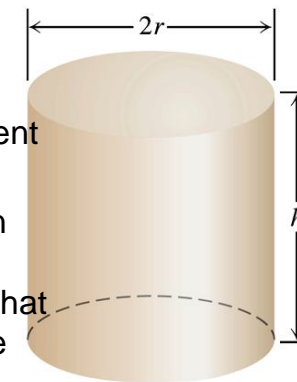
**FIGURE 4.33** The volume of the box in Figure 4.32 graphed as a function of  $x$ .

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## Example 2

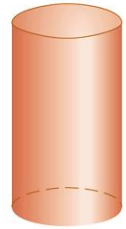
- Designing an efficient cylindrical can
- Design a 1-liter can shaped like a right circular cylinder. What dimensions will use the least material?



**FIGURE 4.34** This 1-L can uses the least material when  $h = 2r$  (Example 2).

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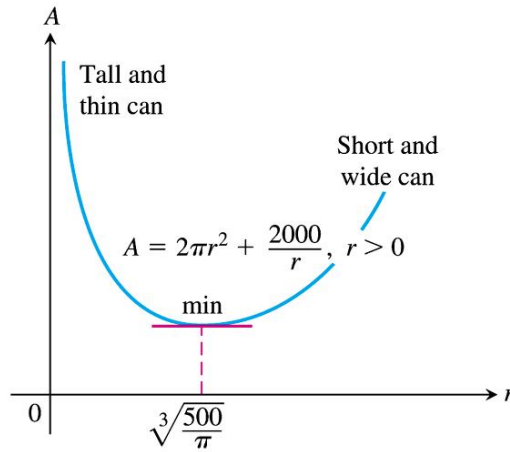
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Tall and thin



Short and wide



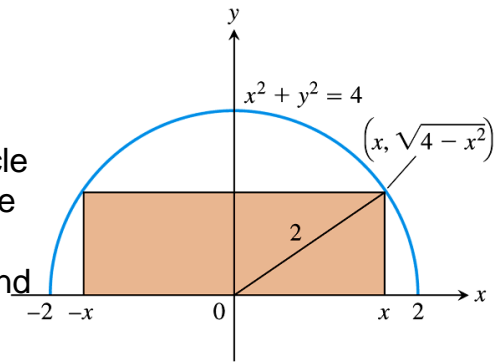
**FIGURE 4.35** The graph of  $A = 2\pi r^2 + 2000/r$  is concave up.

## Solution

- Form the function of the area  $A$  as a function of  $x$ :  $A=A(x)=x(4-x^2)^{1/2}$ ;  $x > 0$ .
- Seek the maximum of  $A$ :

## Example 3

- Inscribing rectangles
- A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what is its dimension?



**FIGURE 4.36** The rectangle inscribed in the semicircle in Example 3.

## 4.6

### Indeterminate Forms and L' Hopital's Rule

# Indeterminate forms 0/0

## THEOREM 6 L'Hôpital's Rule (First Form)

Suppose that  $f(a) = g(a) = 0$ , that  $f'(a)$  and  $g'(a)$  exist, and that  $g'(a) \neq 0$ . Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

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# Example 1

## ■ Using L' Hopital's Rule

### ■ (a)

$$\lim_{x \rightarrow 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1} \Big|_{x=0} = 2$$

### ■ (b)

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} = \frac{1}{2\sqrt{1+x}} \Big|_{x=0} = \frac{1}{2}$$

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# Example 2(a)

## ■ Applying the stronger form of L' Hopital's rule

### ■ (a)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2} &= \lim_{x \rightarrow 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x} \\ &= \lim_{x \rightarrow 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = \frac{-1}{8} \end{aligned}$$

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75

## Example 2(b)

- Applying the stronger form of L' Hopital's rule
- (b)

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$$

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### THEOREM 8 Cauchy's Mean Value Theorem

Suppose functions  $f$  and  $g$  are continuous on  $[a, b]$  and differentiable throughout  $(a, b)$  and also suppose  $g'(x) \neq 0$  throughout  $(a, b)$ . Then there exists a number  $c$  in  $(a, b)$  at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

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### Using L'Hôpital's Rule

To find

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate  $f$  and  $g$ , so long as we still get the form  $0/0$  at  $x = a$ . But as soon as one or the other of these derivatives is different from zero at  $x = a$  we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

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## Example 3

- Incorrect application of the stronger form of L' Hopital's

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2}$$

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## Example 4

- Using L' Hopital's rule with one-sided limits

$$(a) \lim_{x \rightarrow 0^+} \frac{\sin x}{x^2} = \lim_{x \rightarrow 0^+} \frac{\cos x}{2x} = \dots$$

$$(b) \lim_{x \rightarrow 0^-} \frac{\sin x}{x^2} = \lim_{x \rightarrow 0^-} \frac{\cos x}{2x} = \dots$$

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## Example 5

Working with the indeterminate form

$\infty/\infty$

$$(a) \lim_{x \rightarrow \pi/2} \frac{\sec x}{1 + \tan x}$$

$$\lim_{x \rightarrow (\pi/2)^-} \frac{\sec x}{1 + \tan x} = \lim_{x \rightarrow (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \rightarrow (\pi/2)^-} \sin x = 1$$

$$\lim_{x \rightarrow (\pi/2)^+} \frac{\sec x}{1 + \tan x} = \dots$$

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Indeterminate forms  $\infty/\infty$ ,  $\infty \cdot 0$ ,  $\infty - \infty$

- If  $f \rightarrow \pm\infty$  and  $g \rightarrow \pm\infty$  as  $x \rightarrow a$ , then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

- $a$  may be finite or infinite

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## Example 5(b)

$$(b) \lim_{x \rightarrow \infty} \frac{x - 2x^2}{3x^2 + 5x} = \dots$$

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## Example 6

- Working with the indeterminate form  $\infty \cdot 0$

$$\lim_{x \rightarrow \infty} \left( x \sin \frac{1}{x} \right)$$

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## 4.8

### Antiderivatives

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## Example 7

- Working with the indeterminate form  $\infty - \infty$

$$\lim_{x \rightarrow 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left( \frac{x - \sin x}{x \sin x} \right) = \dots$$

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## Finding antiderivatives

### DEFINITION Antiderivative

A function  $F$  is an **antiderivative** of  $f$  on an interval  $I$  if  $F'(x) = f(x)$  for all  $x$  in  $I$ .

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## Example 1

- Finding antiderivatives
- Find an antiderivative for each of the following functions
- (a)  $f(x) = 2x$
- (b)  $f(x) = \cos x$
- (c)  $h(x) = 2x + \cos x$

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## The most general antiderivative

If  $F$  is an antiderivative of  $f$  on an interval  $I$ , then the most general antiderivative of  $f$  on  $I$  is

$$F(x) + C$$

where  $C$  is an arbitrary constant.

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## Example 2 Finding a particular antiderivative

- Find an antiderivative of  $f(x) = \sin x$  that satisfies  $F(0) = 3$
- Solution:  $F(x) = \cos x + C$  is the most general form of the antiderivative of  $f(x)$ .
- We require  $F(x)$  to fulfill the condition that when  $x=3$  (in unit of radian),  $F(x)=0$ . This will fix the value of  $C$ , as per
- $F(3) = 0 = \cos 3 + C \Rightarrow C = -\cos 3$
- Hence,  $F(x) = \cos x - \cos 3$  is the antiderivative sought

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TABLE 4.2 Antiderivative formulas

	Function	General antiderivative
1.	$x^n$	$\frac{x^{n+1}}{n+1} + C, \quad n \neq -1, n \text{ rational}$
2.	$\sin kx$	$-\frac{\cos kx}{k} + C, \quad k \text{ a constant, } k \neq 0$
3.	$\cos kx$	$\frac{\sin kx}{k} + C, \quad k \text{ a constant, } k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

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### Example 3 Finding antiderivatives using table 4.2

- Find the general antiderivative of each of the following functions.
- (a)  $f(x) = x^5$
- (b)  $g(x) = 1/x^{1/2}$
- (c)  $h(x) = \sin 2x$
- (d)  $i(x) = \cos (x/2)$

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### Example 4 Using the linearity rules for antiderivatives

- Find the general antiderivative of
- $f(x) = 3/x^{1/2} + \sin 2x$

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#### DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of  $f$  is the **indefinite integral** of  $f$  with respect to  $x$ , denoted by

$$\int f(x) dx.$$

The symbol  $\int$  is an **integral sign**. The function  $f$  is the **integrand** of the integral, and  $x$  is the **variable of integration**.

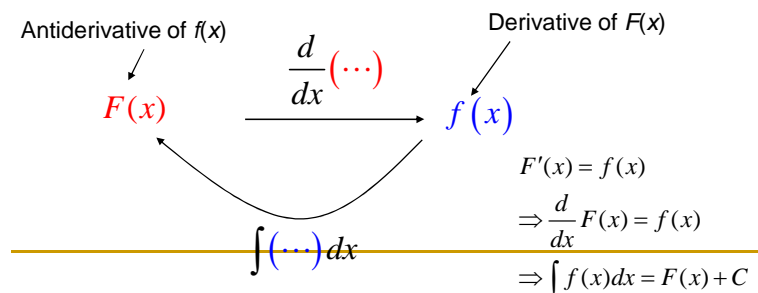
- In other words, given a function  $f(x)$ , the most general form of its antiderivative, previously represented by the symbol  $F(x) + C$ , where  $C$  denotes an arbitrary constant, is now being represented in the form of an indefinite integral, namely,

$$\int f(x) dx \equiv F(x) + C$$

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### Operationally, the indefinite integral of $f(x)$ means ...

The indefinite integral of  $f(x)$  is the inverse of the operation of derivative taking of  $f(x)$



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## Example of indefinite integral notation

$$\int 2x \, dx = x^2 + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int (2x + \cos x) \, dx = x^2 + \sin x + C$$

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## Example 7 Indefinite integration done term-by term and rewriting the constant of integration

### ■ Evaluate

$$\int (x^2 - 2x + 5) \, dx = \int x^2 \, dx - \int 2x \, dx + \int 5 \, dx = \dots$$

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## Chapter 5

### Integration

1

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## 5.1

### Estimating with Finite Sums

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# Riemann Sums

## Approximating area bounded by the graph between $[a, b]$

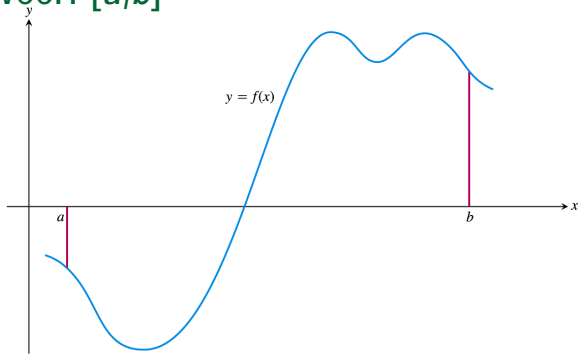


FIGURE 5.8 A typical continuous function  $y = f(x)$  over a closed interval  $[a, b]$ .

Area is approximately given by

$$f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \dots + f(c_n)\Delta x_n$$

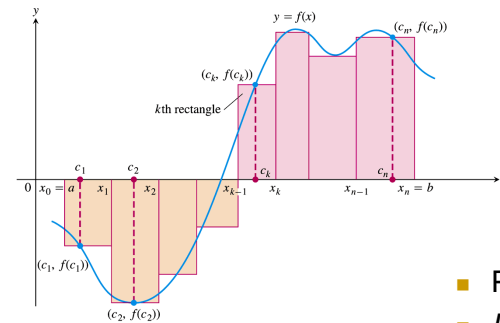
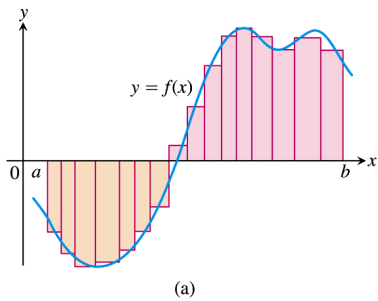


FIGURE 5.9 The rectangles approximate the region between the graph of the function  $y = f(x)$  and the  $x$ -axis.

- Partition of  $[a, b]$  is the set of
- $P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$
- $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$
- $c_n \in [x_{n-1}, x_n]$
- $\|P\| = \text{norm of } P = \text{the largest of all subinterval width}$



Riemann sum for  $f$  on  $[a, b]$

$$R_n = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \dots + f(c_n)\Delta x_n$$

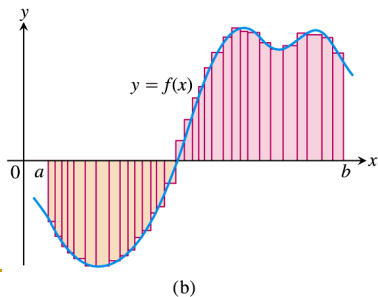
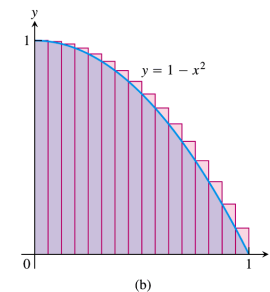
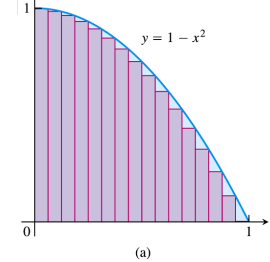


FIGURE 5.10 The curve of Figure 5.9 with rectangles from finer partitions of  $[a, b]$ . Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of  $f$  and the  $x$ -axis with increasing accuracy.

Figure 5.4



- Let the true value of the area is  $R$
- Two approximations to  $R$ :
- $c_n = x_n$  corresponds to case (a). This under estimates the true value of the area  $R$  if  $n$  is finite.
- $c_n = x_{n-1}$  corresponds to case (b). This over estimates the true value of the area  $S$  if  $n$  is finite.

[go back](#)

## Limits of finite sums

- Example 5 The limit of finite approximation to an area
- Find the limiting value of lower sum approximation to the area of the region  $R$  below the graphs  $f(x) = 1 - x^2$  on the interval  $[0,1]$  based on [Figure 5.4\(a\)](#)

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- Taking the limit of  $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} R_n = R = \left( 1 - \frac{2n^3 + 3n^2 + n}{6n^3} \right) = 1 - 2/6 = 2/3$$

- The same limit is also obtained if  $c_n = x_{n-1}$  is chosen instead.
- For all choice of  $c_n \in [x_{n-1}, x_n]$  and partition of  $P$ , the same limit for  $S$  is obtained when  $n \rightarrow \infty$

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## Solution

- $\Delta x_k = (1 - 0)/n = 1/n \equiv \Delta x$ ;  $k = 1, 2, \dots, n$
  - Partition on the  $x$ -axis:  $[0, 1/n], [1/n, 2/n], \dots, [(n-1)/n, 1]$ .
  - $c_k = x_k = k\Delta x = k/n$
  - The sum of the stripes is
$$\begin{aligned} R_n &= \Delta x_1 f(c_1) + \Delta x_2 f(c_2) + \Delta x_3 f(c_3) + \dots + \Delta x_n f(c_n) \\ &= \Delta x f(1/n) + \Delta x f(2/n) + \Delta x f(3/n) + \dots + \Delta x_n f(1) \\ &= \sum_{k=1}^n \Delta x f(k\Delta x) = \Delta x \sum_{k=1}^n f(k/n) \\ &= (1/n) \sum_{k=1}^n [1 - (k/n)^2] \\ &= \sum_{k=1}^n 1/n - k^2/n^3 = 1 - (\sum_{k=1}^n k^2)/n^3 \\ &= 1 - [(n)(n+1)(2n+1)/6]/n^3 = 1 - [2n^3 + 3n^2 + n]/(6n^3) \end{aligned}$$
- $$\sum_{k=1}^n k^2 = (n)(n+1)(2n+1)/6$$

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## 5.3

### The Definite Integral

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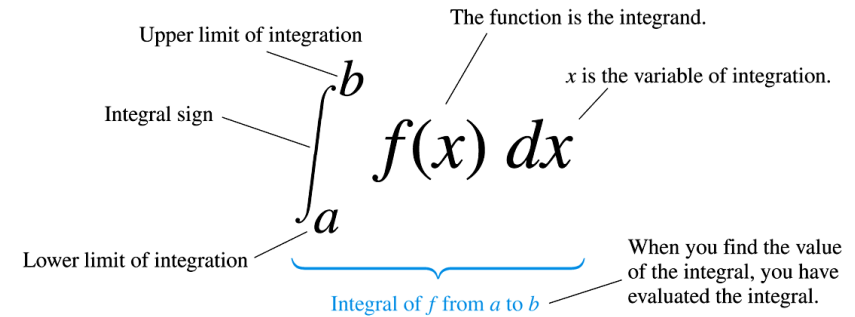
### DEFINITION The Definite Integral as a Limit of Riemann Sums

Let  $f(x)$  be a function defined on a closed interval  $[a, b]$ . We say that a number  $I$  is the **definite integral of  $f$  over  $[a, b]$**  and that  $I$  is the limit of the Riemann sums  $\sum_{k=1}^n f(c_k) \Delta x_k$  if the following condition is satisfied:

Given any number  $\epsilon > 0$  there is a corresponding number  $\delta > 0$  such that for every partition  $P = \{x_0, x_1, \dots, x_n\}$  of  $[a, b]$  with  $\|P\| < \delta$  and any choice of  $c_k$  in  $[x_{k-1}, x_k]$ , we have

$$\left| \sum_{k=1}^n f(c_k) \Delta x_k - I \right| < \epsilon.$$

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“ The integral from  $a$  to  $b$  of  $f$  of  $x$  with respect to  $x$ ”

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- The limit of the Riemann sums of  $f$  on  $[a, b]$  converges to the finite integral  $I$

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k = I = \int_a^b f(x) dx$$

- We say  $f$  is integrable over  $[a, b]$
- Can also write the definite integral as

$$\begin{aligned} I &= \int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(u) du \\ &= \int_a^b f(\text{what ever}) d(\text{what ever}) \end{aligned}$$

- The variable of integration is what we call a ‘dummy variable’

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### THEOREM 1 The Existence of Definite Integrals

A continuous function is integrable. That is, if a function  $f$  is continuous on an interval  $[a, b]$ , then its definite integral over  $[a, b]$  exists.

Question: is a non continuous function integrable?

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# Integral and nonintegrable functions

- Example 1
- A nonintegrable function on  $[0,1]$

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

- Not integrable

# Properties of definite integrals

## THEOREM 2

When  $f$  and  $g$  are integrable, the definite integral satisfies Rules 1 to 7 in Table 5.3.

TABLE 5.3 Rules satisfied by definite integrals

1. <i>Order of Integration:</i>	$\int_b^a f(x) dx = -\int_a^b f(x) dx$	A Definition
2. <i>Zero Width Interval:</i>	$\int_a^a f(x) dx = 0$	Also a Definition
3. <i>Constant Multiple:</i>	$\int_a^b kf(x) dx = k \int_a^b f(x) dx$	Any Number $k$
	$\int_a^b -f(x) dx = -\int_a^b f(x) dx$	$k = -1$
4. <i>Sum and Difference:</i>	$\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$	
5. <i>Additivity:</i>	$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$	
6. <i>Max-Min Inequality:</i>	If $f$ has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$ , then	
	$\min f \cdot (b - a) \leq \int_a^b f(x) dx \leq \max f \cdot (b - a).$	
7. <i>Domination:</i>	$f(x) \geq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x) dx \geq \int_a^b g(x) dx$	
	$f(x) \geq 0$ on $[a, b] \Rightarrow \int_a^b f(x) dx \geq 0$ (Special Case)	

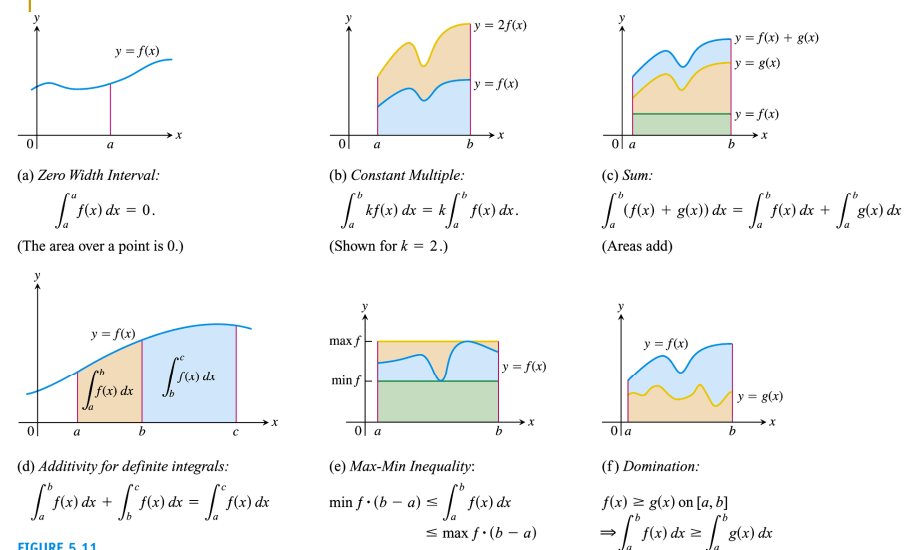


FIGURE 5.11



## Example 3 Finding bounds for an integral

- Show that the value of  $\int_0^1 \sqrt{1 + \cos x} dx$  is less than  $3/2$
- Solution**
- Use rule 6 Max-Min Inequality

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## Area under the graphs of a nonnegative function

### DEFINITION Area Under a Curve as a Definite Integral

If  $y = f(x)$  is nonnegative and integrable over a closed interval  $[a, b]$ , then the **area under the curve  $y = f(x)$  over  $[a, b]$**  is the integral of  $f$  from  $a$  to  $b$ ,

$$A = \int_a^b f(x) dx.$$

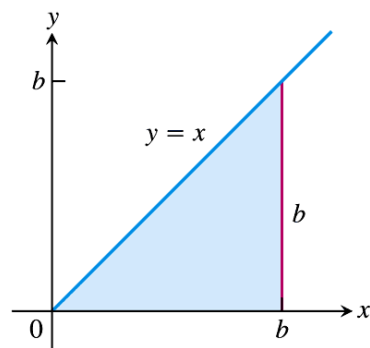
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## Example 4 Area under the line $y = x$

- Compute  $\int_0^b x dx$  (the Riemann sum) and find the area  $A$  under  $y = x$  over the interval  $[0, b]$ ,  $b > 0$



**FIGURE 5.12** The region in Example 4 is a triangle.

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## Solution

By geometrical consideration:

$$A = (1/2) \times \text{high} \times \text{width} = (1/2) \times b \times b = b^2/2$$

Choose partition of  $n$  subinterval with equal width:

$$\{0 = x_0, x_1, x_2, \dots, x_n = b\}, \Delta x_k = x_k - x_{k-1} = \Delta x = b/n$$

Riemann sum:

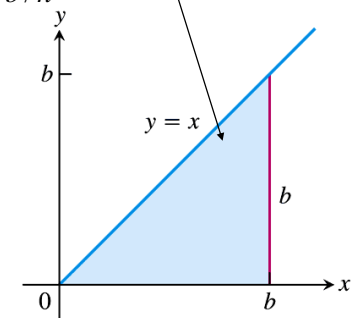
$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta x f(c_k) = \lim_{n \rightarrow \infty} \Delta x \sum_{k=1}^n f(x_k)$$

$$= \lim_{n \rightarrow \infty} \Delta x \sum_{k=1}^n x_k = \lim_{n \rightarrow \infty} \Delta x \sum_{k=1}^n k \Delta x$$

$$= \lim_{n \rightarrow \infty} (\Delta x)^2 \sum_{k=1}^n k = \lim_{n \rightarrow \infty} \left(\frac{b}{n}\right)^2 \sum_{k=1}^n k$$

$$= \lim_{n \rightarrow \infty} \left(\frac{b}{n}\right)^2 \frac{n(n+1)}{2} = \lim_{n \rightarrow \infty} \left(\frac{b}{n}\right)^2 \frac{n(n+1)}{2}$$

$$= \lim_{n \rightarrow \infty} \frac{b^2}{2} \left(1 + \frac{1}{n}\right) = \frac{b^2}{2}$$



**FIGURE 5.12** The region in Example 4 is a triangle.

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Using the additivity rule for definite integration:

$$\int_0^b x dx = \int_0^a x dx + \int_a^b x dx$$

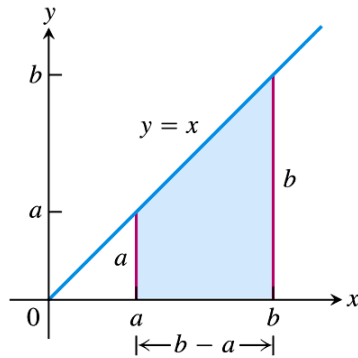
$$\rightarrow \int_a^b x dx = \int_0^b x dx - \int_0^a x dx = \frac{b^2}{2} - \frac{a^2}{2}, a < b$$

Using geometry, the area is the area of a trapezium

$$A = (1/2)(b-a)(b+a)$$

$$= b^2/2 - a^2/2$$

Both approaches to evaluate the area agree



**FIGURE 5.13** The area of this trapezoidal region is  $A = (b^2 - a^2)/2$ .

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- One can prove the following Riemannian sum of the functions  $f(x)=c$  and  $f(x)=x^2$ :

$$\int_a^b c dx = c(b - a), \quad c \text{ any constant} \quad (2)$$

$$\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}, \quad a < b \quad (3)$$

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## Average value of a continuous function revisited

- Average value of nonnegative continuous function  $f$  over an interval  $[a,b]$  is

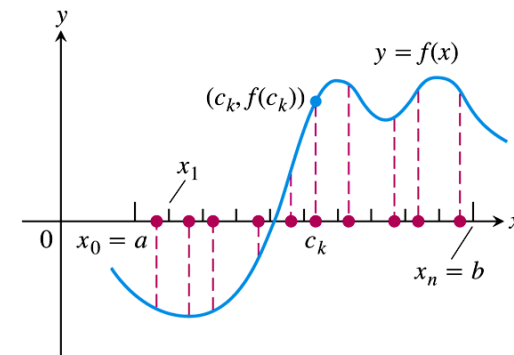
$$\frac{f(c_1) + f(c_2) + \dots + f(c_n)}{n} = \frac{1}{n} \sum_{k=1}^n f(c_k)$$

$$= \frac{\Delta x}{b-a} \sum_{k=1}^n f(c_k) = \frac{1}{b-a} \sum_{k=1}^n \Delta x f(c_k)$$

- In the limit of  $n \rightarrow \infty$ , the average =

$$\frac{1}{b-a} \int_a^b f(x) dx$$

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**FIGURE 5.14** A sample of values of a function on an interval  $[a, b]$ .

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### DEFINITION The Average or Mean Value of a Function

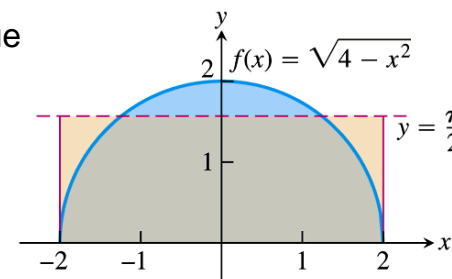
If  $f$  is integrable on  $[a, b]$ , then its **average value on  $[a, b]$** , also called its **mean value**, is

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) dx.$$

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## Example 5 Finding average value

- Find the average value of  $f(x) = \sqrt{4-x^2}$  over  $[-2, 2]$



**FIGURE 5.15** The average value of  $f(x) = \sqrt{4-x^2}$  on  $[-2, 2]$  is  $\pi/2$  (Example 5).

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## 5.4

### The Fundamental Theorem of Calculus

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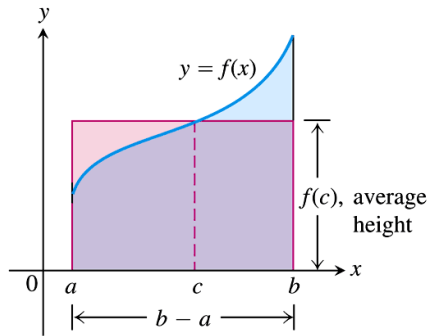
## Mean value theorem for definite integrals

### THEOREM 3 The Mean Value Theorem for Definite Integrals

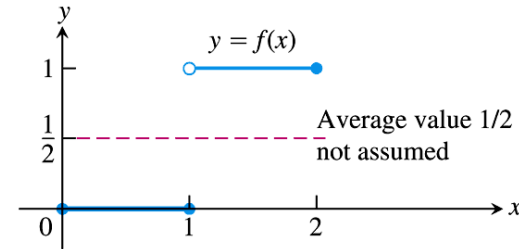
If  $f$  is continuous on  $[a, b]$ , then at some point  $c$  in  $[a, b]$ ,

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx.$$

30



**FIGURE 5.16** The value  $f(c)$  in the Mean Value Theorem is, in a sense, the average (or *mean*) height of  $f$  on  $[a, b]$ . When  $f \geq 0$ , the area of the rectangle is the area under the graph of  $f$  from  $a$  to  $b$ ,

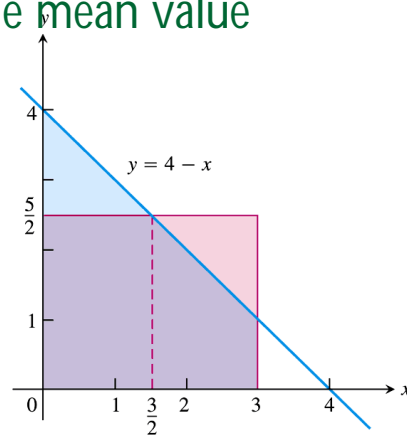


**FIGURE 5.17** A discontinuous function need not assume its average value.

## Example 1 Applying the mean value theorem for integrals

- Find the average value of  $f(x)=4-x$  on  $[0,3]$  and where  $f$  actually takes on this value as some point in the given domain.

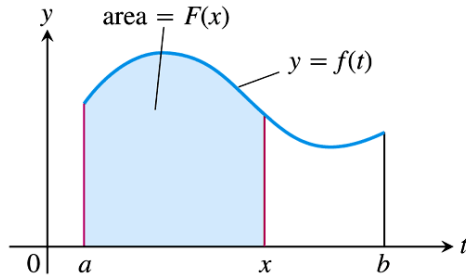
- Solution**
- Average =  $5/2$
- Happens at  $x=3/2$



**FIGURE 5.18** The area of the rectangle with base  $[0, 3]$  and height  $5/2$  (the average value of the function  $f(x) = 4 - x$ ) is equal to the area between the graph of  $f$  and the  $x$ -axis from 0 to 3 (Example 1).

## Fundamental theorem Part 1

- Define a function  $F(x)$ :  $F(x) = \int_a^x f(t)dt$
- $x, a \in I$ , an interval over which  $f(t) > 0$  is integrable.
- The function  $F(x)$  is the area under the graph of  $f(t)$  over  $[a, x]$ ,  $x > a$



**FIGURE 5.19** The function  $F(x)$  defined by Equation (1) gives the area under the graph of  $f$  from  $a$  to  $x$  when  $f$  is nonnegative and  $x > a$ .

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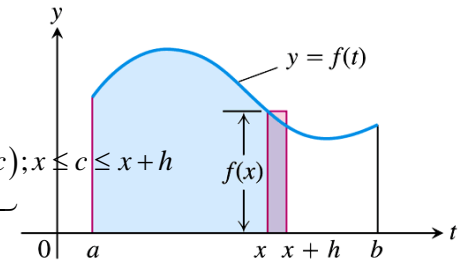
## Fundamental theorem Part 1 (cont.)

$$F(x+h) - F(x) = \int_x^{x+h} f(t) dt$$

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) dt = f(c); x \leq c \leq x+h$$

mean value theorem

$$\lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = F'(x) = f(x)$$



**FIGURE 5.20** In Equation (1),  $F(x)$  is the area to the left of  $x$ . Also,  $F(x+h)$  is the area to the left of  $x+h$ . The difference quotient  $[F(x+h) - F(x)]/h$  is then approximately equal to  $f(x)$ , the height of the rectangle shown here.

The above result holds true even if  $f$  is not positive definite over  $[a, b]$

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### THEOREM 4 The Fundamental Theorem of Calculus Part 1

If  $f$  is continuous on  $[a, b]$  then  $F(x) = \int_a^x f(t) dt$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  and its derivative is  $f(x)$ ;

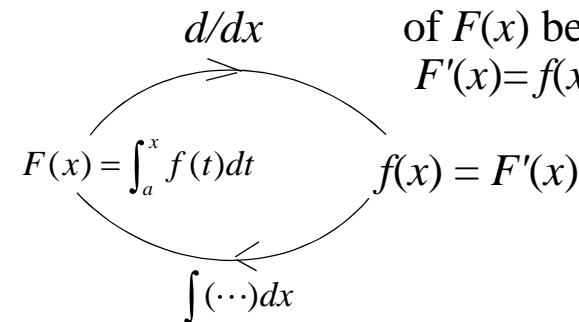
$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x). \quad (2)$$

Note: Convince yourself that

(i)  $F(x)$  is an antiderivative of  $f(x)$

(ii)  $f(x)$  is a derivative of  $F(x)$

$f(x)$  is a derivative of  $F(x)$  because  $F'(x) = f(x)$



$F(x)$  is an antiderivative of  $f(x)$  because  $F(x) = \int_a^x f(t) dt$

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## The main use of theorem 4 is ...

- It tells us that  $\frac{d}{dx} \left( \int_a^x f(t) dt \right) = f(x)$
- In pragmatic terms, if a function is expressed in terms of an integral of the form

$$F(x) = \int_a^x f(t) dt$$

then the derivative of  $F(x)$ ,  $\frac{d}{dx} F(x)$ , is simply  $f(x)$

## Example 3 Applying the fundamental theorem

- Use the fundamental theorem to find

$$(a) \frac{d}{dx} \int_a^x \cos t dt$$

$$(b) \frac{d}{dx} \int_a^x \frac{1}{1+t^2} dt$$

$$(c) \frac{dy}{dx} \text{ if } y = \int_x^5 3t \sin t dt$$

$$(d) \frac{dy}{dx} \text{ if } y = \int_1^{x^2} \cos t dt$$

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## Solution for (d): you have to invoke chain rule

$$\frac{d}{dx} F(x), \text{ where } F(x) \equiv \int_1^{x^2} \cos t dt$$

- Chain rule says if  $F(x) = (f \circ u)(x) = f[u(x)]$ ,

$$\frac{d}{dx} F(x) = \frac{d}{dx} (f \circ u)(x) = \frac{d}{dx} f[u(x)] = \frac{d}{du} f(u) \cdot \frac{d}{dx} u(x)$$

## Solution for (d): you have to invoke chain rule

$$F(x) = \int_1^{x^2} \cos t dt \text{ is a composite function of the form } F(x) = f[u(x)]$$

$F(x) = f[u(x)]$ , where

$$f(u) = \int_1^u \cos t dt, \quad u(x) = x^2$$

so that

$$\begin{aligned} \frac{d}{dx} F(x) &= \frac{d}{du} f(u) \cdot \frac{d}{dx} u(x) = \frac{d}{du} \left( \int_1^u \cos t dt \right) \cdot \frac{d}{dx} (x^2) \\ &= \frac{d}{du} \left( \int_1^u \cos t dt \right) \cdot 2x = \cos u \cdot 2x = 2x \cos(x^2) \end{aligned}$$

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## Example 4 Constructing a function with a given derivative and value

- Find a function  $y = f(x)$  on the domain  $(-\pi/2, \pi/2)$  with derivative  $dy/dx = \tan x$  that satisfies  $f(3)=5$ .
- The strategy:
- Use the fundamental theorem of calculus.
- Think along this line: find a function  $F(x)$  of the form

$$F(x) = \int_a^x q(t) dt$$

such that  $\frac{d}{dx} F(x) = q(x)$ , with  $q(x) = \tan x$

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## Example 4 (Cont. 2)

$$f(x) = \int_a^x \tan t dt + \text{constant}$$

- Find the values of  $a$  and constant so that  $f(3)=5$
- This can be done by choosing  $a = 3$ , constant =5.
- Verify this:

$$f(3) = \int_{a=3}^{x=3} \tan t dt + 5 = 0 + 5 = 5$$

- So, finally, the function we are seeking is

$$f(x) = \int_3^x \tan t dt + 5$$

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## Example 4 (Cont. 1)

### Solution

- Stage 1: If  $F(x) = \int_a^x \tan t dt$ , then  $\frac{dF}{dx} = \tan x$ .

- Stage 2: construct the function  $f(x)$  using  $F(x)$ , and then try to make  $f(x)$  so constructed fulfills the condition of  $f(3)=5$ .

- The way to construct  $f(x)$  from  $F(x)$  is obviously

$$y = f(x) = F(x) + \text{constant} \quad \left( \text{so that } \frac{dy}{dx} = \tan x \right)$$

$$= \int_a^x \tan t dt + \text{constant}$$

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## Fundamental theorem, part 2 (The evaluation theorem)

### THEOREM 4 (Continued) The Fundamental Theorem of Calculus Part 2

If  $f$  is continuous at every point of  $[a, b]$  and  $F$  is any antiderivative of  $f$  on  $[a, b]$ , then

$$\int_a^b f(x) dx = F(b) - F(a).$$

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To calculate the definite integral of  $f$  over  $[a,b]$ , do the following

- 1. Find an antiderivative  $F$  of  $f$ , and
- 2. Calculate the number

$$\int_a^b f(x)dx = F(b) - F(a) = F(x) \Big|_a^b \quad \text{or} \quad \left[ F(x) \right]_a^b$$

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## Example 5 Evaluating integrals

$$(a) \int_0^{\pi} \cos x dx$$

$$(b) \int_{-\pi/4}^0 \sec x \tan x dx$$

$$(c) \int_1^4 \left( \frac{3}{2} \sqrt{x} - \frac{4}{x^2} \right) dx$$

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To summarise

$$\frac{d}{dx} \int_a^x f(t)dt = \frac{dF(x)}{dx} = f(x)$$

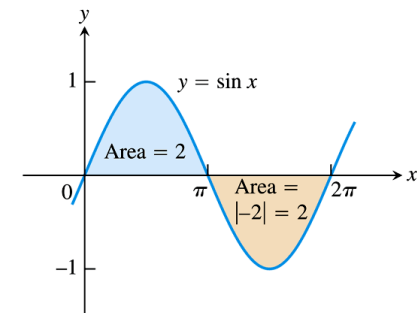
$$\int_a^x \left( \frac{dF(t)}{dt} \right) dt = \int_a^x f(t)dt = F(x) - F(a)$$

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## Example 7 Canceling areas

- Compute
- (a) the definite integral of  $f(x)$  over  $[0,2\pi]$
- (b) the area between the graph of  $f(x)$  and the  $x$ -axis over  $[0,2\pi]$



**FIGURE 5.22** The total area between  $y = \sin x$  and the  $x$ -axis for  $0 \leq x \leq 2\pi$  is the sum of the absolute values of two integrals (Example 7).

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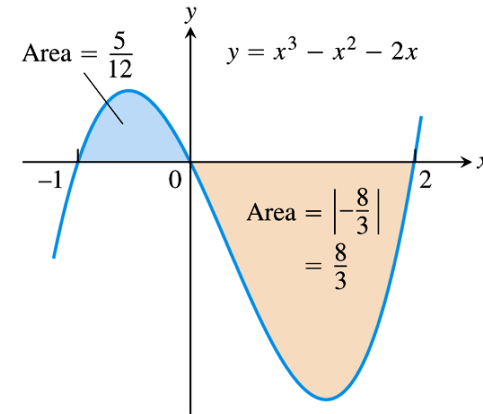


## Example 8 Finding area using antiderivative

- Find the area of the region between the  $x$ -axis and the graph of  $f(x) = x^3 - x^2 - 2x$ ,  $-1 \leq x \leq 2$ .

### Solution

- First find the zeros of  $f$ .
- $f(x) = x(x+1)(x-2)$



**FIGURE 5.23** The region between the curve  $y = x^3 - x^2 - 2x$  and the  $x$ -axis (Example 8).

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## 5.5

### Indefinite Integrals and the Substitution Rule

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## Note

- The indefinite integral of  $f$  with respect to  $x$ ,  
$$\int f(x)dx$$
is a function plus an arbitrary constant
- A definite integral  $\int_a^b f(x)dx$  is a number.

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## Antiderivative and indefinite integral in terms of variable $x$

- If  $F(x)$  is an antiderivative of  $f(x)$ ,

$$\frac{d}{dx} F(x) = f(x)$$



- the indefinite integral of  $f(x)$  is

$$\int f(x) dx = F(x) + C$$

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## Antiderivative and indefinite integral with chain rule

$\frac{d}{dx} F(x) = f(x)$ , i.e.,  $F(x)$  antiderivative of  $f(x)$ ,

⇒  $\frac{d}{du} F[u] = f(u)$ , where  $u = u(x)$ .

Applying chain rule to  $\frac{d}{dx} F[u]$ :

$$\frac{d}{dx} F[u] = \frac{du(x)}{dx} \cdot \frac{dF(u)}{du} = \frac{du}{dx} \cdot f(u) \Rightarrow \frac{d}{dx} F[u] = \frac{du}{dx} \cdot f(u)$$

In other words,  $F[u]$  is an antiderivative of  $\frac{du}{dx} \cdot f(u)$ , so that we can write

$$\int \left( \frac{du}{dx} \cdot f(u) \right) dx = F[u] + C$$

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## A useful mnemonic

$$\frac{d}{dx} (\star + \text{constant}) = \text{😊}$$



$$\int \text{😊} dx = \star + \text{constant}$$

Example:  $\frac{d}{dx} (\tan x + \text{constant}) = \sec^2 x$   
 $\int \sec^2 x dx = \tan x + \text{constant}$

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## The power rule in integral form

$$\frac{d}{dx} \left( \frac{u^{n+1}}{n+1} \right) = u^n \frac{du}{dx} \rightarrow \int \left( u^n \frac{du}{dx} \right) dx = \left( \frac{u^{n+1}}{n+1} \right) + C$$

$$\int \left( u^n \frac{du}{dx} \right) dx = \int u^n \left( \frac{du}{dx} dx \right) = \int u^n du$$

differential of  $u(x)$ ,  $du$  is  $du = \frac{du}{dx} dx$

If  $u$  is any differentiable function, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1, n \text{ rational}). \quad (1)$$

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## Example 1 Using the power rule

The strategy is to convert the integral into the form

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

$$\int \sqrt{1+y^2} \cdot 2y dy = ?$$

$$\text{Let } u = 1 + y^2, du = \frac{du}{dy} dy = 2y dy.$$

$$\int \sqrt{1+y^2} \cdot 2y dy = \int \sqrt{u} \cdot du = \dots$$

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## Example 2 Adjusting the integrand by a constant

$$\int \sqrt{4t-1} dt = ?$$

$$\text{Let } u = 4t - 1, du = 4dt,$$

$$\int \sqrt{4t-1} dt = \int u dt = \frac{1}{4} \int u 4dt = \frac{1}{4} \int \sqrt{u} du = \dots$$

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## Substitution: Running the chain rule backwards

### THEOREM 5 The Substitution Rule

If  $u = g(x)$  is a differentiable function whose range is an interval  $I$  and  $f$  is continuous on  $I$ , then

$$\int f(g(x))g'(x) dx = \int f(u) du.$$

$$\text{let } u = g(x); \int f[g(x)] \cdot g'(x) dx = \int f(u) \cdot \frac{du}{dx} dx = \int f(u) du$$

Used to find the integration with the integrand in the form of the product of  $f[g(x)] \cdot g'(x)$

$$\int \underbrace{f[g(x)]}_{f(u)} \cdot \underbrace{g'(x) dx}_{du} = \int f(u) du$$

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## Example 3 Using substitution

$$\int \cos(\underbrace{7x+5}_u) \frac{dx}{\frac{1}{7} du} = \int \cos u \cdot \frac{du}{7} = \frac{1}{7} \sin u + C = \frac{1}{7} \sin(7x+5) + C$$

62

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## Example 4 Using substitution

$$\int x^2 \sin x^3 dx = ?$$

$$u = x^3; du = 3x^2 dx$$

$$\int \sin \underbrace{x^3}_u \underbrace{x^2 dx}_{\frac{1}{3} du} = \int \sin u \cdot \frac{1}{3} du = -\frac{1}{3} \cos u + C$$

$$= \frac{1}{3} \cos x^3 + C$$

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## Example 5 Using Identities and substitution

$$\int \frac{1}{\cos^2 2x} dx = \int \sec^2 2x dx = \int \sec^2 \underbrace{2x}_u \underbrace{dx}_{\frac{1}{2} du} =$$

$$\frac{1}{2} \int \underbrace{\sec^2 u}_{\frac{d}{du} \tan u} du = \frac{1}{2} \tan u + C = \frac{1}{2} \tan 2x + C$$

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## Example 6 Using different substitutions

$$\int \frac{2z}{\sqrt[3]{z^2+1}} dz = \int \underbrace{(z^2+1)^{-1/3}}_{u^{-1/3}} \underbrace{2z dz}_{du} = \int u^{-1/3} du = \dots$$

65

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## The integrals of $\sin^2 x$ and $\cos^2 x$

### ■ Example 7

$$\begin{aligned} \int \sin^2 x dx &= \frac{1}{2} \int 1 - \cos 2x dx \\ &= \frac{x}{2} - \frac{1}{2} \int \cos \underbrace{2x}_u \underbrace{dx}_{\frac{1}{2} du} \\ &= \frac{x}{2} - \frac{1}{4} \int \cos u du = \dots \end{aligned}$$

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## The integrals of $\sin^2 x$ and $\cos^2 x$

### ■ Example 7(b)

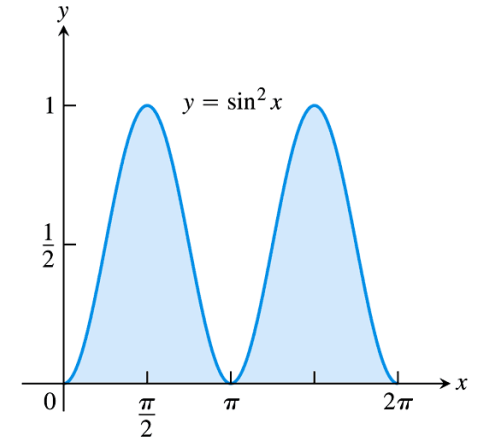
$$\int \cos^2 x \, dx = \frac{1}{2} \int \cos 2x + 1 \, dx = \dots$$

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## Example 8 Area beneath the curve $y = \sin^2 x$

- For Figure 5.24, find
- (a) the definite integral of  $y(x)$  over  $[0, 2\pi]$ .
- (b) the area between the graph and the  $x$ -axis over  $[0, 2\pi]$ .



**FIGURE 5.24** The area beneath the curve  $y = \sin^2 x$  over  $[0, 2\pi]$  equals  $\pi$  square units (Example 8).

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## 5.6

### Substitution and Area Between Curves

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## Substitution formula

### **THEOREM 6** Substitution in Definite Integrals

If  $g'$  is continuous on the interval  $[a, b]$  and  $f$  is continuous on the range of  $g$ , then

$$\int_a^b f(g(x)) \cdot g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$$

$$\text{let } u = g(x); \int_{x=a}^{x=b} f[g(x)] \cdot g'(x) \, dx = \int_{u=g(a)}^{u=g(b)} f[u] \cdot \frac{du}{dx} \, dx = \int_{u=g(a)}^{u=g(b)} f(u) \, du$$

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## Example 1 Substitution

■ Evaluate

$$\int_{-1}^1 3x^2 \sqrt{x^3 + 1} \, dx$$

$$\int_{x=-1}^{x=1} \underbrace{\sqrt{x^3 + 1}}_{u^{1/2}} \cdot \underbrace{3x^2 dx}_{du} = \int_{u(x=-1)}^{u(x=1)} u^{1/2} \cdot du = \dots$$

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## Example 2 Using the substitution formula

$$\int_{x=\pi/4}^{x=\pi/2} \cot x \csc^2 x \, dx = ?$$

$$\int \cot x \csc^2 x \, dx = \int \underbrace{\cot x}_u \cdot \underbrace{\csc^2 x \, dx}_{-du} = -\int u \, du = -\frac{u^2}{2} + c$$

$$= -\frac{\cot^2 x}{2} + c$$

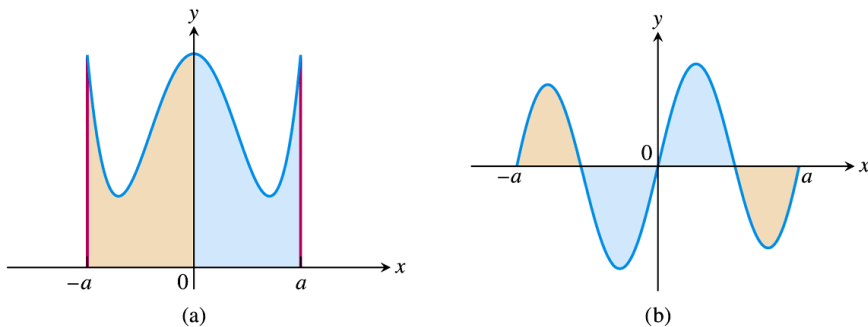
$$\int_{\pi/4}^{\pi/2} \cot x \csc^2 x \, dx = -\frac{\cot^2 x}{2} \Big|_{\pi/4}^{\pi/2} = \frac{\cot^2 x}{2} \Big|_{\pi/2}^{\pi/4} = \frac{1}{2} \left[ \underbrace{\cot^2(\pi/4)}_1 - \underbrace{\cot^2(\pi/2)}_0 \right] = \frac{1}{2}$$

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## Definite integrals of symmetric functions



**FIGURE 5.26** (a)  $f$  even,  $\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx$  (b)  $f$  odd,  $\int_{-a}^a f(x) \, dx = 0$

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### Theorem 7

Let  $f$  be continuous on the symmetric interval  $[-a, a]$ .

(a) If  $f$  is even, then  $\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx$ .

(b) If  $f$  is odd, then  $\int_{-a}^a f(x) \, dx = 0$ .

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## Example 3 Integral of an even function

Evaluate  $\int_{-2}^2 (x^4 - 4x^2 + 6) dx$

Solution:

$$f(x) = x^4 - 4x^2 + 6;$$

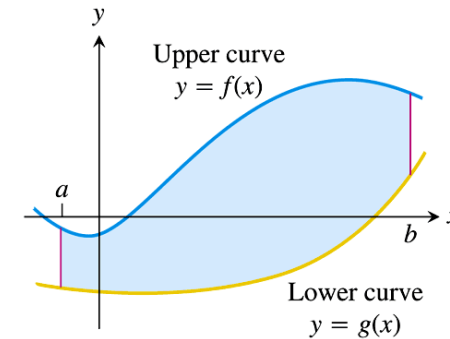
$$f(-x) = (-x)^4 - 4(-x)^2 + 6 = x^4 - 4x^2 + 6 = f(x)$$

even function

How about integration of the same function from  $x=-1$  to  $x=2$

75

## Area between curves

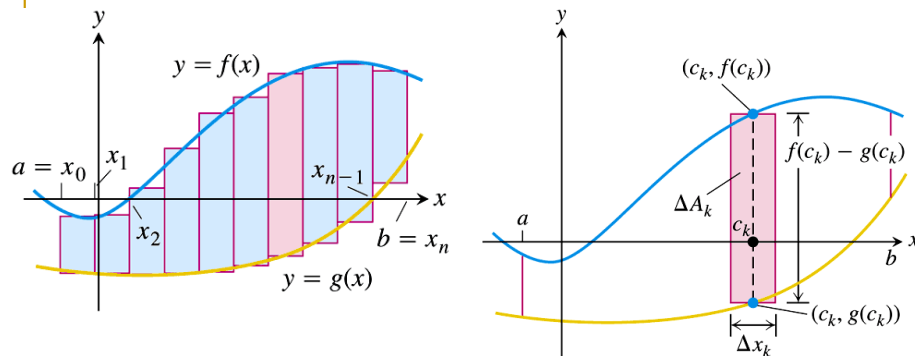


**FIGURE 5.27** The region between the curves  $y = f(x)$  and  $y = g(x)$  and the lines  $x = a$  and  $x = b$ .

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**FIGURE 5.28** We approximate the region with rectangles perpendicular to the  $x$ -axis.

**FIGURE 5.29** The area  $\Delta A_k$  of the  $k$ th rectangle is the product of its height,  $f(c_k) - g(c_k)$ , and its width,  $\Delta x_k$ .

$$A \approx \sum_{k=1}^n \Delta A_k = \sum_{k=1}^n \Delta x_k [f(c_k) - g(c_k)]$$

$$A = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \Delta x_k [f(c_k) - g(c_k)] = \int_a^b [f(x) - g(x)] dx$$

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### DEFINITION Area Between Curves

If  $f$  and  $g$  are continuous with  $f(x) \geq g(x)$  throughout  $[a, b]$ , then the **area of the region between the curves  $y = f(x)$  and  $y = g(x)$  from  $a$  to  $b$**  is the integral of  $(f - g)$  from  $a$  to  $b$ :

$$A = \int_a^b [f(x) - g(x)] dx.$$

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## Example 4 Area between intersecting curves

- Find the area of the region enclosed by the parabola  $y = 2 - x^2$  and the line  $y = -x$ .

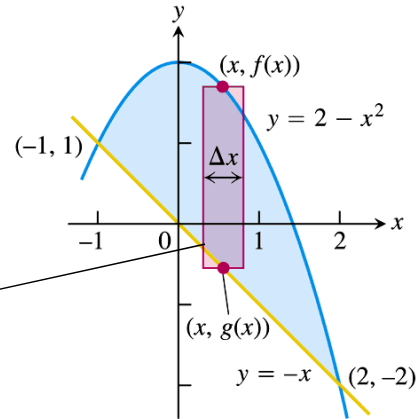


FIGURE 5.30 The region in Example 4 with a typical approximating rectangle.

$$\Delta A = (f(x) - g(x)) \cdot \Delta x$$

$$A = \lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta A_k = \int_0^A dA;$$

$$A = \int_{a=-1}^{b=2} \underbrace{[f(x)]}_{2-x^2} - \underbrace{[g(x)]}_x dx$$

$$= \int_{-1}^2 (2 - x^2 - x) dx = \dots$$

## Example 5 Changing the integral to match a boundary change

- Find the area of the shaded region

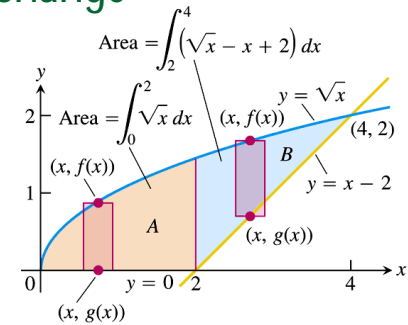


FIGURE 5.31 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 5.

$$Area = A + B$$

$$A = \int_0^2 \sqrt{x} dx;$$

$$B = \int_2^4 \sqrt{x} - (x - 2) dx$$

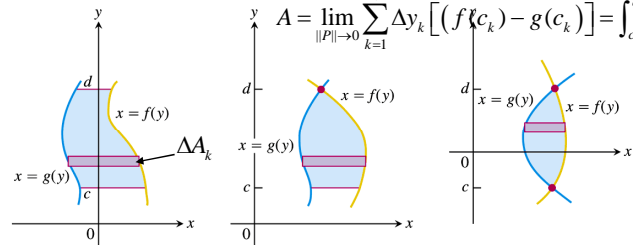
## Integration with Respect to y

If a region's bounding curves are described by functions of  $y$ , the approximating rectangles are horizontal instead of vertical and the basic formula has  $y$  in place of  $x$ .

For regions like these

$$A \approx \sum_{k=1}^n \Delta A_k = \sum_{k=1}^n \Delta y_k [(f(c_k) - g(c_k))]$$

$$A = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \Delta y_k [(f(c_k) - g(c_k))] = \int_c^d [f(y) - g(y)] dy$$

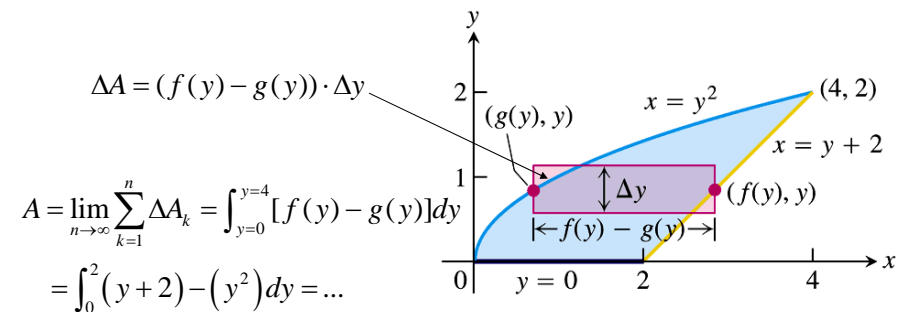


use the formula

$$A = \int_c^d [f(y) - g(y)] dy.$$

In this equation  $f$  always denotes the right-hand curve and  $g$  the left-hand curve, so  $f(y) - g(y)$  is nonnegative.

## Example 6 Find the area of the region in Example 5 by integrating with respect to y



$$\Delta A = (f(y) - g(y)) \cdot \Delta y$$

$$A = \lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta A_k = \int_{y=0}^{y=2} [f(y) - g(y)] dy$$

$$= \int_0^2 (y + 2) - (y^2) dy = \dots$$

FIGURE 5.32 It takes two integrations to find the area of this region if we integrate with respect to  $x$ . It takes only one if we integrate with respect to  $y$  (Example 6).

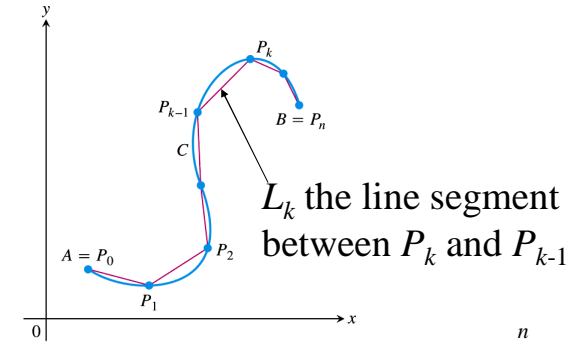


# 6.3

## Lengths of Plane Curves

1

# Length of a parametrically defined curve



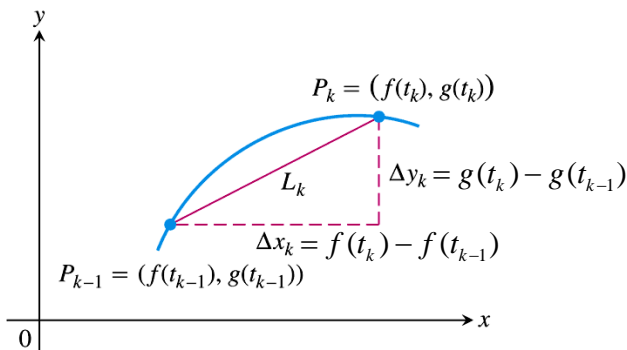
**FIGURE 6.24** The curve  $C$  defined parametrically by the equations  $x = f(t)$  and  $y = g(t)$ ,  $a \leq t \leq b$ . The length of the curve from  $A$  to  $B$  is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at  $A = P_0$ , then to  $P_1$ , and so on, ending at  $B = P_n$ .

$$L = \lim_{\|P\| \rightarrow 0} \sum_k^n L_k$$

2

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**FIGURE 6.25** The arc  $P_{k-1}P_k$  is approximated by the straight line segment shown here, which has length  $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$ .

3

$y$  is parametrized by  $t$  via  $y = g(t)$ ;

$x$  is parametrized by  $t$  via  $x = f(t)$ .

$$\Delta y_k = g(t_k) - g(t_{k-1}) = g'(t_k^*) \cdot (t_k - t_{k-1}) = g'(t_k^*) \cdot \Delta t;$$

$$\Delta x_k = f(t_k) - f(t_{k-1}) = f'(t_k^{**}) \cdot (t_k - t_{k-1}) = f'(t_k^{**}) \cdot \Delta t$$

due to mean value theorem

$$L_k = \sqrt{(\Delta y_k)^2 + (\Delta x_k)^2} = \Delta t \sqrt{(g'(t_k^*))^2 + (f'(t_k^{**}))^2}$$

$$L = \lim_{n \rightarrow \infty} \sum_k^n L_k = \lim_{\|P\| \rightarrow 0} \sum_k^n L_k$$

$$= \lim_{\|P\| \rightarrow 0} \sum_k^n \Delta t \sqrt{(g'(t_k^*))^2 + (f'(t_k^{**}))^2}$$

$$= \int_a^b \sqrt{(g'(t))^2 + (f'(t))^2} dt = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt$$

4

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**DEFINITION** Length of a Parametric Curve

If a curve  $C$  is defined parametrically by  $x = f(t)$  and  $y = g(t)$ ,  $a \leq t \leq b$ , where  $f'$  and  $g'$  are continuous and not simultaneously zero on  $[a, b]$ , and  $C$  is traversed exactly once as  $t$  increases from  $t = a$  to  $t = b$ , then the length of  $C$  is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

5

**Example 1** The circumference of a circle

- Find the length of the circle of radius  $r$  defined parametrically by
- $x = r \cos t$  and  $y = r \sin t$ ,  $0 \leq t \leq 2\pi$

$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt \equiv \int_0^{2\pi} \sqrt{(r \cos t)^2 + (r \sin t)^2} dt \\ &= r \int_0^{2\pi} dt = 2\pi r \end{aligned}$$

6

**Length of a curve  $y = f(x)$** 

Assign the parameter  $x = t$ , the length of the curve  $y = f(x)$  is then given by

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt$$

$$y = y[x(t)] \Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx} \quad \left(\because \frac{dx}{dt} = 1\right)$$

$$\begin{aligned} L &= \int_a^b dt \sqrt{\left(\frac{dy}{dx} \cdot \frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \int_a^b dx \sqrt{\left(\frac{dy}{dx}\right)^2 + 1} \\ &= \int_a^b dx \sqrt{[f'(x)]^2 + 1} \end{aligned}$$

7

**Formula for the Length of  $y = f(x)$ ,  $a \leq x \leq b$** 

If  $f$  is continuously differentiable on the closed interval  $[a, b]$ , the length of the curve (graph)  $y = f(x)$  from  $x = a$  to  $x = b$  is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + [f'(x)]^2} dx. \quad (2)$$

8

### Example 3 Applying the arc length formula for a graph

- Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \leq x \leq 1$$

9

### Dealing with discontinuity in $dy/dx$

- At a point on a curve where  $dy/dx$  fails to exist and we may be able to find the curve's length by expressing  $x$  as a function of  $y$  and applying the following

**Formula for the Length of  $x = g(y)$ ,  $c \leq y \leq d$**

If  $g$  is continuously differentiable on  $[c, d]$ , the length of the curve  $x = g(y)$  from  $y = c$  to  $y = d$  is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d \sqrt{1 + [g'(y)]^2} dy. \quad (3)$$

10

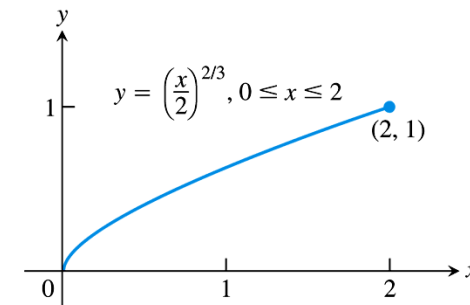
### Example 4 Length of a graph which has a discontinuity in $dy/dx$

- Find the length of the curve  $y = (x/2)^{2/3}$  from  $x = 0$  to  $x = 2$ .

- Solution**

- $dy/dx = (1/3)(2/x)^{1/3}$  is not defined at  $x=0$ .
- $dx/dy = 3y^{1/2}$  is continuous on  $[0, 1]$ .

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**FIGURE 6.27** The graph of  $y = (x/2)^{2/3}$  from  $x = 0$  to  $x = 2$  is also the graph of  $x = 2y^{3/2}$  from  $y = 0$  to  $y = 1$  (Example 4).

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# Chapter 7

## Transcendental Functions

1

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### DEFINITION One-to-One Function

A function  $f(x)$  is **one-to-one** on a domain  $D$  if  $f(x_1) \neq f(x_2)$  whenever  $x_1 \neq x_2$  in  $D$ .

3

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# 7.1

## Inverse Functions and Their Derivatives

2

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### Example 1 Domains of one-to-one functions

- (a)  $f(x) = x^{1/2}$  is one-to-one on any domain of nonnegative numbers
- (b)  $g(x) = \sin x$  is NOT one-to-one on  $[0, \pi]$  but one-to-one on  $[0, \pi/2]$ .

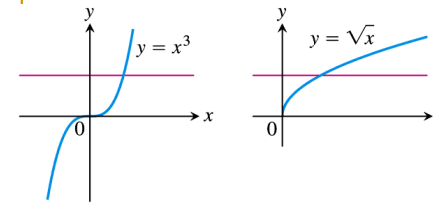
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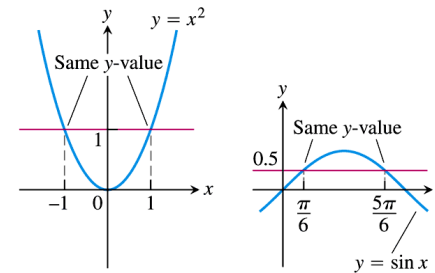
### The Horizontal Line Test for One-to-One Functions

A function  $y = f(x)$  is one-to-one if and only if its graph intersects each horizontal line at most once.

5



One-to-one: Graph meets each horizontal line at most once.



Not one-to-one: Graph meets one or more horizontal lines more than once.

**FIGURE 7.1** Using the horizontal line test, we see that  $y = x^3$  and  $y = \sqrt{x}$  are one-to-one on their domains  $(-\infty, \infty)$  and  $[0, \infty)$ , but  $y = x^2$  and  $y = \sin x$  are not one-to-one on their domains  $(-\infty, \infty)$ .

6

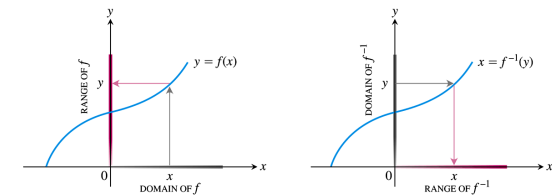
### DEFINITION Inverse Function

Suppose that  $f$  is a one-to-one function on a domain  $D$  with range  $R$ . The **inverse function**  $f^{-1}$  is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a.$$

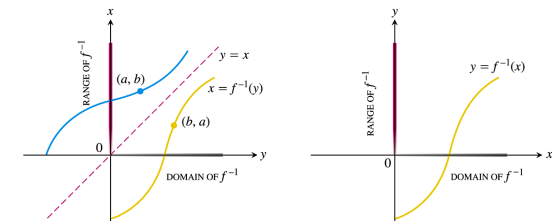
The domain of  $f^{-1}$  is  $R$  and the range of  $f^{-1}$  is  $D$ .

7



(a) To find the value of  $f$  at  $x$ , we start at  $x$ , go up to the curve, and then over to the  $y$ -axis.

(b) The graph of  $f$  is already the graph of  $f^{-1}$ , but with  $x$  and  $y$  interchanged. To find the  $x$  that gave  $y$ , we start at  $y$  and go over to the curve and down to the  $x$ -axis. The domain of  $f^{-1}$  is the range of  $f$ . The range of  $f^{-1}$  is the domain of  $f$ .



(c) To draw the graph of  $f^{-1}$  in the more usual way, we reflect the system in the line  $y = x$ .

(d) Then we interchange the letters  $x$  and  $y$ . We now have a normal-looking graph of  $f^{-1}$  as a function of  $x$ .

**FIGURE 7.2** Determining the graph of  $y = f^{-1}(x)$  from the graph of  $y = f(x)$ .

8

## Finding inverses

- 1. Solve the equation  $y = f(x)$  for  $x$ . This gives a formula  $x = f^{-1}(y)$  where  $x$  is expressed as a function of  $y$ .
- 2. Interchange  $x$  and  $y$ , obtaining a formula  $y = f^{-1}(x)$  where  $f^{-1}(x)$  is expressed in the conventional format with  $x$  as the independent variable and  $y$  as the dependent variables.

9

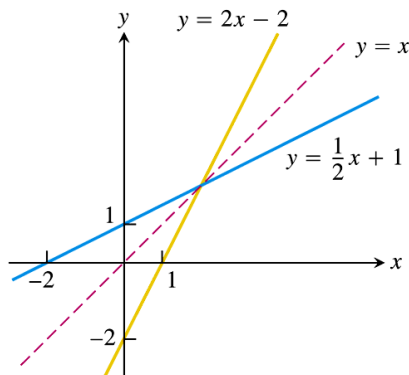
## Example 2 Finding an inverse function

- Find the inverse of  $y = x/2 + 1$ , expressed as a function of  $x$ .
- Solution**
  - 1. solve for  $x$  in terms of  $y$ :  $x = 2(y - 1)$
  - 2. interchange  $x$  and  $y$ :  $y = 2(x - 1)$
  - The inverse function  $f^{-1}(x) = 2(x - 1)$
  - Check:
    - $f^{-1}[f(x)] = 2[f(x) - 1] = 2[(x/2 + 1) - 1] = x = f[f^{-1}(x)]$

10

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**FIGURE 7.3** Graphing  $f(x) = (1/2)x + 1$  and  $f^{-1}(x) = 2x - 2$  together shows the graphs' symmetry with respect to the line  $y = x$ . The slopes are reciprocals of each other (Example 2).

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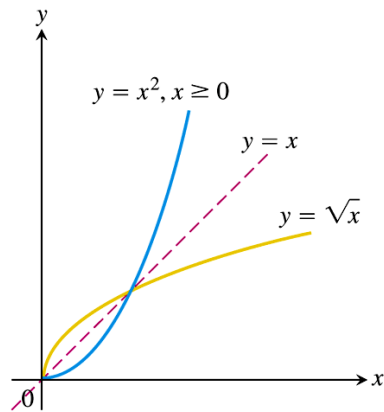
## Example 3 Finding an inverse function

- Find the inverse of  $y = x^2$ ,  $x \geq 0$ , expressed as a function of  $x$ .
- Solution**
  - 1. solve for  $x$  in terms of  $y$ :  $x = \sqrt{y}$
  - 2. interchange  $x$  and  $y$ :  $y = \sqrt{x}$
  - The inverse function  $f^{-1}(x) = \sqrt{x}$

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**FIGURE 7.4** The functions  $y = \sqrt{x}$  and  $y = x^2, x \geq 0$ , are inverses of one another (Example 3).

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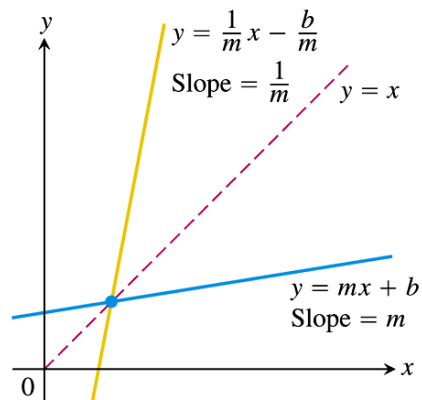
## Derivatives of inverses of differentiable functions

- From example 2 (a linear function)
- $f(x) = x/2 + 1; f^{-1}(x) = 2(x + 1);$
- $df(x)/dx = 1/2; df^{-1}(x)/dx = 2,$
- i.e.  $df(x)/dx = 1/df^{-1}(x)/dx$
- Such a result is obvious because their graphs are obtained by reflecting on the  $y = x$  line.
- Does the reciprocal relationship between the slopes of  $f$  and  $f^{-1}$  holds for other functions as well?

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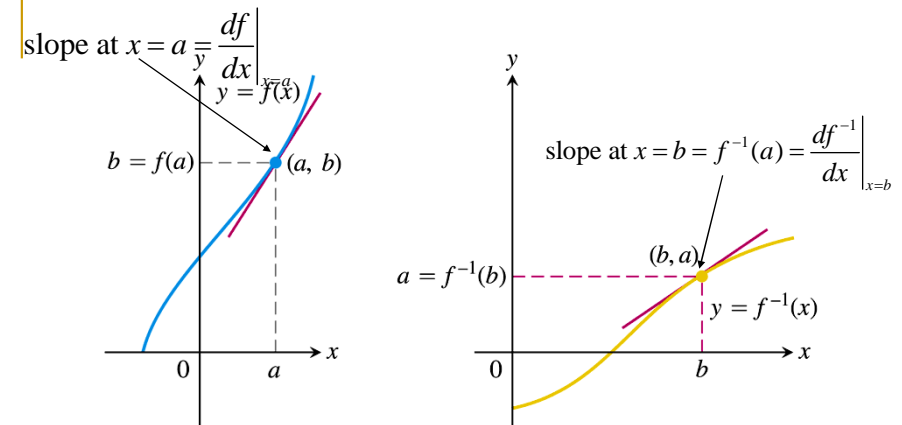
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**FIGURE 7.5** The slopes of nonvertical lines reflected through the line  $y = x$  are reciprocals of each other.

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The slopes are reciprocal:  $(f^{-1})'(b) = \frac{1}{f'(a)}$  or  $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$

**FIGURE 7.6** The graphs of inverse functions have reciprocal slopes at corresponding points.

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## Example 4 Applying theorem 1

### THEOREM 1 The Derivative Rule for Inverses

If  $f$  has an interval  $I$  as domain and  $f'(x)$  exists and is never zero on  $I$ , then  $f^{-1}$  is differentiable at every point in its domain. The value of  $(f^{-1})'$  at a point  $b$  in the domain of  $f^{-1}$  is the reciprocal of the value of  $f'$  at the point  $a = f^{-1}(b)$ :

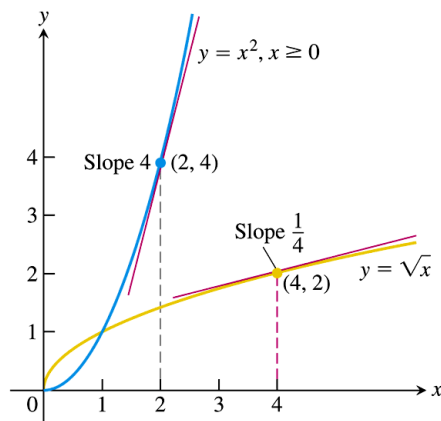
$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left. \frac{df^{-1}}{dx} \right|_{x=b} = \frac{1}{\left. \frac{df}{dx} \right|_{x=f^{-1}(b)}} \quad (1)$$

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**FIGURE 7.7** The derivative of  $f^{-1}(x) = \sqrt{x}$  at the point  $(4, 2)$  is the reciprocal of the derivative of  $f(x) = x^2$  at  $(2, 4)$  (Example 4).

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- The function  $f(x) = x^2$ ,  $x \geq 0$  and its inverse  $f^{-1}(x) = \sqrt{x}$  have derivatives  $f'(x) = 2x$ , and  $(f^{-1})'(x) = 1/(2\sqrt{x})$ .
- Theorem 1 predicts that the derivative of  $f^{-1}(x)$  is  $(f^{-1})'(x) = 1/f'[f^{-1}(x)] = 1/f'[\sqrt{x}] = 1/(2\sqrt{x})$

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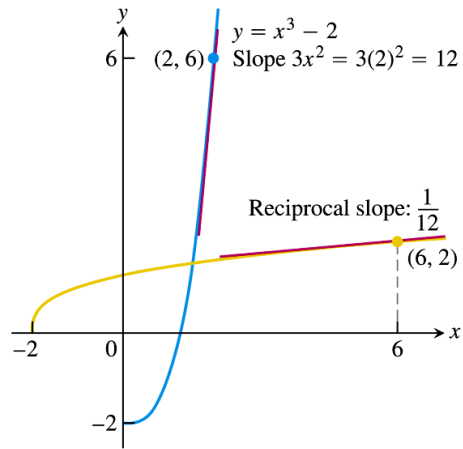
## Example 5 Finding a value of the inverse derivative

- Let  $f(x) = x^3 - 2$ . Find the value of  $df^{-1}/dx$  at  $x = 6 = f(2)$  without a formula for  $f^{-1}$ .
- The point for  $f$  is  $(2, 6)$ ; The corresponding point for  $f^{-1}$  is  $(6, 2)$ .
- **Solution**
- $df/dx = 3x^2$
- $df^{-1}/dx|_{x=6} = 1/(df/dx|_{x=2}) = 1/3x^2|_{x=2} = 1/12$

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**FIGURE 7.8** The derivative of  $f(x) = x^3 - 2$  at  $x = 2$  tells us the derivative of  $f^{-1}$  at  $x = 6$  (Example 5).

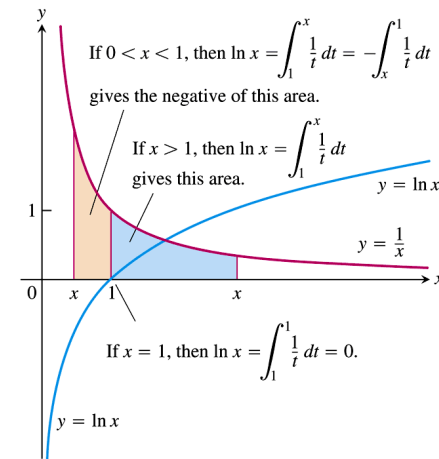
## 7.2

### Natural Logarithms

## Definition of natural logarithmic function

### DEFINITION The Natural Logarithm Function

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0$$



**FIGURE 7.9** The graph of  $y = \ln x$  and its relation to the function  $y = 1/x, x > 0$ . The graph of the logarithm rises above the  $x$ -axis as  $x$  moves from 1 to the right, and it falls below the axis as  $x$  moves from 1 to the left.

- Domain of  $\ln x = (0, \infty)$
- Range of  $\ln x = (-\infty, \infty)$
- $\ln x$  is an increasing function since  $dy/dx = 1/x > 0$

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$$\ln x = \int_1^x \frac{1}{x} dx$$

TABLE 7.1 Typical 2-place values of  $\ln x$

$x$	$\ln x$
0	undefined
0.05	-3.00
0.5	-0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

$e$  lies between 2 and 3

$\ln x = 1$

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**DEFINITION The Number  $e$**

The number  $e$  is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1$$

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By definition, the antiderivative of  $\ln x$  is just  $1/x$

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

$$\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}, \quad u > 0 \quad (1)$$

Let  $u = u(x)$ . By chain rule,

$$d/dx [\ln u(x)] = d/du(\ln u) \cdot du(x)/dx$$

$$= (1/u) \cdot du(x)/dx$$

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## Example 1 Derivatives of natural logarithms

$$(a) \frac{d}{dx} \ln 2x =$$

$$(b) u = x^2 + 3; \frac{d}{dx} \ln u = \frac{du}{dx} \frac{1}{u} =$$

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## Example 2 Interpreting the properties of logarithms

$$(a) \ln 6 = \ln(2 \cdot 3) = \ln 2 + \ln 3;$$

$$(b) \ln 4 - \ln 5 = \ln(4/5) = \ln 0.8$$

$$(c) \ln(1/8) = \ln 1 - \ln 2^3 = -3 \ln 2$$

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## Properties of logarithms

### THEOREM 2 Properties of Logarithms

For any numbers  $a > 0$  and  $x > 0$ , the natural logarithm satisfies the following rules:

1. *Product Rule:*  $\ln ax = \ln a + \ln x$
2. *Quotient Rule:*  $\ln \frac{a}{x} = \ln a - \ln x$
3. *Reciprocal Rule:*  $\ln \frac{1}{x} = -\ln x$  Rule 2 with  $a = 1$
4. *Power Rule:*  $\ln x^r = r \ln x$   $r$  rational

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## Example 3 Applying the properties to function formulas

$$(a) \ln 4 + \ln \sin x = \ln(4 \sin x);$$

$$(b) \ln \frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$$

$$(c) \ln(\sec x) = \ln \frac{1}{\cos x} = -\ln \cos x$$

$$(d) \ln \sqrt[3]{x+1} = \ln(x+1)^{1/3} = (1/3) \ln(x+1)$$

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## Proof of $\ln ax = \ln a + \ln x$

- $\ln ax$  and  $\ln x$  have the same derivative:

$$\frac{d}{dx} \ln ax = \frac{d(ax)}{dx} \frac{1}{ax} = a \frac{1}{ax} = \frac{1}{x} = \frac{d}{dx} \ln x$$

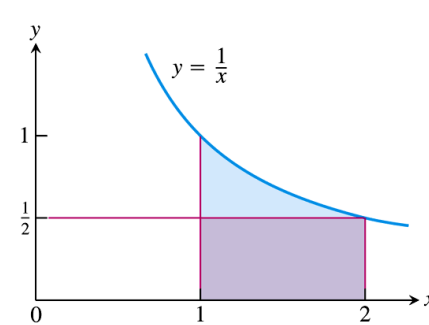
- Hence, by the corollary 2 of the mean value theorem, they differ by a constant  $C$

$$\ln ax = \ln x + C$$

- We will prove that  $C = \ln a$  by applying the definition  $\ln x$  at  $x = 1$ .

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## Estimate the value of $\ln 2$



$$\begin{aligned} \ln 2 &= \int_1^2 \frac{1}{x} dx \\ \frac{1}{2} \cdot (2-1) &< \int_1^2 \frac{1}{x} dx < 1 \cdot (2-1) = 1 \\ \frac{1}{2} &< \ln 2 < 1 \end{aligned}$$

**FIGURE 7.10** The rectangle of height  $y = 1/2$  fits beneath the graph of  $y = 1/x$  for the interval  $1 \leq x \leq 2$ .

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## The integral $\int (1/u) du$

$$\text{From } \frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}$$

For  $u > 0$

Taking the integration on both sides gives

$$\int \frac{d}{dx} \ln u dx = \int \frac{1}{u} \frac{du}{dx} dx.$$

$$\text{Let } y = \ln u \rightarrow \frac{d}{dx} \ln u dx = \frac{dy}{dx} dx = dy \rightarrow \int \frac{d}{dx} \ln u dx = \int dy = \int d \ln u$$

$$\int d \ln u = \int \frac{du}{u} \rightarrow \ln u + C' = \int \frac{du}{u};$$

For  $u < 0$ :

$-u > 0$ ,

$$\int \frac{d}{dx} \ln(-u) dx = \int \frac{1}{(-u)} \frac{d(-u)}{dx} dx$$

$$\int d \ln(-u) = \int \frac{du}{u} \rightarrow \ln(-u) + C'' = \int \frac{du}{u}$$

Combining both cases of  $u > 0, u < 0$ ,

$$\int \frac{du}{u} = \ln |u| + C$$

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$$\text{recall: } \int u^n du = \frac{u^{n+1}}{n+1} + C, n \text{ rational, } \neq -1$$

If  $u$  is a differentiable function that is never zero,

$$\int \frac{1}{u} du = \ln |u| + C. \quad (5)$$

$$\text{From } \int u^{-1} du = \ln |u| + C.$$

let  $u = f(x)$ .

$$\int u^{-1} du = \int \frac{du}{u} = \int \frac{df(x)}{f(x)} = \int \frac{\frac{df(x)}{dx} dx}{f(x)}$$

$$\Rightarrow \int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

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## Example 4 Applying equation (5)

$$(a) \int \frac{2x dx}{x^2 - 5} = \int \frac{d(x^2 - 5)}{x^2 - 5} = \ln |x^2 - 5| + C$$

$$(b) \int_{-\pi/2}^{\pi/2} \frac{4 \cos x}{3 + 2 \sin x} dx = \dots$$

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## The integrals of $\tan x$ and $\cot x$

$$\int \tan u \, du = -\ln |\cos u| + C = \ln |\sec u| + C$$

$$\int \cot u \, du = \ln |\sin u| + C = -\ln |\csc x| + C$$

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## Example 5

$$\begin{aligned} \int \tan 2x dx &= \int \frac{\sin 2x}{\cos 2x} dx = \int -\frac{1}{2} \frac{d \cos 2x}{\cos 2x} dx \\ &= -\frac{1}{2} \int \frac{d \cos 2x}{\cos 2x} = -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln |u| + C \\ &= -\frac{1}{2} \ln |\cos 2x| + C \\ &= \frac{1}{2} \ln |\sec 2x| + C \end{aligned}$$

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## Example 6 Using logarithmic differentiation

■ Find  $dy/dx$  if  $y = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}, x > 1$

$$\ln y = \ln(x^2 + 1) + (1/2) \ln(x + 3) - \ln(x - 1)$$

$$\frac{d}{dx} \ln y = \frac{d}{dx} \ln(x^2 + 1) + \frac{1}{2} \frac{d}{dx} \ln(x + 3) - \frac{d}{dx} \ln(x - 1)$$

$$\frac{1}{y} \frac{dy}{dx} = \dots$$

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# 7.3

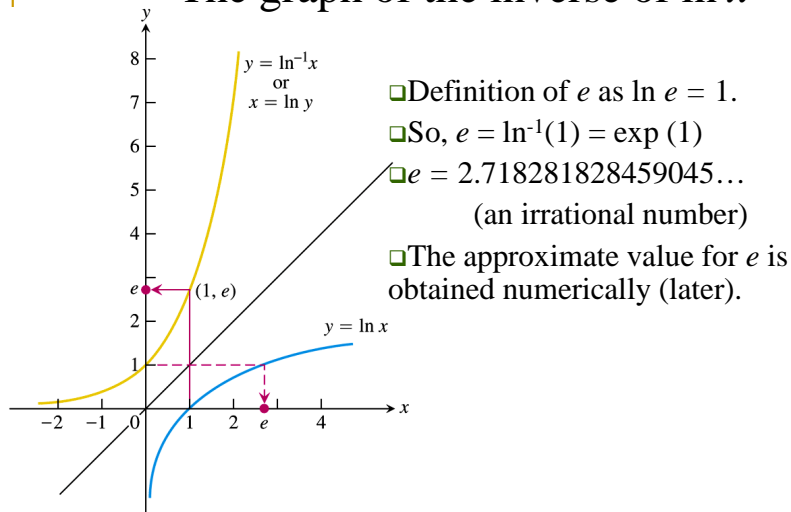
## The Exponential Function

## The inverse of $\ln x$ and the number $e$

- $\ln x$  is one-to-one, hence it has an inverse. We name the inverse of  $\ln x$ ,  $\ln^{-1} x$  as  $\exp(x)$

$$\lim_{x \rightarrow \infty} \ln^{-1} x = \infty, \lim_{x \rightarrow -\infty} \ln^{-1} x = 0$$

## The graph of the inverse of $\ln x$



**FIGURE 7.11** The graphs of  $y = \ln x$  and  $y = \ln^{-1} x = \exp x$ . The number  $e$  is  $\ln^{-1} 1 = \exp(1)$ .

## The function $y = e^x$ in terms of the exponential function $\exp$

- We can raise the number  $e$  to a rational power  $r$ ,  $e^r$
- $e^r$  is positive since  $e$  is positive, hence  $e^r$  has a logarithm (recall that logarithm is defined only for positive number).
- From the power rule of theorem 2 on the properties of natural logarithm,  $\ln x^r = r \ln x$ , where  $r$  is rational, we have

$$\ln e^r = r$$

- We take the inverse to obtain

$$\ln^{-1}(\ln e^r) = \ln^{-1}(r)$$

$$e^r = \ln^{-1}(r) = \exp r, \text{ for } r \text{ rational.}$$

## The number $e$ to a real (possibly irrational) power $x$

- How do we define  $e^x$  where  $x$  is irrational?
- This can be defined by assigning  $e^x$  as  $\exp x$  since  $\ln^{-1}(x)$  is defined (because the inverse function of  $\ln x$  is defined for all real  $x$ ).

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Typical values of  $e^x$

$x$	$e^x$ (rounded)
-1	0.37
0	1
1	2.72
2	7.39
10	22026
100	$2.6881 \times 10^{43}$

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### DEFINITION The Natural Exponential Function

For every real number  $x$ ,  $e^x = \ln^{-1} x = \exp x$ .

Note: please do make a distinction between  $e^x$  and  $\exp x$ . They have different definitions.

$e^x$  is the number  $e$  raised to the power of real number  $x$ .

$\exp x$  is defined as the inverse of the logarithmic function,  $\exp x = \ln^{-1} x$

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### Inverse Equations for $e^x$ and $\ln x$

$$e^{\ln x} = x \quad (\text{all } x > 0) \quad (2)$$

$$\ln(e^x) = x \quad (\text{all } x) \quad (3)$$

- (2) follows from the definition of the exponent function:
- From  $e^x = \exp x$ , let  $x \rightarrow \ln x$
- $e^{\ln x} = \exp[\ln x] = x$  (by definition).
- For (3): From  $e^x = \exp x$ , take logarithm both sides,  $\rightarrow \ln e^x = \ln [\exp x] = x$  (by definition)

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## Example 1 Using inverse equations

$$(a) \ln e^2 = \dots$$

$$(b) \ln e^{-1} = \dots$$

$$(c) \ln \sqrt{e} = \ln e^{1/2} = \dots$$

$$(d) \ln e^{\sin x} = \dots$$

$$(f) e^{\ln 2} = \dots$$

$$(g) e^{\ln(x^2+1)} = \dots$$

$$(h) e^{3 \ln 2} = e^{\ln 2^3} = \dots$$

$$(i) e^{3 \ln 2} = e^{3 \cdot \ln 2} = (e^{\ln 2})^3 = \dots$$

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## Example 2 Solving for an exponent

- Find  $k$  if  $e^{2k}=10$ .

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## The general exponential function $a^x$

- Since  $a = e^{\ln a}$  for any positive number  $a$
- $a^x = (e^{\ln a})^x = e^{x \ln a}$

### DEFINITION General Exponential Functions

For any numbers  $a > 0$  and  $x$ , the exponential function with base  $a$  is

$$a^x = e^{x \ln a}.$$

For the first time we have a precise meaning for an irrational exponent. (previously  $a^x$  is defined for only rational  $x$  and  $a$ )

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## Example 3 Evaluating exponential functions

$$(a) 2^{\sqrt{3}} = (e^{\ln 2})^{\sqrt{3}} = e^{\sqrt{3} \ln 2} \approx e^{1.20} \approx 3.32$$

$$(b) 2^\pi = (e^{\ln 2})^\pi = e^{\pi \ln 2} \approx e^{2.18} \approx 8.8$$

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## Laws of exponents

### THEOREM 3 Laws of Exponents for $e^x$

For all numbers  $x, x_1$ , and  $x_2$ , the natural exponential  $e^x$  obeys the following laws:

1.  $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2.  $e^{-x} = \frac{1}{e^x}$
3.  $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4.  $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Theorem 3 also valid for  $a^x$

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## Proof of law 1

$$y_1 = e^{x_1}, y_2 = e^{x_2}$$

$$\Rightarrow x_1 = \ln y_1, x_2 = \ln y_2$$

$$\Rightarrow x_1 + x_2 = \ln y_1 + \ln y_2 = \ln y_1 y_2$$

$$\Rightarrow \exp(x_1 + x_2) = \exp(\ln y_1 y_2)$$

$$e^{x_1+x_2} = y_1 y_2 = e^{x_1} e^{x_2}$$

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## Example 4 Applying the exponent laws

$$(a) e^{x+\ln 2} =$$

$$(b) e^{-\ln x} =$$

$$(c) \frac{e^{2x}}{e} =$$

$$(d) (e^3)^x =$$

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## The derivative and integral of $e^x$

$$f(x) = \ln x, y = e^x = \ln^{-1} x = f^{-1}(x)$$

$$\frac{dy}{dx} = \frac{d}{dx} e^x = \frac{d}{dx} f^{-1}(x) = \frac{1}{\left. \frac{df(x)}{dx} \right|_{x=f^{-1}(x)}}$$

$$= \frac{1}{(1/x)|_{x=f^{-1}(x)}} = \frac{1}{(1/x)|_{x=y}} = y = e^x$$

$$\frac{d}{dx} e^x = e^x \quad (5)$$

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## Example 5 Differentiating an exponential

$$\frac{d}{dx}(5e^x) =$$

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By the virtue of the chain rule, we obtain

If  $u$  is any differentiable function of  $x$ , then

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}. \quad (6)$$

$$f(u) = e^u; u = u(x);$$

$$\frac{d}{dx}(e^{u(x)}) = \frac{d}{dx} f(u) = \frac{df(u)}{du} \frac{du(x)}{dx} = e^u \frac{du}{dx}$$

$$\int e^u du = e^u + C.$$

This is the integral equivalent of (6)

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## Example 7 Integrating exponentials

$$(a) \int_0^{\ln 2} e^{3x} dx =$$

$$(b) \int_0^{\pi/2} e^{\sin x} \cos x dx = \int_0^{\pi/2} \underbrace{e^{\sin x}}_{e^u} \underbrace{\cos x dx}_{du}$$

$$= \int_{u(0)}^{u(\pi/2)} e^u du$$

$$= e^u \Big|_{u(0)}^{u(\pi/2)} = e^{u(\pi/2)} - e^{u(0)} = e^{\sin(\pi/2)} - e^{\sin(0)} = e - 1$$

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## The number $e$ expressed as a limit

### THEOREM 4 The Number $e$ as a Limit

The number  $e$  can be calculated as the limit

$$e = \lim_{x \rightarrow 0} (1 + x)^{1/x}.$$

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# Proof

- If  $f(x) = \ln x$ , then  $f'(x) = 1/x$ , so  $f'(1) = 1$ .  
But by definition of derivative,

- $$f'(y) = \lim_{h \rightarrow 0} \frac{f(y+h) - f(y)}{h}$$
$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \rightarrow 0} \frac{f(1+x) - f(x)}{x}$$
$$= \lim_{x \rightarrow 0} \frac{\ln(1+x) - \ln(1)}{x} = \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}$$
$$= \lim_{x \rightarrow 0} [\ln(1+x)]^{\frac{1}{x}} = \ln \left[ \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} \right] = 1 \quad (\text{since } f'(1) = 1)$$
$$\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = \lim_{y \rightarrow \infty} \left(1 + \frac{1}{y}\right)^y = e$$

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Define  $x^n$  for any real  $x > 0$  as  $x^n = e^{n \ln x}$ .

Here  $n$  need not be rational but can be any real number as long as  $x$  is positive.

Then we can take the logarithm of  $x^n$  :

$$\ln x^n = \ln(e^{n \ln x}) = n \ln x.$$

Note : c.f the power rule in theorem 2.

Can you tell the difference?

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Once  $x^n$  is defined via  $x^n = e^{n \ln x}$ , we can take its differentiation :

$$\frac{d}{dx} x^n = \frac{d}{dx} \left( e^{\frac{u(x)}{n \ln x}} \right) = \frac{du}{dx} \frac{de^u}{du} = \frac{n}{x} e^{n \ln x} = \frac{n}{x} x^n = nx^{n-1}$$
$$\Rightarrow \frac{d}{dx} x^n = nx^{n-1}$$

Note : Can you tell the difference between this formula and the one we discussed in earlier chapters (Theorem 4, Chapter 3)?

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- By virtue of chain rule,

$$u = u(x);$$

$$\frac{d}{dx} u^n = \frac{du(x)}{dx} \frac{du^n}{du} = \frac{du(x)}{dx} n u^{n-1}$$

### Power Rule (General Form)

If  $u$  is a positive differentiable function of  $x$  and  $n$  is any real number, then  $u^n$  is a differentiable function of  $x$  and

$$\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx}.$$

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## Example 9 using the power rule with irrational powers

$$(a) \frac{d}{dx} x^{\sqrt{2}} \equiv \frac{du^n}{dx} = \frac{du}{dx} nu^{n-1}$$

$$\frac{du}{dx} nu^{n-1} \equiv \frac{dx}{dx} \sqrt{2} x^{\sqrt{2}-1} = \sqrt{2} x^{\sqrt{2}-1}$$

$$(b) \frac{d}{dx} (2 + \sin 3x)^\pi \equiv \frac{du^n}{dx} = \frac{du}{dx} nu^{n-1}$$

$$\frac{du}{dx} nu^{n-1} \equiv \frac{d(2 + \sin 3x)}{dx} \pi u^{\pi-1} = 3\pi (2 + \sin 3x)^{\pi-1} \cos 3x$$

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## The derivative of $a^x$

$$a^x = e^{x \ln a}$$

$$\frac{d}{dx} a^x = \frac{d}{dx} \left( e^{x \ln a} \right) = \frac{d}{dx} (x \ln a) \frac{d}{du} (e^u)$$

$$= e^u \ln a = e^{x \ln a} \ln a = a^x \ln a$$

By virtue of the chain rule,

$$\frac{d}{dx} a^{u(x)} = \frac{du}{dx} \frac{d}{du} (a^u) = a^u \ln a \frac{du}{dx}$$

If  $a > 0$  and  $u$  is a differentiable function of  $x$ , then  $a^u$  is a differentiable function of  $x$  and

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx} \quad (1)$$

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## 7.4

### $a^x$ and $\log_a x$

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## Example 1: Differentiating general exponential functions

$$(a) \frac{d}{dx} 3^x = \frac{d}{dx} \left( e^{x \ln 3} \right) = \frac{d}{dx} (x \ln 3) \frac{d}{du} (e^u)$$

$$= \ln 3 \cdot e^{x \ln 3} = 3^x \ln 3$$

$$(b) \frac{d}{dx} 3^{-x} = -\frac{d}{d(-x)} 3^{(-x)} = -\frac{d}{du} 3^u = -\frac{d}{du} 3^u$$

$$= -3^u \ln 3 = -3^{(-x)} \ln 3 = -\ln 3 / 3^x$$

$$(c) \frac{d}{dx} 3^{\sin x} = \frac{du}{dx} \frac{d}{du} 3^u = \frac{d(\sin x)}{dx} 3^u \ln 3 = 3^{\sin x} \ln 3 \cdot \cos x$$

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## Other power functions

- Example 2 Differentiating a general power function
- Find  $dy/dx$  if  $y = x^x$ ,  $x > 0$ .
- **Solution:** Write  $x^x$  as a power of  $e$
- $x^x = e^{x \ln x}$

$$\frac{d}{dx} \left( e^{x \ln x} \right) = \frac{du}{dx} \frac{d}{du} (e^u) = \frac{d}{dx} (x \ln x) \cdot (e^u) = \dots$$

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## Example 3 Integrating general exponential functions

$$(a) \int 2^x dx = \frac{2^x}{\ln 2} + C$$

$$(b) \int 2^{\sin x} \overbrace{\cos dx}^{du} = \int 2^u du = \dots$$

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## Integral of $a^u$

From  $\frac{d}{dx} a^{u(x)} = a^u \ln a \frac{du}{dx}$ , divide by  $\ln a$ :

$$\Rightarrow \frac{1}{\ln a} \frac{d}{dx} a^{u(x)} = a^u \frac{du}{dx}$$

$\Rightarrow \frac{d}{dx} a^{u(x)} = a^u \ln a \frac{du}{dx}$ , integrate both sides wrp to  $dx$ :

$$\Rightarrow \int \left( \frac{d}{dx} a^u \right) dx = \int \left( a^u \ln a \frac{du}{dx} \right) dx:$$

$$\Rightarrow \int da^u = \ln a \int a^u du + C$$

$$\Rightarrow \int a^u du = \frac{1}{\ln a} \int da^u = \frac{a^u}{\ln a}$$

$$\int a^u du = \frac{a^u}{\ln a} + C. \quad (2)$$

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## Logarithm with base $a$

### DEFINITION $\log_a x$

For any positive number  $a \neq 1$ ,

$\log_a x$  is the inverse function of  $a^x$ .

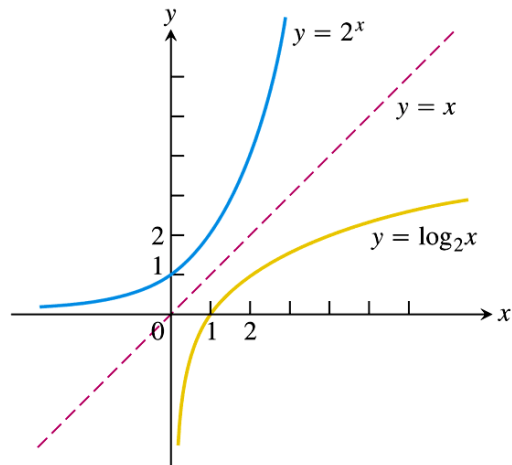
### Inverse Equations for $a^x$ and $\log_a x$

$$a^{\log_a x} = x \quad (x > 0) \quad (3)$$

$$\log_a (a^x) = x \quad (\text{all } x) \quad (4)$$

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**FIGURE 7.13** The graph of  $2^x$  and its inverse,  $\log_2 x$ .

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## Example 4 Applying the inverse equations

$$(a) \log_2 2^5 = 5$$

$$(b) 2^{\log_2 3} = 3$$

$$(c) \log_{10} 10^{(-7)} = -7$$

$$(d) 10^{\log_{10} 4} = 4$$

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## Evaluation of $\log_a x$

Taking  $\ln$  on both sides of  $a^{\log_a x} = x$  gives

$$\ln(a^{\log_a x}) = \ln x$$

$$\text{LHS, } \ln(a^{\log_a x}) = \log_a x \ln a.$$

Equating LHS to RHS yields

$$\log_a x \ln a = \ln x$$

$$\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a} \quad (5)$$

- Example:  $\log_{10} 2 = \ln 2 / \ln 10$

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### TABLE 7.2 Rules for base $a$ logarithms

For any numbers  $x > 0$  and  $y > 0$ ,

1. *Product Rule:*  
 $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule:*  
 $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Reciprocal Rule:*  
 $\log_a \frac{1}{y} = -\log_a y$
4. *Power Rule:*  
 $\log_a x^y = y \log_a x$

- Proof of rule 1:

$$\ln xy = \ln x + \ln y$$

divide both sides by  $\ln a$

$$\frac{\ln(xy)}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a}$$

$$\log_a(xy) = \log_a x + \log_a y$$

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## Derivatives and integrals involving $\log_a x$

$$\frac{d}{dx}(\log_a u) = \frac{du}{dx} \frac{d(\log_a u)}{du}$$

$$\frac{d}{du}(\log_a u) = \frac{d}{du}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a} \frac{d}{du}(\ln u) = \frac{1}{\ln a} \frac{1}{u}$$

$$\frac{d}{dx}(\log_a u) = \frac{du}{dx} \cdot \left(\frac{1}{\ln a} \frac{1}{u}\right) = \frac{1}{\ln a} \left(\frac{1}{u}\right) \cdot \frac{du}{dx}$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} \frac{du}{dx}$$

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## 7.5

### Exponential Growth and Decay

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## Example 5

$$(a) \frac{d}{dx} \left( \log_{10} \overbrace{(3x+1)}^u \right) = \frac{du}{dx} \frac{d(\log_{10} u)}{du}$$

$$= \frac{d}{dx}(3x+1) \frac{1}{\ln 10} \frac{d(\ln u)}{du} = \frac{3}{\ln 10} \frac{1}{(3x+1)}$$

$$(b) \int \frac{\log_2 x}{x} dx = \frac{1}{\ln 2} \int \underbrace{\frac{\ln x}{u}} \cdot \underbrace{\frac{dx}{x}}_{d(\ln x)=du} = \frac{1}{\ln 2} \int u du = \dots$$

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## The law of exponential change

- For a quantity  $y$  increases or decreases at a rate proportional to its size at a given time  $t$  follows the law of exponential change, as per

$$\frac{dy}{dt} \propto y(t) \Rightarrow \frac{dy}{dt} = ky(t).$$

$k$  is the proportional constant.

Very often we have to specify the value of  $y$  at some specified time, for example the initial condition

$$y(t=0) = y_0$$

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Rearrange the equation  $\frac{dy}{dt} = ky$  :

$$\frac{1}{y} \frac{dy}{dt} = k \rightarrow \int \frac{1}{y} \frac{dy}{dt} dt = \int k dt$$

$$\rightarrow \int \frac{1}{y} dy = k \int dt = kt \rightarrow \ln |y| = kt + \ln C$$

$$\rightarrow y = \pm Ce^{kt} = Ae^{kt}, A = \pm C.$$

Put in the initial value of  $y$  at  $t = 0$  is  $y_0$  :

$$\rightarrow y(0) = y_0 = Ae^{k \cdot 0} = A \rightarrow y = y_0 e^{kt}$$

#### The Law of Exponential Change

$$y = y_0 e^{kt} \quad (2)$$

Growth:  $k > 0$     Decay:  $k < 0$

The number  $k$  is the **rate constant** of the equation.

## Example 1 Reducing the cases of infectious disease

- Suppose that in the course of any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1000? Assume the law of exponential change applies.

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## Example 3 Half-life of a radioactive element

- The effective radioactive lifetime of polonium-210 is very short (in days). The number of radioactive atoms remaining after  $t$  days in a sample that starts with  $y_0$  radioactive atoms is  $y = y_0 \exp(-5 \times 10^{-3} t)$ . Find the element's half life.

## Solution

- Radioactive elements decay according to the exponential law of change. The half life of a given radioactive element can be expressed in term of the rate constant  $k$  that is specific to a given radioactive species. Here  $k = -5 \times 10^{-3}$ .
- At the half-life,  $t = t_{1/2}$ ,  
 $y(t_{1/2}) = y_0/2 = y_0 \exp(-5 \times 10^{-3} t_{1/2})$   
 $\exp(-5 \times 10^{-3} t_{1/2}) = 1/2$   
 $\rightarrow \ln(1/2) = -5 \times 10^{-3} t_{1/2}$   
 $\rightarrow t_{1/2} = -\ln(1/2)/5 \times 10^{-3} = \ln(2)/5 \times 10^{-3} = \dots$

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# 7.7

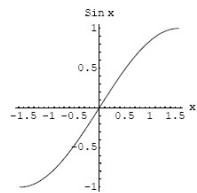
## Inverse Trigonometric Functions

# Defining the inverses

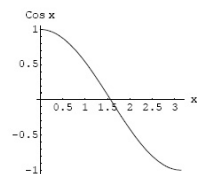
- Trigo functions are periodic, hence not one-to-one in the their domains.
- If we restrict the trigonometric functions to intervals on which they are one-to-one, then we can define their inverses.

Domain restriction that makes the trigonometric functions one-to-one

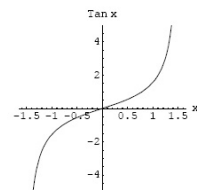
Function :  $\sin x$  Domain :  $[-\pi/2, \pi/2]$  Range :  $[-1, 1]$



Function :  $\cos x$  Domain :  $[0, \pi]$  Range :  $[-1, 1]$

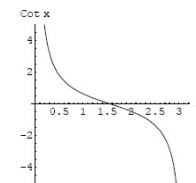


Function :  $\tan x$  Domain :  $[-\pi/2, \pi/2]$  Range :  $(-\infty, \infty)$

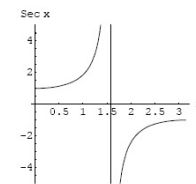


Domain restriction that makes the trigonometric functions one-to-one

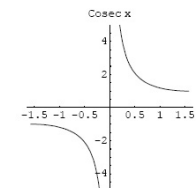
Function :  $\cot x$  Domain :  $[0, \pi]$  Range :  $(-\infty, \infty)$



Function :  $\sec x$  Domain :  $[0, \pi/2) \cup (\pi/2, \pi]$  Range :  $(-\infty, -1] \cup [1, \infty)$



Function :  $\operatorname{cosec} x$  Domain :  $[-\pi/2, 0) \cup (0, \pi/2]$  Range :  $(-\infty, -1] \cup [1, \infty)$



# Inverses for the restricted trigo functions

$$y = \sin^{-1} x = \arcsin x$$

$$y = \cos^{-1} x = \arccos x$$

$$y = \tan^{-1} x = \arctan x$$

$$y = \cot^{-1} x = \operatorname{arccot} x$$

$$y = \sec^{-1} x = \operatorname{arcsec} x$$

$$y = \csc^{-1} x = \operatorname{arccsc} x$$

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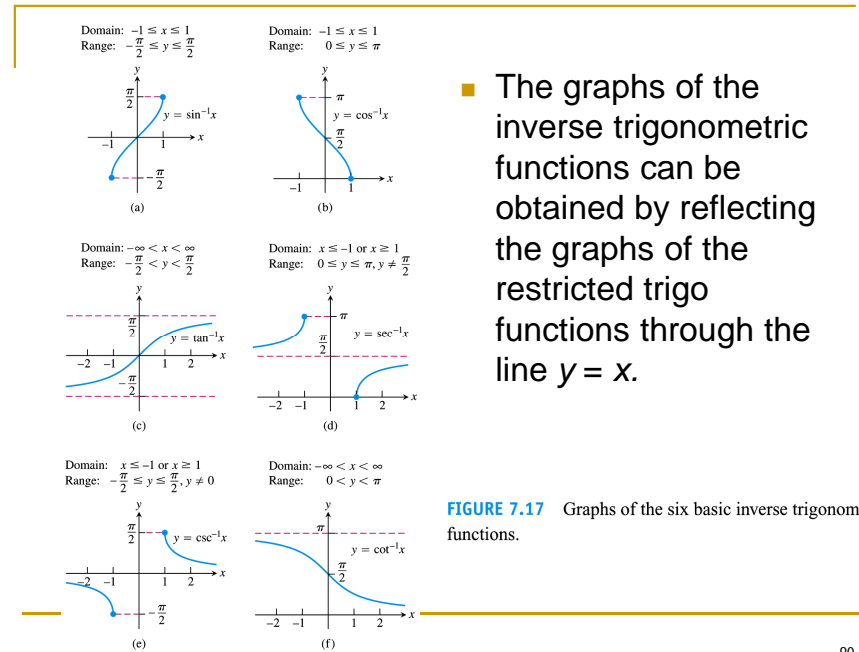
## DEFINITION Arcsine and Arccosine Functions

$y = \sin^{-1} x$  is the number in  $[-\pi/2, \pi/2]$  for which  $\sin y = x$ .

$y = \cos^{-1} x$  is the number in  $[0, \pi]$  for which  $\cos y = x$ .

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■ The graphs of the inverse trigonometric functions can be obtained by reflecting the graphs of the restricted trigo functions through the line  $y = x$ .

FIGURE 7.17 Graphs of the six basic inverse trigonometric functions.

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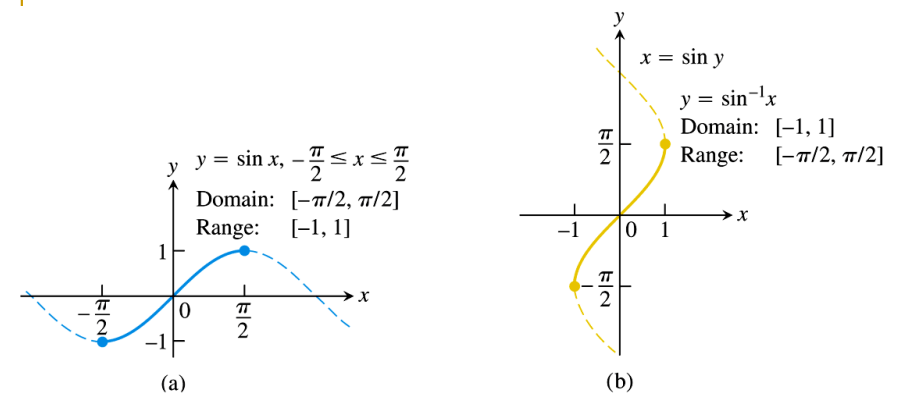
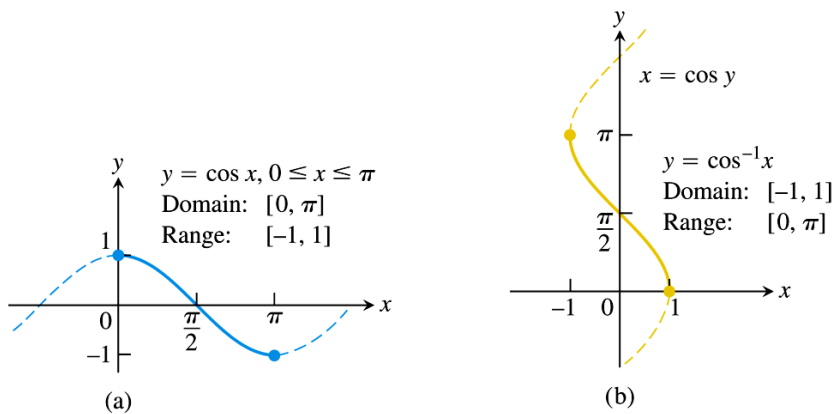


FIGURE 7.18 The graphs of (a)  $y = \sin x$ ,  $-\pi/2 \leq x \leq \pi/2$ , and (b) its inverse,  $y = \sin^{-1} x$ . The graph of  $\sin^{-1} x$ , obtained by reflection across the line  $y = x$ , is a portion of the curve  $x = \sin y$ .

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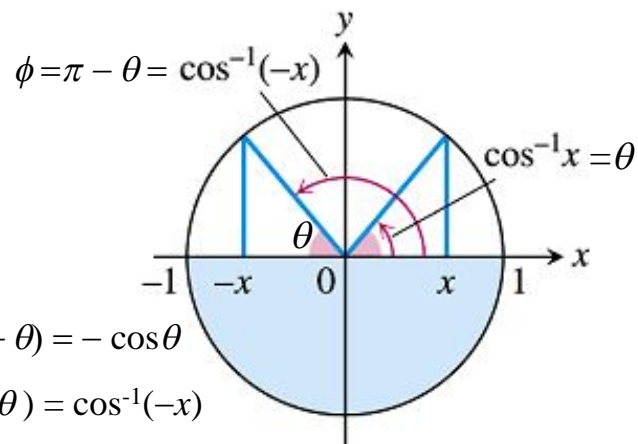
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**FIGURE 7.19** The graphs of (a)  $y = \cos x, 0 \leq x \leq \pi$ , and (b) its inverse,  $y = \cos^{-1} x$ . The graph of  $\cos^{-1} x$ , obtained by reflection across the line  $y = x$ , is a portion of the curve  $x = \cos y$ .

## Some specific values of $\sin^{-1} x$ and $\cos^{-1} x$

$x$	$\sin^{-1} x$	$x$	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/3$	$\sqrt{3}/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/6$	$1/2$	$\pi/3$
$-1/2$	$-\pi/6$	$-1/2$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$-\sqrt{2}/2$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$-\sqrt{3}/2$	$5\pi/6$



$$\theta = \cos^{-1} x;$$

$$\cos \phi = \cos(\pi - \theta) = -\cos \theta$$

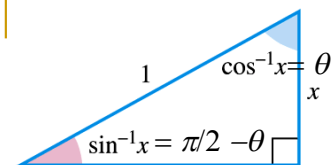
$$\phi = \cos^{-1}(-\cos \theta) = \cos^{-1}(-x)$$

Add up  $\theta$  and  $\phi$ :

$$\theta + \phi = \cos^{-1} x + \cos^{-1}(-x)$$

**FIGURE 7.20**  $\cos^{-1} x$  and  $\cos^{-1}(-x)$  are supplementary angles (so their sum is  $\pi$ ).

$$\pi = \cos^{-1} x + \cos^{-1}(-x)$$



**FIGURE 7.21**  $\sin^{-1} x$  and  $\cos^{-1} x$  are complementary angles (so their sum is  $\pi/2$ ).

$$\cos^{-1} x = \theta; \sin^{-1} x = \left(\frac{\pi}{2} - \theta\right);$$

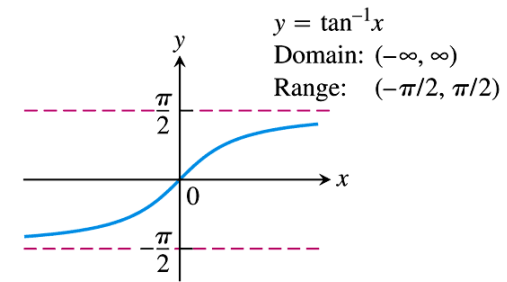
$$\cos^{-1} x + \sin^{-1} x = \theta + \left(\frac{\pi}{2} - \theta\right) = \frac{\pi}{2}$$

link to slide derivatives of the other three

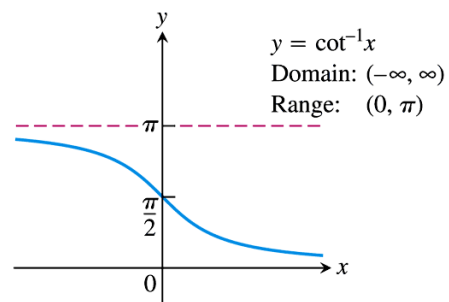
**DEFINITION** Arctangent and Arccotangent Functions

$y = \tan^{-1} x$  is the number in  $(-\pi/2, \pi/2)$  for which  $\tan y = x$ .

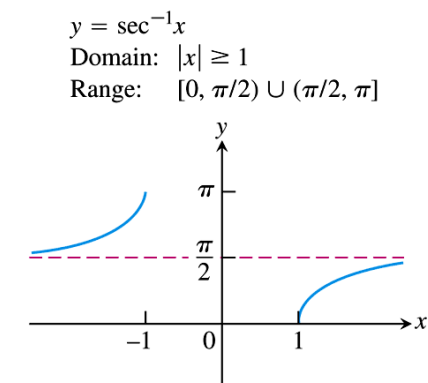
$y = \cot^{-1} x$  is the number in  $(0, \pi)$  for which  $\cot y = x$ .



**FIGURE 7.22** The graph of  $y = \tan^{-1} x$ .

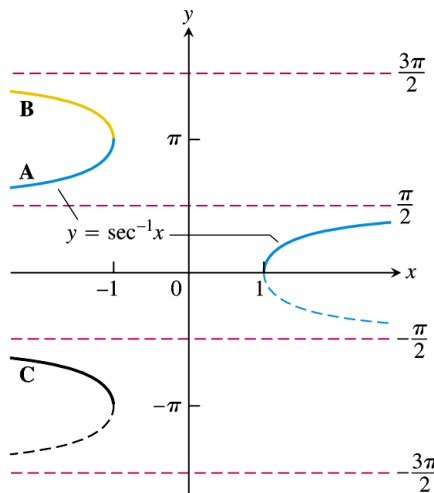


**FIGURE 7.23** The graph of  $y = \cot^{-1} x$ .



**FIGURE 7.24** The graph of  $y = \sec^{-1} x$ .

Domain:  $|x| \geq 1$   
 Range:  $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$

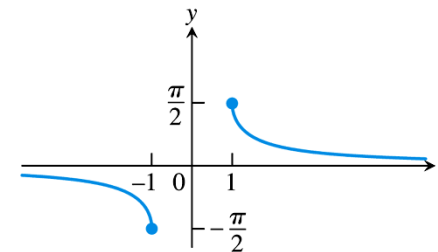


**FIGURE 7.26** There are several logical choices for the left-hand branch of  $y = \sec^{-1} x$ . With choice **A**,  $\sec^{-1} x = \cos^{-1}(1/x)$ , a useful identity employed by many calculators.

## Some specific values of $\tan^{-1} x$

$x$	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

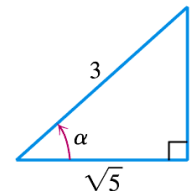
$y = \csc^{-1} x$   
 Domain:  $|x| \geq 1$   
 Range:  $[-\pi/2, 0) \cup (0, \pi/2]$



**FIGURE 7.25** The graph of  $y = \csc^{-1} x$ .

## Example 4

- Find  $\cos \alpha$ ,  $\tan \alpha$ ,  $\sec \alpha$ ,  $\csc \alpha$  if  $\alpha = \sin^{-1}(2/3)$ .
- $\sin \alpha = 2/3$
- ...



**FIGURE 7.27** If  $\alpha = \sin^{-1}(2/3)$ , then the values of the other basic trigonometric functions of  $\alpha$  can be read from this triangle (Example 4).

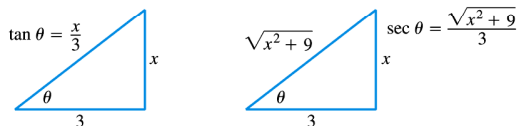
**EXAMPLE 5** Find  $\sec(\tan^{-1} \frac{x}{3})$ .

**Solution** We let  $\theta = \tan^{-1}(x/3)$  (to give the angle a name) and picture  $\theta$  in a right triangle with

$$\tan \theta = \text{opposite/adjacent} = x/3.$$

The length of the triangle's hypotenuse is

$$\sqrt{x^2 + 3^2} = \sqrt{x^2 + 9}.$$



Thus,

$$\begin{aligned} \sec\left(\tan^{-1} \frac{x}{3}\right) &= \sec \theta \\ &= \frac{\sqrt{x^2 + 9}}{3}. \end{aligned} \quad \text{sec } \theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

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## The derivative of $y = \sin^{-1} x$

$$f(x) = \sin^{-1} x \Rightarrow f^{-1}(x) = \sin x;$$

$$\frac{df(x)}{dx} = \frac{1}{\left. \frac{df^{-1}(x)}{dx} \right|_{x=f(x)}} = \frac{1}{\cos x \Big|_{x=f(x)}} = \frac{1}{\cos f(x)}$$

$$\text{Let } y = f(x) = \sin^{-1} x \rightarrow x = \sin y \Rightarrow \cos y = \sqrt{1 - x^2}$$

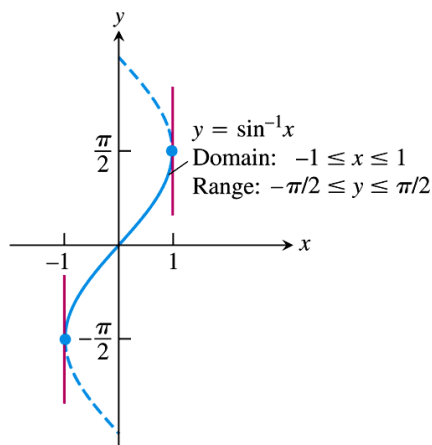
$$\frac{1}{\cos(f(x))} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

$$\therefore \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

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$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

Note that the graph is not differentiable at the end points of  $x = \pm 1$  because the tangents at these points are vertical.

**FIGURE 7.29** The graph of  $y = \sin^{-1} x$ .

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## The derivative of $y = \sin^{-1} u$

If  $u = u(x)$  is a differentiable function of  $x$ ,

$$\frac{d}{dx} \sin^{-1} u = ?$$

Use chain rule: Let  $y = \sin^{-1} u$

$$\frac{d}{dx} \sin^{-1} u = \frac{du}{dx} \frac{d}{du} (\sin^{-1} u) = \frac{du}{dx} \frac{1}{\sqrt{1 - u^2}}$$

Note that  $|u| < 1$  for the formula to apply

$$\frac{d}{dx} (\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1.$$

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## Example 7 Applying the derivative formula

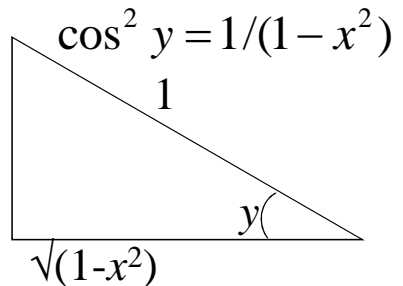
$$\frac{d}{dx} \sin^{-1} x^2 = \dots$$

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## The derivative of $y = \tan^{-1} u$

$$y = \tan^{-1} x \Rightarrow x = \tan y$$

$$1 = \frac{d}{dx}(\tan y) = \frac{dy}{dx} \sec^2 y$$

$$\frac{dy}{dx} = \cos^2 y = 1/(1-x^2)$$


By virtue of chain rule, we obtain

$$\frac{d}{dx} (\tan^{-1} u) = \frac{1}{1+u^2} \frac{du}{dx}$$

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## Example 8

$$x(t) = \tan^{-1} \sqrt{t}$$

$$\left. \frac{dx}{dt} \right|_{t=16} = ?$$

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## The derivative of $y = \sec^{-1} x$

$$y = \sec^{-1} x \Rightarrow x = \sec y$$

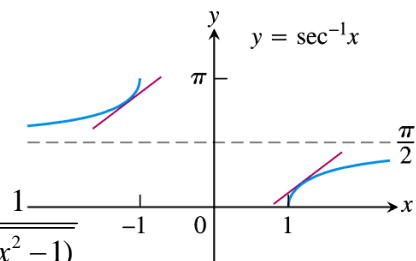
$$1 = \frac{d}{dx}(\sec y) = \frac{dy}{dx} \sec y \tan y$$

$$\tan y = \pm \sqrt{\sec^2 y - 1} = \pm \sqrt{x^2 - 1}$$

$$\frac{d}{dx} \sec^{-1} x = \cos y \cot y = \pm \frac{1}{x} \frac{1}{\sqrt{x^2 - 1}}$$

$$\therefore \frac{dy}{dx} > 0 \text{ (from Figure 7.30),}$$

$$\frac{dy}{dx} = \frac{1}{|x| \sqrt{x^2 - 1}}$$



**FIGURE 7.30** The slope of the curve  $y = \sec^{-1} x$  is positive for both  $x < -1$  and  $x > 1$ .

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## The derivative of $y = \sec^{-1} u$

By virtue of chain rule, we obtain

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1.$$

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## Example 5 Using the formula

$$\frac{d}{dx} \sec^{-1}(5x^4) = \dots$$

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## Derivatives of the other three

- The derivative of  $\cos^{-1}x$ ,  $\cot^{-1}x$ ,  $\csc^{-1}x$  can be easily obtained thanks to the following identities:

### Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$

$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$

$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

Link to fig. 7.21

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TABLE 7.3 Derivatives of the inverse trigonometric functions

1.  $\frac{d(\sin^{-1} u)}{dx} = \frac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$
2.  $\frac{d(\cos^{-1} u)}{dx} = -\frac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$
3.  $\frac{d(\tan^{-1} u)}{dx} = \frac{du/dx}{1+u^2}$
4.  $\frac{d(\cot^{-1} u)}{dx} = -\frac{du/dx}{1+u^2}$
5.  $\frac{d(\sec^{-1} u)}{dx} = \frac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$
6.  $\frac{d(\csc^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

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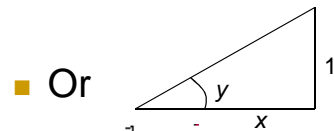
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## Example 10 A tangent line to the arccotangent curve

- Find an equation for the tangent to the graph of  $y = \cot^{-1} x$  at  $x = -1$ .

- Use either  $\frac{df^{-1}(x)}{dx} = \frac{1}{\left. \frac{df(x)}{dx} \right|_{x=f^{-1}(x)}}$



$$\text{Ans} = -\frac{1}{1+x^2}$$

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## Example 11 Using the integral formulas

$$(a) \int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} =$$

$$(b) \int_0^1 \frac{dx}{1+x^2} =$$

$$(c) \int_{2/\sqrt{3}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} =$$

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## Integration formula

- By integrating both sides of the derivative formulas in Table 7.3, we obtain three useful integration formulas in Table 7.4.

**TABLE 7.4** Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant  $a \neq 0$ .

- $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \left( \frac{u}{a} \right) + C$  (Valid for  $u^2 < a^2$ )
- $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \left( \frac{u}{a} \right) + C$  (Valid for all  $u$ )
- $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{u}{a} \right| + C$  (Valid for  $|u| > a > 0$ )

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## Example 13 Completing the square

$$\begin{aligned} \int \frac{dx}{\sqrt{4x-x^2}} &= \int \frac{dx}{\sqrt{-(x^2-4x)}} = \int \frac{dx}{\sqrt{-[(x-2)^2-4]}} \\ &= \int \frac{dx}{\sqrt{4-(x-2)^2}} = \int \frac{du}{\sqrt{2^2-u^2}} = \dots \end{aligned}$$

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## Example 15 Using substitution

$$\int \frac{dx}{\sqrt{e^{2x} - 6}} = \int \frac{dx}{\sqrt{(e^x)^2 - (\sqrt{6})^2}} =$$
$$\int \frac{1}{\sqrt{(e^x)^2 - (\sqrt{6})^2}} \frac{de^x}{e^x} = \int \frac{1}{\sqrt{u^2 - (\sqrt{6})^2}} \frac{du}{u} = \dots$$

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## Even and odd parts of the exponential function

- In general:
- $f(x) = \frac{1}{2} [f(x) + f(-x)] + \frac{1}{2} [f(x) - f(-x)]$
- $\frac{1}{2} [f(x) + f(-x)]$  is the even part
- $\frac{1}{2} [f(x) - f(-x)]$  is the odd part
  
- Specifically:
- $f(x) = e^x = \frac{1}{2} (e^x + e^{-x}) + \frac{1}{2} (e^x - e^{-x})$
- The odd part  $\frac{1}{2} (e^x - e^{-x}) \equiv \cosh x$  (hyperbolic cosine of  $x$ )
- The even part  $\frac{1}{2} (e^x + e^{-x}) \equiv \sinh x$  (hyperbolic sine of  $x$ )

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## 7.8

### Hyperbolic Functions

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**TABLE 7.6** Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\operatorname{coth}^2 x = 1 + \operatorname{csch}^2 x$$

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# Proof of

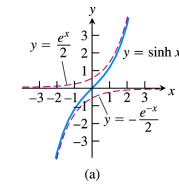
$$\sinh 2x = 2 \cosh x \sinh x$$

$$\begin{aligned} \sinh 2x &= \frac{1}{2}(e^{2x} - e^{-2x}) = \frac{1}{2} \frac{(e^{4x} - 1)}{e^{2x}} \\ &= \frac{1}{2} \frac{(e^{2x} - 1)(e^{2x} + 1)}{e^x} = \frac{2}{2} \cdot \frac{1}{2} (e^x - e^{-x})(e^x + e^{-x}) \\ &= 2 \cdot \frac{1}{2} (e^x - e^{-x}) \cdot \frac{1}{2} (e^x + e^{-x}) = 2 \sinh x \cosh x \end{aligned}$$

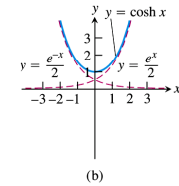
TABLE 7.5 The six basic hyperbolic functions

FIGURE 7.31

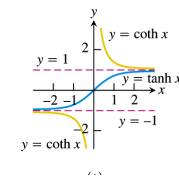
Hyperbolic sine of  $x$ :  $\sinh x = \frac{e^x - e^{-x}}{2}$



Hyperbolic cosine of  $x$ :  $\cosh x = \frac{e^x + e^{-x}}{2}$



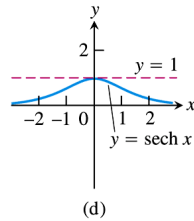
Hyperbolic tangent:  $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$



Hyperbolic cotangent:  $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

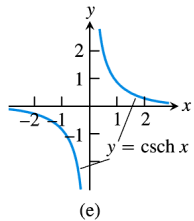
Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$



# Derivatives and integrals

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.8 Integral formulas for hyperbolic functions

$$\int \sinh u \, du = \cosh u + C$$

$$\int \cosh u \, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

$$\frac{d}{dx} \sinh u = \frac{du}{dx} \frac{d}{dx} \sinh x$$

$$\frac{d}{dx} \sinh x = \frac{d}{dx} \frac{1}{2} (e^x - e^{-x}) = \frac{1}{2} (e^x + e^{-x}) = \cosh x$$

$$\therefore \frac{d}{dx} \sinh u = \frac{du}{dx} \cosh x$$

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## Example 1 Finding derivatives and integrals

$$(a) \frac{d}{dx} \tanh \sqrt{1+t^2} = \frac{du}{dx} \frac{d}{du} \tanh u$$

$$(b) \int \coth 5x dx = \frac{1}{5} \int \coth u du = \frac{1}{5} \int \frac{\cosh u}{\sinh u} du$$

$$= \frac{1}{5} \int \frac{d(\sinh u)}{\sinh u} = \frac{1}{5} \int \frac{dv}{v} = \frac{1}{5} \ln |v| + C = \frac{1}{5} \ln |\sinh 5x| + C$$

$$(c) \int \sinh^2 x dx = \frac{1}{2} \int (\cosh 2x - 1) dx = \dots$$

$$(d) \int 4e^x \sinh x dx = 4 \int \frac{e^x - e^{-x}}{2} de^x = 2 \int u - u^{-1} du$$

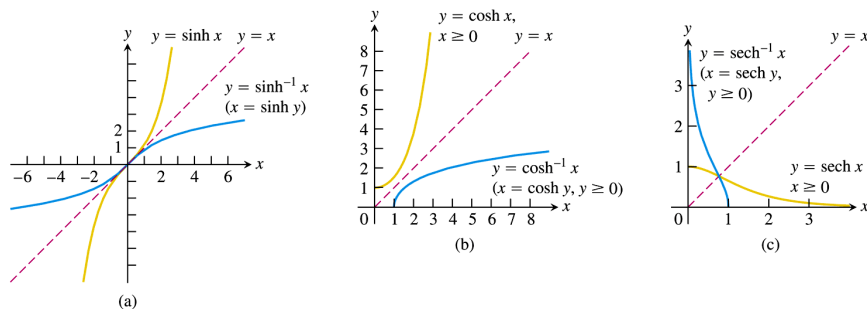
$$= 2 \left( \frac{u^2}{2} - \ln |u| \right) + C = (e^x)^2 - \ln e^{2x} + C = e^{2x} - 2x + C$$

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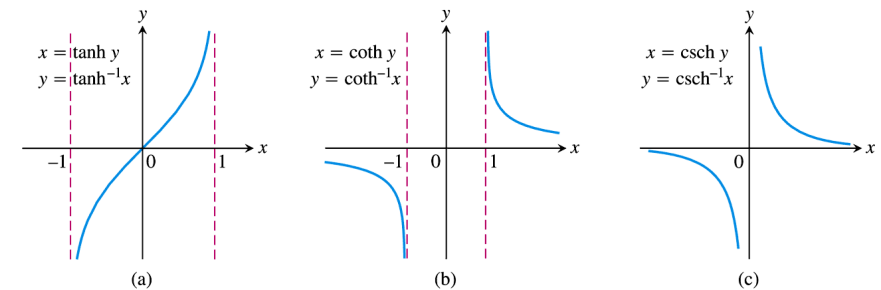
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## Inverse hyperbolic functions



**FIGURE 7.32** The graphs of the inverse hyperbolic sine, cosine, and secant of  $x$ . Notice the symmetries about the line  $y = x$ .



**FIGURE 7.33** The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of  $x$ .

The inverse is useful in integration.

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# Useful Identities

**TABLE 7.9** Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\operatorname{coth}^{-1} x = \tanh^{-1} \frac{1}{x}$$

# Proof

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

Take sech of  $\cosh^{-1} \frac{1}{x}$ .

$$\operatorname{sech} \left( \cosh^{-1} \frac{1}{x} \right) = \frac{1}{\cosh \left( \cosh^{-1} \frac{1}{x} \right)} = \frac{1}{\frac{1}{x}} = x$$

$$\operatorname{sech} \left( \cosh^{-1} \frac{1}{x} \right) = x$$

Take  $\operatorname{sech}^{-1}$  on both sides:

$$\operatorname{sech}^{-1} \left( \operatorname{sech} \left( \cosh^{-1} \frac{1}{x} \right) \right) = \operatorname{sech}^{-1} x \Rightarrow \left( \cosh^{-1} \frac{1}{x} \right) = \operatorname{sech}^{-1} x$$

**TABLE 7.10** Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}, \quad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx}, \quad |u| < 1$$

$$\frac{d(\operatorname{coth}^{-1} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx}, \quad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = \frac{-du/dx}{u\sqrt{1-u^2}}, \quad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{1+u^2}}, \quad u \neq 0$$

Integrating these formulas will allow us to obtain a list of useful integration formulas involving hyperbolic functions

e.g.

$$\int \frac{1}{\sqrt{1+x^2}} dx = \frac{d}{dx} \sinh^{-1} x$$

$$\rightarrow \int \frac{1}{\sqrt{1+x^2}} dx = \int \frac{d}{dx} \sinh^{-1} x dx$$

$$\int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + C$$

# Proof

$$\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$$

let  $y = \sinh^{-1} x$

$$x = \sinh y \rightarrow \frac{d}{dx} x = \frac{d}{dx} \sinh y = \frac{dy}{dx} \cosh y$$

$$\rightarrow \frac{dy}{dx} = \operatorname{sech} y = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

$\Rightarrow$  By virtue of chain rule,

$$\frac{d}{dx} \sinh^{-1} u = \frac{du}{dx} \frac{1}{\sqrt{1+u^2}}$$

## Example 2 Derivative of the inverse hyperbolic cosine

■ Show that

$$\frac{d}{dx} \cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}}.$$

Let  $y = \cosh^{-1} x \dots$

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$$\sinh^{-1}(2/\sqrt{3}) = ?$$

$$\text{Let } q = \sinh^{-1}(2/\sqrt{3})$$

$$\sinh q = 2/\sqrt{3} \rightarrow \frac{1}{2}(e^q - e^{-q}) = \frac{2}{\sqrt{3}}$$

$$e^{2q} - \frac{4}{\sqrt{3}}e^q - 1 = 0$$

$$e^q = \frac{\frac{4}{\sqrt{3}} + \sqrt{\left(-\frac{4}{\sqrt{3}}\right)^2 - 4(-1)}}{2} = \frac{\frac{4}{\sqrt{3}} + \sqrt{\frac{296}{9}}}{2} = 2.682$$

$$\sinh^{-1}(2/\sqrt{3}) = q = \ln 2.682 = 0.9866$$

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## Example 3 Using table 7.11

$$\int_0^1 \frac{2dx}{\sqrt{3+4x^2}}$$

Let  $y = 2x$

$$\int_0^1 \frac{2dx}{\sqrt{3+4x^2}} = \int_0^2 \frac{dy}{\sqrt{3+y^2}}$$

Scale it again to normalise the constant 3 to 1

$$\text{Let } z = \frac{y}{\sqrt{3}} \rightarrow \int_0^2 \frac{dy}{\sqrt{3+y^2}} = \int_0^{2/\sqrt{3}} \frac{\sqrt{3}dz}{\sqrt{3+3z^2}} = \int_0^{2/\sqrt{3}} \frac{dz}{\sqrt{1+z^2}}$$

$$= \sinh^{-1} z \Big|_0^{2/\sqrt{3}} = \sinh^{-1}(2/\sqrt{3}) - \sinh^{-1}(0) = \sinh^{-1}(2/\sqrt{3}) - 0$$

$$= \sinh^{-1}(2/\sqrt{3})$$

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**TABLE 7.11** Integrals leading to inverse hyperbolic functions

1.  $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \quad a > 0$
2.  $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \quad u > a > 0$
3.  $\int \frac{du}{a^2 - u^2} = \begin{cases} \frac{1}{a} \tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \frac{1}{a} \coth^{-1}\left(\frac{u}{a}\right) + C, & \text{if } u^2 > a^2 \end{cases}$
4.  $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a} \operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \quad 0 < u < a$
5.  $\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a} \operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \quad u \neq 0 \text{ and } a > 0$

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# Chapter 8

## Techniques of Integration

1

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# 8.1

## Basic Integration Formulas

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TABLE 8.1 Basic integration formulas

1. $\int du = u + C$	13. $\int \cot u \, du = \ln  \sin u  + C$ $= -\ln  \csc u  + C$
2. $\int k \, du = ku + C$ (any number $k$ )	14. $\int e^u \, du = e^u + C$
3. $\int (du + dv) = \int du + \int dv$	15. $\int a^u \, du = \frac{a^u}{\ln a} + C$ ( $a > 0, a \neq 1$ )
4. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C$ ( $n \neq -1$ )	16. $\int \sinh u \, du = \cosh u + C$
5. $\int \frac{du}{u} = \ln  u  + C$	17. $\int \cosh u \, du = \sinh u + C$
6. $\int \sin u \, du = -\cos u + C$	18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \left( \frac{u}{a} \right) + C$
7. $\int \cos u \, du = \sin u + C$	19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \left( \frac{u}{a} \right) + C$
8. $\int \sec^2 u \, du = \tan u + C$	20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left  \frac{u}{a} \right  + C$
9. $\int \csc^2 u \, du = -\cot u + C$	21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1} \left( \frac{u}{a} \right) + C$ ( $a > 0$ )
10. $\int \sec u \tan u \, du = \sec u + C$	22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1} \left( \frac{u}{a} \right) + C$ ( $u > a > 0$ )
11. $\int \csc u \cot u \, du = -\csc u + C$	
12. $\int \tan u \, du = -\ln  \cos u  + C$ $= \ln  \sec u  + C$	

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## Example 1 Making a simplifying substitution

$$\begin{aligned} \int \frac{2x-9}{\sqrt{x^2-9x+1}} dx &= \int \frac{\overbrace{d(x^2-9x)}^u}{\sqrt{x^2-9x+1}} \\ &= \int \frac{du}{\sqrt{u+1}} = \int \frac{d(u+1)}{\sqrt{u+1}} = \int \frac{dv}{\sqrt{v}} = 2v^{1/2} + C \\ &= 2(u+1)^{1/2} + C = 2(x^2-9x+1)^{1/2} + C \end{aligned}$$

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## Example 2 Completing the square

$$\begin{aligned}\int \frac{dx}{\sqrt{8x-x^2}} &= \int \frac{dx}{\sqrt{16-(x-4)^2}} = \\ \int \frac{d(x-4)}{\sqrt{16-(x-4)^2}} &= \int \frac{du}{\sqrt{4^2-u^2}} \\ &= \sin^{-1} \frac{u}{4} + C = \sin^{-1} \left( \frac{x-4}{4} \right) + C\end{aligned}$$

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## Example 3 Expanding a power and using a trigonometric identity

$$\begin{aligned}\int (\sec x + \tan x)^2 dx &= \int (\sec^2 x + \tan^2 x + 2 \sec x \tan x) dx. \\ \text{Recall: } \tan^2 x &= \sec^2 x - 1; \frac{d}{dx} \tan x = \sec^2 x; \frac{d}{dx} \sec x = \tan x \sec x; \\ &= \int (2 \sec^2 x - 1 + 2 \sec x \tan x) dx \\ &= 2 \tan x + -x + 2 \sec x + C\end{aligned}$$

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## Example 4 Eliminating a square root

$$\begin{aligned}\int_0^{\pi/4} \sqrt{1+\cos 4x} dx &= \\ \cos 4x &= \cos 2(2x) = 2 \cos^2(2x) - 1 \\ \int_0^{\pi/4} \sqrt{1+\cos 4x} dx &= \int_0^{\pi/4} \sqrt{2 \cos^2 2x} dx = \sqrt{2} \int_0^{\pi/4} |\cos 2x| dx \\ &= \sqrt{2} \int_0^{\pi/4} \cos 2x dx = \dots\end{aligned}$$

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## Example 5 Reducing an improper fraction

$$\begin{aligned}\int \frac{3x^2 - 7x}{3x + 2} dx &= \int x - 3 + \frac{6}{3x + 2} dx \\ &= \int x - 3 + \frac{2}{x + 2/3} dx \\ &= \frac{1}{2} x^2 - 3x + 2 \ln \left| x + \frac{2}{3} \right| + C\end{aligned}$$

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## Example 6 Separating a fraction

$$\begin{aligned}
 & \int \frac{3x+2}{\sqrt{1-x^2}} dx \\
 &= 3 \int \frac{x}{\sqrt{1-x^2}} dx + \int \frac{2}{\sqrt{1-x^2}} dx \\
 &= 3 \int \frac{\frac{1}{2} d(x^2)}{\sqrt{1-x^2}} + 2 \int \frac{1}{\sqrt{1-x^2}} dx \\
 &= \frac{3}{2} \int \frac{du}{\sqrt{1-u}} + 2 \sin^{-1} x + C \qquad \int \frac{du}{(1-u)^{1/2}} = -2(1-u)^{1/2} + C' \\
 &= \frac{3}{2} [-2(1-u)^{1/2}] + 2 \sin^{-1} x + C'' \\
 &= -3\sqrt{1-x^2} + 2 \sin^{-1} x + C''
 \end{aligned}$$

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## Example 7 Integral of $y = \sec x$

$$\begin{aligned}
 \int \sec x dx &= ? \\
 d \sec x &= \sec x \tan x dx \\
 d \tan x &= \sec^2 x dx = \sec x \sec x dx \\
 d(\sec x + \tan x) &= \sec x(\sec x + \tan x) dx \\
 \sec x dx &= \frac{d(\sec x + \tan x)}{\sec x + \tan x} \\
 \int \sec x dx &= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} = \ln |\sec x + \tan x| + C
 \end{aligned}$$

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**TABLE 8.2** The secant and cosecant integrals

1.  $\int \sec u du = \ln |\sec u + \tan u| + C$
2.  $\int \csc u du = -\ln |\csc u + \cot u| + C$

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### Procedures for Matching Integrals to Basic Formulas

#### PROCEDURE

#### EXAMPLE

Making a simplifying substitution

$$\frac{2x-9}{\sqrt{x^2-9x+1}} dx = \frac{du}{\sqrt{u}}$$

Completing the square

$$\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$$

Using a trigonometric identity

$$\begin{aligned}
 (\sec x + \tan x)^2 &= \sec^2 x + 2 \sec x \tan x + \tan^2 x \\
 &= \sec^2 x + 2 \sec x \tan x \\
 &\quad + (\sec^2 x - 1) \\
 &= 2 \sec^2 x + 2 \sec x \tan x - 1
 \end{aligned}$$

Eliminating a square root

$$\sqrt{1+\cos 4x} = \sqrt{2 \cos^2 2x} = \sqrt{2} |\cos 2x|$$

Reducing an improper fraction

$$\frac{3x^2-7x}{3x+2} = x-3 + \frac{6}{3x+2}$$

Separating a fraction

$$\frac{3x+2}{\sqrt{1-x^2}} = \frac{3x}{\sqrt{1-x^2}} + \frac{2}{\sqrt{1-x^2}}$$

Multiplying by a form of 1

$$\begin{aligned}
 \sec x &= \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \\
 &= \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}
 \end{aligned}$$

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## 8.2

### Integration by Parts

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### Alternative form of Eq. (1)

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx \quad (1)$$

$$g'(x) \equiv \frac{dg(x)}{dx} = h(x) \rightarrow \int g'(x)dx = \int h(x)dx$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$
$$\rightarrow \int f(x)h(x)dx = f(x)\left[\int h(x)dx\right] - \int \left\{f'(x)\left[\int h(x)dx\right]\right\}dx$$

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## Product rule in integral form

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$$
$$\int \frac{d}{dx}[f(x)g(x)]dx = \int g(x)\frac{d}{dx}[f(x)]dx + \int f(x)\frac{d}{dx}[g(x)]dx$$
$$f(x)g(x) = \int g(x)f'(x)dx + \int f(x)g'(x)dx$$

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx \quad (1)$$

### Integration by parts formula

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### Alternative form of the integration by parts formula

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$$
$$\int \frac{d}{dx}[f(x)g(x)]dx = \int g(x)\frac{d}{dx}[f(x)]dx + \int f(x)\frac{d}{dx}[g(x)]dx$$
$$f(x)g(x) = \int g(x)df(x) + \int f(x)dg(x)$$

Let  $u = f(x); v = g(x)$ . The above formula is recast into the form  
 $uv = \int vdu + \int u dv$

#### Integration by Parts Formula

$$\int u dv = uv - \int v du \quad (2)$$

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## Example 4 Repeated use of integration by parts

$$\int x^2 e^x dx = ?$$

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## Evaluating by parts for definite integrals

### Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x) dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x) dx \quad (3)$$

or, equivalently

$$\int_a^b f(x)h(x) dx = f'(x)h(x)\Big|_a^b - \int_a^b \left\{ f'(x) \left[ \int h(x) dx \right] \right\} dx$$

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## Example 5 Solving for the unknown integral

$$\int e^x \cos x dx = ?$$

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## Example 6 Finding area

- Find the area of the region in Figure 8.1

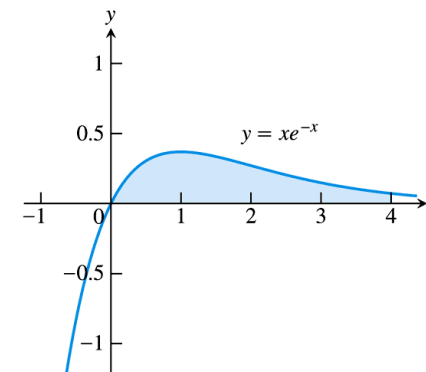


FIGURE 8.1 The region in Example 6.

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## Solution

$$\int_0^4 xe^{-x} dx = \dots$$

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## 8.3

### Integration of Rational Functions by Partial Fractions

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## Example 9 Using a reduction formula

■ Evaluate  $\int \cos^3 x dx$

■ Use

$$\begin{aligned} \int \cos^n x dx &= \int \overbrace{\cos^{n-1} x}^u \cdot \overbrace{\cos x dx}^{dv} \\ &= \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx \end{aligned}$$

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## General description of the method

- A rational function  $f(x)/g(x)$  can be written as a sum of partial fractions. To do so:
- (a) The degree of  $f(x)$  must be less than the degree of  $g(x)$ . That is, the fraction must be proper. If it isn't, divide  $f(x)$  by  $g(x)$  and work with the remainder term.
- We must know the factors of  $g(x)$ . In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors.

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## Reducibility of a polynomial

- A polynomial is said to be **reducible** if it is the product of two polynomials of lower degree.
- A polynomial is **irreducible** if it is not the product of two polynomials of lower degree.
- THEOREM (Ayers, Schaum's series, pg. 305)
- Consider a polynomial  $g(x)$  of order  $n \geq 2$  (with leading coefficient 1). Two possibilities:
  1.  $g(x) = (x-r) h_1(x)$ , where  $h_1(x)$  is a polynomial of degree  $n-1$ , or
  2.  $g(x) = (x^2+px+q) h_2(x)$ , where  $h_2(x)$  is a polynomial of degree  $n-2$ , and  $(x^2+px+q)$  is the irreducible quadratic factor.

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## Quadratic polynomial

- A quadratic polynomial (polynomial of order  $n = 2$ ) is either reducible or not reducible.
- Consider:  $g(x) = x^2 + px + q$ .
- If  $(p^2 - 4q) \geq 0$ ,  $g(x)$  is reducible, i.e.  $g(x) = (x+r_1)(x+r_2)$ .
- If  $(p^2 - 4q) < 0$ ,  $g(x)$  is irreducible.

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## Example

$$g(x) = x^3 - 4x = \underbrace{(x-2)}_{\text{linear factor}} \cdot \underbrace{x(x+2)}_{\text{poly. of degree 2}}$$

$$g(x) = x^3 + 4x = \underbrace{(x^2 + 4)}_{\text{irreducible quadratic factor}} \cdot \underbrace{x}_{\text{poly. of degree 1}}$$

$$g(x) = x^4 - 9 = \underbrace{(x^2 + 3)}_{\text{irreducible quadratic factor}} \cdot \underbrace{(x + \sqrt{3})(x - \sqrt{3})}_{\text{poly. of degree 2}}$$

$$g(x) = x^3 - 3x^2 - x + 3 = \underbrace{(x+1)}_{\text{linear factor}} \cdot \underbrace{(x-2)^2}_{\text{poly. of degree 2}}$$

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- In general, a polynomial of degree  $n$  can always be expressed as the product of linear factors and irreducible quadratic factors:

$$P_n(x) = (x - r_1)^{n_1} (x - r_2)^{n_2} \dots (x - r_l)^{n_l} \times (x^2 + p_1x + q_1)^{m_1} (x^2 + p_2x + q_2)^{m_2} \dots (x^2 + p_kx + q_k)^{m_k}$$

$$n = (n_1 + n_2 + \dots + n_l) + 2(m_1 + m_2 + \dots + m_k)$$

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## Integration of rational functions by partial fractions

### Method of Partial Fractions ( $f(x)/g(x)$ Proper)

1. Let  $x - r$  be a linear factor of  $g(x)$ . Suppose that  $(x - r)^m$  is the highest power of  $x - r$  that divides  $g(x)$ . Then, to this factor, assign the sum of the  $m$  partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \dots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of  $g(x)$ .

2. Let  $x^2 + px + q$  be a quadratic factor of  $g(x)$ . Suppose that  $(x^2 + px + q)^n$  is the highest power of this factor that divides  $g(x)$ . Then, to this factor, assign the sum of the  $n$  partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \dots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of  $g(x)$  that cannot be factored into linear factors with real coefficients.

3. Set the original fraction  $f(x)/g(x)$  equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of  $x$ .
4. Equate the coefficients of corresponding powers of  $x$  and solve the resulting equations for the undetermined coefficients.

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## Example 1 Distinct linear factors

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} dx = \dots$$

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3} = \dots$$

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## Example 2 A repeated linear factor

$$\int \frac{6x + 7}{(x + 2)^2} dx = \dots$$

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$

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## Example 3 Integrating an improper fraction

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx = \dots$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{5x - 3}{(x - 3)(x + 1)} = \frac{A}{x - 3} + \frac{B}{x + 1} = \dots$$

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## Example 4 Integrating with an irreducible quadratic factor in the denominator

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2} dx = \dots$$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2} = \dots$$

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## Other ways to determine the coefficients

- Example 8 Using differentiation
- Find  $A$ ,  $B$  and  $C$  in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$\frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3} = \frac{x-1}{(x+1)^3}$$

$$\Rightarrow A(x+1)^2 + B(x+1) + C = x-1$$

$$x = -1 \Rightarrow C = -2$$

$$\Rightarrow A(x+1)^2 + B(x+1) = x+1$$

$$\Rightarrow A(x+1) + B = 1$$

$$\frac{d}{dx}[A(x+1) + B] = \frac{d}{dx}(1) = 0$$

$$A = 0$$

$$B = 1$$

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## Example 5 A repeated irreducible quadratic factor

$$\int \frac{1}{x(x^2+1)^2} dx = ?$$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} = \dots$$

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## Example 9 Assigning numerical values to $x$

- Find  $A$ ,  $B$  and  $C$  in

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \equiv f(x)$$

$$= x^2 + 1$$

$$f(1) = 2A + 1^2 + 1 = 2 \Rightarrow A = 1$$

$$f(2) = -B + 2^2 + 1 = 5 \Rightarrow B = -5$$

$$f(3) = 2C + 3^2 + 1 = 10 \Rightarrow C = 5$$

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# 8.4

## Trigonometric Integrals

### Example 1 $m$ is odd

$$\int \sin^3 x \cos^2 x \, dx = ?$$

$$\int \sin^3 x \cos^2 x \, dx = -\int \sin^2 x \cos^2 x \, d(\cos x)$$

$$= \int (\cos^2 x - 1) \cos^2 x \, d(\cos x)$$

$$= \int (u^2 - 1)u^2 \, du = \dots$$

### Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where  $m$  and  $n$  are nonnegative integers (positive or zero). We can divide the work into three cases.

**Case 1** If  $m$  is odd, we write  $m$  as  $2k + 1$  and use the identity  $\sin^2 x = 1 - \cos^2 x$  to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Then we combine the single  $\sin x$  with  $dx$  in the integral and set  $\sin x \, dx$  equal to  $-d(\cos x)$ .

**Case 2** If  $m$  is even and  $n$  is odd in  $\int \sin^m x \cos^n x \, dx$ , we write  $n$  as  $2k + 1$  and use the identity  $\cos^2 x = 1 - \sin^2 x$  to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single  $\cos x$  with  $dx$  and set  $\cos x \, dx$  equal to  $d(\sin x)$ .

**Case 3** If both  $m$  and  $n$  are even in  $\int \sin^m x \cos^n x \, dx$ , we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

to reduce the integrand to one in lower powers of  $\cos 2x$ .

### Example 2 $m$ is even and $n$ is odd

$$\int \cos^5 x \, dx = ?$$

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (\cos^2 x)^2 \, d(\sin x) =$$

$$= \int (1 - \sin^2 x)^2 \, d \underbrace{\sin x}_u$$

$$= \int (1 - u^2)^2 \, du = \int 1 + u^4 - 2u^2 \, du = \dots$$



### Example 3 $m$ and $n$ are both even

$$\int \cos^2 x \sin^4 x \, dx = ?$$

$$\int \cos^2 x \sin^4 x \, dx =$$

$$\int \left( \frac{1 - \cos 2x}{2} \right) \left( \frac{1 + \cos 2x}{2} \right)^2 dx$$

$$= \frac{1}{4} \int (1 - \cos 2x)(1 + \cos 2x)^2 dx$$

$$= \frac{1}{4} \int (1 + \cos^2 2x - \cos 2x - \cos^3 2x) dx = \dots$$

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### Example 6 Integrals of powers of $\tan x$ and $\sec x$

$$\int \sec^3 x dx = ?$$

Use integration by parts.

$$\int \sec^3 x dx = \int \underbrace{\sec x}_u \cdot \underbrace{\sec^2 x dx}_{dv};$$

$$dv = \sec^2 x dx \rightarrow v = \int \sec^2 x dx = \tan x$$

$$u = \sec x \rightarrow du = \sec x \tan x dx$$

$$\int \underbrace{\sec x}_u \cdot \underbrace{\sec^2 x dx}_{dv}$$

$$= \sec x \tan x - \int \tan x \cdot \underbrace{\sec x \tan x dx}_{du}$$

$$= \sec x \tan x - \int \tan^2 x \sec x dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx$$

$$\int \sec^3 x dx = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \dots$$

$$\int \sec x dx = \int \sec x \frac{(\tan x + \sec x)}{\tan x + \sec x} dx$$

$$= \int \frac{(\sec x \tan x + \sec^2 x)}{\tan x + \sec x} dx$$

$$= \int \frac{d(\sec x + \tan x)}{\tan x + \sec x}$$

$$= \ln |\sec x + \tan x| + C$$

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### Example 7 Products of sines and cosines

$$\int \cos 5x \sin 3x dx = ?$$

$$\sin mx \sin nx = \frac{1}{2} [\cos(m-n)x - \cos(m+n)x];$$

$$\sin mx \cos nx = \frac{1}{2} [\sin(m-n)x + \sin(m+n)x];$$

$$\cos mx \cos nx = \frac{1}{2} [\cos(m-n)x + \cos(m+n)x]$$

$$\int \cos 5x \sin 3x dx$$

$$= \frac{1}{2} \int [\sin(-2x) + \sin 8x] dx$$

= ...

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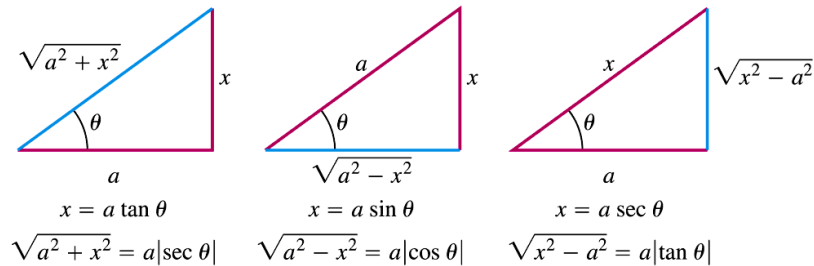
## 8.5

### Trigonometric Substitutions

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## Three basic substitutions



**FIGURE 8.2** Reference triangles for the three basic substitutions identifying the sides labeled  $x$  and  $a$  for each substitution.

Useful for integrals involving  $\sqrt{a^2 - x^2}$ ,  $\sqrt{a^2 + x^2}$ ,  $\sqrt{x^2 - a^2}$  in the denominator of the integrand.

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## Example 2 Using the substitution $x = a \sin \theta$

$$\int \frac{x^2 dx}{\sqrt{9 - x^2}} = ?$$

$$x = 3 \sin y \rightarrow dx = 3 \cos y \, dy$$

$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{9 - x^2}} &= \int \frac{9 \sin^2 y \cdot 3 \cos y \, dy}{\sqrt{9 - 9 \sin^2 y}} = \\ &= 9 \int \frac{\sin^2 y \cdot \cos y \, dy}{\sqrt{1 - \sin^2 y}} \\ &= 9 \int \sin^2 y \, dy = \dots \end{aligned}$$

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## Example 1 Using the substitution $x = a \tan \theta$

$$\int \frac{dx}{\sqrt{4 + x^2}} = ?$$

$$x = 2 \tan y \rightarrow dx = 2 \sec^2 y \, dy = 2(\tan^2 y + 1) \, dy$$

$$\begin{aligned} \int \frac{dx}{\sqrt{4 + 4 \tan^2 y}} &= \int \frac{2(\tan^2 y + 1)}{\sqrt{4 + 4 \tan^2 y}} \, dy \\ &= \int \frac{(\tan^2 y + 1)}{\sqrt{1 + \tan^2 y}} \, dy = \int \sqrt{\sec^2 y} \, dy = \int |\sec y| \, dy \\ &= \ln |\sec y + \tan y| + C \end{aligned}$$

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## Example 3 Using the substitution $x = a \sec \theta$

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = ?$$

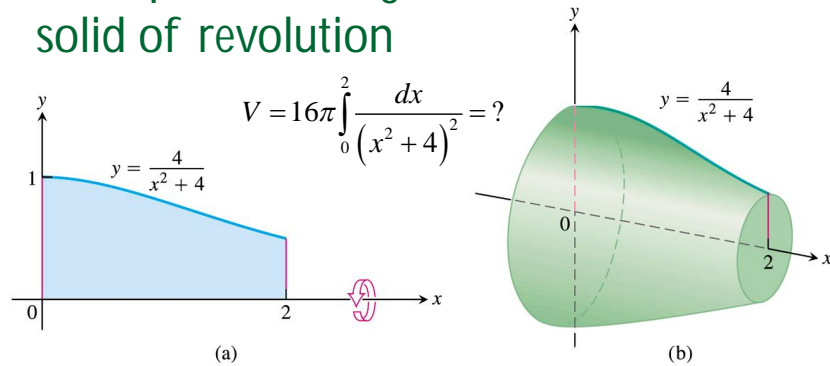
$$x = \frac{2}{5} \sec y \rightarrow dx = \frac{2}{5} \sec y \tan y \, dy$$

$$\begin{aligned} \int \frac{dx}{\sqrt{25x^2 - 4}} &= \frac{2}{5} \int \frac{\sec y \tan y \, dy}{\sqrt{4 \sec^2 y - 4}} = \frac{1}{5} \int \frac{\sec y \tan y \, dy}{\sqrt{\sec^2 y - 1}} \\ &= \frac{1}{5} \int \frac{\sec y \tan y \, dy}{\sqrt{\sec^2 y - 1}} = \frac{1}{5} \int \sec y \, dy \\ &= \frac{1}{5} \ln |\sec y + \tan y| + C = \dots \end{aligned}$$

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## Example 4 Finding the volume of a solid of revolution



$$V = 16\pi \int_0^2 \frac{dx}{(x^2 + 4)^2} = ?$$

FIGURE 8.7 The region (a) and solid (b) in Example 4.

## Solution

$$V = 16\pi \int_0^2 \frac{dx}{(x^2 + 4)^2} = ?$$

$$\text{Let } x = 2 \tan y \rightarrow dx = 2 \sec^2 y dy$$

$$\begin{aligned} V &= \pi \int_0^{\pi/4} \frac{2 \sec^2 y dy}{(\tan^2 y + 1)^2} = \pi \int_0^{\pi/4} \frac{2 \sec^2 y dy}{(\sec^2 y)^2} \\ &= 2\pi \int_0^{\pi/4} \cos^2 y dy = \dots \end{aligned}$$

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## 8.6

### Integral Tables

## Integral tables is provided at the back of Thomas'

- T-4 A brief tables of integrals
- Integration can be evaluated using the tables of integral.

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**EXAMPLE 1** Find

$$\int x(2x + 5)^{-1} dx.$$

**Solution** We use Formula 8 (not 7, which requires  $n \neq -1$ ):

$$\int x(ax + b)^{-1} dx = \frac{x}{a} - \frac{b}{a^2} \ln |ax + b| + C.$$

With  $a = 2$  and  $b = 5$ , we have

$$\int x(2x + 5)^{-1} dx = \frac{x}{2} - \frac{5}{4} \ln |2x + 5| + C.$$

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**EXAMPLE 2** Find

$$\int \frac{dx}{x\sqrt{2x + 4}}.$$

**Solution** We use Formula 13(b):

$$\int \frac{dx}{x\sqrt{ax + b}} = \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{ax + b} - \sqrt{b}}{\sqrt{ax + b} + \sqrt{b}} \right| + C, \quad \text{if } b > 0.$$

With  $a = 2$  and  $b = 4$ , we have

$$\begin{aligned} \int \frac{dx}{x\sqrt{2x + 4}} &= \frac{1}{\sqrt{4}} \ln \left| \frac{\sqrt{2x + 4} - \sqrt{4}}{\sqrt{2x + 4} + \sqrt{4}} \right| + C \\ &= \frac{1}{2} \ln \left| \frac{\sqrt{2x + 4} - 2}{\sqrt{2x + 4} + 2} \right| + C. \end{aligned}$$

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**EXAMPLE 3** Find

$$\int \frac{dx}{x\sqrt{2x - 4}}.$$

**Solution** We use Formula 13(a):

$$\int \frac{dx}{x\sqrt{ax - b}} = \frac{2}{\sqrt{b}} \tan^{-1} \sqrt{\frac{ax - b}{b}} + C.$$

With  $a = 2$  and  $b = 4$ , we have

$$\int \frac{dx}{x\sqrt{2x - 4}} = \frac{2}{\sqrt{4}} \tan^{-1} \sqrt{\frac{2x - 4}{4}} + C = \tan^{-1} \sqrt{\frac{x - 2}{2}} + C.$$

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**EXAMPLE 4** Find

$$\int \frac{dx}{x^2\sqrt{2x - 4}}.$$

**Solution** We begin with Formula 15:

$$\int \frac{dx}{x^2\sqrt{ax + b}} = -\frac{\sqrt{ax + b}}{bx} - \frac{a}{2b} \int \frac{dx}{x\sqrt{ax + b}} + C.$$

With  $a = 2$  and  $b = -4$ , we have

$$\int \frac{dx}{x^2\sqrt{2x - 4}} = -\frac{\sqrt{2x - 4}}{-4x} + \frac{2}{2 \cdot 4} \int \frac{dx}{x\sqrt{2x - 4}} + C.$$

We then use Formula 13(a) to evaluate the integral on the right (Example 3) to obtain

$$\int \frac{dx}{x^2\sqrt{2x - 4}} = \frac{\sqrt{2x - 4}}{4x} + \frac{1}{4} \tan^{-1} \sqrt{\frac{x - 2}{2}} + C.$$

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**EXAMPLE 5** Find

$$\int x \sin^{-1} x \, dx.$$

**Solution** We use Formula 99:

$$\int x^n \sin^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \sin^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1} dx}{\sqrt{1-a^2x^2}}, \quad n \neq -1.$$

With  $n = 1$  and  $a = 1$ , we have

$$\int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2 dx}{\sqrt{1-x^2}}.$$

The integral on the right is found in the table as Formula 33:

$$\int \frac{x^2}{\sqrt{a^2-x^2}} dx = \frac{a^2}{2} \sin^{-1} \left( \frac{x}{a} \right) - \frac{1}{2} x \sqrt{a^2-x^2} + C.$$

With  $a = 1$ ,

$$\int \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1-x^2} + C.$$

The combined result is

$$\begin{aligned} \int x \sin^{-1} x \, dx &= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \left( \frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1-x^2} + C \right) \\ &= \left( \frac{x^2}{2} - \frac{1}{4} \right) \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} + C'. \end{aligned}$$

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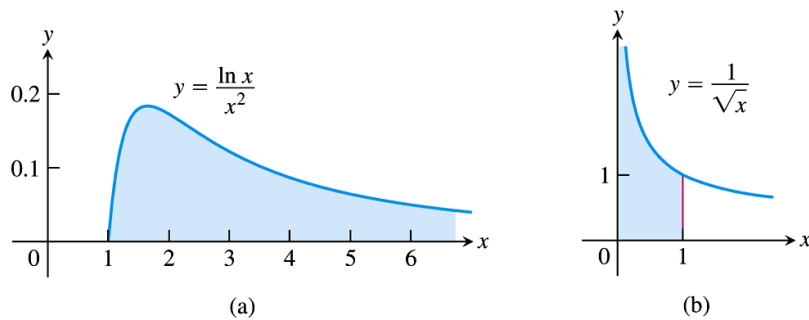
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# 8.8

## Improper Integrals

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**FIGURE 8.17** Are the areas under these infinite curves finite?

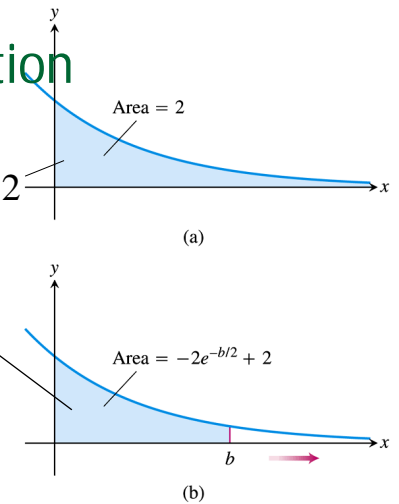
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## Infinite limits of integration

$$A(a) = \lim_{b \rightarrow \infty} A(b) = \lim_{b \rightarrow \infty} 2 - 2e^{-b/2} = 2$$

$$A(b) = \int_0^b e^{-x/2} dx = \dots = 2 - 2e^{-b/2}$$



**FIGURE 8.18** (a) The area in the first quadrant under the curve  $y = e^{-x/2}$  is (b) an improper integral of the first type.

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### DEFINITION Type I Improper Integrals

Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If  $f(x)$  is continuous on  $[a, \infty)$ , then

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

2. If  $f(x)$  is continuous on  $(-\infty, b]$ , then

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx.$$

3. If  $f(x)$  is continuous on  $(-\infty, \infty)$ , then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx,$$

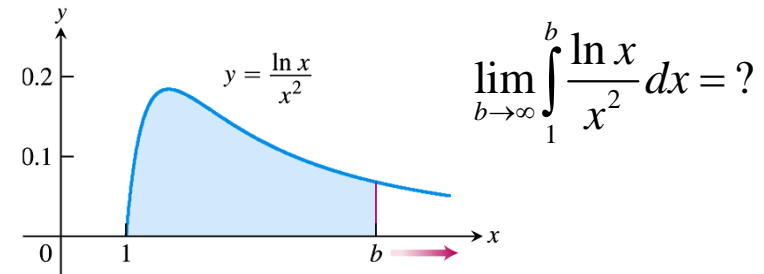
where  $c$  is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

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## Example 1 Evaluating an improper integral on $[1, \infty]$

- Is the area under the curve  $y = (\ln x)/x^2$  from 1 to  $\infty$  finite? If so, what is it?



**FIGURE 8.19** The area under this curve is an improper integral (Example 1).

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## Solution

$$\int_1^b \frac{\ln x}{x} \frac{dx}{x} = \int_1^b \frac{\ln x}{x} d(\ln x) = \int_{\ln 1}^{\ln b} \frac{u}{e^u} du \quad ; u = \ln x, x = e^u$$

$$\int_0^{\ln b} u \underbrace{e^{-u} du}_{dw} = u \underbrace{(-e^{-u})}_w \Big|_0^{\ln b} - \int_0^{\ln b} \underbrace{(-e^{-u})}_w du$$

$$= ue^{-u} \Big|_{\ln b}^0 + \int_0^{\ln b} e^{-u} du = ue^{-u} \Big|_{\ln b}^0 - e^{-u} \Big|_0^{\ln b}$$

$$= -\ln b \cdot e^{-\ln b} - (e^{-\ln b} - 1) = -\frac{1}{b} \ln b - \frac{1}{b} + 1$$

$$\lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \left[ -\frac{1}{b} \ln b - \frac{1}{b} + 1 \right] = 1$$

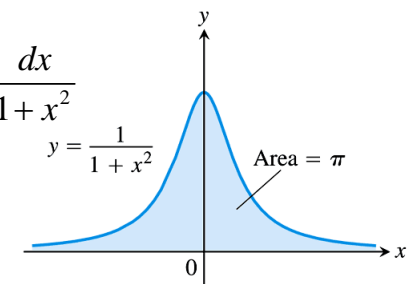
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## Example 2 Evaluating an integral on $[-\infty, \infty]$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = ?$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b \rightarrow \infty} \int_{-b}^0 \frac{dx}{1+x^2} + \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{1+x^2}$$

$$= 2 \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{1+x^2}$$



**FIGURE 8.20** The area under this curve is finite (Example 2).

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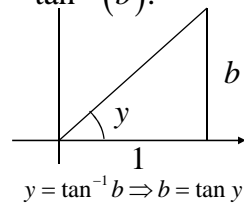
# Solution

Using the integral table (Eq. 16)

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$\int_0^b \frac{dx}{1+x^2} = \left[ \tan^{-1} x \right]_0^b = \tan^{-1}(b) - \tan^{-1} 0 = \tan^{-1}(b).$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2 \lim_{b \rightarrow \infty} \tan^{-1} b = 2 \cdot \frac{\pi}{2} = \pi$$



$$y = \tan^{-1} b \Rightarrow b = \tan y$$

$$\lim_{b \rightarrow \infty} \tan^{-1} b = \frac{\pi}{2}$$

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## DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If  $f(x)$  is continuous on  $(a, b]$  and is discontinuous at  $a$  then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx.$$

2. If  $f(x)$  is continuous on  $[a, b)$  and is discontinuous at  $b$ , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx.$$

3. If  $f(x)$  is discontinuous at  $c$ , where  $a < c < b$ , and continuous on  $[a, c) \cup (c, b]$ , then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

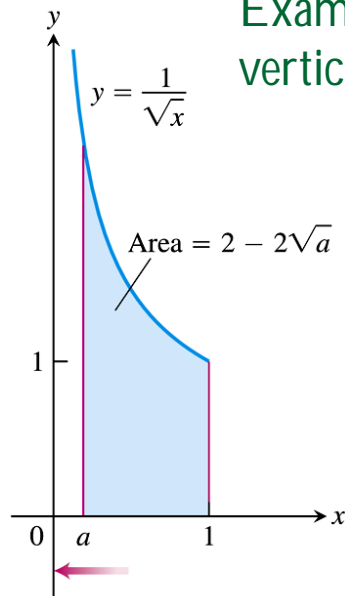
In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

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## Example 3 Integrands with vertical asymptotes



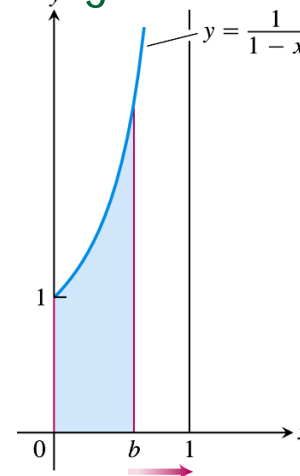
**FIGURE 8.21** The area under this curve is

$$\lim_{a \rightarrow 0^+} \int_a^1 \left( \frac{1}{\sqrt{x}} \right) dx = 2,$$

an improper integral of the second kind.

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## Example 4 A divergent improper integral



- Investigate the convergence of

$$\int_0^1 \frac{dx}{1-x}$$

**FIGURE 8.22** The limit does not exist:

$$\int_0^1 \left( \frac{1}{1-x} \right) dx = \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{1-x} dx = \infty$$

The area beneath the curve and above the  $x$ -axis for  $[0, 1)$  is not a real number (Example 4).

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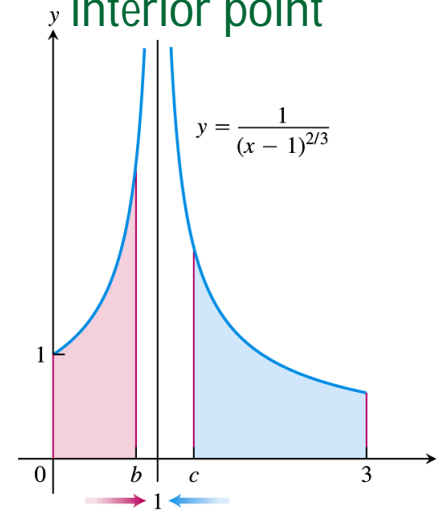
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## Solution

$$\begin{aligned} \int_0^1 \frac{dx}{1-x} &= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{1-x} = -\lim_{b \rightarrow 1^-} [\ln |x-1|]_0^b \\ &= -\lim_{b \rightarrow 1^-} [\ln |b-1| - \ln |0-1|] \\ &= -\lim_{b \rightarrow 1^-} [\ln |b-1| - \ln |0-1|] = \lim_{b \rightarrow 1^-} [\ln |b-1|^{-1}] \\ &= \lim_{\varepsilon \rightarrow 0} \left[ \ln \frac{1}{\varepsilon} \right] = \infty \end{aligned}$$

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## Example 5 Vertical asymptote at an interior point



$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = ?$$

**FIGURE 8.23** Example 5 shows the convergence of

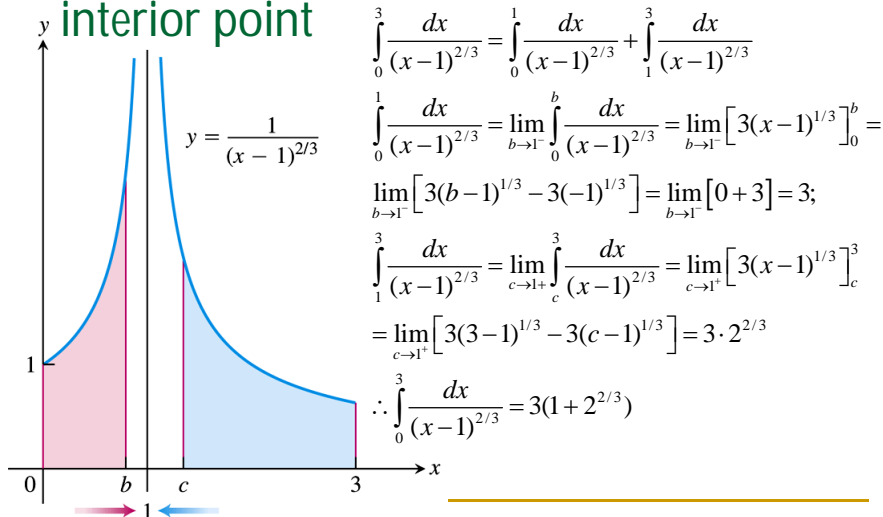
$$\int_0^3 \frac{1}{(x-1)^{2/3}} dx = 3 + 3\sqrt[3]{2},$$

so the area under the curve exists (so it is a real number).

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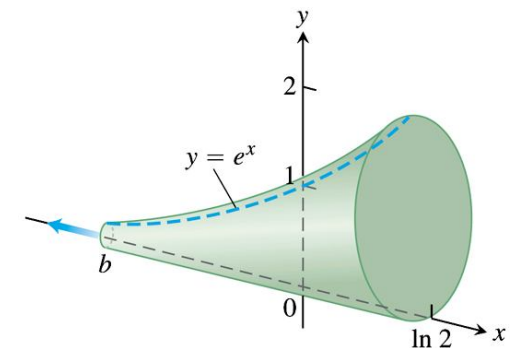
## Example 5 Vertical asymptote at an interior point



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## Example 7 Finding the volume of an infinite solid

- The cross section of the solid in Figure 8.24 perpendicular to the x-axis are circular disks with diameters reaching from the x-axis to the curve  $y = e^x$ ,  $-\infty < x < \ln 2$ . Find the volume of the horn.



**FIGURE 8.24** The calculation in Example 7 shows that this infinite horn has a finite volume.

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## Example 7 Finding the volume of an infinite solid

volume of a slice of disk of thickness  $dx$ , diameter  $y$

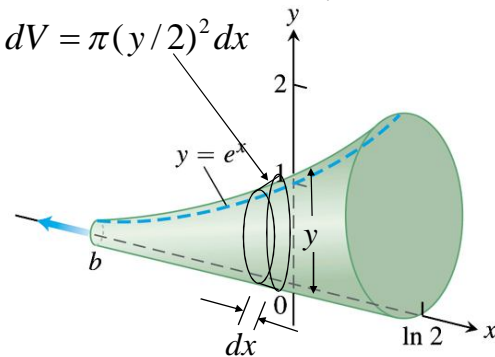
$$V = \int_0^V dV = \frac{1}{4} \lim_{b \rightarrow -\infty} \int_b^{\ln 2} \pi y(x)^2 dx$$

$$= \frac{1}{4} \lim_{b \rightarrow -\infty} \int_b^{\ln 2} \pi e^{2x} dx$$

$$= \frac{1}{8} \lim_{b \rightarrow -\infty} \left[ \pi e^{2x} \right]_b^{\ln 2}$$

$$= \frac{1}{8} \lim_{b \rightarrow -\infty} \left[ 4\pi - \pi e^{2b} \right]$$

$$= \frac{1}{8} \pi \lim_{b \rightarrow -\infty} (4 - e^{2b}) = \frac{\pi}{2}$$



**FIGURE 8.24** The calculation in Example 7 shows that this infinite horn has a finite volume.

## Chapter 11

### Infinite Sequences and Series

## 11.1

### Sequences

## What is a sequence

- A sequence is a list of numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

in a given order.

- Each  $a$  is a **term** of the sequence.
- Example of a sequence:
- $2, 4, 6, 8, 10, 12, \dots, 2n, \dots$
- $n$  is called the **index** of  $a_n$

## Other example of sequences

### DEFINITION Infinite Sequence

An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

- In the previous example, a general term  $a_n$  of index  $n$  in the sequence is described by the formula

$$a_n = 2n.$$

- We denote the sequence in the previous example by  $\{a_n\} = \{2, 4, 6, 8, \dots\}$
- In a sequence the order is important:
- $2, 4, 6, 8, \dots$  and  $\dots, 8, 6, 4, 2$  are not the same

$$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \dots, \sqrt{n}, \dots\}; a_n = \sqrt{n};$$

$$\{b_n\} = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1} \frac{1}{n}, \dots\}; b_n = (-1)^{n+1} \frac{1}{n};$$

$$\{c_n\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n-1}{n}, \dots\}; c_n = \frac{n-1}{n};$$

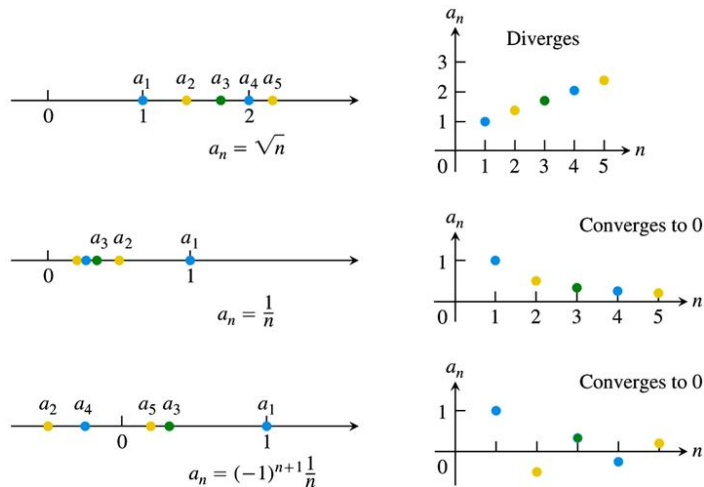
$$\{d_n\} = \{1, -1, 1, -1, 1, \dots, (-1)^{n+1}, \dots\}; d_n = (-1)^{n+1};$$

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**FIGURE 11.1** Sequences can be represented as points on the real line or as points in the plane where the horizontal axis  $n$  is the index number of the term and the vertical axis  $a_n$  is its value.

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### DEFINITIONS Converges, Diverges, Limit

The sequence  $\{a_n\}$  **converges** to the number  $L$  if to every positive number  $\epsilon$  there corresponds an integer  $N$  such that for all  $n$ ,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

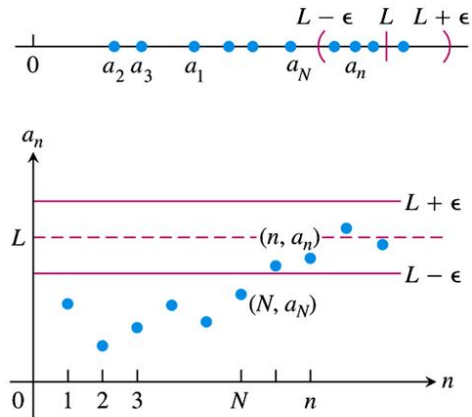
If no such number  $L$  exists, we say that  $\{a_n\}$  **diverges**.

If  $\{a_n\}$  converges to  $L$ , we write  $\lim_{n \rightarrow \infty} a_n = L$ , or simply  $a_n \rightarrow L$ , and call  $L$  the **limit** of the sequence (Figure 11.2).

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**FIGURE 11.2**  $a_n \rightarrow L$  if  $y = L$  is a horizontal asymptote of the sequence of points  $\{(n, a_n)\}$ . In this figure, all the  $a_n$ 's after  $a_N$  lie within  $\epsilon$  of  $L$ .

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### DEFINITION Diverges to Infinity

The sequence  $\{a_n\}$  **diverges to infinity** if for every number  $M$  there is an integer  $N$  such that for all  $n$  larger than  $N$ ,  $a_n > M$ . If this condition holds we write

$$\lim_{n \rightarrow \infty} a_n = \infty \quad \text{or} \quad a_n \rightarrow \infty.$$

Similarly if for every number  $m$  there is an integer  $N$  such that for all  $n > N$  we have  $a_n < m$ , then we say  $\{a_n\}$  **diverges to negative infinity** and write

$$\lim_{n \rightarrow \infty} a_n = -\infty \quad \text{or} \quad a_n \rightarrow -\infty.$$

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### THEOREM 1

Let  $\{a_n\}$  and  $\{b_n\}$  be sequences of real numbers and let  $A$  and  $B$  be real numbers. The following rules hold if  $\lim_{n \rightarrow \infty} a_n = A$  and  $\lim_{n \rightarrow \infty} b_n = B$ .

1. **Sum Rule:**  $\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$
2. **Difference Rule:**  $\lim_{n \rightarrow \infty} (a_n - b_n) = A - B$
3. **Product Rule:**  $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = A \cdot B$
4. **Constant Multiple Rule:**  $\lim_{n \rightarrow \infty} (k \cdot b_n) = k \cdot B$  (Any number  $k$ )
5. **Quotient Rule:**  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$  if  $B \neq 0$

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### EXAMPLE 3 Applying Theorem 1

By combining Theorem 1 with the limits of Example 1, we have:

- (a)  $\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = -1 \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = -1 \cdot 0 = 0$  Constant Multiple Rule and Example 1a
- (b)  $\lim_{n \rightarrow \infty} \left(\frac{n-1}{n}\right) = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{n} = 1 - 0 = 1$  Difference Rule and Example 1a
- (c)  $\lim_{n \rightarrow \infty} \frac{5}{n^2} = 5 \cdot \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = 5 \cdot 0 \cdot 0 = 0$  Product Rule
- (d)  $\lim_{n \rightarrow \infty} \frac{4 - 7n^6}{n^6 + 3} = \lim_{n \rightarrow \infty} \frac{(4/n^6) - 7}{1 + (3/n^6)} = \frac{0 - 7}{1 + 0} = -7.$  Sum and Quotient Rules

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### THEOREM 2 The Sandwich Theorem for Sequences

Let  $\{a_n\}$ ,  $\{b_n\}$ , and  $\{c_n\}$  be sequences of real numbers. If  $a_n \leq b_n \leq c_n$  holds for all  $n$  beyond some index  $N$ , and if  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ , then  $\lim_{n \rightarrow \infty} b_n = L$  also.

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### EXAMPLE 4 Applying the Sandwich Theorem

Since  $1/n \rightarrow 0$ , we know that

(a)  $\frac{\cos n}{n} \rightarrow 0$  because  $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$ ;

(b)  $\frac{1}{2^n} \rightarrow 0$  because  $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$ ;

(c)  $(-1)^n \frac{1}{n} \rightarrow 0$  because  $-\frac{1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$ .

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### THEOREM 3 The Continuous Function Theorem for Sequences

Let  $\{a_n\}$  be a sequence of real numbers. If  $a_n \rightarrow L$  and if  $f$  is a function that is continuous at  $L$  and defined at all  $a_n$ , then  $f(a_n) \rightarrow f(L)$ .

- Example 6: Applying theorem 3 to show that the sequence  $\{2^{1/n}\}$  converges to 0.
- Taking  $a_n = 1/n$ ,  $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0 \equiv L$
- Define  $f(x) = 2^x$ . Note that  $f(x)$  is continuous on  $x=L$ , and is defined for all  $x = a_n = 1/n$
- According to Theorem 3,
- $\lim_{n \rightarrow \infty} f(a_n) = f(L)$
- LHS:  $\lim_{n \rightarrow \infty} f(a_n) = \lim_{n \rightarrow \infty} f(1/n) = \lim_{n \rightarrow \infty} 2^{1/n}$
- RHS =  $f(L) = 2^L = 2^0 = 1$
- Equating LHS = RHS, we have  $\lim_{n \rightarrow \infty} 2^{1/n} = 1$
- $\Rightarrow$  the sequence  $\{2^{1/n}\}$  converges to 1

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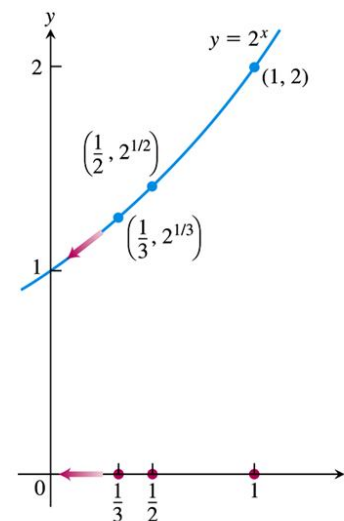


FIGURE 11.3 As  $n \rightarrow \infty$ ,  $1/n \rightarrow 0$  and  $2^{1/n} \rightarrow 2^0$  (Example 6).

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#### THEOREM 4

Suppose that  $f(x)$  is a function defined for all  $x \geq n_0$  and that  $\{a_n\}$  is a sequence of real numbers such that  $a_n = f(n)$  for  $n \geq n_0$ . Then

$$\lim_{x \rightarrow \infty} f(x) = L \quad \Rightarrow \quad \lim_{n \rightarrow \infty} a_n = L.$$

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### Example 9 Applying l'Hopital rule to determine convergence

Does the sequence whose  $n$ th term is  $a_n = \left(\frac{n+1}{n-1}\right)^n$  converge?

If so, find  $\lim_{n \rightarrow \infty} a_n$ .

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#### ■ Example 7: Applying l'Hopital rule

■ Show that  $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$

■ Solution: The function  $f(x) = \frac{\ln x}{x}$  is defined for  $x \geq 1$  and agrees with the sequence  $\{a_n = (\ln n)/n\}$  for  $n \geq 1$ .

■ Applying l'Hopital rule on  $f(x)$ :

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

■ By virtue of Theorem 4,

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0 \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

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#### Solution: Use l'Hopital rule

Let  $f(x) = \left(\frac{x+1}{x-1}\right)^x$  so that  $f(n) = a_n$  for  $n \geq 1$ .

$$\rightarrow \ln f(x) = x \ln \left(\frac{x+1}{x-1}\right)$$

$$\lim_{x \rightarrow \infty} \ln f(x) = \lim_{x \rightarrow \infty} x \ln \left(\frac{x+1}{x-1}\right) = \lim_{x \rightarrow \infty} \frac{\ln \left(\frac{x+1}{x-1}\right)}{1/x}$$

$$= \lim_{x \rightarrow \infty} \frac{\left(\frac{-2}{x^2-1}\right)}{-1/x^2} = \lim_{x \rightarrow \infty} \frac{2x^2}{x^2-1} = 2$$

By virtue of Theorem 4,  $\lim_{x \rightarrow \infty} \ln f(x) = 2 \Rightarrow$

$$\lim_{x \rightarrow \infty} f(x) = \exp(2) \Rightarrow \lim_{n \rightarrow \infty} a_n = \exp(2)$$

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### THEOREM 5

The following six sequences converge to the limits listed below:

1.  $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$
2.  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$
3.  $\lim_{n \rightarrow \infty} x^{1/n} = 1 \quad (x > 0)$
4.  $\lim_{n \rightarrow \infty} x^n = 0 \quad (|x| < 1)$
5.  $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$
6.  $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

All of the results in Theorem 5 can be proven using Theorem 4. See if you can show some of them yourself.

In Formulas (3) through (6),  $x$  remains fixed as  $n \rightarrow \infty$ .

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### DEFINITION Nondecreasing Sequence

A sequence  $\{a_n\}$  with the property that  $a_n \leq a_{n+1}$  for all  $n$  is called a **nondecreasing sequence**.

- Example 12 Nondecreasing sequence
- (a)  $1, 2, 3, 4, \dots, n, \dots$
- (b)  $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots$   
(nondecreasing because  $a_{n+1} - a_n \geq 0$ )
- (c)  $\{3\} = \{3, 3, 3, \dots\}$
- Two kinds of nondecreasing sequences: bounded and non-bounded.

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## Example 10

- (a)  $(\ln n^2)/n = 2 (\ln n) / n \rightarrow 2 \cdot 0 = 0$
- (b)  $\sqrt[n]{n^2} = n^{2/n} = (n^{1/n})^2 \rightarrow (1)^2$
- (c)  $\sqrt[n]{3n} = \sqrt[n]{3} \cdot \sqrt[n]{n} = 3^{1/n} \cdot n^{1/n} \rightarrow 1 \cdot 1 = 1$
- (d)  $\left(-\frac{1}{2}\right)^n \rightarrow 0$
- (e)  $\left(\frac{n-2}{n}\right)^n = \left(1 + \frac{(-2)}{n}\right)^n \rightarrow e^{-2}$
- (f)  $\frac{100^n}{n!} \rightarrow 0$

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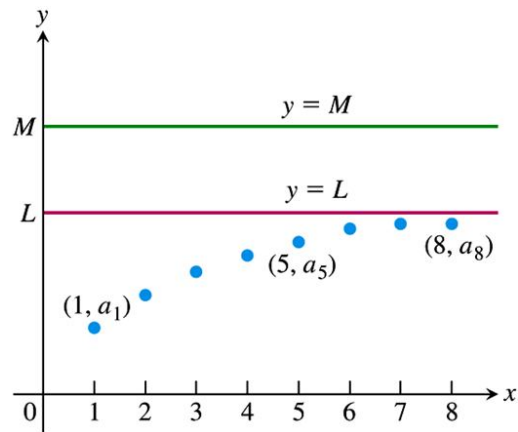
### DEFINITIONS Bounded, Upper Bound, Least Upper Bound

A sequence  $\{a_n\}$  is **bounded from above** if there exists a number  $M$  such that  $a_n \leq M$  for all  $n$ . The number  $M$  is an **upper bound** for  $\{a_n\}$ . If  $M$  is an upper bound for  $\{a_n\}$  but no number less than  $M$  is an upper bound for  $\{a_n\}$ , then  $M$  is the **least upper bound** for  $\{a_n\}$ .

- Example 13 Applying the definition for boundedness
- (a)  $1, 2, 3, \dots, n, \dots$  has no upper bound
- (b)  $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots$  is bounded from above by  $M = 1$ .
- Since no number less than 1 is an upper bound for the sequence, so 1 is the least upper bound.

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**FIGURE 11.4** If the terms of a nondecreasing sequence have an upper bound  $M$ , they have a limit  $L \leq M$ .

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### THEOREM 6 The Nondecreasing Sequence Theorem

A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

- If a non-decreasing sequence converges it is bounded from above.
- If a non-decreasing sequence is bounded from above it converges.
- In Example 13 (b)  $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots\}$  is bounded by the least upper bound  $M = 1$ . Hence according to Theorem 6, the sequence converges, and the limit of convergence is the least upper bound 1.

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## 11.2

### Infinite Series

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### DEFINITIONS Infinite Series, $n$ th Term, Partial Sum, Converges, Sum

Given a sequence of numbers  $\{a_n\}$ , an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number  $a_n$  is the  **$n$ th term** of the series. The sequence  $\{s_n\}$  defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^n a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number  $s_n$  being the  **$n$ th partial sum**. If the sequence of partial sums converges to a limit  $L$ , we say that the series **converges** and that its **sum** is  $L$ . In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

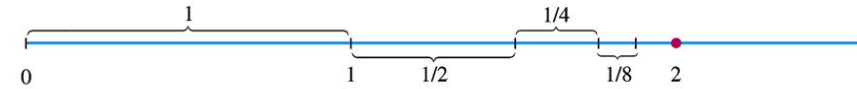
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## Example of a partial sum formed by a sequence $\{a_n = 1/2^{n-1}\}$

Partial sum		Suggestive expression for partial sum	Value
First:	$s_1 = 1$	$2 - 1$	1
Second:	$s_2 = 1 + \frac{1}{2}$	$2 - \frac{1}{2}$	$\frac{3}{2}$
Third:	$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$2 - \frac{1}{4}$	$\frac{7}{4}$
⋮	⋮	⋮	⋮
$n$ th:	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$

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**FIGURE 11.5** As the lengths  $1, 1/2, 1/4, 1/8, \dots$  are added one by one, the sum approaches 2.

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## Short hand notation for infinite series

$$\sum_n a_n, \sum_{k=1}^{\infty} a_k \text{ or } \sum a_n$$

- The infinite series is either converge or diverge

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## Geometric series

- Geometric series are the series of the form  $a + ar + ar^2 + ar^3 + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1}$
- $a$  and  $r = a_{n+1}/a_n$  are fixed numbers and  $a \neq 0$ .  $r$  is called the ratio.
- Three cases can be classified:  $r < 1, r > 1, r = 1$ .

If  $|r| < 1$ , the geometric series  $a + ar + ar^2 + \dots + ar^{n-1} + \dots$  converges to  $a/(1 - r)$ :

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}, \quad |r| < 1.$$

If  $|r| \geq 1$ , the series diverges.

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## Proof of $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ for $|r| < 1$

Assume  $r \neq 1$ .

$$s_n = \sum_{k=1}^{k=n} ar^{k-1} = a + ar + ar^2 + \dots + ar^{n-1}$$

$$rs_n = r(a + ar + ar^2 + \dots + ar^{n-1}) = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n$$

$$s_n - rs_n = a - ar^n = a(1 - r^n)$$

$$s_n = a(1 - r^n)/(1 - r)$$

$$\text{If } |r| < 1: \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} \quad (\text{By theorem 5.4, } \lim_{n \rightarrow \infty} r^n = 1 \text{ for } |r| < 1)$$

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## For cases $|r| \geq 1$

$$\text{If } |r| > 1: \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{a(1 - r^n)}{1 - r} = \infty \quad (\text{Because } |r^n| \rightarrow \infty \text{ if } |r| > 1)$$

$$\text{If } r = 1: s_n = a + ar^1 + ar^2 + \dots + ar^{n-1} = na$$

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} na = a \lim_{n \rightarrow \infty} n = \infty$$

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### EXAMPLE 1 Index Starts with $n = 1$

The geometric series with  $a = 1/9$  and  $r = 1/3$  is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots = \sum_{n=1}^{\infty} \frac{1}{9} \left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1 - (1/3)} = \frac{1}{6}$$

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## Example 2 Index starts with $n=0$

- The series  $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \frac{5}{4^0} - \frac{5}{4^1} + \frac{5}{4^2} - \frac{5}{4^3} + \dots$

is a geometric series with  $a=5$ ,  $r=-(1/4)$ .

- It converges to  $s_{\infty} = a/(1-r) = 5/(1+1/4) = 4$

- Note: Be reminded that no matter how complicated the expression of a geometric series is, the series is simply completely specified by  $r$  and  $a$ . In other words, if you know  $r$  and  $a$  of a geometric series, you know almost everything about the series.

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## Example 4

$$5.232323\dots = 5.\overline{23} = 5.\dot{2}\dot{3}$$

Express the above decimal as a ratio of two integers.

$$5.\dot{2}\dot{3} = 5 + [\dots]$$

$$[\dots] = 0.23 + 0.0023 + 0.000023 + \dots$$

$$= 0.23(\dots) = \frac{23}{100}(\dots)$$

$$(\dots) = 1 + 0.01 + 0.0001 + \dots = \frac{a}{1-r} = \frac{1}{1-0.01} = \frac{1}{1-\frac{1}{100}} = \frac{1}{\frac{99}{100}} = \frac{100}{99}$$

$$5.\dot{2}\dot{3} = \frac{23}{100} \frac{100}{99} = \frac{23}{99}$$

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## Divergent series

### ■ Example 6

$$\sum n^2 = 1 + 2 + 4 + 16 + \dots n^2 + \dots$$

diverges because the partial sums  $s_n$  grows beyond every number  $L$

$$\sum \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \dots + \frac{n+1}{n} + \dots$$

diverges because each term is greater than 1,

$$\Rightarrow \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \dots + \frac{n+1}{n} + \dots > \sum_{n=1}^{\infty} 1 \rightarrow \infty$$

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## Example 5 Telescopic series

- Find the sum of the series  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

### ■ Solution

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)}$$

$$s_k = \sum_{n=1}^k \frac{1}{n(n+1)} = \sum_{n=1}^k \left( \frac{1}{n} - \frac{1}{(n+1)} \right) =$$

$$\left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \dots + \left( \frac{1}{k-1} - \frac{1}{k} \right) + \left( \frac{1}{k} - \frac{1}{k+1} \right)$$

$$= 1 - \frac{1}{k+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k \rightarrow \infty} s_k = 1$$

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## Note

- In general, when we deal with a series, there are two questions we would like to answer:
  - (1) the existence of the limit of the series  $s_{\infty} = \sum_{k=1}^{\infty} a_k$
  - (2) In the case where the limit of the series exists, what is the value of this limit?
- The tests that will be discussed in the following only provide the answer to question (1) but not necessarily question (2).

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## Theorem 7 (not very useful to test the convergence of a series)

### THEOREM 7

If  $\sum_{n=1}^{\infty} a_n$  converges, then  $a_n \rightarrow 0$ .

- Let  $S$  be the convergent limit of the series, i.e.  $\lim_{n \rightarrow \infty} s_n = \sum_{n=1}^{\infty} a_n = S$
- When  $n$  is large,  $s_n$  and  $s_{n-1}$  are close to  $S$
- This means  $a_n = s_n - s_{n-1} \rightarrow S - S = 0$  as  $n \rightarrow \infty$

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## Example 7 Applying the $n$ th-term test

(a)  $\sum_{n=1}^{\infty} n^2$  diverges because  $\lim_{n \rightarrow \infty} n^2 = \infty$ , i.e.  $\lim_{n \rightarrow \infty} a_n$  fail to exist.

(b)  $\sum_{n=1}^{\infty} \frac{n+1}{n}$  diverges because  $\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1 \neq 0$ .

(c)  $\sum_{n=1}^{\infty} (-1)^{n+1}$  diverges because  $\lim_{n \rightarrow \infty} (-1)^{n+1}$  fail to exist.

(d)  $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$  diverges because  $\lim_{n \rightarrow \infty} \frac{-n}{2n+5} = \frac{-1}{2} \neq 0$  (l'Hopital rule)

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### The $n$ th-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$  diverges if  $\lim_{n \rightarrow \infty} a_n$  fails to exist or is different from zero.

Comment: useful to spot almost instantly if a series is divergent.

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## A question

- Will the series converge if  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ ?

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## Example 8 $a_n \rightarrow 0$ but the series diverges

$$1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{2 \text{ terms}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{4 \text{ terms}} + \dots + \underbrace{\frac{1}{2^n} + \frac{1}{2^n} + \frac{1}{2^n} + \dots + \frac{1}{2^n}}_{2^n \text{ terms}} + \dots$$

- The terms are grouped into clusters that add up to 1, so the partial sum increases without bound  $\rightarrow$  the series diverges
- Yet  $a_n = 2^{-n} \rightarrow 0$

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- Question:
- If  $\sum a_n$  and  $\sum b_n$  both diverges, must  $\sum(a_n \pm b_n)$  diverge?

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### THEOREM 8

If  $\sum a_n = A$  and  $\sum b_n = B$  are convergent series, then

1. *Sum Rule:*  $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:*  $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:*  $\sum ka_n = k\sum a_n = kA$  (Any number  $k$ ).

- Corollary:
- Every nonzero constant multiple of a divergent series diverges
- If  $\sum a_n$  converges and  $\sum b_n$  diverges, then  $\sum(a_n + b_n)$  and  $\sum(a_n - b_n)$  both diverges.

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**EXAMPLE 9** Find the sums of the following series.

$$\begin{aligned} \text{(a)} \quad \sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} &= \sum_{n=1}^{\infty} \left( \frac{1}{2^{n-1}} - \frac{1}{6^{n-1}} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} && \text{Difference Rule} \\ &= \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/6)} && \text{Geometric series with } a = 1 \text{ and } r = 1/2, 1/6 \\ &= 2 - \frac{6}{5} \\ &= \frac{4}{5} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \sum_{n=0}^{\infty} \frac{4}{2^n} &= 4 \sum_{n=0}^{\infty} \frac{1}{2^n} && \text{Constant Multiple Rule} \\ &= 4 \left( \frac{1}{1 - (1/2)} \right) && \text{Geometric series with } a = 1, r = 1/2 \\ &= 8 \end{aligned}$$

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# 11.3

## The Integral Test

### Corollary of Theorem 6

A series  $\sum_{n=1}^{\infty} a_n$  of nonnegative terms converges if and only if its partial sums are bounded from above.

Comment: To test whether a non-decreasing sequence converges, check whether its partial sum is bounded from above. If it is, the sequence converges.

This is particularly useful for sequences with

$$a_n \rightarrow 0 \text{ as } n \rightarrow \infty$$

for which neither the  $n$ -term test nor Theorem 7 can be used to conclude the divergence / convergence.

## Nondecreasing partial sums

- Suppose  $\{a_n\}$  is a sequence with  $a_n > 0$  for all  $n$
- Then, the partial sum  $s_{n+1} = s_n + a_n \geq s_n$
- $\Rightarrow$  The partial sums form a nondecreasing sequence

$$\{s_n = \sum_{k=1}^n a_k\} = \{s_1, s_2, s_3, \dots, s_n, \dots\}$$

- Theorem 6, the Nondecreasing Sequence Theorem tells us that the series  $\sum_{n=1}^{\infty} a_n$  converges if and only if the partial sums are bounded from above.

### Example 1 The harmonic series

- The series  $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{2}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{4}{8} = \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}\right)}_{> \frac{8}{16} = \frac{1}{2}} + \dots$  diverges.

- Consider the sequence of partial sums  $\{s_1, s_2, s_4, s_{16}, \dots, s_{2^k}, \dots\}$ 
  - $s_1 = 1$
  - $s_2 = s_1 + 1/2 > 1 \cdot (1/2)$
  - $s_4 = s_2 + (1/3 + 1/4) > 2 \cdot (1/2)$
  - $s_8 = s_4 + (1/5 + 1/6 + 1/7 + 1/8) > 3 \cdot (1/2)$
  - ...
  - $s_{2^k} > k \cdot (1/2)$

- The partial sum of the first  $2^k$  terms in the series,  $s_n > k/2$ , where  $k=0, 1, 2, 3, \dots$
- This means the partial sum,  $s_n$ , is not bounded from above.
- Hence, by the virtue of Corollary 6, the harmonic series diverges

### THEOREM 9 The Integral Test

Let  $\{a_n\}$  be a sequence of positive terms. Suppose that  $a_n = f(n)$ , where  $f$  is a continuous, positive, decreasing function of  $x$  for all  $x \geq N$  ( $N$  a positive integer). Then the series  $\sum_{n=N}^{\infty} a_n$  and the integral  $\int_N^{\infty} f(x) dx$  both converge or both diverge.

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If  $p > 1$ , then  $f(x) = 1/x^p$  is a positive decreasing function of  $x$ . Since

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^p} dx &= \int_1^{\infty} x^{-p} dx = \lim_{b \rightarrow \infty} \left[ \frac{x^{-p+1}}{-p+1} \right]_1^b \\ &= \frac{1}{1-p} \lim_{b \rightarrow \infty} \left( \frac{1}{b^{p-1}} - 1 \right) \\ &= \frac{1}{1-p} (0 - 1) = \frac{1}{p-1}, \quad \begin{array}{l} b^{p-1} \rightarrow \infty \text{ as } b \rightarrow \infty \\ \text{because } p-1 > 0. \end{array}\end{aligned}$$

the series converges by the Integral Test.

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### EXAMPLE 3 The $p$ -Series

Show that the  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

( $p$  a real constant) converges if  $p > 1$ , and diverges if  $p \leq 1$ .

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If  $p < 1$ , then  $1 - p > 0$  and

$$\int_1^{\infty} \frac{1}{x^p} dx = \frac{1}{1-p} \lim_{b \rightarrow \infty} (b^{1-p} - 1) = \infty.$$

The series diverges by the Integral Test.

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If  $p = 1$ , we have the (divergent) harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

The  $p$ -series with  $p = 1$  is the **harmonic series**

We have convergence for  $p > 1$  but divergence for every other value of  $p$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

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## Caution

- The integral test only tells us whether a given series converges or otherwise
- The test DOES NOT tell us what the convergent limit of the series is (in the case where the series converges), as the series and the integral need not have the same value in the convergent case.

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## Example 4 A convergent series

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$  is convergent by the integral test:

Let  $f(x) = \frac{1}{x^2+1}$ , so that  $f(n) = a_n = \frac{1}{n^2+1}$ .  $f(x)$  is continuous, positive, decreasing for all  $x \geq 1$ .

$$\int_1^{\infty} f(x) dx = \int_1^{\infty} \frac{1}{x^2+1} dx = \dots = \lim_{b \rightarrow \infty} \left[ \tan^{-1} x \right]_1^b = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Hence,  $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$  converges by the integral test.

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## 11.4

### Comparison Tests

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## THEOREM 10 The Comparison Test

Let  $\sum a_n$  be a series with no negative terms.

(a)  $\sum a_n$  converges if there is a convergent series  $\sum c_n$  with  $a_n \leq c_n$  for all  $n > N$ , for some integer  $N$ .

(b)  $\sum a_n$  diverges if there is a divergent series of nonnegative terms  $\sum d_n$  with  $a_n \geq d_n$  for all  $n > N$ , for some integer  $N$ .

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## EXAMPLE 1 Applying the Comparison Test

(a) The series

$$\sum_{n=1}^{\infty} \frac{5}{5n-1}$$

diverges because its  $n$ th term

$$\frac{5}{5n-1} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n}$$

is greater than the  $n$ th term of the divergent harmonic series.

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## EXAMPLE 1 Applying the Comparison Test

(b) The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

converges because its terms are all positive and less than or equal to the corresponding terms of

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

The geometric series on the left converges and we have

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1 - (1/2)} = 3.$$

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## Caution

- The comparison test only tell us whether a given series converges or otherwise
- The test DOES NOT tell us what the convergent limit of the series is (in the case where the series converges), as the two series need not have the same value in the convergent case

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### THEOREM 11 Limit Comparison Test

Suppose that  $a_n > 0$  and  $b_n > 0$  for all  $n \geq N$  ( $N$  an integer).

1. If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$ , then  $\sum a_n$  and  $\sum b_n$  both converge or both diverge.
2. If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$  and  $\sum b_n$  converges, then  $\sum a_n$  converges.
3. If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$  and  $\sum b_n$  diverges, then  $\sum a_n$  diverges.

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## Example 2 continued

- (b) Let  $a_n = 1/(2^n - 1)$ . For large  $n$ , we expect  $a_n$  to behave like  $1/2^n$ , so we let  $b_n = 1/2^n$ . Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{2^n}{2^n - 1} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 - (1/2^n)} \\ &= 1, \end{aligned}$$

$\sum a_n$  converges by Part 1 of the Limit Comparison Test.

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### EXAMPLE 2 Using the Limit Comparison Test

Which of the following series converge, and which diverge?

- (a)  $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$
- (b)  $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

#### Solution

- (a) Let  $a_n = (2n+1)/(n^2+2n+1)$ . For large  $n$ , we expect  $a_n$  to behave like  $2n/n^2 = 2/n$  since the leading terms dominate for large  $n$ , so we let  $b_n = 1/n$ . Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2n^2 + n}{n^2 + 2n + 1} = 2,$$

$\sum a_n$  diverges by Part 1 of the Limit Comparison Test. We could just as well have taken  $b_n = 2/n$ , but  $1/n$  is simpler.

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## Caution

- The limit comparison test only tell us whether a given series converges or otherwise
- The test DOES NOT tell us what the convergent limit of the series is (in the case where the series converges)

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## 11.5

### The Ratio and Root Tests

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#### EXAMPLE 1 Applying the Ratio Test

Investigate the convergence of the following series.

$$(a) \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \quad (b) \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$$

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## THEOREM 12 The Ratio Test

Let  $\sum a_n$  be a series with positive terms and suppose that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then

- (a) the series *converges* if  $\rho < 1$ ,
- (b) the series *diverges* if  $\rho > 1$  or  $\rho$  is infinite,
- (c) the test is *inconclusive* if  $\rho = 1$ .

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### Solution

(a)

For the series  $\sum_{n=0}^{\infty} (2^n + 5)/3^n$ ,

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} \\ &= \frac{1}{3} \cdot \left( \frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}} \right) \rightarrow \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}. \end{aligned}$$

The series converges because  $\rho = 2/3$  is less than 1.

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## Solution

(b)

$$\text{If } a_n = \frac{(2n)!}{n!n!}, \text{ then } a_{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!}$$

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{n!n!(2n+2)(2n+1)(2n)!}{(n+1)!(n+1)!(2n)!} \\ &= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{4n+2}{n+1} \rightarrow 4. \end{aligned}$$

The series diverges because  $\rho = 4$  is greater than 1.

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## Caution

- The ratio test only tell us whether a given series converges or otherwise
- The test DOES NOT tell us what the convergent limit of the series is (in the case where the series converges)

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### THEOREM 13 The Root Test

Let  $\sum a_n$  be a series with  $a_n \geq 0$  for  $n \geq N$ , and suppose that

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \rho.$$

Then

- (a) the series *converges* if  $\rho < 1$ ,
- (b) the series *diverges* if  $\rho > 1$  or  $\rho$  is infinite,
- (c) the test is *inconclusive* if  $\rho = 1$ .

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### EXAMPLE 3 Applying the Root Test

Which of the following series converges, and which diverges?

(a)  $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$     (b)  $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$     (c)  $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

**Solution**

(a)  $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$  converges because  $\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{(\sqrt[n]{n})^2}{2} \rightarrow \frac{1}{2} < 1$ .

(b)  $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$  diverges because  $\sqrt[n]{\frac{2^n}{n^2}} = \frac{2}{(\sqrt[n]{n})^2} \rightarrow \frac{2}{1} > 1$ .

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# 11.6

## Alternating Series, Absolute and Conditional Convergence

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### THEOREM 14 The Alternating Series Test (Leibniz's Theorem)

The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$$

converges if all three of the following conditions are satisfied:

1. The  $u_n$ 's are all positive.
2.  $u_n \geq u_{n+1}$  for all  $n \geq N$ , for some integer  $N$ .
3.  $u_n \rightarrow 0$ .

The alternating harmonic series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges because it satisfies the three requirements of Leibniz's theorem.

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# Alternating series

- A series in which the terms are alternately positive and negative

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots + \frac{(-1)^{n+1}}{n} + \dots$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots + \frac{(-1)^n 4}{2^n} + \dots$$

$$1 - 2 + 3 - 4 + 5 - 6 + \dots + (-1)^{n+1} n + \dots$$

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### THEOREM 15 The Alternating Series Estimation Theorem

If the alternating series  $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$  satisfies the three conditions of Theorem 14, then for  $n \geq N$ ,

$$s_n = u_1 - u_2 + \dots + (-1)^{n+1} u_n$$

approximates the sum  $L$  of the series with an error whose absolute value is less than  $u_{n+1}$ , the numerical value of the first unused term. Furthermore, the remainder,  $L - s_n$ , has the same sign as the first unused term.

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**EXAMPLE 2** We try Theorem 15 on a series whose sum we know:

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} - \dots$$

The theorem says that if we truncate the series after the eighth term, we throw away a total that is positive and less than  $1/256$ . The sum of the first eight terms is  $0.6640625$ . The sum of the series is

$$\frac{1}{1 - (-1/2)} = \frac{1}{3/2} = \frac{2}{3}.$$

The difference,  $(2/3) - 0.6640625 = 0.0026041666\dots$ , is positive and less than  $(1/256) = 0.00390625$ .

**DEFINITION** **Conditionally Convergent**

A series that converges but does not converge absolutely **converges conditionally**.

Example:

The alternative harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \text{ converges (by virtue of Leibniz Theorem)}$$

But the corresponding absolute series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \text{ diverges (a harmonic series)}$$

Hence, by definition, the alternating harmonic series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges conditionally.

**DEFINITION** **Absolutely Convergent**

A series  $\sum a_n$  **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values,  $\sum |a_n|$ , converges.

Example:

The geometric series

$$\sum_{n=1}^{\infty} 1 \left( -\frac{1}{2} \right)^{n-1} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots \text{ converges absolutely since}$$

the corresponding absolute series

$$\sum_{n=1}^{\infty} \left| 1 \left( -\frac{1}{2} \right)^{n-1} \right| = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \text{ converges}$$

**THEOREM 16** **The Absolute Convergence Test**

If  $\sum_{n=1}^{\infty} |a_n|$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges.

In other words, if a series converges absolutely, it converges.

In the previous example, we shown that the geometric series  $\sum_{n=1}^{\infty} 1 \left( -\frac{1}{2} \right)^{n-1}$  converges absolutely. Hence, by virtue of the absolute convergent test, the series  $\sum_{n=1}^{\infty} 1 \left( -\frac{1}{2} \right)^{n-1}$  converges.

## Caution

- All series that are absolutely convergent converges.
- But the converse is not true, namely, not all convergent series are absolutely convergent.
- Think of series that is conditionally convergent. These are convergent series that are not absolutely convergent.

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## EXAMPLE 3 Applying the Absolute Convergence Test

(a)

For  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$ , the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

The original series converges because it converges absolutely.

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## EXAMPLE 3 Applying the Absolute Convergence Test

(b)

$$\text{For } \sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots,$$

the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \cdots,$$

which converges by comparison with  $\sum_{n=1}^{\infty} (1/n^2)$  because  $|\sin n| \leq 1$  for every  $n$ .

The original series converges absolutely; therefore it converges.

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### THEOREM 17 The Rearrangement Theorem for Absolutely Convergent Series

If  $\sum_{n=1}^{\infty} a_n$  converges absolutely, and  $b_1, b_2, \dots, b_n, \dots$  is any arrangement of the sequence  $\{a_n\}$ , then  $\sum b_n$  converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

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1. **The  $n$ th-Term Test:** Unless  $a_n \rightarrow 0$ , the series diverges.
2. **Geometric series:**  $\sum ar^n$  converges if  $|r| < 1$ ; otherwise it diverges.
3.  **$p$ -series:**  $\sum 1/n^p$  converges if  $p > 1$ ; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test.
5. **Series with some negative terms:** Does  $\sum |a_n|$  converge? If yes, so does  $\sum a_n$ , since absolute convergence implies convergence.
6. **Alternating series:**  $\sum a_n$  converges if the series satisfies the conditions of the Alternating Series Test.

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## 11.7

### Power Series

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#### DEFINITIONS Power Series, Center, Coefficients

A **power series about  $x = 0$**  is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots \quad (1)$$

A **power series about  $x = a$**  is a series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots + c_n (x - a)^n + \cdots \quad (2)$$

in which the **center**  $a$  and the **coefficients**  $c_0, c_1, c_2, \dots, c_n, \dots$  are constants.

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#### EXAMPLE 1 A Geometric Series

Taking all the coefficients to be 1 in Equation (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This is the geometric series with first term 1 and ratio  $x$ . It converges to  $1/(1 - x)$  for  $|x| < 1$ . We express this fact by writing

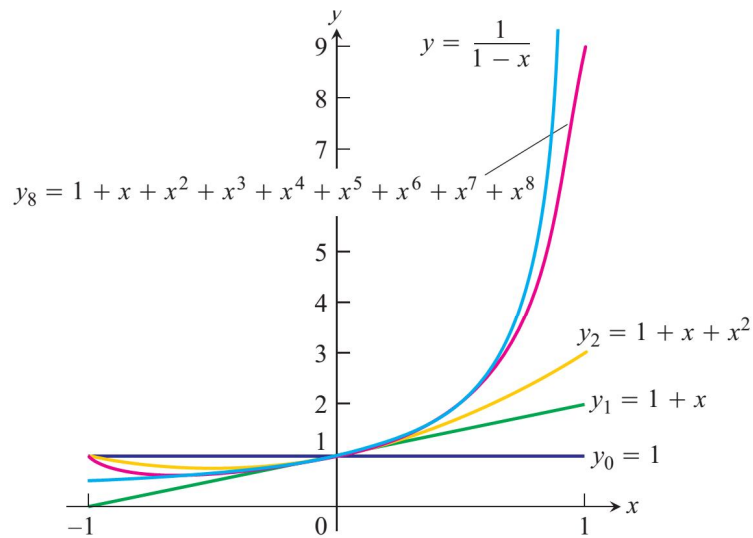
$$\frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1. \quad (3)$$

Mathematica simulation

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**FIGURE 11.10** The graphs of  $f(x) = 1/(1 - x)$  and four of its polynomial approximations (Example 1).

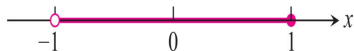
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**Solution** Apply the Ratio Test to the series  $\sum |u_n|$ , where  $u_n$  is the  $n$ th term of the series in question.

$$(a) \left| \frac{u_{n+1}}{u_n} \right| = \frac{n}{n+1} |x| \rightarrow |x|.$$

The series converges absolutely for  $|x| < 1$ . It diverges if  $|x| > 1$  because the  $n$ th term does not converge to zero. At  $x = 1$ , we get the alternating harmonic series  $1 - 1/2 + 1/3 - 1/4 + \dots$ , which converges. At  $x = -1$  we get  $-1 - 1/2 - 1/3 - 1/4 - \dots$ , the negative of the harmonic series; it diverges. Series (a) converges for  $-1 < x \leq 1$  and diverges elsewhere.



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### EXAMPLE 3 Testing for Convergence Using the Ratio Test

For what values of  $x$  do the following power series converge?

- (a)  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$
- (b)  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$
- (c)  $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
- (d)  $\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \dots$

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**(b)**

$$\left| \frac{u_{n+1}}{u_n} \right| = \frac{2n-1}{2n+1} x^2 \rightarrow x^2.$$

The series converges absolutely for  $x^2 < 1$ . It diverges for  $x^2 > 1$  because the  $n$ th term does not converge to zero. At  $x = 1$  the series becomes  $1 - 1/3 + 1/5 - 1/7 + \dots$ , which converges by the Alternating Series Theorem. It also converges at  $x = -1$  because it is again an alternating series that satisfies the conditions for convergence. The value at  $x = -1$  is the negative of the value at  $x = 1$ . Series (b) converges for  $-1 \leq x \leq 1$  and diverges elsewhere.



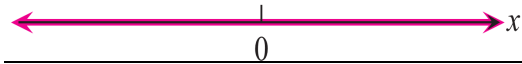
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(c)

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \rightarrow 0 \text{ for every } x.$$

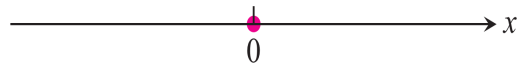
The series converges absolutely for all  $x$ .



(d)

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = (n+1)|x| \rightarrow \infty \text{ unless } x = 0.$$

The series diverges for all values of  $x$  except  $x = 0$ .



## The radius of convergence of a power series

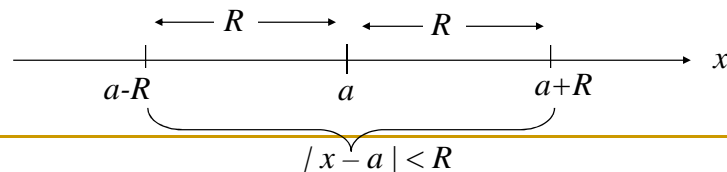
### THEOREM 18 The Convergence Theorem for Power Series

If the power series  $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$  converges for  $x = c \neq 0$ , then it converges absolutely for all  $x$  with  $|x| < |c|$ . If the series diverges for  $x = d$ , then it diverges for all  $x$  with  $|x| > |d|$ .

### COROLLARY TO THEOREM 18

The convergence of the series  $\sum c_n(x-a)^n$  is described by one of the following three possibilities:

1. There is a positive number  $R$  such that the series diverges for  $x$  with  $|x-a| > R$  but converges absolutely for  $x$  with  $|x-a| < R$ . The series may or may not converge at either of the endpoints  $x = a - R$  and  $x = a + R$ .
2. The series converges absolutely for every  $x$  ( $R = \infty$ ).
3. The series converges at  $x = a$  and diverges elsewhere ( $R = 0$ ).



- $R$  is called the radius of convergence of the power series
- The interval of radius  $R$  centered at  $x = a$  is called the interval of convergence
- The interval of convergence may be open, closed, or half-open:  $[a-R, a+R]$ ,  $(a-R, a+R)$ ,  $[a-R, a+R)$  or  $(a-R, a+R]$
- A power series converges for all  $x$  that lies within the interval of convergence.

## How to Test a Power Series for Convergence

1. Use the Ratio Test (or  $n$ th-Root Test) to find the interval where the series converges absolutely. Ordinarily, this is an open interval

$$|x - a| < R \quad \text{or} \quad a - R < x < a + R.$$

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. If the interval of absolute convergence is  $a - R < x < a + R$ , the series diverges for  $|x - a| > R$  (it does not even converge conditionally), because the  $n$ th term does not approach zero for those values of  $x$ .

See example 3 (previous slides, where we determined their interval of convergence)

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## THEOREM 19 The Term-by-Term Differentiation Theorem

If  $\sum c_n(x - a)^n$  converges for  $a - R < x < a + R$  for some  $R > 0$ , it defines a function  $f$ :

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n, \quad a - R < x < a + R.$$

Such a function  $f$  has derivatives of all orders inside the interval of convergence. We can obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} n c_n(x - a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x - a)^{n-2},$$

and so on. Each of these derived series converges at every interior point of the interval of convergence of the original series.

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## EXAMPLE 4 Applying Term-by-Term Differentiation

Find series for  $f'(x)$  and  $f''(x)$  if

$$\begin{aligned} f(x) &= \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots \\ &= \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1 \end{aligned}$$

**Solution**

$$\begin{aligned} f'(x) &= \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots \\ &= \sum_{n=1}^{\infty} nx^{n-1}, \quad -1 < x < 1 \\ f''(x) &= \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots \\ &= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \quad -1 < x < 1 \end{aligned}$$

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## Caution

- Power series is term-by-term differentiable
- However, in general, not all series is term-by-term differentiable, e.g. the trigonometric series  $\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$  is not (it's not a power series)

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# A power series can be integrated term by term throughout its interval of convergence

## THEOREM 20 The Term-by-Term Integration Theorem

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

converges for  $a - R < x < a + R$  ( $R > 0$ ). Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for  $a - R < x < a + R$  and

$$\int f(x) dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for  $a - R < x < a + R$ .

## EXAMPLE 5 A Series for $\tan^{-1}x$ , $-1 \leq x \leq 1$

Identify the function

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \quad -1 \leq x \leq 1.$$

**Solution** We differentiate the original series term by term and get

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots, \quad -1 < x < 1.$$

This is a geometric series with first term 1 and ratio  $-x^2$ , so

$$f'(x) = \frac{1}{1 - (-x^2)} = \frac{1}{1 + x^2}.$$

We can now integrate  $f'(x) = 1/(1 + x^2)$  to get

$$\int f'(x) dx = \int \frac{dx}{1 + x^2} = \tan^{-1}x + C.$$

The series for  $f(x)$  is zero when  $x = 0$ , so  $C = 0$ . Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1}x, \quad -1 < x < 1. \quad (7)$$

— In Section 11.10, we will see that the series also converges to  $\tan^{-1}x$  at  $x = \pm 1$ . —

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## EXAMPLE 6 A Series for $\ln(1+x)$ , $-1 < x \leq 1$

The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval  $-1 < t < 1$ . Therefore,

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big|_0^x \quad \text{Theorem 20}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 < x < 1.$$

It can also be shown that the series converges at  $x = 1$  to the number  $\ln 2$ , but that was not guaranteed by the theorem.

## THEOREM 21 The Series Multiplication Theorem for Power Series

If  $A(x) = \sum_{n=0}^{\infty} a_n x^n$  and  $B(x) = \sum_{n=0}^{\infty} b_n x^n$  converge absolutely for  $|x| < R$ , and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^n a_k b_{n-k},$$

then  $\sum_{n=0}^{\infty} c_n x^n$  converges absolutely to  $A(x)B(x)$  for  $|x| < R$ :

$$\left( \sum_{n=0}^{\infty} a_n x^n \right) \cdot \left( \sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n.$$

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**EXAMPLE 7** Multiply the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots = \frac{1}{1-x}, \quad \text{for } |x| < 1,$$

by itself to get a power series for  $1/(1-x)^2$ , for  $|x| < 1$ .

**Solution** Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + \dots + x^n + \dots = 1/(1-x)$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = 1 + x + x^2 + \dots + x^n + \dots = 1/(1-x)$$

and

$$\begin{aligned} c_n &= \underbrace{a_0 b_n + a_1 b_{n-1} + \dots + a_k b_{n-k} + \dots + a_n b_0}_{n+1 \text{ terms}} \\ &= \underbrace{1 + 1 + \dots + 1}_{n+1 \text{ ones}} = n + 1. \end{aligned}$$

Then, by the Series Multiplication Theorem,

$$\begin{aligned} A(x) \cdot B(x) &= \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1)x^n \\ &= 1 + 2x + 3x^2 + 4x^3 + \dots + (n+1)x^n + \dots \end{aligned}$$

is the series for  $1/(1-x)^2$ . The series all converge absolutely for  $|x| < 1$ .

Notice that Example 4 gives the same answer because

$$\frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{1}{(1-x)^2}.$$

# 11.8

## Taylor and Maclaurin Series

## Series Representation

- In the previous topic we see that an infinite series represents a function. The converse is also true, namely:

A function that is infinitely differentiable  $f(x)$  can be

expressed as a power series  $\sum_{n=1}^{\infty} b_n (x-a)^n$

- We say: The function  $f(x)$  generates the power series  $\sum_{n=1}^{\infty} b_n (x-a)^n$
- The power series generated by the infinitely differentiable function is called Taylor series.
- The Taylor series provide useful polynomial approximations of the generating functions

## Finding the Taylor series representation

- In short, given an infinitely differentiable function  $f(x)$ , we would like to find out what is the Taylor series representation of  $f(x)$ , i.e. what is the coefficients of  $b_n$  in  $\sum_{n=1}^{\infty} b_n (x-a)^n$
- In addition, we would also need to work out the interval of  $x$  in which the Taylor series representation of  $f(x)$  converges.
- In generating the Taylor series representation of a generating function, we need to specify the point  $x=a$  at which the Taylor series is to be generated.

## Example 1 Finding a Taylor series

### DEFINITIONS Taylor Series, Maclaurin Series

Let  $f$  be a function with derivatives of all orders throughout some interval containing  $a$  as an interior point. Then the **Taylor series generated by  $f$  at  $x = a$**  is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n + \dots$$

The **Maclaurin series generated by  $f$**  is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n + \dots,$$

the Taylor series generated by  $f$  at  $x = 0$ .

Note: Maclaurin series is effectively a special case of Taylor series with  $a = 0$ .

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- Find the Taylor series generated by  $f(x)=1/x$  at  $a=2$ . Where, if anywhere, does the series converge to  $1/x$ ?
- $f(x) = x^{-1}$ ;  $f'(x) = -x^{-2}$ ;  $f^{(n)}(x) = (-1)^n n! x^{-(n+1)}$
- The Taylor series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!} (x-2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k k! x^{-(k+1)}}{k!} \Big|_{x=2} (x-2)^k = (-1)^0 2^{-1} (x-2)^0 + (-1)^1 2^{-2} (x-2)^1 + (-1)^2 2^{-3} (x-2)^2 + \dots (-1)^k 2^{-(k+1)} (x-2)^k + \dots = 1/2 - (x-2)/4 + (x-2)^2/8 + \dots (-1)^k (x-2)^k / 2^{(k+1)} + \dots$$

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$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!} (x-2)^k = 1/2 - (x-2)/4 + (x-2)^2/8 + \dots (-1)^k (x-2)^k / 2^{(k+1)} + \dots$$

This is a geometric series with  $r = -(x-2)/2$ ,

Hence, the Taylor series converges for  $|r| = |(x-2)/2| < 1$ ,

or equivalently,  $0 < x < 4$ .

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!} (x-2)^k = \frac{a}{1-r} = \frac{1/2}{1 - (-(x-2)/2)} = \frac{1}{x}$$

$\Rightarrow$  the Taylor series  $1/2 - (x-2)/4 + (x-2)^2/8 + \dots (-1)^k (x-2)^k / 2^{(k+1)} + \dots$

converges to  $\frac{1}{x}$  for  $0 < x < 4$ .

*\*Mathematica simulation*

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## Taylor polynomials

- Given an infinitely differentiable function  $f$ , we can approximate  $f(x)$  at values of  $x$  near  $a$  by the Taylor polynomial of  $f$ , i.e.  $f(x)$  can be approximated by  $f(x) \approx P_n(x)$ , where

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k = \frac{f(a)}{0!} + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f^{(3)}(a)}{3!} (x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n$$

- $P_n(x)$  = Taylor polynomial of degree  $n$  of  $f$  generated at  $x=a$ .
- $P_n(x)$  is simply the first  $n$  terms in the Taylor series of  $f$ .
- The remainder,  $|R_n(x)| = |f(x) - P_n(x)|$  becomes smaller if higher order approximation is used
- In other words, the higher the order  $n$ , the better is the approximation of  $f(x)$  by  $P_n(x)$
- In addition, the Taylor polynomial gives a close fit to  $f$  near the point  $x = a$ , but the error in the approximation can be large at points that are far away.

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## Example 2 Finding Taylor polynomial for $e^x$ at $x = 0$

$$f(x) = e^x \rightarrow f^{(n)}(x) = e^x$$

$$P_n(x) = \sum_{k=0}^{k=n} \frac{f^{(k)}(x)}{k!} \Big|_{x=0} \quad x^k = \frac{e^0}{0!} x^0 + \frac{e^0}{1!} x^1 + \frac{e^0}{2!} x^2 + \frac{e^0}{3!} x^3 + \dots + \frac{e^0}{n!} x^n$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} \quad \text{This is the Taylor polynomial of order } n \text{ for } e^x$$

If the limit  $n \rightarrow \infty$  is taken,  $P_n(x) \rightarrow$  Taylor series.

$$\text{The Taylor series for } e^x \text{ is } 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

In this special case, the Taylor series for  $e^x$  converges to  $e^x$  for all  $x$ .  
(To be proven later)

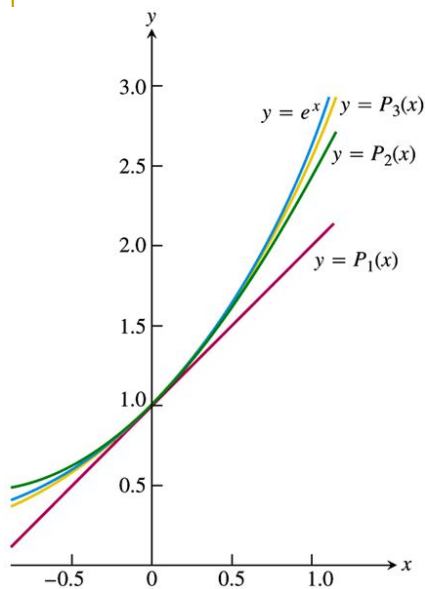
### DEFINITION Taylor Polynomial of Order $n$

Let  $f$  be a function with derivatives of order  $k$  for  $k = 1, 2, \dots, N$  in some interval containing  $a$  as an interior point. Then for any integer  $n$  from 0 through  $N$ , the **Taylor polynomial of order  $n$**  generated by  $f$  at  $x = a$  is the polynomial

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x - a)^k + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

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**FIGURE 11.12** The graph of  $f(x) = e^x$  and its Taylor polynomials  
 $P_1(x) = 1 + x$   
 $P_2(x) = 1 + x + (x^2/2!)$   
 $P_3(x) = 1 + x + (x^2/2!) + (x^3/3!)$ .  
 Notice the very close agreement near the center  $x = 0$  (Example 2).

\*Mathematica simulation

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### EXAMPLE 3 Finding Taylor Polynomials for $\cos x$

Find the Taylor series and Taylor polynomials generated by  $f(x) = \cos x$  at  $x = 0$ .

**Solution** The cosine and its derivatives are

$$\begin{aligned} f(x) &= \cos x, & f'(x) &= -\sin x, \\ f''(x) &= -\cos x, & f^{(3)}(x) &= \sin x, \\ &\vdots & &\vdots \\ f^{(2n)}(x) &= (-1)^n \cos x, & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x. \end{aligned}$$

At  $x = 0$ , the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \quad f^{(2n+1)}(0) = 0.$$

The Taylor series generated by  $f$  at 0 is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots \\ = 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots \\ = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}. \end{aligned}$$

This is also the Maclaurin series for  $\cos x$ . In Section 11.9, we will see that the series converges to  $\cos x$  at every  $x$ .

Because  $f^{(2n+1)}(0) = 0$ , the Taylor polynomials of orders  $2n$  and  $2n + 1$  are identical:

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Figure 11.13 shows how well these polynomials approximate  $f(x) = \cos x$  near  $x = 0$ . Only the right-hand portions of the graphs are given because the graphs are symmetric about the  $y$ -axis.

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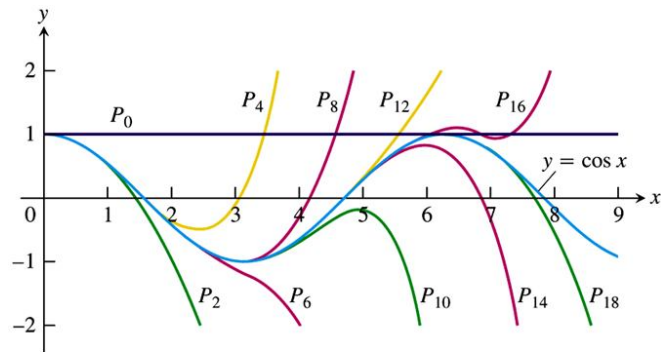


FIGURE 11.13 The polynomials

$$P_{2n}(x) = \sum_{k=0}^n \frac{(-1)^k x^{2k}}{(2k)!}$$

converge to  $\cos x$  as  $n \rightarrow \infty$ . We can deduce the behavior of  $\cos x$  arbitrarily far away solely from knowing the values of the cosine and its derivatives at  $x = 0$  (Example 3).

\*Mathematica simulation<sub>120</sub>

## 11.9

### Convergence of Taylor Series; Error Estimates

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- When does a Taylor series converge to its generating function?
- ANS:  
The Taylor series converge to its generating function if the |remainder| =  $|R_n(x)| = |f(x) - P_n(x)| \rightarrow 0$  as  $n \rightarrow \infty$

#### Taylor's Formula

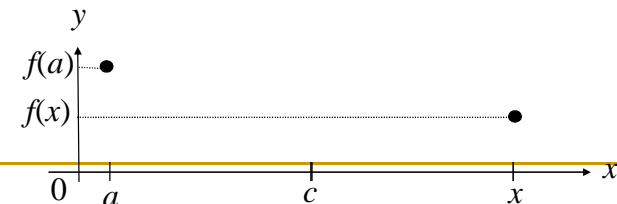
If  $f$  has derivatives of all orders in an open interval  $I$  containing  $a$ , then for each positive integer  $n$  and for each  $x$  in  $I$ ,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x), \quad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \quad (2)$$

$R_n(x)$  is called the remainder of order  $n$



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$$f(x) = P_n(x) + R_n(x) \text{ for each } x \text{ in } I.$$

If  $R_n(x) \rightarrow 0$  as  $n \rightarrow \infty$ ,  $P_n(x)$  converges to  $f(x)$ , then we can write

$$f(x) = \lim_{n \rightarrow \infty} P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \text{ for some } c \text{ between } 0 \text{ and } x$$

$$|R_n(x)| = \left| \frac{e^c}{(n+1)!} x^{n+1} \right|$$

If  $x > 0, 0 < c < x$

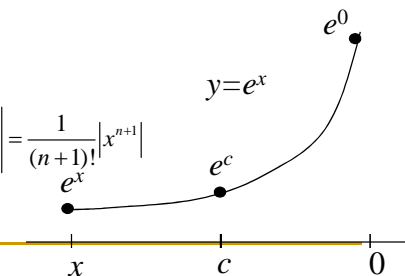
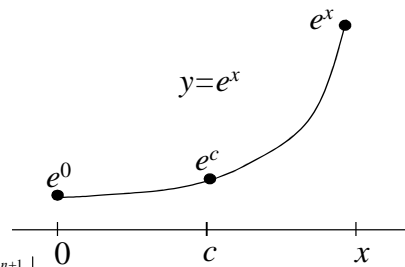
$$\Rightarrow 1 = e^0 < e^c < e^x \rightarrow \left| \frac{x^{n+1}}{(n+1)!} \right| < \left| \frac{e^c x^{n+1}}{(n+1)!} \right| < \left| \frac{e^x x^{n+1}}{(n+1)!} \right|$$

$$\rightarrow |R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \text{ for } x > 0.$$

If  $x < 0, x < c < 0$

$$\Rightarrow e^x < e^c < e^0 \rightarrow \left| \frac{e^c x^{n+1}}{(n+1)!} \right| < \left| \frac{e^0 x^{n+1}}{(n+1)!} \right| = \left| \frac{x^{n+1}}{(n+1)!} \right| = \frac{1}{(n+1)!} |x^{n+1}|$$

$$\rightarrow |R_n(x)| < \frac{|x^{n+1}|}{(n+1)!} \text{ for } x < 0$$



## Example 1 The Taylor series for $e^x$ revisited

- Show that the Taylor series generated by  $f(x)=e^x$  at  $x=0$  converges to  $f(x)$  for every value of  $x$ .
- Note: This can be proven by showing that  $|R_n| \rightarrow 0$  when  $n \rightarrow \infty$

Combining the result of both  $x > 0$  and  $x < 0$ ,

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \text{ when } x > 0,$$

$$|R_n(x)| < \frac{|x^{n+1}|}{(n+1)!} \text{ when } x < 0$$

Hence, irrespective of the sign of  $x$ ,  $\lim_{n \rightarrow \infty} |R_n(x)| = 0$  and the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ converge to } e^x \text{ for every } x.$$

### THEOREM 5

The following six sequences converge to the limits listed below:

1.  $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$
2.  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$
3.  $\lim_{n \rightarrow \infty} x^{1/n} = 1 \quad (x > 0)$
4.  $\lim_{n \rightarrow \infty} x^n = 0 \quad (|x| < 1)$
5.  $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$
6.  $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

In Formulas (3) through (6),  $x$  remains fixed as  $n \rightarrow \infty$ .

### THEOREM 1

Let  $\{a_n\}$  and  $\{b_n\}$  be sequences of real numbers and let  $A$  and  $B$  be real numbers. The following rules hold if  $\lim_{n \rightarrow \infty} a_n = A$  and  $\lim_{n \rightarrow \infty} b_n = B$ .

1. **Sum Rule:**  $\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$
2. **Difference Rule:**  $\lim_{n \rightarrow \infty} (a_n - b_n) = A - B$
3. **Product Rule:**  $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = A \cdot B$
4. **Constant Multiple Rule:**  $\lim_{n \rightarrow \infty} (k \cdot b_n) = k \cdot B \quad (\text{Any number } k)$
5. **Quotient Rule:**  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B} \quad \text{if } B \neq 0$



**THEOREM 23 The Remainder Estimation Theorem**

If there is a positive constant  $M$  such that  $|f^{(n+1)}(t)| \leq M$  for all  $t$  between  $x$  and  $a$ , inclusive, then the remainder term  $R_n(x)$  in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x - a|^{n+1}}{(n + 1)!}.$$

If this condition holds for every  $n$  and the other conditions of Taylor's Theorem are satisfied by  $f$ , then the series converges to  $f(x)$ .

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**EXAMPLE 6** Calculate  $e$  with an error of less than  $10^{-6}$ .

**Solution** We can use the result of Example 1 with  $x = 1$  to write

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1),$$

with

$$R_n(1) = e^c \frac{1}{(n + 1)!} \quad \text{for some } c \text{ between } 0 \text{ and } 1.$$

For the purposes of this example, we assume that we know that  $e < 3$ . Hence, we are certain that

$$\frac{1}{(n + 1)!} < R_n(1) < \frac{3}{(n + 1)!}$$

because  $1 < e^c < 3$  for  $0 < c < 1$ .

By experiment we find that  $1/9! > 10^{-6}$ , while  $3/10! < 10^{-6}$ . Thus we should take  $(n + 1)$  to be at least 10, or  $n$  to be at least 9. With an error of less than  $10^{-6}$ ,

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{9!} \approx 2.718282. \quad \blacksquare$$

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**EXAMPLE 3** The Taylor Series for  $\cos x$  at  $x = 0$  Revisited

Show that the Taylor series for  $\cos x$  at  $x = 0$  converges to  $\cos x$  for every value of  $x$ .

**Solution** We add the remainder term to the Taylor polynomial for  $\cos x$  (Section 11.8, Example 3) to obtain Taylor's formula for  $\cos x$  with  $n = 2k$ :

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + R_{2k}(x).$$

Because the derivatives of the cosine have absolute value less than or equal to 1, the Remainder Estimation Theorem with  $M = 1$  gives

$$|R_{2k}(x)| \leq 1 \cdot \frac{|x|^{2k+1}}{(2k + 1)!}.$$

For every value of  $x$ ,  $R_{2k} \rightarrow 0$  as  $k \rightarrow \infty$ . Therefore, the series converges to  $\cos x$  for every value of  $x$ . Thus,

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \quad (5)$$

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**EXAMPLE 7** For what values of  $x$  can we replace  $\sin x$  by  $x - (x^3/3!)$  with an error of magnitude no greater than  $3 \times 10^{-4}$ ?

**Solution** Here we can take advantage of the fact that the Taylor series for  $\sin x$  is an alternating series for every nonzero value of  $x$ . According to the Alternating Series Estimation Theorem (Section 11.6), the error in truncating

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

after  $(x^3/3!)$  is no greater than

$$\left| \frac{x^5}{5!} \right| = \frac{|x|^5}{120}.$$

Therefore the error will be less than or equal to  $3 \times 10^{-4}$  if

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \quad \text{or} \quad |x| < \sqrt[5]{360 \times 10^{-4}} \approx 0.514. \quad \text{Rounded down, to be safe}$$

The Alternating Series Estimation Theorem tells us something that the Remainder Estimation Theorem does not: namely, that the estimate  $x - (x^3/3!)$  for  $\sin x$  is an underestimate when  $x$  is positive because then  $x^5/120$  is positive.

Figure 11.15 shows the graph of  $\sin x$ , along with the graphs of a number of its approximating Taylor polynomials. The graph of  $P_3(x) = x - (x^3/3!)$  is almost indistinguishable from the sine curve when  $-1 \leq x \leq 1$ .

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# 11.10

## Applications of Power Series

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# The binomial series for powers and roots

- Consider the Taylor series generated by  $f(x) = (1+x)^m$ , where  $m$  is a constant:

$$\begin{aligned}
 f(x) &= (1+x)^m \\
 f'(x) &= m(1+x)^{m-1}, f''(x) = m(m-1)(1+x)^{m-2}, \\
 f'''(x) &= m(m-1)(m-2)(1+x)^{m-3}, \\
 &\vdots \\
 f^{(k)}(x) &= m(m-1)(m-2)\dots(m-k+1)(1+x)^{m-k}; \\
 \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k &= \sum_{k=0}^{\infty} \frac{m(m-1)(m-2)\dots(m-k+1)}{k!} x^k \\
 &= 1 + mx + m(m-1)x^2 + m(m-1)(m-2)x^3 + \dots + \frac{m(m-1)(m-2)\dots(m-k+1)}{k!} x^k + \dots
 \end{aligned}$$

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# The binomial series for powers and roots

$$\begin{aligned}
 f(x) &= (1+x)^m \\
 &= 1 + mx + m(m-1)x^2 + m(m-1)(m-2)x^3 + \dots + \frac{m(m-1)(m-2)\dots(m-k+1)}{k!} x^k + \dots
 \end{aligned}$$

- This series is called the binomial series, converges absolutely for  $|x| < 1$ . (The convergence can be determined by using Ratio test,  $\left| \frac{u_{k+1}}{u_k} \right| = \left| \frac{m-k}{k+1} x \right| \rightarrow |x|$ )

In short, the binomial series is the Taylor series for  $f(x) = (1+x)^m$ , where  $m$  a constant

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### The Binomial Series

For  $-1 < x < 1$ ,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\dots(m-k+1)}{k!} \quad \text{for } k \geq 3.$$

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**EXAMPLE 2** Using the Binomial Series

We know from Section 3.8, Example 1, that  $\sqrt{1+x} \approx 1 + (x/2)$  for  $|x|$  small. With  $m = 1/2$ , the binomial series gives quadratic and higher-order approximations as well, along with error estimates that come from the Alternating Series Estimation Theorem:

$$(1+x)^{1/2} = 1 + \frac{x}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \dots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \dots$$

Substitution for  $x$  gives still other approximations. For example,

$$\sqrt{1-x^2} \approx 1 - \frac{x^2}{2} - \frac{x^4}{8} \quad \text{for } |x^2| \text{ small}$$

$$\sqrt{1-\frac{1}{x}} \approx 1 - \frac{1}{2x} - \frac{1}{8x^2} \quad \text{for } \left|\frac{1}{x}\right| \text{ small, that is, } |x| \text{ large.}$$

**EXAMPLE 7** Limits Using Power Series

Evaluate

$$\lim_{x \rightarrow 1} \frac{\ln x}{x-1}$$

**Solution** We represent  $\ln x$  as a Taylor series in powers of  $x - 1$ . This can be accomplished by calculating the Taylor series generated by  $\ln x$  at  $x = 1$  directly or by replacing  $x$  by  $x - 1$  in the series for  $\ln(1+x)$  in Section 11.7, Example 6. Either way, we obtain

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \dots,$$

from which we find that

$$\lim_{x \rightarrow 1} \frac{\ln x}{x-1} = \lim_{x \rightarrow 1} \left(1 - \frac{1}{2}(x-1) + \dots\right) = 1.$$

# Taylor series representation of $\ln x$ at $x = 1$

- $f(x) = \ln x; f'(x) = x^{-1};$
- $f''(x) = (-1)(1)x^{-2}; f'''(x) = (-1)^2(2)(1)x^{-3} \dots$
- $f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n};$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} \Big|_{x=1} (x-1)^n = \frac{f^{(0)}(x)}{0!} \Big|_{x=1} (x-1)^0 + \sum_{n=1}^{\infty} \frac{f^{(n)}(x)}{n!} \Big|_{x=1} (x-1)^n$$

$$= \frac{\ln 1}{0!} + \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}(n-1)!x^{-n}}{n!} \Big|_{x=1} (x-1)^n = 0 + \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}(1)^{-n}}{n} (x-1)^n$$

$$= \frac{(-1)^0}{1}(x-1)^1 + \frac{(-1)^1}{2}(x-1)^2 + \frac{(-1)^2}{3}(x-1)^3 + \dots$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots + (-1)^n \frac{1}{n}(x-1)^n + \dots$$

\*Mathematica simulation 137

**EXAMPLE 8** Limits Using Power Series

Evaluate

$$\lim_{x \rightarrow 0} \frac{\sin x - \tan x}{x^3}$$

**Solution** The Taylor series for  $\sin x$  and  $\tan x$ , to terms in  $x^5$ , are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \quad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

Hence,

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \dots = x^3 \left(-\frac{1}{2} - \frac{x^2}{8} - \dots\right)$$

and

$$\lim_{x \rightarrow 0} \frac{\sin x - \tan x}{x^3} = \lim_{x \rightarrow 0} \left(-\frac{1}{2} - \frac{x^2}{8} - \dots\right)$$

$$= -\frac{1}{2}.$$

TABLE 11.1 Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \dots + (-x)^n + \dots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \leq 1$$

$$\ln \frac{1+x}{1-x} = 2 \tanh^{-1} x = 2 \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2n+1}}{2n+1} + \dots \right) = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}, \quad |x| < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \leq 1$$

**Binomial Series**

$$(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \dots + \frac{m(m-1)(m-2)\dots(m-k+1)x^k}{k!} + \dots$$

$$= 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k, \quad |x| < 1,$$

where

$$\binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2!}, \quad \binom{m}{k} = \frac{m(m-1)\dots(m-k+1)}{k!} \quad \text{for } k \geq 3.$$

*Note:* To write the binomial series compactly, it is customary to define  $\binom{m}{0}$  to be 1 and to take  $x^0 = 1$  (even in the usually excluded case where  $x = 0$ ), yielding  $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$ . If  $m$  is a positive integer, the series terminates at  $x^m$  and the result converges for all  $x$ .

# 11.11

## Fourier Series

## 'Weakness' of power series approximation

- In the previous lesson, we have learnt to approximate a given function using power series approximation, which give good fit if the approximated power series representation is evaluated near the point it is generated
- For point far away from the point the power series being generated, the approximation becomes poor
- In addition, the series approximation works only within the interval of convergence. Outside the interval of convergence, the series representation fails to represent the generating function
- Furthermore, power series approximation can not represent satisfactorily a function that has a jump discontinuity.
- Fourier series, our next topic, provide an alternative to overcome such shortage

Suppose we wish to approximate a function  $f$  on the interval  $[0, 2\pi]$  by a sum of sine and cosine functions,

$$f_n(x) = a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots + (a_n \cos nx + b_n \sin nx)$$

or, in sigma notation,

$$f_n(x) = a_0 + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx). \tag{1}$$

We would like to choose values for the constants  $a_0, a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  that make  $f_n(x)$  a "best possible" approximation to  $f(x)$ . The notion of "best possible" is defined as follows:

1.  $f_n(x)$  and  $f(x)$  give the same value when integrated from 0 to  $2\pi$ .
2.  $f_n(x) \cos kx$  and  $f(x) \cos kx$  give the same value when integrated from 0 to  $2\pi$  ( $k = 1, \dots, n$ ).
3.  $f_n(x) \sin kx$  and  $f(x) \sin kx$  give the same value when integrated from 0 to  $2\pi$  ( $k = 1, \dots, n$ ).

We chose  $f_n$  so that the integrals on the left remain the same when  $f_n$  is replaced by  $f$ , so we can use these equations to find  $a_0, a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  from  $f$ :

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx \quad (2)$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx dx, \quad k = 1, \dots, n \quad (3)$$

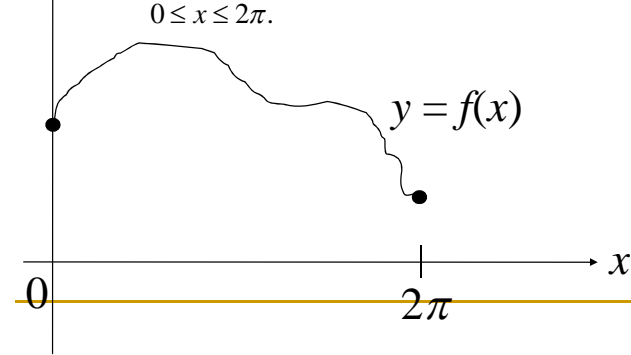
$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx, \quad k = 1, \dots, n \quad (4)$$

A function  $f(x)$  defined on  $[0, 2\pi]$  can be represented by a Fourier series

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^n f_k(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k \cos kx + b_k \sin kx$$

$$= a_0 + \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k \cos kx + b_k \sin kx, \quad \leftarrow \text{Fourier series representation of } f(x)$$

$0 \leq x \leq 2\pi.$



## Orthogonality of sinusoidal functions

$m, k$  nonzero integer.

If  $m=k$ ,

$$\int_0^{2\pi} \cos mx \cos kx dx = \int_0^{2\pi} \cos mx \cos mx dx = \frac{1}{2} \int_0^{2\pi} (1 + \cos(2mx)) dx = \frac{1}{2} \left[ x + \frac{\sin 2mx}{2m} \right]_0^{2\pi} = \pi.$$

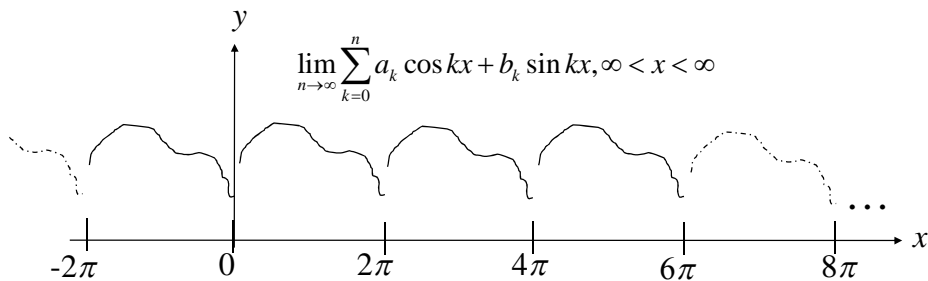
$$\int_0^{2\pi} \sin mx \sin kx dx = \int_0^{2\pi} \sin^2 mx dx = \pi$$

If  $m \neq k$ ,

$$\int_0^{2\pi} \cos mx \cos kx dx = 0, \int_0^{2\pi} \sin mx \sin kx dx = 0. \text{ (can be proven using, say, integration by parts or formula for the product of two sinusoidal functions).}$$

$$\text{In addition, } \int_0^{2\pi} \sin mx dx = \int_0^{2\pi} \cos mx dx = 0.$$

Also,  $\int_0^{2\pi} \sin mx \cos kx dx = 0$  for all  $m, k$ . We say sin and cos functions are orthogonal to each other.



If  $-\infty < x < \infty$ , the Fourier series  $\lim_{n \rightarrow \infty} \sum_{k=0}^n f_k(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k \cos kx + b_k \sin kx$

actually represents a periodic function  $f(x)$  of a period of  $L = 2\pi$ ,

## Derivation of $a_0$

$$f_n(x) = a_0 + \sum_{k=1}^n a_k \cos kx + b_k \sin kx$$

Integrate both sides with respect to  $x$  from  $x = 0$  to  $x = 2\pi$

$$\int_0^{2\pi} f_n(x) dx = \int_0^{2\pi} a_0 dx + \sum_{k=1}^n \int_0^{2\pi} a_k \cos kx dx + \sum_{k=1}^n \int_0^{2\pi} b_k \sin kx dx =$$

$$\int_0^{2\pi} a_0 dx + \sum_{k=1}^n a_k \int_0^{2\pi} \cos kx dx + \sum_{k=1}^n b_k \int_0^{2\pi} \sin kx dx$$

$$= 2\pi a_0 + 0 + 0 = 2\pi a_0$$

$$\Rightarrow 2\pi a_0 = \int_0^{2\pi} f_n(x) dx.$$

For large enough  $n$ ,  $f_n$  gives a good representation of  $f$ , hence we can replace  $f_n$  by  $f$ :

$$\Rightarrow a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx$$

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## Derivation of $b_k$ , $k \geq 1$

$b_k$  is similarly derived by multiplying both sides by  $\sin mx$  ( $m$  nonzero integer), and integrate with respect to  $x$  from  $x = 0$  to  $x = 2\pi$ .

$$\int_0^{2\pi} f(x) \sin mx dx =$$

$$\int_0^{2\pi} a_0 \sin mx dx + \sum_{k=1}^n a_k \int_0^{2\pi} \cos kx \sin mx dx + \sum_{k=1}^n b_k \int_0^{2\pi} \sin kx \sin mx dx$$

$$= 0 + 0 + b_m \int_0^{2\pi} \sin mx \sin mx dx = \pi b_m$$

$$\Rightarrow b_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin mx dx.$$

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## Derivation of $a_k$ , $k \geq 1$

$$f_n(x) = a_0 + \sum_{k=1}^n a_k \cos kx + b_k \sin kx$$

Multiply both sides by  $\cos mx$  ( $m$  nonzero integer), and integrate with respect to  $x$  from  $x = 0$  to  $x = 2\pi$ . By doing so, the integral  $\int_0^{2\pi} \cos mx \sin kx dx$  get 'killed off' due to the orthogonality property of the sinusoidal functions.

In addition,  $\int_0^{2\pi} \cos mx \cos kx dx$  will also get 'killed off' except for the case  $m = k$ .

$$\int_0^{2\pi} f(x) \cos mx dx =$$

$$\int_0^{2\pi} a_0 \cos mx dx + \sum_{k=1}^n a_k \int_0^{2\pi} \cos kx \cos mx dx + \sum_{k=1}^n b_k \int_0^{2\pi} \sin kx \cos mx dx$$

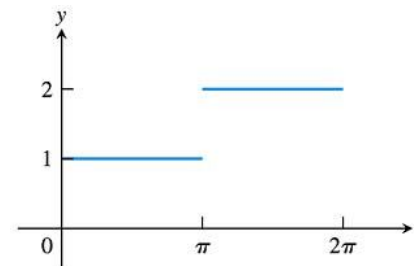
$$= 0 + a_m \int_0^{2\pi} \cos mx \cos mx dx + 0 = \pi a_m$$

$$\Rightarrow a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx dx.$$

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- Fourier series can represent some functions that cannot be represented by Taylor series, e.g. step function such as



(a)

FIGURE 11.16 (a) The step function

$$f(x) = \begin{cases} 1, & 0 \leq x \leq \pi \\ 2, & \pi < x \leq 2\pi \end{cases}$$

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**EXAMPLE 1** Finding a Fourier Series Expansion

Fourier series can be used to represent some functions that cannot be represented by Taylor series; for example, the step function  $f$  shown in Figure 11.16a.

$$f(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \pi \\ 2, & \text{if } \pi < x \leq 2\pi. \end{cases}$$

The coefficients of the Fourier series of  $f$  are computed using Equations (2), (3), and (4).

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx \\ &= \frac{1}{2\pi} \left( \int_0^\pi 1 \, dx + \int_\pi^{2\pi} 2 \, dx \right) = \frac{3}{2} \\ a_k &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx \\ &= \frac{1}{\pi} \left( \int_0^\pi \cos kx \, dx + \int_\pi^{2\pi} 2 \cos kx \, dx \right) \\ &= \frac{1}{\pi} \left( \left[ \frac{\sin kx}{k} \right]_0^\pi + \left[ \frac{2 \sin kx}{k} \right]_\pi^{2\pi} \right) = 0, \quad k \geq 1 \\ b_k &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx \\ &= \frac{1}{\pi} \left( \int_0^\pi \sin kx \, dx + \int_\pi^{2\pi} 2 \sin kx \, dx \right) \\ &= \frac{1}{\pi} \left( \left[ -\frac{\cos kx}{k} \right]_0^\pi + \left[ -\frac{2 \cos kx}{k} \right]_\pi^{2\pi} \right) \\ &= \frac{\cos k\pi - 1}{k\pi} = \frac{(-1)^k - 1}{k\pi}. \end{aligned}$$

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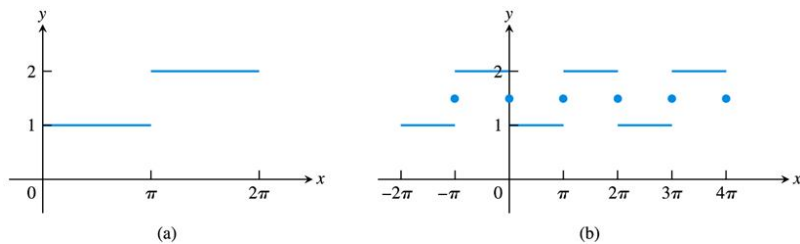


FIGURE 11.16 (a) The step function

$$f(x) = \begin{cases} 1, & 0 \leq x \leq \pi \\ 2, & \pi < x \leq 2\pi \end{cases}$$

(b) The graph of the Fourier series for  $f$  is periodic and has the value  $3/2$  at each point of discontinuity (Example 1).

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So

$$a_0 = \frac{3}{2}, \quad a_1 = a_2 = \dots = 0,$$

and

$$b_1 = -\frac{2}{\pi}, \quad b_2 = 0, \quad b_3 = -\frac{2}{3\pi}, \quad b_4 = 0, \quad b_5 = -\frac{2}{5\pi}, \quad b_6 = 0, \dots$$

The Fourier series is

$$\frac{3}{2} - \frac{2}{\pi} \left( \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right).$$

Notice that at  $x = \pi$ , where the function  $f(x)$  jumps from 1 to 2, all the sine terms vanish, leaving  $3/2$  as the value of the series. This is not the value of  $f$  at  $\pi$ , since  $f(\pi) = 1$ . The Fourier series also sums to  $3/2$  at  $x = 0$  and  $x = 2\pi$ . In fact, all terms in the Fourier series are periodic, of period  $2\pi$ , and the value of the series at  $x + 2\pi$  is the same as its value at  $x$ . The series we obtained represents the periodic function graphed in Figure 11.16b, with domain the entire real line and a pattern that repeats over every interval of width  $2\pi$ . The function jumps discontinuously at  $x = n\pi, n = 0, \pm 1, \pm 2, \dots$  and at these points has value  $3/2$ , the average value of the one-sided limits from each side. The convergence of the Fourier series of  $f$  is indicated in Figure 11.17.

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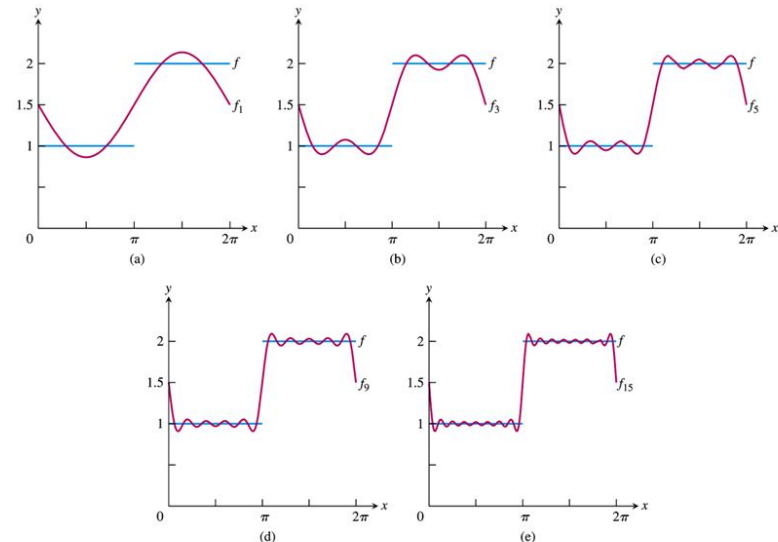


FIGURE 11.17 The Fourier approximation functions  $f_1, f_3, f_5, f_9,$  and  $f_{15}$  of the function  $f(x) = \begin{cases} 1, & 0 \leq x \leq \pi \\ 2, & \pi < x \leq 2\pi \end{cases}$  in Example 1.

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## Fourier series representation of a function defined on the general interval $[a,b]$

**THEOREM 24** Let  $f(x)$  be a function such that  $f$  and  $f'$  are piecewise continuous on the interval  $[0, 2\pi]$ . Then  $f$  is equal to its Fourier series at all points where  $f$  is continuous. At a point  $c$  where  $f$  has a discontinuity, the Fourier series converges to

$$\frac{f(c^+) + f(c^-)}{2}$$

where  $f(c^+)$  and  $f(c^-)$  are the right- and left-hand limits of  $f$  at  $c$ .

- For a function defined on the interval  $[0, 2\pi]$ , the Fourier series representation of  $f(x)$  is defined as  $f(x) = a_0 + \sum_{k=1}^n a_k \cos kx + b_k \sin kx$
- How about a function defined on an general interval of  $[a,b]$  where the period is  $L=b-a$  instead of  $2\pi$ ? Can we still use  $a_0 + \sum_{k=1}^n a_k \cos kx + b_k \sin kx$  to represent  $f(x)$  on  $[a,b]$ ?

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## Fourier series representation of a function defined on the general interval $[a,b]$

- For a function defined on the interval of  $[a,b]$  the Fourier series representation on  $[a,b]$  is actually

$$a_0 + \sum_{k=1}^n a_k \cos \frac{2\pi kx}{L} + b_k \sin \frac{2\pi kx}{L}$$

$$a_0 = \frac{1}{L} \int_a^b f(x) dx$$

$$a_m = \frac{2}{L} \int_a^b f(x) \cos \frac{2\pi mx}{L} dx$$

$$b_m = \frac{2}{L} \int_a^b f(x) \sin \frac{2\pi mx}{L} dx, m \text{ positive integer}$$

- $L=b-a$

## Derivation of $a_0$

$$f(x) = a_0 + \sum_{k=1}^n a_k \cos \frac{2\pi kx}{L} + b_k \sin \frac{2\pi kx}{L}$$

$$\int_a^b f(x) dx = \int_a^b a_0 dx + \sum_{k=1}^n \int_a^b a_k \cos \frac{2\pi kx}{L} dx + b_k \sin \frac{2\pi kx}{L} dx =$$

$$\int_a^b a_0 dx + \sum_{k=1}^n a_k \int_a^b \cos \frac{2\pi kx}{L} dx + \sum_{k=1}^n b_k \int_a^b \sin \frac{2\pi kx}{L} dx$$

$$= a_0(b-a)$$

$$\Rightarrow a_0 = \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{L} \int_a^b f(x) dx$$

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# Derivation of $a_k$

$$f(x) = a_0 + \sum_{k=1}^n a_k \cos \frac{2\pi kx}{L} + b_k \sin \frac{2\pi kx}{L}$$

$$\int_a^b f(x) \cos \frac{2\pi mx}{L} dx$$

$$= \int_a^b a_0 \cos \frac{2\pi mx}{L} dx + \sum_{k=1}^n \int_a^b \left( a_k \cos \frac{2\pi kx}{L} \cos \frac{2\pi mx}{L} + b_k \sin \frac{2\pi kx}{L} \cos \frac{2\pi mx}{L} \right) dx$$

$$= 0 + a_m \int_a^b \cos^2 \frac{2\pi mx}{L} dx + 0 = a_m \frac{L}{2}$$

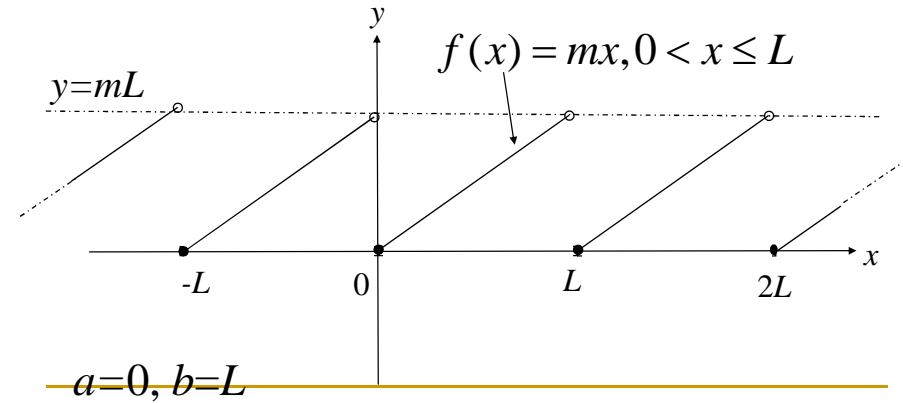
$$\Rightarrow a_m = \frac{2}{L} \int_a^b f(x) \cos \frac{2\pi mx}{L} dx$$

Similarly,

$$b_m = \frac{2}{L} \int_a^b f(x) \sin \frac{2\pi mx}{L} dx$$

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# Example:



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$$a_0 = \frac{1}{L} \int_a^b f(x) dx = \frac{1}{L} \int_a^b mx dx = \frac{m}{2L} (b^2 - a^2) = \frac{mL}{2}$$

$$a_k = \frac{2}{L} \int_a^b mx \cos \frac{2\pi kx}{L} dx = \frac{2m}{L} \int_a^b x \cos \frac{2\pi kx}{L} dx = \frac{2m}{L} \frac{L^2 (\cos 2k\pi - 1)}{4k^2 \pi^2} = 0;$$

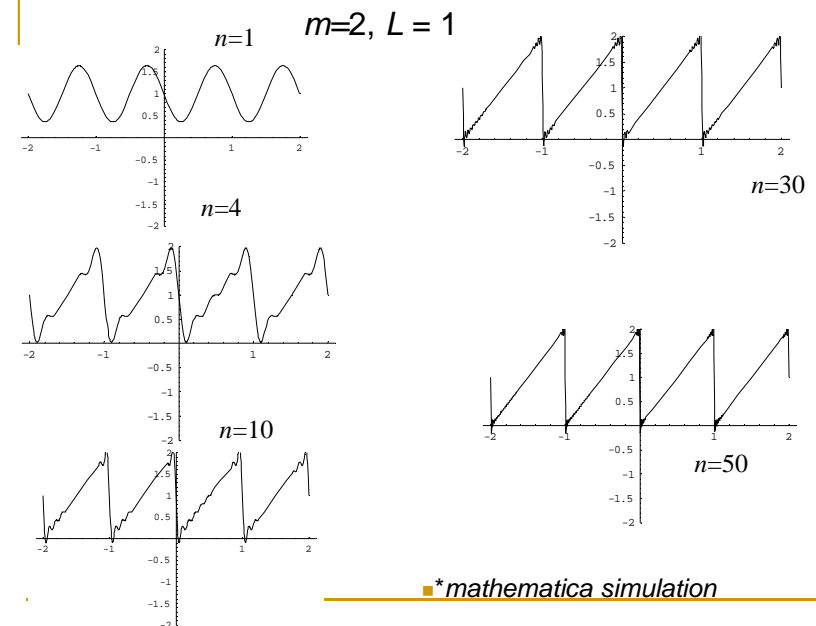
$$b_k = \frac{2}{L} \int_a^b f(x) \sin \frac{2\pi kx}{L} dx = \frac{2m}{L} \int_0^L x \sin \frac{2\pi kx}{L} dx$$

$$= \frac{2m}{L} \cdot L^2 \left( \frac{-2k\pi \cos(2k\pi) + \sin 2k\pi}{4k^2 \pi^2} \right) = \frac{-mL}{k\pi};$$

$$f(x) = mx = \frac{mL}{2} - \frac{mL}{\pi} \sum_{k=1}^n \frac{\sin 2\pi kx}{k}$$

$$= mL \left( \frac{1}{2} - \frac{\sin 2\pi x}{\pi} - \frac{\sin 4\pi x}{2\pi} - \frac{\sin 6\pi x}{3\pi} - \dots - \frac{\sin 2n\pi x}{n\pi} + \dots \right)$$

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\*mathematica simulation

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# CALCULUS ASSIGNMENT

## QUESTIONS FOR ZCA 101

Tutorial 1 (Chapter 1)  
Thomas' Calculus 11<sup>th</sup> edition

Exercise 1.3

**Functions and Their Graphs**

find the domain and range of each function.

1.  $f(x) = 1 + x^2$

3.  $F(t) = \frac{1}{\sqrt{t}}$

5.  $g(z) = \sqrt{4 - z^2}$

**Finding Formulas for Functions**

13. Express the edge length of a cube as a function of the cube's diagonal length  $d$ . Then express the surface area and volume of the cube as a function of the diagonal length.

**Functions and Graphs**

Find the domain and graph the functions

16.  $f(x) = 1 - 2x - x^2$

18.  $g(x) = \sqrt{-x}$

22. Graph the following equations and explain why they are not graphs of functions of  $x$ .

a.  $|x| + |y| = 1$

b.  $|x + y| = 1$

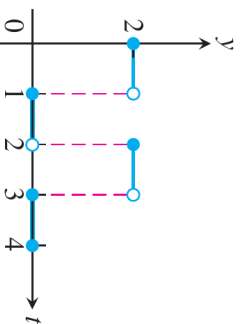
**Piecewise-Defined Functions**

Graph the function

$$24. g(x) = \begin{cases} 1 - x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$$

27. Find a formula for each function graphed.

b.



**Exercise 1.4**

**Recognizing Functions**

In Exercises 3, identify each function as a constant function, linear function, power function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function. Remember that some functions can fall into more than one category.

3. a.  $y = \frac{3 + 2x}{x - 1}$

b.  $y = x^{5/2} - 2x + 1$

c.  $y = \tan \pi x$

d.  $y = \log_{\pi} x$

**Increasing and Decreasing Functions**

Graph the functions. What symmetries, if any, do the graphs have? Specify the intervals over which the function is increasing and

16.  $y = (-x)^{3/2}$  the intervals where it is decreasing.

18.  $y = -x^{2/3}$

**Even and Odd Functions**

Say whether the function is even, odd, or neither.

Give reasons for your answer.

19.  $f(x) = 3$

21.  $f(x) = x^2 + 1$

23.  $g(x) = x^3 + x$

### EXERCISES 1.5

Sums, Differences, Products, and Quotients

Find the domains and ranges of  $f$ ,  $g$ ,  $f + g$ , and

$$f \cdot g.$$

$$2. f(x) = \sqrt{x+1}, \quad g(x) = \sqrt{x-1}$$

Graph the functions

$$39. y = \sqrt[3]{x-1} - 1$$

$$48. y = \frac{1}{(x+1)^2}$$

### Vertical and Horizontal Scaling

Exercises below tell by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph.

$$51. y = x^2 - 1, \quad \text{stretched vertically by a factor of 3}$$

$$55. y = \sqrt{x+1}, \quad \text{compressed horizontally by a factor of 4}$$

### Composites of Functions

6. If  $f(x) = x - 1$  and  $g(x) = 1/(x + 1)$ , find  
 a.  $f(g(1/2))$   
 b.  $g(f(1/2))$

### EXERCISES 1.6

#### Radians, Degrees, and Circular Arcs

4. If you roll a 1-m-diameter wheel forward 30 cm over level ground, through what angle will the wheel turn? Answer in radians (to the nearest tenth) and degrees (to the nearest degree).

#### Evaluating Trigonometric Functions

5. Copy and complete the following table of function values. If the function is undefined at a given angle, enter “UND.” Do not use a calculator or tables.

$\theta$	$-\pi$	$-2\pi/3$	$0$	$\pi/2$	$3\pi/4$
$\sin \theta$					
$\cos \theta$					
$\tan \theta$					
$\cot \theta$					

Find the other two if  $x$  lies in the specified interval.

$$7. \sin x = \frac{3}{5}, \quad x \in \left[ \frac{\pi}{2}, \pi \right]$$

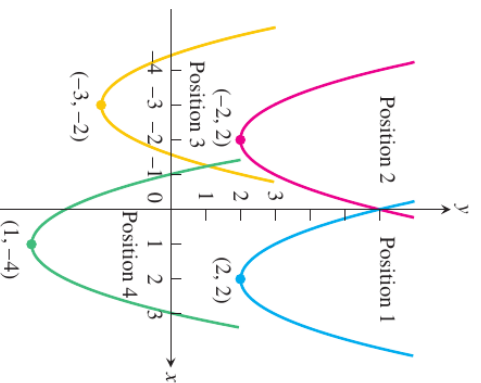
12. Copy and complete the following table.

$g(x)$	$f(x)$	$(f \circ g)(x)$
a. $\frac{1}{x-1}$	$ x $	?
b. ?	$\frac{x-1}{x}$	$\frac{x}{x+1}$
c. ?	$\sqrt{x}$	$ x $
d. $\sqrt{x}$	?	$ x $

### Shifting Graphs

17. Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

- a.  $y = (x - 1)^2 - 4$   
 b.  $y = (x - 2)^2 + 2$   
 c.  $y = (x + 2)^2 + 2$   
 d.  $y = (x + 3)^2 - 2$



### Graphing Trigonometric Functions

Graph the functions the  $ts$ -plane ( $t$ -axis horizontal,  $s$ -axis vertical). What is the period of each function? What symmetries do the graphs have?

23.  $s = \cot 2t$

25.  $s = \sec\left(\frac{\pi t}{2}\right)$

### Additional Trigonometric Identities

Use the addition formulas to derive the identity.

31.  $\cos\left(x - \frac{\pi}{2}\right) = \sin x$

### Using the Addition Formulas

Express the given quantity in terms of  $\sin x$  and  $\cos x$ .

39.  $\cos(\pi + x)$

### Using the Double-Angle Formulas

Find the function values

47.  $\cos^2 \frac{\pi}{8}$

49.  $\sin^2 \frac{\pi}{12}$

**Tutorial 2 (Chapter 2)**  
**Thomas' Calculus 11<sup>th</sup> edition**

**Exercise 2.2**

**Limit Calculations**

Find the limits.

17.  $\lim_{h \rightarrow 0} \frac{\sqrt{3h + 1} - 1}{h}$

19.  $\lim_{x \rightarrow 5} \frac{x - 5}{x^2 - 25}$

27.  $\lim_{u \rightarrow 1} \frac{u^4 - 1}{u^3 - 1}$

**Using Limit Rules**

39. Suppose  $\lim_{x \rightarrow c} f(x) = 5$  and  $\lim_{x \rightarrow c} g(x) = -2$ . Find

a.  $\lim_{x \rightarrow c} f(x)g(x)$

**Limits of Average Rates of Change**

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

occur frequently in calculus. Evaluate this limit for the given value of  $x$  and function  $f$ .

47.  $f(x) = \sqrt{x}$ ,  $x = 7$

**Using the Sandwich Theorem**

49. If  $\sqrt{5 - 2x^2} \leq f(x) \leq \sqrt{5 - x^2}$  for  $-1 \leq x \leq 1$ , find  $\lim_{x \rightarrow 0} f(x)$ .

**EXERCISES 2.3**

**Centering Intervals About a Point**

In Exercises 1–6, sketch the interval  $(a, b)$  on the  $x$ -axis with the point  $x_0$  inside. Then find a value of  $\delta > 0$  such that for all  $x$ ,  $0 < |x - x_0| < \delta \Rightarrow a < x < b$ .

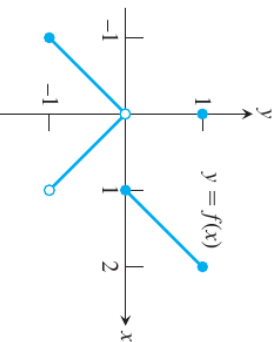
4.  $a = -7/2$ ,  $b = -1/2$ ,  $x_0 = -3/2$

**Exercise 2.1**

**Limits from Graphs**

3. Which of the following statements about the function  $y = f(x)$  graphed here are true, and which are false?

- a.  $\lim_{x \rightarrow 0} f(x)$  exists.
- b.  $\lim_{x \rightarrow 0} f(x) = 0$ .
- c.  $\lim_{x \rightarrow 0} f(x) = 1$ .
- d.  $\lim_{x \rightarrow 1} f(x) = 1$ .
- e.  $\lim_{x \rightarrow 1} f(x) = 0$ .
- f.  $\lim_{x \rightarrow x_0} f(x)$  exists at every point  $x_0$  in  $(-1, 1)$ .



**Existence of Limits**

- 9. If  $\lim_{x \rightarrow 1} f(x) = 5$ , must  $f$  be defined at  $x = 1$ ? If it is, must  $f(1) = 5$ ? Can we conclude *anything* about the values of  $f$  at  $x = 1$ ? Explain.
- 10. If  $f(1) = 5$ , must  $\lim_{x \rightarrow 1} f(x)$  exist? If it does, then must  $\lim_{x \rightarrow 1} f(x) = 5$ ? Can we conclude *anything* about  $\lim_{x \rightarrow 1} f(x)$ ? Explain.

**Limits by Substitution**

Find the limits by substitution.

27.  $\lim_{x \rightarrow \pi/2} x \sin x$

**Average Rates of Change**

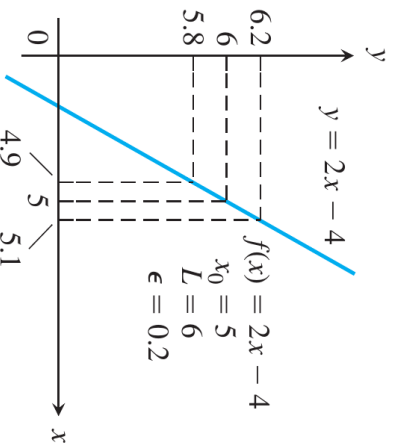
Find the average rate of change of the function over the given interval or intervals.

33.  $R(\theta) = \sqrt{4\theta + 1}$ ;  $[0, 2]$

## Finding Deltas Graphically

use the graphs to find a  $\delta > 0$  such that for all  $x$

7.



## EXERCISES 2.4

### Finding Limits Graphically

7. a. Graph  $f(x) = \begin{cases} x^3, & x \neq 1 \\ 0, & x = 1. \end{cases}$
- b. Find  $\lim_{x \rightarrow 1^-} f(x)$  and  $\lim_{x \rightarrow 1^+} f(x)$ .
- c. Does  $\lim_{x \rightarrow 1} f(x)$  exist? If so, what is it? If not, why not?

### Finding One-Sided Limits Algebraically

15.  $\lim_{h \rightarrow 0^+} \frac{\sqrt{h^2 + 4h} + 5 - \sqrt{5}}{h}$  Find the limit

$$\text{Using } \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

Find the limits

33.  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta}$

### Calculating Limits as $x \rightarrow \pm\infty$

In Exercises 37–42, find the limit of each function (a) as  $x \rightarrow \infty$  and (b) as  $x \rightarrow -\infty$ . (You may wish to visualize your answer with a graphing calculator or computer.)

41.  $h(x) = \frac{-5 + (7/x)}{3 - (1/x^2)}$

### Limits of Rational Functions

In Exercises 47–56, find the limit of each rational function (a)  $x \rightarrow \infty$  and (b) as  $x \rightarrow -\infty$ .

49.  $f(x) = \frac{x+1}{x^2+3}$

## EXERCISES 2.5

### Infinite Limits

Find the limits.

13.  $\lim_{x \rightarrow (\pi/2)^-} \tan x$

## Finding Deltas Algebraically

Each of Exercises 15–30 gives a function  $f(x)$  and numbers  $L$ ,  $x_0$  and  $\epsilon > 0$ . In each case, find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \epsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \epsilon$  holds.

15.  $f(x) = x + 1$ ,  $L = 5$ ,  $x_0 = 4$ ,

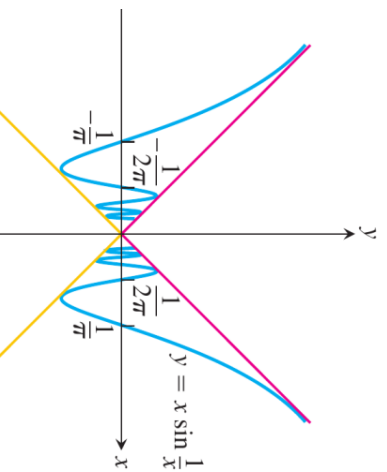
## More on Formal Limits

31.  $f(x) = 3 - 2x$ ,  $x_0 = 3$ ,  $\epsilon = 0.02$

Prove the limit statements

43.  $\lim_{x \rightarrow 1} \frac{1}{x} = 1$

49.  $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$



## Additional Calculations

Find the limits.

25.  $\lim \left( \frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}} \right)$  as

- a.  $x \rightarrow 0^+$   
c.  $x \rightarrow 1^+$

## Graphing Rational Functions

Graph the rational functions in Exercises 27–38. Include the graphs and equations of the asymptotes and dominant terms.

36.  $y = \frac{x^2 - 1}{2x + 4}$

## Inventing Functions

Find a function that satisfies the given conditions and sketch its graph. (The answers here are not unique. Any function that satisfies the conditions is acceptable. Feel free to use formulas defined in pieces if that will help.)

43.  $\lim_{x \rightarrow \pm\infty} f(x) = 0$ ,  $\lim_{x \rightarrow 2^-} f(x) = \infty$ , and  $\lim_{x \rightarrow 2^+} f(x) = \infty$

## Graphing Terms

The function is given as the sum or difference of two terms. First graph the terms (with the same set of axes). Then, using these graphs as guides, sketch in the graph of the function.

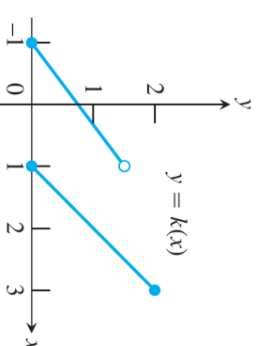
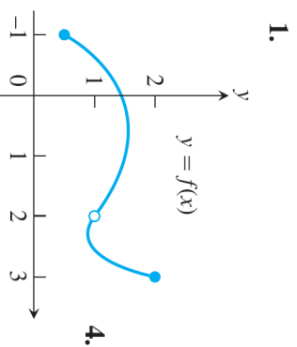
59.  $y = \tan x + \frac{1}{x^2}$ ,  $-\frac{\pi}{2} < x < \frac{\pi}{2}$

## EXERCISES 2.6

### Continuity from Graphs

In the exercises below, say whether the function graphed is continuous on  $[-1, 3]$ . If not, where does it fail to be

continuous and why?



## Applying the Continuity Test

At which points do the functions fail to be continuous? At which points, if any, are the discontinuities removable? Not removable? Give reasons for your

17.  $y = |x - 1| + \sin x$

20.  $y = \frac{x + 2}{\cos x}$

## Composite Functions

Find the limits. Are the functions continuous at the point being approached?

29.  $\lim_{x \rightarrow \pi} \sin(x - \sin x)$

## EXERCISES 2.7

### Slopes and Tangent Lines

Find an equation for the tangent to the curve at the

5.  $y = 4 - x^2$ ,  $(-1, 3)$  given point. Then sketch the curve and tangent together.



Find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

18.  $f(x) = \sqrt{x + 1}$ ,  $(8, 3)$

22.  $y = \frac{x - 1}{x + 1}$ ,  $x = 0$

### Tangent Lines with Specified Slopes

26. Find an equation of the straight line having slope  $1/4$  that is tangent to the curve  $y = \sqrt{x}$ .

### Rates of Change

30. **Ball's changing volume** What is the rate of change of the volume of a ball ( $V = (4/3)\pi r^3$ ) with respect to the radius when the radius is  $r = 2$ ?

**Tutorial 3 (Chapter 3)**  
**Thomas' Calculus 11<sup>th</sup> edition**

**EXERCISES 3.1**

**Finding Derivative Functions and Values**

Using the definition, calculate the derivatives of the functions. Then find the values of the derivatives as specified.

6.  $r(s) = \sqrt{2s + 1}$ ;  $r'(0), r'(1), r'(1/2)$

Find the indicated derivatives.

10.  $\frac{dv}{dt}$  if  $v = t - \frac{1}{t}$

12.  $\frac{dz}{dw}$  if  $z = \frac{1}{\sqrt{3w - 2}}$

**Slopes and Tangent Lines**

Differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

18.  $w = g(z) = 1 + \sqrt{4 - z}$ ,  $(z, w) = (3, 2)$

Find the values of the derivative.

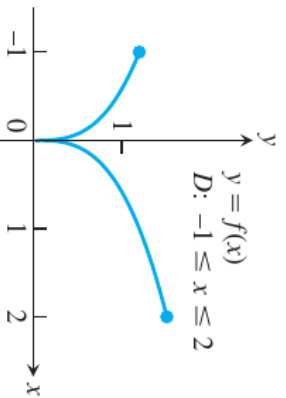
21.  $\left. \frac{dr}{d\theta} \right|_{\theta=0}$  if  $r = \frac{2}{\sqrt{4 - \theta}}$

**Differentiability and Continuity on an Interval**

The figure below shows the graph of a function over a closed interval D. At what domain points does the function appear to be

- a. differentiable?
- b. continuous but not differentiable?
- c. neither continuous nor differentiable?

43.



**EXERCISES 3.2**

**Derivative Calculations**

Find the derivatives of the functions

24.  $u = \frac{5x + 1}{2\sqrt{x}}$

28.  $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

Find the first and second derivatives.

38.  $p = \frac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$

**Using Numerical Values**

39. Suppose  $u$  and  $v$  are functions of  $x$  that are differentiable at  $x = 0$  and that

$u(0) = 5, u'(0) = -3, v(0) = -1, v'(0) = 2.$

Find the values of the following derivatives at  $x = 0.$

a.  $\frac{d}{dx}(uv)$     b.  $\frac{d}{dx}\left(\frac{u}{v}\right)$     c.  $\frac{d}{dx}\left(\frac{v}{u}\right)$     d.  $\frac{d}{dx}(7v - 2u)$

**Slopes and Tangents**

- 41. a. **Normal to a curve** Find an equation for the line perpendicular to the tangent to the curve  $y = x^3 - 4x + 1$  at the point  $(2, 1)$ .
- b. **Smallest slope** What is the smallest slope on the curve? At what point on the curve does the curve have this slope?
- c. **Tangents having specified slope** Find equations for the tangents to the curve at the points where the slope of the curve is 8.

**EXERCISES 3.3**

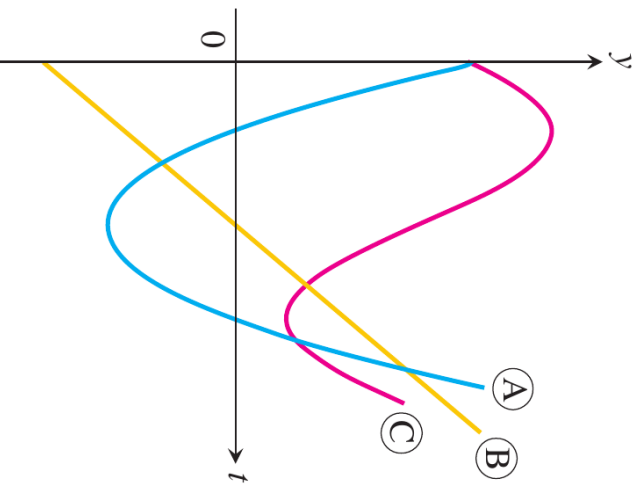
**Motion Along a Coordinate Line**

- 7. **Particle motion** At time  $t$ , the position of a body moving along the  $s$ -axis is  $s = t^3 - 6t^2 + 9t$  m.
  - a. Find the body's acceleration each time the velocity is zero.
  - b. Find the body's speed each time the acceleration is zero.
  - c. Find the total distance traveled by the body from  $t = 0$  to  $t = 2.$

## Free-Fall Applications

- 10. Lunar projectile motion** A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of  $s = 24t - 0.8t^2$  meters in  $t$  sec.
- Find the rock's velocity and acceleration at time  $t$ . (The acceleration in this case is the acceleration of gravity on the moon.)
  - How long does it take the rock to reach its highest point?
  - How high does the rock go?
  - How long does it take the rock to reach half its maximum height?
  - How long is the rock aloft?

## Conclusions About Motion from Graphs



**FIGURE 3.21** The graphs for Exercise 21.

- 35.** Find all points on the curve  $y = \tan x$ ,  $-\pi/2 < x < \pi/2$ , where the tangent line is parallel to the line  $y = 2x$ . Sketch the curve and tangent(s) together, labeling each with its equation.

- 27. Draining a tank** It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth  $y$  of fluid in the tank  $t$  hours after the valve is opened is given by the formula

$$y = 6 \left( 1 - \frac{t}{12} \right)^2 \text{ m.}$$

- Find the rate  $dy/dt$  (m/h) at which the tank is draining at time  $t$ .
- When is the fluid level in the tank falling fastest? Slowest? What are the values of  $dy/dt$  at these times?

## EXERCISES 3.4

### Derivatives

find  $dy/dx$

- $y = (\sec x + \tan x)(\sec x - \tan x)$
- $y = x^2 \sin x + 2x \cos x - 2 \sin x$
- Find  $y''$  if
  - $y = \csc x$ .

### Tangent Lines

Graph the curves over the given intervals, together with their tangents at the given values of  $x$ . Label each curve and tangent with its equation.

- $y = \sec x$ ,  $-\pi/2 < x < \pi/2$   
 $x = -\pi/3, \pi/4$

### Trigonometric Limits

Find the limits

- $\lim_{x \rightarrow 2} \sin \left( \frac{1}{x} - \frac{1}{2} \right)$
- $\lim_{x \rightarrow -\pi/6} \sqrt{1 + \cos(\pi \csc x)}$

## Differentiating Implicitly

Use implicit differentiation to find  $dy/dx$

$$19. x^2y + xy^2 = 6$$

## EXERCISES 3.5

### Derivative Calculations

In Exercises 9–18, write the function in the form  $y = f(u)$  and  $u = g(x)$ . Then find  $dy/dx$  as a function of  $x$ .

$$14. y = \left(\frac{x}{5} + \frac{1}{5x}\right)^5$$

Find the derivatives of the functions

$$26. y = \frac{1}{x} \sin^{-5} x - \frac{x}{3} \cos^3 x$$

### Second Derivatives

Find  $y''$

$$49. y = \left(1 + \frac{1}{x}\right)^3$$

$$51. y = \frac{1}{9} \cot(3x - 1)$$

### Finding Numerical Values of Derivatives

61. Find  $ds/dt$  when  $\theta = 3\pi/2$  if  $s = \cos \theta$  and  $d\theta/dt = 5$ .

### Tangents to Parametrized Curves

$$94. x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\pi/4$$

## EXERCISES 3.6

### Derivatives of Rational Powers

### Find $dy/dx$

$$13. y = \sin [(2t + 5)^{-2/3}]$$

**Second Derivatives**  
In Exercises 37–42, use implicit differentiation to find  $dy/dx$  and then  $d^2y/dx^2$ .

$$41. 2\sqrt{y} = x - y$$

### Slopes, Tangents, and Normals

Verify that the given point is on the curve and find the lines that are (a) tangent and (b) normal to the curve at the given point.

$$45. y^2 + x^2 = y^4 - 2x \quad \text{at } (-2, 1) \text{ and } (-2, -1)$$

### Implicitly Defined Parametrizations

$$63. x^2 - 2tx + 2t^2 = 4, \quad 2y^3 - 3t^2 = 4, \quad t = 2$$

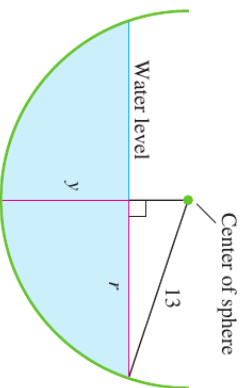
## EXERCISES 3.7

12. **Changing dimensions in a rectangular box** Suppose that the edge lengths  $x$ ,  $y$ , and  $z$  of a closed rectangular box are changing at the following rates:

$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec.}$$

20. **A growing raindrop** Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's

19. **A draining hemispherical reservoir** Water is flowing at the rate of  $6 \text{ m}^3/\text{min}$  from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions, given that the volume of water in a hemispherical bowl of radius  $R$  is  $V = (\pi/3)y^2(3R - y)$  when the water is  $y$  meters deep.



### Linearizing Trigonometric Functions

Find the linearization of  $f$  at  $x = a$ .

11.  $f(x) = \sin x$  at (a)  $x = 0$ , (b)  $x = \pi$

### The Approximation $(1 + x)^k \approx 1 + kx$

17. **Faster than a calculator** Use the approximation  $(1 + x)^k \approx 1 + kx$  to estimate the following.

a.  $(1.0002)^{50}$       b.  $\sqrt[3]{1.009}$

### Derivatives in Differential Form

Find  $dy$ .

28.  $y = \sec(x^2 - 1)$

### Approximation Error

The function  $f(x)$  changes value when  $x$  changes from  $x_0$  to  $x_0 + dx$ . Find

31.  $f(x) = x^2 + 2x$ ,  $x_0 = 1$ ,  $dx = 0.1$

## EXERCISES 3.8

### Finding Linearizations

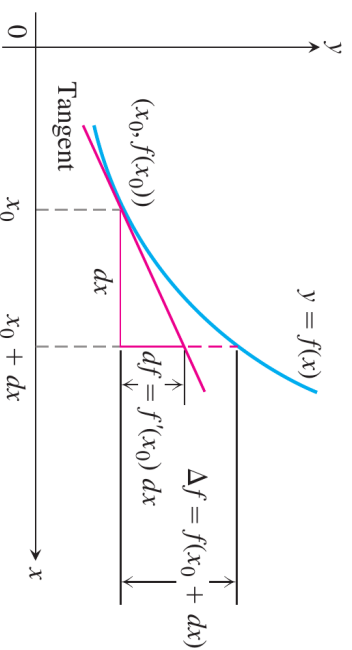
find the linearization  $L(x)$  of  $f(x)$  at  $x = a$ .

4.  $f(x) = \sqrt[3]{x}$ ,  $a = -8$

### Linearization for Approximation

You want linearizations that will replace the functions in the following over intervals that include the given points  $x_0$ . To make your subsequent work as simple as possible, you want to center each linearization not at  $x_0$  but at a nearby integer  $x = a$  at which the given function and its derivative are easy to evaluate. What linearization do you use in each case?

5.  $f(x) = x^2 + 2x$ ,  $x_0 = 0.1$



**Tutorial 4 (Chapter 4)**  
**Thomas' Calculus 11<sup>th</sup> edition**

**EXERCISES 4.1**

**Absolute Extrema on Finite Closed Intervals**

Find the absolute maximum and minimum values of the function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

**24.**  $g(x) = -\sqrt{5-x^2}$ ,  $-\sqrt{5} \leq x \leq 0$

**Finding Extreme Values**

Find the function's absolute maximum and minimum values and say where they are assumed.

**44.**  $y = \frac{x+1}{x^2+2x+2}$

**Local Extrema and Critical Points**

Find the derivative at each critical point and determine the local extreme values.

**48.**  $y = x^2\sqrt{3-x}$

**54.** Let  $f(x) = |x^3 - 9x|$ .

- a. Does  $f'(0)$  exist?
- b. Does  $f'(3)$  exist?
- c. Does  $f'(-3)$  exist?
- d. Determine all extrema of  $f$ .

**Optimization Applications**

**Area of an athletic field**

**62.** An athletic field is to be built in the shape of a rectangle  $x$  units long capped by semicircular regions of radius  $r$  at the two ends. The field is to be bounded by a 400-m racetrack.

- a. Express the area of the rectangular portion of the field as a function of  $x$  alone or  $r$  alone (your choice).
- b. What values of  $x$  and  $r$  give the rectangular portion the largest possible area?

**EXERCISES 4.2**

**Finding  $c$  in the Mean Value Theorem**

Find the value or values of  $c$  that satisfy the equation  $\frac{f(b) - f(a)}{b - a} = f'(c)$  in the conclusion of the Mean Value Theorem for the functions and intervals.

**2.**  $f(x) = x^{2/3}$ ,  $[0, 1]$

**Checking and Using Hypotheses**

**10.** For what values of  $a$ ,  $m$  and  $b$  does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \leq x \leq 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval  $[0, 2]$ ?

**Roots (Zeros)**

Show that the function has exactly one zero in the given interval.

**15.**  $f(x) = x^4 + 3x + 1$ ,  $[-2, -1]$

**Finding Functions from Derivatives**

**25.** Suppose that  $f'(x) = 2x$  for all  $x$ . Find  $f(2)$  if

a.  $f(0) = 0$     b.  $f(1) = 0$     c.  $f(-2) = 3$ .

**Finding Position from Acceleration**

Exercise 43 give the acceleration  $a = d^2s/dt^2$ , initial velocity and initial position of a body moving on a coordinate line. Find the body's position at time  $t$ .

**43.**  $a = -4 \sin 2t$ ,  $v(0) = 2$ ,  $s(0) = -3$

## EXERCISES 4.3

### Analyzing $f$ Given $f'$

Answer the following questions about the functions whose derivatives are given below:

- What are the critical points of  $f$ ?
- On what intervals is  $f$  increasing or decreasing?
- At what points, if any, does  $f$  assume local maximum and minimum values?

7.  $f'(x) = x^{-1/3}(x + 2)$

8.  $f'(x) = x^{-1/2}(x - 3)$

### Extremes of Given Functions

- Find the intervals on which the function is increasing and decreasing.
- Then identify the function's local extreme values, if any, saying where they are taken on.
- Which, if any, of the extreme values are absolute?

24.  $f(x) = \frac{x^3}{3x^2 + 1}$

### Extreme Values on Half-Open Intervals

- Identify the function's local extreme values in the given domain, and say where they are assumed.
  - Which of the extreme values, if any, are absolute?
32.  $g(x) = -x^2 - 6x - 9, \quad -4 \leq x < \infty$
36.  $k(x) = x^3 + 3x^2 + 3x + 1, \quad -\infty < x \leq 0$

### Theory and Examples

47. As  $x$  moves from left to right through the point  $c = 2$ , is the graph of  $f(x) = x^3 - 3x + 2$  rising, or is it falling? Give reasons for your answer.

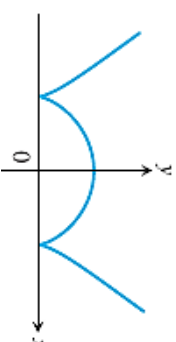
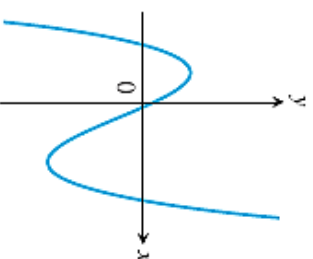
## EXERCISES 4.4

### Analyzing Graphed Functions

Identify the inflection points and local maxima and minima of the functions graphed below. Identify the intervals on which the functions are concave up and concave down.

1.  $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$

3.  $y = \frac{3}{4}(x^2 - 1)^{2/3}$



### Graph Equations

Use the steps of the graphing procedure to graph the equations below. Include the coordinates of any local extreme points and inflection points.

40.  $y = \sqrt{|x - 4|}$

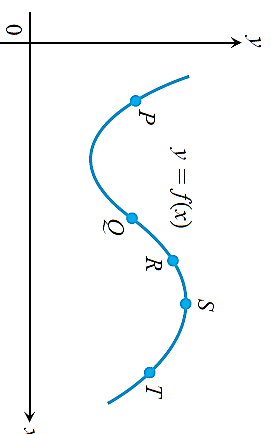
### Sketching the General Shape Knowing $y'$

Each of Exercises below gives the first derivative of a continuous function  $= f(x)$ . Find  $y''$  and sketch the general shape of the graph of  $f$ .

41.  $y' = 2 + x - x^2$

### Theory and Examples

67. The accompanying figure shows a portion of the graph of a twice-differentiable function  $y = f(x)$ . At each of the five labelled points, classify  $y'$  and  $y''$  as positive, negative, or zero.



75. Suppose the derivative of the function  $y = f(x)$  is

$$y' = (x - 1)^2(x - 2)$$

At what points, if any, does the graph of  $f$  have a local minimum, local maximum, or point of inflection?

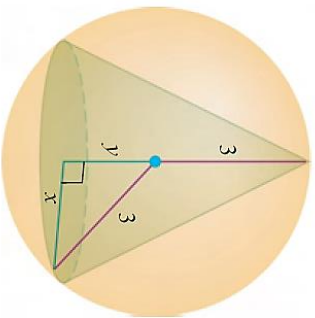
(Hint: Draw the sign pattern for  $y'$ .)



## EXERCISES 4.5

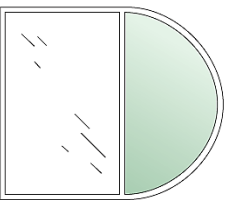
### Applications in Geometry

6. You are planning to close off a corner of the first quadrant with a line segment 20 units long running from  $(a, 0)$  to  $(0, b)$ . Show that the area of the triangle enclosed by the segment is largest when  $a = b$ .
12. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



18. A rectangle is to be inscribed under the arch of the curve  $y = 4\cos(0.5x)$  from  $x = -\pi$  to  $x = \pi$ . What are the dimensions of the rectangle with largest area, and what is the largest area?

22. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.



## EXERCISES 4.6

### Finding Limits

In Exercises 1 and 5, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

1.  $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4}$

### Applying l'Hôpital's Rule

Use l'Hôpital's Rule to find the limits in Exercises 22 and 25.

22.  $\lim_{x \rightarrow 0^+} \left( \frac{1}{x} - \frac{1}{\sqrt{x}} \right)$

25.  $\lim_{x \rightarrow \pm\infty} \frac{3x - 5}{2x^2 - x + 2}$

### Theory and Applications

#### 32. $\infty/\infty$ Form

Give an example of two differentiable functions  $f$  and  $g$  with  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = \infty$  that satisfy the following.

a.  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 3$

b.  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$

c.  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty$

## EXERCISES 4.8

### Finding Antiderivatives

In Exercises 8 and 14, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

8. a.  $\frac{4}{3} \sqrt[3]{x}$

b.  $\frac{1}{3 \sqrt[3]{x}}$

c.  $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$

14. a.  $\csc^2 x$

b.  $-\frac{3}{2} \csc^2 \frac{3x}{2}$

c.  $1 - 8 \csc^2 2x$

### Finding Indefinite Integrals

In Exercise 31 and 46, find the most general antiderivative or indefinite integral. Check your answer by differentiation.

31.  $\int 2x(1 - x^{-3}) dx$

46.  $\int (2 \cos 2x - 3 \sin 3x) dx$

### Checking Antiderivative Formulas



Verify the formulas in Exercises 60 by differentiation.

$$60. \int \frac{1}{(x+1)^2} dx = \frac{x}{x+1} + C$$

### Theory and Examples

**101.** Suppose that

$$f(x) = \frac{d}{dx}(1 - \sqrt{x}) \quad \text{and} \quad g(x) = \frac{d}{dx}(x + 2)$$

Find:

$$\mathbf{a.} \int f(x) dx \qquad \mathbf{b.} \int g(x) dx$$

$$\mathbf{c.} \int [-f(x)] dx \qquad \mathbf{d.} \int [-g(x)] dx$$

$$\mathbf{e.} \int [f(x) + g(x)] dx \qquad \mathbf{f.} \int [f(x) - g(x)] dx$$

**Tutorial 5 (Chapter 5 and 6)**  
**Thomas' Calculus 11<sup>th</sup> edition**

**EXERCISES 5.1**

**Area**

In Exercise 1 use finite approximations to estimate the area under the graph of the function using

- a. a lower sum with two rectangles of equal width.
- b. a lower sum with four rectangles of equal width.
- c. an upper sum with two rectangles of equal width.
- d. an upper sum with four rectangles of equal width.

1.  $f(x) = x^2$  between  $x = 0$  and  $x = 1$

**Area of a Circle**

21. Inscribe a regular  $n$ -sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of  $n$ :

- a. 4 (square)      b. 8 (octagon)      c. 16
- d. Compare the areas in parts (a), (b) and (c) with the area of the circle.

**EXERCISES 5.2**

**Sigma Notation**

Write the sums in Exercises 1 without sigma notation. Then evaluate them.

1.  $\sum_{k=1}^2 \frac{6k}{k+1}$

**Values of Finite Sums**

17. Suppose that  $\sum_{k=1}^n a_k = -5$  and  $\sum_{k=1}^n b_k = 6$ . Find the values of

c.  $\sum_{k=1}^n (a_k + b_k)$

Evaluate the sums in Exercise 24.

24.  $\sum_{k=1}^6 (k^2 - 5)$

**Limits of Upper Sums**

For the functions in Exercise 36, find a formula for the upper sum obtained by dividing the interval  $[a, b]$  into  $n$  equal subintervals. Then take a limit of this sum as  $n \rightarrow \infty$  to calculate the area under the curve over  $[a, b]$ .

36.  $f(x) = 2x$  over the interval  $[0, 3]$

**EXERCISES 5.3**

**Expressing Limits as Integrals**

Express the limits in Exercise 1 as definite integrals.

1.  $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n c_k^2 \Delta x_k$ , where  $P$  is a partition of  $[0, 2]$

**Using Properties and Known Values to Find Other Integrals**

12. Suppose that  $\int_{-3}^0 g(t) dt = \sqrt{2}$ . Find

a.  $\int_0^{-3} g(t) dt$       b.  $\int_{-3}^0 g(u) du$

c.  $\int_{-3}^0 [-g(x)] dx$       d.  $\int_{-3}^0 \frac{g(r)}{\sqrt{2}} dr$

**Using Area to Evaluate Definite Integrals**

In Exercise 15, graph the integrands and use areas to evaluate the integrals.

15.  $\int_{-2}^4 \left( \frac{x}{2} + 3 \right) dx$

**Evaluations**

Use the results of Equations (1) and (3) to evaluate the integrals in Exercise 38.

38.  $\int_0^{3b} x^2 dx$

### Average Value

In Exercise 55, graph the function and find its average value over the given interval.

55.  $f(x) = x^2 - 1$  on  $[0, \sqrt{3}]$

### EXERCISES 5.4

#### Evaluating Integrals

Evaluate the integrals in Exercises 23 and 25.

23.  $\int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2} ds$

25.  $\int_{-4}^4 |x| dx$

#### Derivatives of Integrals

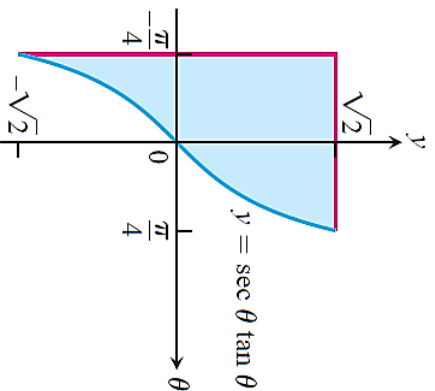
Find  $dy/dx$  in Exercise 36.

36.  $y = \int_{\tan x}^0 \frac{dt}{1 + t^2}$

#### Area

Find the areas of the shaded regions in Exercise 45.

45.



### Theory and Examples

62. Find

$$\lim_{x \rightarrow 0} \frac{1}{x^3} \int_0^x \frac{t^2}{t^4 + 1} dt$$

### EXERCISES 5.5

#### Evaluating Integrals

Evaluate the indefinite integrals in Exercise 4 and 11 by using the given substitutions to reduce the integrals to standard form.

4.  $\int \left(1 - \cos \frac{t}{2}\right)^2 \sin \frac{t}{2} dt$ ,  $u = 1 - \cos \frac{t}{2}$

11.  $\int \csc^2 2\theta \cot 2\theta d\theta$

- a. Using  $u = \cot 2\theta$     b. Using  $u = \csc 2\theta$

Evaluate the integrals in Exercises 36 and 48.

36.  $\int \frac{6 \cos t}{(2 + \sin t)^3} dt$

48.  $\int 3x^5 \sqrt{x^3 + 1} dx$

#### Simplifying Integrals Step by Step

Evaluate the integrals in Exercise 51.

51.  $\int \frac{(2r - 1) \cos \sqrt{3(2r - 1)^2 + 6}}{\sqrt{3(2r - 1)^2 + 6}} dr$

### EXERCISES 5.6

#### Evaluating Definite Integrals

Use the substitution formula in Theorem 6 to evaluate the integrals in Exercises 7 and 14.

7. a.  $\int_{-1}^1 \frac{5r}{(4 + r^2)^2} dr$     b.  $\int_0^1 \frac{5r}{(4 + r^2)^2} dr$

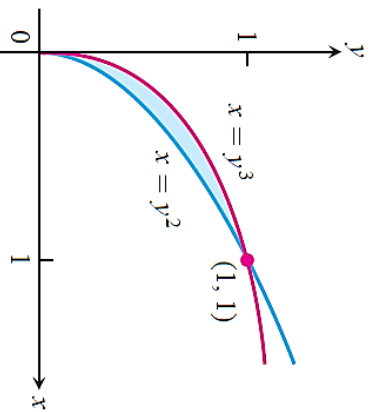
14. a.  $\int_{-\pi/2}^0 \frac{\sin w}{(3 + 2 \cos w)^2} dw$

b.  $\int_0^{\pi/2} \frac{\sin w}{(3 + 2 \cos w)^2} dw$

## Area

Find the total areas of the shaded regions in Exercise 32.

32.



73. Find the area of the region in the first quadrant bounded by the line  $y = x$ , the line  $x = 2$ , the curve  $y = 1/x^2$ , and the x-axis.

## EXERCISES 6.3

### Length of Parametrized Curves

Find the lengths of the curves in Exercise 1.

1.  $x = 1 - t$ ,  $y = 2 + 3t$ ,  $-2/3 \leq t \leq 1$

### Finding Lengths of Curves

Find the lengths of the curves in Exercises 7 and 16. If you have a grapher, you may want to graph these curves to see what they look like.

7.  $y = (1/3)(x^2 + 2)^{3/2}$  from  $x = 0$  to  $x = 3$

16.  $y = \int_{-2}^x \sqrt{3t^4 - 1} dt$ ,  $-2 \leq x \leq -1$

### Theory and Applications

27. a. Find a curve through the point (1, 1) whose length integral is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}} dx$$

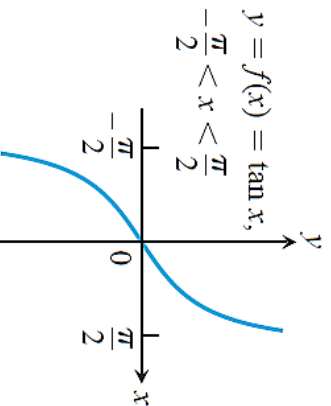
b. How many such curves are there?  
Give reasons for your answer.

### EXERCISES 7.1

#### Graphing Inverse Functions

Exercise 10 shows the graph of a function  $y = f(x)$ . Copy the graph and draw in the line  $y = x$ . Then use symmetry with respect to the line  $y = x$  to add the graph of  $f^{-1}$  to your sketch. (It is not necessary to find a formula for  $f^{-1}$ .) Identify the domain and range of  $f^{-1}$ .

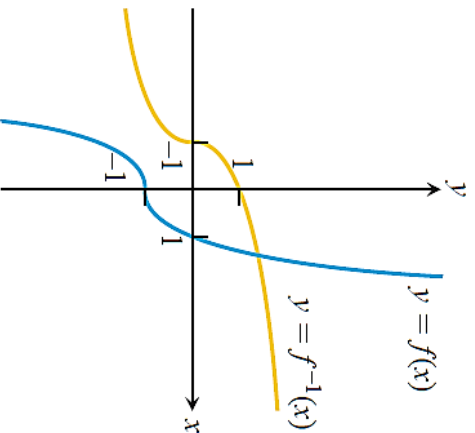
10.



#### Formulas for Inverse Functions

Exercise 15 gives a formula for a function  $y = f(x)$  and shows the graphs of  $f$  and  $f^{-1}$ . Find a formula for  $f^{-1}$  in each case.

15.  $f(x) = x^3 - 1$



21.  $f(x) = x^3 + 1$

#### Derivatives of Inverse Functions

In Exercises 25 and 30:

- a. Find  $f^{-1}(x)$ .
  - b. Graph  $f$  and  $f^{-1}$  together.
  - c. Evaluate  $df/dx$  at  $x = a$  and  $\frac{df^{-1}}{dx}$  at  $x = f(a)$  to show that at these points  $\frac{df^{-1}}{dx} = 1/(\frac{df}{dx})$ .
25.  $f(x) = 2x + 3,$   $a = -1$
- 30.
- a. Show that  $h(x) = x^3/4$  and  $k(x) = (4x)^{1/3}$  are inverses of one another.
  - b. Graph  $h$  and  $k$  over an  $x$ -interval large enough to show the graphs intersecting at  $(2, 2)$  and  $(-2, -2)$ . Be sure the picture shows the required symmetry about the line  $y = x$ .
  - c. Find the slopes of the tangents to the graphs at  $h$  and  $k$  at  $(2, 2)$  and  $(-2, -2)$ .
  - d. What lines are tangent to the curves at the origin?

### EXERCISES 7.2

#### Using the Properties of Logarithms

1. Express the following logarithms in terms of  $\ln 2$  and  $\ln 3$ .

a. $\ln 0.75$	b. $\ln (4/9)$
d. $\ln \sqrt[3]{9}$	e. $\ln 3\sqrt{2}$
	c. $\ln (1/2)$
	f. $\ln \sqrt{13.5}$

#### Derivatives of Logarithms

In Exercise 22, find the derivative of  $y$  with respect to  $x$ ,  $t$ , or  $\theta$ , as appropriate.

22.  $y = \frac{x \ln x}{1 + \ln x}$

#### Integration

Evaluate the integrals in Exercise 39.

39.  $\int \frac{2y \, dy}{y^2 - 25}$

## Logarithmic Differentiation

In Exercise 64, use logarithmic differentiation to find the derivative of  $y$  with respect to the given independent variable.

$$64. y = \frac{\theta \sin \theta}{\sqrt{\sec \theta}}$$

## Theory and Applications

69. Locate and identify the absolute extreme values of

a.  $\ln(\cos x)$  on  $[-\frac{\pi}{4}, \frac{\pi}{3}]$ ,

b.  $\cos(\ln x)$  on  $[\frac{1}{2}, 2]$ .

## EXERCISES 7.3

### Algebraic Calculations with the Exponential and Logarithm

Find simpler expressions for the quantities in Exercise 2.

2. a.  $e^{\ln(x^2+y^2)}$       b.  $e^{-\ln 0.3}$       c.  $e^{\ln \pi x - \ln 2}$

### Solving Equations with Logarithmic or Exponential Terms

In Exercise 10, solve for  $y$  in terms of  $t$  or  $x$ , as appropriate.

10.  $\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x)$

In Exercise 16, solve for  $t$ .

16.  $e^{(x^2)}e^{(2x+1)} = e^t$

## Derivatives

In Exercises 23 and 36, find the derivative of  $y$  with respect to  $x$ ,  $t$ , or  $\theta$ , as appropriate.

23.  $y = (x^2 - 2x + 2)e^x$

36.  $y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt$

## Integrals

Evaluate the integrals in Exercises 49 and 56.

49.  $\int \frac{e^{\sqrt{t}}}{\sqrt{t}} dt$

56.  $\int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \csc^2 \theta \, d\theta$

## Theory and Applications

67. Find the absolute maximum and minimum values of  $f(x) = e^x - 2x$  on  $[0, 1]$ .

## EXERCISES 7.4

### Algebraic Calculations With $a^x$ and $\log_a x$

Simplify the expressions in Exercise 4.

4. a.  $25^{\log_5(3x^2)}$       b.  $\log_e(e^x)$       c.  $\log_4(2e^x \sin x)$

## Derivatives

In Exercises 18 and 29, find the derivative of  $y$  with respect to the given independent variable.

18.  $y = (\ln \theta)^\pi$

29.  $y = \log_3 \left( \left( \frac{x+1}{x-1} \right)^{\ln 3} \right)$

## Logarithmic Differentiation

In Exercises 41 and 46, use logarithmic differentiation to find the derivative of  $y$  with respect to the given independent variable.

41.  $y = (\sqrt{t})^t$

46.  $y = (\ln x)^{\ln x}$

## Integration

Evaluate the integrals in Exercise 65.

65.  $\int_0^2 \frac{\log_2(x+2)}{x+2} dx$

Evaluate the integrals in Exercise 72.

$$72. \int_1^{e^x} \frac{1}{t} dt$$

### Theory and Applications

75. Find the area of the region between the curve  $y = 2x/(1 + x^2)$  and the interval  $-2 \leq x \leq 2$  of the  $x$ -axis.

### EXERCISES 7.5

#### 6. Voltage in a discharging capacitor

Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage  $V$  across its terminals and that, if  $t$  is measured in seconds,

$$\frac{dV}{dt} = -\frac{1}{40}V.$$

Solve this equation for  $V$ , using  $V_0$  to denote the value of  $V$  when  $t = 0$ . How long will it take the voltage to drop to 10% of its original value?

#### 8. Growth of bacteria

A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours there are 10,000 bacteria. At the end of 5 hours there are 40,000. How many bacteria were present initially?

### EXERCISES 7.7

#### Common Values of Inverse Trigonometric Functions

Use reference triangles to find the angles in Exercise 6.

$$6. \text{ a. } \cos^{-1}\left(\frac{-1}{2}\right) \qquad \text{b. } \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\text{c. } \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$$

### Trigonometric Function Values

13. Given that  $\alpha = \sin^{-1}(5/13)$ , find  $\cos \alpha$ ,  $\tan \alpha$ ,  $\sec \alpha$ ,  $\csc \alpha$ , and  $\cot \alpha$ .

### Evaluating Trigonometric and Inverse Trigonometric Terms

Find the values in Exercise 26.

$$26. \sec(\cot^{-1} \sqrt{3} + \csc^{-1}(-1))$$

### Finding Derivatives

In Exercise 51, find the derivative of  $y$  with respect to the appropriate variable.

$$51. y = \sin^{-1} \sqrt{2} t$$

### Evaluating Integrals

Evaluating the integrals in Exercise 72.

$$72. \int \frac{dx}{\sqrt{1 - 4x^2}}$$

Evaluate the integrals in Exercise 107.

$$107. \int \frac{(\sin^{-1} x)^2 dx}{\sqrt{1 - x^2}}$$

### Integration Formulas

Verify the integration formulas in Exercise 117.

$$117. \int \frac{\tan^{-1} x}{x^2} dx = \ln x - \frac{1}{2} \ln(1 + x^2) - \frac{\tan^{-1} x}{x} + C$$

### EXERCISES 7.8

#### Hyperbolic Function Values and Identities

Each of Exercise 1 gives a value of  $\sinh x$  or  $\cosh x$ . Use the definitions and the identity  $\cosh^2 x - \sinh^2 x = 1$  to find the values of the remaining five hyperbolic functions.

$$1. \sinh x = -\frac{3}{4}$$

## Derivatives

In Exercise 16, find the derivative of  $y$  with respect to the appropriate variable.

16.  $y = t^2 \tanh \frac{1}{t}$

## Applications and Theory

### 83. Arc length

Find the length of the segment of the curve  $y = (1/2) \cosh 2x$  from  $x = 0$  to  $x = \ln \sqrt{5}$ .

## Indefinite Integrals

Evaluate the integrals in Exercise 43.

43.  $\int 6 \cosh \left( \frac{x}{2} - \ln 3 \right) dx$

## Definite Integrals

Evaluate the integrals in Exercise 60.

60.  $\int_0^{\ln 10} 4 \sinh^2 \left( \frac{x}{2} \right) dx$

## Evaluating Inverse Hyperbolic Functions and Related Integrals

When hyperbolic function keys are not available on a calculator, it is still possible to evaluate the inverse hyperbolic functions by expressing them as logarithms, as shown here.

$\sinh^{-1} x = \ln \left( x + \sqrt{x^2 + 1} \right),$	$-\infty < x < \infty$
$\cosh^{-1} x = \ln \left( x + \sqrt{x^2 + 1} \right),$	$x \geq 1$
$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x},$	$ x  < 1$
$\operatorname{sech}^{-1} x = \ln \left( \frac{1 + \sqrt{1-x^2}}{x} \right),$	$0 < x \leq 1$
$\operatorname{csch}^{-1} x = \ln \left( \frac{1}{x} + \frac{\sqrt{1+x^2}}{ x } \right),$	$x \neq 0$
$\operatorname{coth}^{-1} x = \frac{1}{2} \ln \frac{x+1}{x-1},$	$ x  > 1$

Use the formulas in the box here to express the numbers in Exercise 66 in terms of natural logarithms.

66.  $\operatorname{csch}^{-1} (-1/\sqrt{3})$



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### EXERCISES 8.1

#### Basic Substitutions

Evaluate each integral in Exercise 36 by using a substitution to reduce it to standard form.

$$36. \int \frac{\ln x \, dx}{x + 4x \ln^2 x}$$

#### Completing the Square

Evaluate each integral in Exercise 41 by completing the square and using a substitution to reduce it to standard form.

$$41. \int \frac{dx}{(x + 1)\sqrt{x^2 + 2x}}$$

#### Improper Fractions

Evaluate each integral in Exercise 50 by reducing the improper fraction and using a substitution (if necessary) to reduce it to standard form.

$$50. \int_{-1}^3 \frac{4x^2 - 7}{2x + 3} \, dx$$

#### Separating Fractions

Evaluate each integral in Exercise 56 by separating the fraction and using a substitution (if necessary) to reduce it to standard form.

$$56. \int_0^{1/2} \frac{2 - 8x}{1 + 4x^2} \, dx$$

#### Multiplying by a Form of 1

Evaluate each integral in Exercise 59 by multiplying by a form of 1 and using a substitution (if necessary) to reduce it to standard form.

$$59. \int \frac{1}{\sec \theta + \tan \theta} \, d\theta$$

#### Eliminating Square Roots

Evaluate each integral in Exercise 68 by eliminating the square root.

$$68. \int_{\pi/2}^{\pi} \sqrt{1 - \sin^2 \theta} \, d\theta$$

#### Assorted Integrations

Evaluate each integral in Exercise 82 by using any technique you think is appropriate.

$$82. \int \frac{dx}{x\sqrt{3 + x^2}}$$

#### Trigonometric Powers

83.

- Evaluate  $\int \cos^3 \theta \, d\theta$ . (Hint:  $\cos^2 \theta = 1 - \sin^2 \theta$ .)
- Evaluate  $\int \cos^5 \theta \, d\theta$ .
- Without actually evaluating the integral, explain how you would evaluate  $\int \cos^9 \theta \, d\theta$ .

### EXERCISES 8.2

#### Integration by Parts

Evaluate the integrals in Exercise 1, 19 and 24.

$$1. \int x \sin \frac{x}{2} \, dx$$

$$19. \int_{2/\sqrt{3}}^2 t \sec^{-1} t \, dt$$

$$24. \int e^{-2x} \sin 2x \, dx$$

#### Substitution and Integration by Parts

Evaluate the integrals in Exercise 30 by using a substitution prior to integration by parts.

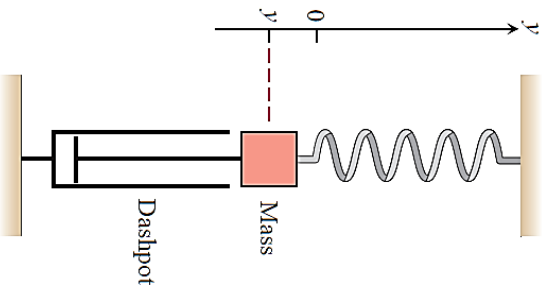
$$30. \int z(\ln z)^2 \, dz$$

### 37. Average value

A retarding force, symbolized by the dashpot in the figure, slows the motion of the weighted spring so that the mass's position at time  $t$  is

$$y = 2e^{-t} \cos t, \quad t \geq 0.$$

Find the average value of  $y$  over the interval  $0 \leq t \leq 2\pi$ .



### Reduction Formulas

In Exercise 41, use integration by parts to establish the reduction formula.

$$41. \int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx, \quad a \neq 0$$

### EXERCISES 8.3

#### Expanding Quotients into Partial Fractions

Expand the quotients in Exercise 6 by partial fractions.

$$6. \frac{z}{z^3 - z^2 - 6z}$$

#### Nonrepeated Linear Factors

In Exercise 12, express the integrands as a sum of partial fractions and evaluate the integrals.

$$12. \int \frac{2x + 1}{x^2 - 7x + 12} dx$$

#### Repeated Linear Factors

In Exercise 20, express the integrands as a sum of partial fractions and evaluate the integrals.

$$20. \int \frac{x^2 dx}{(x - 1)(x^2 + 2x + 1)}$$

#### Irreducible Quadratic Factors

In Exercise 26, express the integrands as a sum of partial fractions and evaluate the integrals.

$$26. \int \frac{s^4 + 81}{s(s^2 + 9)^2} ds$$

#### Improper Fractions

In Exercise 31, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

$$31. \int \frac{9x^3 - 3x + 1}{x^3 - x^2} dx$$

#### Evaluating Integrals

Evaluating the integrals in Exercise 38.

$$38. \int \frac{\sin \theta d\theta}{\cos^2 \theta + \cos \theta - 2}$$

### EXERCISES 8.4

#### Products of Powers of Sines and Cosines

Evaluate the integrals in Exercise 6 and 14.

$$6. \int_0^{\pi/2} 7 \cos^7 t dt$$

$$14. \int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta d\theta$$

### **Integrals with Square Roots**

Evaluate the integrals in Exercise 22.

$$22. \int_{-\pi}^{\pi} (1 - \cos^2 t)^{3/2} dt$$

### **Powers of Tan $x$ and Sec $x$**

Evaluate the integrals in Exercise 26.

$$26. \int_0^{\pi/12} 3 \sec^4 3x dx$$

### **Products of Sines and Cosines**

Evaluate the integrals in Exercise 38.

$$38. \int_{-\pi/2}^{\pi/2} \cos x \cos 7x dx$$

### **EXERCISES 8.5**

#### **Basic Trigonometric Substitutions**

Evaluate the integrals in Exercise 1, 14 and 28.

$$1. \int \frac{dy}{\sqrt{9 + y^2}}$$

$$14. \int \frac{2 dx}{x^3 \sqrt{x^2 - 1}}, \quad x > 1$$

$$28. \int \frac{(1 - r^2)^{5/2}}{r^8} dr$$

In Exercise 32, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

$$32. \int_1^e \frac{dy}{y \sqrt{1 + (\ln y)^2}}$$

#### **Applications**

41. Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve  $y = \sqrt{9 - x^2}/3$ .

### **EXERCISES 8.6**

#### **Using Integral Tables**

Use the table of integrals to evaluate the integrals in Exercise 8 and 20.

$$8. \int \frac{dx}{x^2 \sqrt{4x - 9}}$$

$$20. \int \frac{d\theta}{4 + 5 \sin 2\theta}$$

#### **Substitution and Integral Tables**

In Exercise 45, use a substitution to change the integral into one you can find in the table. Then evaluate the integral.

$$45. \int \cot t \sqrt{1 - \sin^2 t} dt, \quad 0 < t < \pi/2$$

#### **Using Reduction Formulas**

Use reduction formulas to evaluate the integrals in Exercise 60.

$$60. \int \csc^2 y \cos^5 y dy$$

#### **Powers of $x$ Times Exponentials**

Evaluate the integrals in Exercise 80 using table Formulas 103-106. These integrals can also be evaluated using integration (Section 8.2).

$$80. \int x 2^{\sqrt{x}} dx$$

#### **Substitutions with Reduction Formulas**

Evaluate the integrals in Exercise 81 by making a substitution (possibly trigonometric) and then applying a reduction formula.

$$81. \int e^t \sec^3(e^t - 1) dt$$

## Hyperbolic Functions

Use the integral tables to evaluate the integrals in Exercise 90.

$$90. \int x \sinh 5x \, dx$$

## EXERCISES 8.8

### Evaluating Improper Integrals

Evaluate the integrals in Exercises 1 and 26 without using tables.

$$1. \int_0^{\infty} \frac{dx}{x^2 + 1}$$

$$26. \int_0^1 (-\ln x) \, dx$$

### Testing for Convergence

In Exercises 35, 50 and 64, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

$$35. \int_0^{\pi/2} \tan \theta \, d\theta$$

$$50. \int_0^{\infty} \frac{d\theta}{1 + e^\theta}$$

$$64. \int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}}$$

### Theory and Examples

65. Find the values of  $p$  for which each integral converges.

$$\text{b. } \int_2^{\infty} \frac{dx}{x(\ln x)^p}$$

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**EXERCISES 11.1**

**Finding Terms of a Sequence**

Exercise 2 gives a formula for the  $n$ th term  $a_n$  of a sequence  $\{a_n\}$ . Find the values of  $a_1$ ,  $a_2$ ,  $a_3$ , and  $a_4$ .

2.  $a_n = \frac{1}{n!}$

**Finding a Sequence's Formula**

In Exercise 16, find a formula for the  $n$ th term of the sequence.

16. The sequence 1,  $-\frac{1}{4}$ ,  $\frac{1}{9}$ ,  $-\frac{1}{16}$ ,  $\frac{1}{25}$ ,  $\dots$

Reciprocals of squares  
of the positive integers,  
with alternating signs

**Finding Limits**

Which of the sequences  $\{a_n\}$  in Exercises 25, 49 and 80 converge, and which diverge? Find the limit of each convergent sequence.

25.  $a_n = \frac{1 - 2n}{1 + 2n}$

49.  $a_n = \left(1 + \frac{7}{n}\right)^n$

80.  $a_n = \frac{(\ln n)^5}{\sqrt{n}}$

**EXERCISES 11.2**

**Finding  $n$ th Partial Sums**

In Exercise 1, find a formula for the  $n$ th partial sum of each series and use it to find the series' sum if the series converges.

1.  $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \dots + \frac{2}{3^{n-1}} + \dots$

**Series with Geometric Terms**

In Exercise 7, write out the first few terms of each series to show how the series starts. Then find the sum of the series.

7.  $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$

**Telescoping Series**

Find the sum of each series in Exercise 15.

15.  $\sum_{n=1}^{\infty} \frac{4}{(4n - 3)(4n + 1)}$

**Convergence or Divergence**

Is Exercise 23 converge or diverge? Give reasons for your answer. If a series converges, find its sum.

23.  $\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$

**Geometric Series**

In geometric series in Exercise 41, write out the first few terms of the series to find  $a$  and  $r$ , and find the sum of the series. Then express the inequality  $|r| < 1$  in terms of  $x$  and find the values of  $x$  for which the inequality holds and the series converges.

41.  $\sum_{n=0}^{\infty} (-1)^n x^n$

**Repeating Decimals**

Express each of the numbers in Exercise 51 as the ratio of two integers.

51.  $0.\overline{23} = 0.23\ 23\ 23\ \dots$

### EXERCISES 11.3

#### Determining Convergence or Divergence

Which of the series in Exercises 1, 9, 10 and 28 converge, and which diverge? Give reasons for your answers. (When you check an answer, remember that there may be more than one way to determine the series' convergence or divergence.)

1.  $\sum_{n=1}^{\infty} \frac{1}{10^n}$

9.  $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

10.  $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

28.  $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

### EXERCISES 11.4

#### Determining Convergence and Divergence

Which of the series in 1, 10 and 36 converge, and which diverge? Give reasons for your answers.

1.  $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$

10.  $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$

36.  $\sum_{n=1}^{\infty} \frac{1}{1 + 2^2 + 3^2 + \cdots + n^2}$

### EXERCISES 11.6

#### Determining Convergence or Divergence

Is Exercise 1 converge or diverge? Give reasons for your answers.

1.  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

#### Absolute Convergence

Which of the series in Exercises 13 and 30 converge absolutely, which converge, and which diverge? Give reasons for your answers.

13.  $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[n]{n}}$

30.  $\sum_{n=1}^{\infty} (-5)^{-n}$

### EXERCISES 11.7

#### Intervals of Convergence

In Exercise 1, 11 and 22, (a) find the series' radius and interval of convergence. For what values of  $x$  does the series converge (b) absolutely, (c) conditionally?

1.  $\sum_{n=0}^{\infty} x^n$

11.  $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

22.  $\sum_{n=1}^{\infty} (\ln n)x^n$

In Exercise 36, find the series' interval of convergence and, within this interval, the sum of the series as a function of  $x$ .

36.  $\sum_{n=0}^{\infty} (\ln x)^n$

### EXERCISES 11.8

#### Finding Taylor Polynomials

In Exercises 1 and 4, find the Taylor polynomials of orders 0, 1, 2, and 3 generated by  $f$  at  $a$ .

1.  $f(x) = \ln x$ ,  $a = 1$

4.  $f(x) = 1/(x + 2)$ ,  $a = 0$

#### Finding Taylor Series at $x = 0$ (Maclaurin Series)

Find the Maclaurin series for the functions in Exercise 9.

9.  $e^{-x}$

#### Finding Taylor Series

In Exercises 24 and 28, find the Taylor series generated by  $f$  at  $x = a$ .

24.  $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2$ ,

$a = -1$

28.  $f(x) = 2^x$ ,  $a = 1$

### EXERCISES 11.9

#### Taylor Series by Substitution

Use substitution to find the Taylor series at  $x = 0$  of the functions in Exercise 1.

1.  $e^{-5x}$

#### More Taylor Series

Find Taylor series at  $x = 0$  for the functions in Exercise 8.

8.  $x^2 \sin x$

### EXERCISES 11.10

#### Binomial Series

Find the first four terms of the binomial series for the functions in Exercises 1 and 9.

1.  $(1 + x)^{1/2}$

9.  $\left(1 + \frac{1}{x}\right)^{1/2}$

Find the binomial series for the functions in Exercise 11.

11.  $(1 + x)^4$

### EXERCISES 11.11

#### Finding Fourier Series

In Exercises 1 and 8, find the Fourier series associated with the given functions. Sketch each function.

1.  $f(x) = 1 \quad 0 \leq x \leq 2\pi$

8.  $f(x) = \begin{cases} 2, & 0 \leq x \leq \pi \\ -x, & \pi < x \leq 2\pi \end{cases}$