Chapter_10_Linear Equations

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Consider a system of 3 linear equations in 3 unknown, x_1, x_2, x_3.

$$x_1 + x_2 + x_3 = 0$$

$$2x_1 + 2x_2 + 2x_3 = 0$$

$$3x_1 + 3x_2 + 3x_3 = 0$$

(*i*) Give the most trivial solution to the system of equations above. (*ii*) Give a not-most trivial solution to the system of equations above. (*iii*) Is the system consistent? (*iv*) Explain why you say so in (*iii*). (*v*) How many solutions are there for the linear equation system?

Ans:

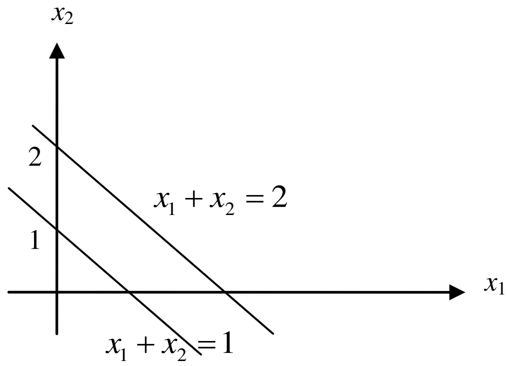
(*i*) $x_1 = 0$, $x_2=0$, $x_3 =0$. (*ii*) $x_1 = -1$, $x_2=1$, $x_3 =0$. (*iii*) Yes. (*iv*) This is because there exists at least one set of values of x_1, x_2, x_3 that satisfy the set of equations. (*v*) Infinitely many, for example, $x_1 = 0$, $x_2 = -x_3$ anything.

2. Consider a system of 2 linear equations in 2 unknown, x_1, x_2: $x_1 + x_2 = 2$, $x_1 + x_2 = 1$

   (*i*) Try sketching the two equations given in the x_2-x_1 plane. (*ii*) Do these two graphs intersect at all? (*iii*) Can you find a solution to the linear equation system given? (*iv*) Is the system consistent? (*v*) Explain why you say so in (*iv*).

Ans:

(*i*)



(*ii*) They never intersect. (*iii*) There is no solution to the equation system. (*iv*)The system is not consistent. (*v*) It's inconsistent because it has no values of x_1, x_2 that satisfy the equation system.

3. Express the linear equation system in (1) and (2) in (*i*) matrix form *AX=H*. (*ii*) augmented matrix form [*A H*].

Ans:

(*i*) (1), $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(*ii*)(2), $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(*iii*)(1), $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{pmatrix}$

(*iii*)(2), $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

4. Consider the following system of equation:

$x_1 - x_2 = 1$ (Eq. 1)

$x_1 + x_2 = 2$ (Eq. 2)

(*i*) Express the equation system above into augmented matrix form.
(*ii*) Perform the following operation: Replace (Eq. 1) by (Eq.1) + (Eq. 2), and call the resultant equation (Eq 1'). At the mean time, leave (Eq. 2) untouched. Write down the resultant equation system. What are the corresponding row elementary operations that have an equivalent effect on the augmented matrix?
(*iii*) Now, perform a follow-up operations: Replace (Eq. 2)

by (Eq. 2) added with (Eq 1') multiplied by (-1/2). Write down the resultant equations. What are the corresponding row elementary operations that have an equivalent effect on the augmented matrix?
(*iv*) Now, multiply (Eq. 1') by a factor of 1/2. Write down the resultant equations. What are the corresponding row elementary operations that have an equivalent effect on the augmented matrix?
(*v*) Now, reduce the augmented matrix in (*i*) into RREF. Do you get the same augmented matrix as resulted in (*iv*)? (*vi*) Can you read off the solutions of x_1, x_2 from the resultant RREF matrix by inspection?

Ans:

(*i*) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

(*ii*) $2x_1 + 0 \cdot x_2 = 3$, (Eq. 1')

$x_1 + x_2 = 2$. (Eq. 2)

The corresponding elementary operation is:

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \xrightarrow{R_1^2(1)} \begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix}.$$

(*iii*) $2x_1 + 0 \cdot x_2 = 3$, (Eq. 1')

$0 + x_2 = 1/2$. (Eq. 2')

The corresponding elementary operation is:

$$\begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix} \xrightarrow{R_2^1(-1/2)} \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 1/2 \end{pmatrix}$$

(*iv*) $x_1 + 0 \cdot x_2 = 3/2$, (Eq. 1')

$0 + x_2 = 1/2$. (Eq. 2')

$$\begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 1/2 \end{pmatrix} \xrightarrow{R_1(1/2)} \begin{pmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 1/2 \end{pmatrix}$$

(*v*) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \end{pmatrix}$ (RREF). Yes.

(*vi*) The solution can be read off from RREF by inspection: x_1= 3/2, x_2= ½.

5. (*i*) Express the following linear equation system in augmented matrix form [*A H*].

$x_1 + x_2 + x_3 = 1$; $2x_1 + x_2 + 3x_3 = 2$;

$3x_1 + x_2 + x_3 = 0$.

(*ii*) Reduce the augmented matrix [*A H*] into reduced row echelon form. (*iii*) Write out the equation system represented by the RREF augmented matrix as obtained in (*ii*). Read off the solutions by inspection.

Ans:

(*i*) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 1 & 1 & 0 \end{pmatrix}$.

(*ii*) Mathematica gives RREF= $\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 1 & \frac{3}{4} \end{pmatrix}$.

(*iii*) $x_1 + 0 \cdot x_2 + 0 \cdot x_3 = -1/2$;

$0 \cdot x_1 + x_2 + 0 \cdot x_3 = 3/4$;

$0 \cdot x_1 + 0 \cdot x_2 + x_3 = 3/4$.

(*iv*) $x_1 = -1/2$; $x_2 = 3/4$; $x_3 = 3/4$

6. Choose the correct answer: (*i*) REF is a (special / general) case of RREF. (*ii*) The procedure you used in (5), in which the augmented matrix is reduced into RREF, to solve for the solutions, is called (Gaussian /Gaussian-Jordan) elimination.

Ans:
(*i*) general (*ii*) Gaussian-Jordan.

7. The equation system in
(*i*) (1) is a (homogeneous/non-homogeneous) system.
(*ii*) (2) is a (homogeneous/non-homogeneous) system.
(*iii*) (5) is a (homogeneous/non-homogeneous) system.

Ans: (*i*) homogeneous (*ii*) non-homogeneous (*iii*) non-homogeneous.

8. Refer to theorem III, page 77, Ayers. Answer the following properties of the equation system in (5):
   (*i*) The system in (5) is (homogeneous or non-homogeneous).
   (*ii*) The number of unknowns in (5) is _______.
   (*iii*) The number of equation in (5) is _______.
   (*iv*) The determinant of $|A|$ in (5) is (zero / non-zero).
   (*v*) The solution as obtained in (5) is (unique / not-unique).

Ans:

(*i*) Non-homogeneous
(*ii*) 3
(*iii*) 3
(*iv*) $4 \neq 0$
(*v*) unique

10. Given an arbitrary linear equation system of the form $AX=H$, there are two possibility on the existence of its solution, i.e. either the solution exists or _________. If the solution exists, it could be __________or ___________, __________ or ____________.

Ans: not-exist; trivial/not trivial; unique/not unique.

11. Give a simple example of a non-homogeneous system that (*i*) has a unique solution. What is the unique solution? (*ii*) has no solution (i.e. not consistent); (*iii*) has non-unique solutions.

Ans:

(*i*) Any non-homogeneous system with same number of equations and unknown, with coefficient matrix A non-singular will do the job, such as that in (4).
(*ii*) Any non-homogeneous system with coefficient matrix A singular will do the job, such as that in (2).
(*iii*) Any non-homogeneous system with number of coefficient larger than the number of equation will do the job, such as

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}.$$

12. Give an example of a 2 by 2 homogeneous system that (*i*) has a unique solution; (*ii*) has no solution (i.e. not consistent); (*iii*) has non-unique solutions.

Ans:

(*i*) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; Any homogeneous system with a coefficient matrix that is not singular will admit a unique solution, which is the trivial solution, $x_1 = x_2 = 0$.
(*ii*) Not possible. All homogeneous system has at least the trivial solution.
(*iii*) Any system with a coefficient matrix that is singular will admit non-unique solution, e.g.

$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $x_1 = -2x_2 =$ anything.

13. Given a non-homogeneous equation $AX=H$, with $|A| \neq 0$, A an n-square coefficient matrix, X column vector of n variables, H non-zero column vector of n components.
   (*i*) List down all the ways you can think of that can be employed to solve the linear equation systems. I can think of 5, how many can you think of?
   (*ii*) As an exercise, solve the given equation system using all the methods you have listed. You should get the same answer with all these different methods.

Ans:

(*i*) Gaussian elimination: Reduce $[A|H]$ using elementary operation to a REF. Obtain the solution by inspection.

(*ii*) Gaussian-Jordan elimination: Reduce $[A|H]$ using elementary operation to a RREF. Obtain the solution by inspection.

(*iii*) First, find the inverse of A, then obtain the solution via $X=A^{-1}H$.

(*iv*) Cramer rule.

(*v*) Reduce A in the augmented matrix $[A|H]$ into an identity matrix via a sequence of elementary operations, $E= E_sE_{s-1}\ldots E_2E_1$. When each step of operation is being operated on A, the same operation will also be operated concurrently on H. When A is finally transformed into I, H will be transformed into the solution column. The resultant augmented matrix becomes $[I \mid X_{sol}]$. The solution can then be read off from the solution column X_{sol}.

14. Consider an equation system $AX=H$, which represent m equations in n unknown. What is the sufficient condition that this equation system is consistent? (*Hint: find the answer in page 76 of Ayers "fundamental theorems."*)

Ans:
If rank of $[A \mid H]$ = rank $[A]$, the system is guaranteed to be consistent.

15. Consider a homogeneous equation system $AX=0$, which has n equations in n unknowns.
(*i*) Does the system is guaranteed consistent? Explain by referring to your answer in (14).
(*ii*) What can you say about the solution X if the rank of A, r, is equal the number of unknowns, n? (Give your answer in terms of its existence, uniqueness and triviality.)
(*iii*) What can you say about the solution X if the rank of A, r, is less than the number of unknowns, n? (Give your answer in terms of its existence, uniqueness and triviality.)
(*iv*) Consider the statement: ALL homogeneous equation system is consistent. Is this statement true?

Ans:
(*i*) Since for the homogenous system, $H=0$, the rank of $[A \mid H] = [A \mid 0]$ is always = rank $[A]$. Hence, by (14) the homogeneous system is always consistent.
(*ii*) If $r = n$, the solution exist, unique, and trivial.
(*iii*) If $r < n$, the solution exist, non-unique, and non-trivial.

16. Consider the homogeneous equation system in (15), $AX=0$. (*i*) If A is not singular, A^{-1} exist. What do you get if you operate A^{-1} on $AX=0$ from the left-hand-side (LHS)? Try to figure out what will happen to the solution X. (*ii*) So, what is your conclusion?

Ans:
(*i*) $A^{-1}(AX) = A^{-1}0 = 0 \rightarrow X = 0$. (*ii*) If $|A|$ is not zero, we conclude that the homogeneous equation system of n equations in n unknowns has only the trivial solution as its unique solution.

17. Consider (16) again. (*i*) If A is singular, i.e. A^{-1} does not exist, can you claim that $AX=0$ has no solution? (*ii*) What's the difference in the solution of a homogeneous equation system of singular coefficient matrix A and one that is not? Give your answer in terms of its existence, uniqueness and triviality.

Ans: (*i*) No. You can't you claim that $AX=0$ has no solution if A^{-1} does not exist. In fact, solutions always exist for non-homogeneous system irrespective of whether A is singular or not. (*ii*) For HE with singular A, solutions do exist, non-trivial, and non-unique; whereas for those with non-singular A, the solution is trivial and unique.

18. Consider the equation system $x_1 - 2x_2 + 3x_3 = 4$; $x_1 + x_2 + 2x_3 = 5$. (*i*) Express the system in matrix form. (*ii*) Is the number of unknowns larger than the number of equation? (*iii*) So, how many solutions would you expect? (*iii*) Solve the equation system using Gaussian-Jordan elimination.

Ans:
(*i*) $\begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ (*ii*) Yes. (*iii*) Infinitely many solutions.
(*iii*)
$$\begin{pmatrix} 1 & -2 & 3 & 4 \\ 1 & 1 & 2 & 5 \end{pmatrix} \xrightarrow{R_2^1(-1)} \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 3 & -1 & 1 \end{pmatrix} \xrightarrow{R_1^2(2/3)}$$
$$\begin{pmatrix} 1 & 0 & 7/3 & 14/3 \\ 0 & 3 & -1 & 1 \end{pmatrix} \xrightarrow{R_2(1/3)} \begin{pmatrix} 1 & 0 & 7/3 & 14/3 \\ 0 & 1 & -1/3 & 1/3 \end{pmatrix}.$$
$\Leftrightarrow x_1 + \frac{7}{3}x_3 = \frac{14}{3}$; $x_2 - \frac{1}{3}x_3 = \frac{1}{3}$. Choose x_3 as an independent parameter, $x_1 = -\frac{7}{3}x_3 + \frac{14}{3}$; $x_2 = \frac{1}{3} + \frac{2}{3}x_3$.

That is $X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{7}{3}x_3 + \frac{14}{3} \\ \frac{2}{3}x_3 + \frac{1}{3} \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} \frac{7}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{7}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}$ is a solution to the above system.

19. Refer to solved problems 2, page 79 of Ayers. We will learn how to 'count' the rank of a matrix in this DQ. Consider a homogeneous system of 2 linear equations in 3 unknown, x_1, x_2, x_3, $AX=H$, where

$$A=\begin{pmatrix}1&1&2&1\\2&3&-1&-2\\4&5&3&0\end{pmatrix}, X=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},\quad H=\begin{pmatrix}5\\2\\7\end{pmatrix}$$

(i) Transform the augmented matrix [A H] into [A_1 | H_1], the RREF form of [A | H] using row elementary operations. Express [A_1 | H_1] explicitly.

(ii) Is the rank of [A_1 | H_1], [A | H] equal?

(iii) Count the number of non-zero row in [A_1 | H_1]. The number of non-zero row equals the rank of the augmented matrix [A_1 | H_1]. This number is the same as the number of leading 1 in the RREF. (Note that this simple fact is not mentioned explicitly in Ayers.) Hence, what is the rank of [A | H]?

(iv) From the expression of [A_1 | H_1], what is the rank of the matrix A_1? What is the rank of matrix A?

(v) By referring to the fundamental theorem I, page 76, Ayers, is the system consistent?

Ans:

(i) [A_1 | H_1] =

$$\begin{pmatrix}1&0&7&5&0\\0&1&-5&-4&0\\0&0&0&0&1\end{pmatrix}$$

(ii) Yes, since row elementary operation leave the rank of the matrix unchanged.

(iii) The number of non-zero row in [A_1 | H_1] is 3. Hence the rank of [A | H] is 3.

(iv) The rank of the matrix A_1 = $\begin{pmatrix}1&0&7&5\\0&1&-5&-4\\0&0&0&0\end{pmatrix}$ is 2, hence rank (A) =2.

(v) Since rank(A) ≠ rank([A | H]), the system is not consistent.

20. Consider the homogeneous equation system of

$$x_1+2x_2+2x_4=0\,, x_2+3x_3=0\,.$$

(i) Express the system in the matrix form of $AX=H$.

(ii) This is an equation system with _____equations and _______ unknowns.

(iii) Determine the rank of A and [A | 0].

(iv) Is this system consistent?

(v) By comparing the rank of A and the number of unknowns, can you determine whether the system will admit non-trivial solution? State explicitly whether the non-trivial solutions are expected. (*Hint: refer to theorem IV, page 78, Ayers.*)

(vi) From the answer to (*ii*), state your expectation whether the solutions will be unique or otherwise.

(vii) Based on your answer to (*v*) you know what the rank of the matrix A is. Hence, how many linearly independent solutions do you expect for the HE system? (*Hint: refer to theorem VI, page 78, Ayers.*)

Ans:

(*i*) $A=\begin{pmatrix}1&2&0&2\\0&1&3&0\end{pmatrix}, X=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}, H=\begin{pmatrix}0\\0\end{pmatrix}$.

(*ii*) 2 equations with 4 unknowns.

(*iii*) RREF of A = $\begin{bmatrix}1&0&-6&2\\0&1&3&0\end{bmatrix}$. Hence Rank ($A$), r =2. So is the rank of the augmented matrix [A | H].

(*iv*) All homogeneous system is necessarily consistent.

(*v*) Since $r = 2 < n = 4$, by theorem IV, page 78, Ayers, the homogeneous system shall admit non-trivial solution.

(*vi*) Since there are more unknowns then the number of equation, of course the solution is expected to be non-unique, and possibly infinite.

(*vii*) There are n-r = 4 – 2 = 2 linearly independent solutions expected.

21. Solve the HE system in the previous DQ. Your solution should agree with the number of linearly independent solutions is as given in (*vi*) in the same question.

Ans:

From A ~ $\begin{bmatrix}1&0&-6&2\\0&1&3&0\end{bmatrix}$, by inspection, we write down:

$$x_1-6x_3+2x_4=0\,, x_2+3x_3=0\,.$$

We will choose $x_3=a, x_4=b$ as arbitrary parameters.

Then $x_1=6a-2b$, $x_2=-3a$. Hence,

$$X=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}6a-2b\\-3a\\a\\b\end{pmatrix}=a\begin{pmatrix}6\\-3\\1\\0\end{pmatrix}+b\begin{pmatrix}-2\\0\\0\\1\end{pmatrix}=av_1+bv_2.$$

Here, $v_1 = \begin{pmatrix} 6 \\ -3 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ are respectively two solutions to the HE system AX=0, i.e. Av_1= Av_2=0, which can be checked. By theorem VI, page 78 in Ayers, the linear combination of v_1, v_2, av_1+ bv_2 are also solutions to the HE system. The number of linearly independent solutions is 2, which agrees with what was 'predicted' in previous question, n-r = 4 – 2 = 2.

22. You may like to refer to Example 5, page 79, Ayers. Consider a non homogeneous equation system:

    $x_1 - 2x_2 + 3x_3 = 4,\ \ x_1 + x_2 + 2x_3 = 5$.

    (i) Express the system in the matrix form of *AX=H*.

    (ii) This is an equation system with ______equations and ________ unknowns.

    (iii) Determine the rank of [A | H] and A.

    (iv) Is this system consistent?

    (v) From the answer to (*ii*), state your expectation whether the solutions will be unique or otherwise.

    (vi) Based on your answer to (*iii*) you know what the rank of the matrix A is. Hence, how many linearly independent solutions do you expect for the HE system? (*Hint: refer to theorem VI, page 78, Ayers.*)

    (vii) Obtain the solution.

Ans:

(*i*) $A = \begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, H = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

(*ii*) 2 equations with 3 unknowns.

(*iii*) RREF of [A |H] = $\begin{pmatrix} 1 & 0 & \frac{7}{3} & \frac{14}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$. Hence Rank ([$A$ |H]), r =2. So is the rank of the augmented matrix A.

(*iv*) Due to (*iii*), the system is consistent.

(*v*) Since there are more unknowns then the number of equation, of course the solution is expected to be non-unique, and possibly infinite.

(*vi*) There is n-r = 3 – 2 = 1 linearly independent solution to be expected.

(*vii*) From [A |H] ~ $\begin{pmatrix} 1 & 0 & \frac{7}{3} & \frac{14}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$, by inspection, we write down: $x_1 + \frac{7}{3}x_3 = \frac{14}{3},\ \ x_2 - \frac{1}{3}x_3 = \frac{1}{3}$.

We will choose $x_3 = a$ as arbitrary parameter. Then

$$x_1 = \frac{14}{3} - \frac{7}{3}a = \frac{7}{3}(2-a),\ \ x_2 = \frac{1}{3} + \frac{1}{3}a = \frac{1}{3}(1+a).$$

Hence, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{14}{3} - \frac{7}{3}a \\ \frac{1}{3} + \frac{1}{3}a \\ a \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 14 \\ 1 \\ 0 \end{pmatrix} + \frac{a}{3}\begin{pmatrix} -7 \\ 1 \\ 3 \end{pmatrix}$.

Here, $\begin{pmatrix} \frac{14}{3} - \frac{7}{3}a \\ \frac{1}{3} + \frac{1}{3}a \\ a \end{pmatrix}$ is the solution to the HE system *AX=H*, with a an arbitrary parameter. The number of linearly independent solutions is 1, which agrees with what was 'predicted' in previous question, n-r = 3 – 2 = 1.

23. You may like to try out the trick you have learnt from the above DQ on the solved problem 1 in page 79, Ayers.

Ans: [A|H]= $\begin{pmatrix} 1 & 1 & -2 & 1 & 3 & 1 \\ 2 & -1 & 2 & 2 & 6 & 2 \\ 3 & 2 & -4 & -3 & -9 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{pmatrix}$.

Rank [A|H] = 3; Rank [A] = 3; system is consistent; n-r-=5-3=2 =number of linearly independent solutions expected.

From the RREF of [A|H],

$x_1 = 1; x_2 - 2x_3 = 0; x_4 + 3x_5 = 0$ .Choose n-r=2 arbitrary parameters, say, $x_3 = a$, x_5=b.

The solution, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2a \\ a \\ -3b \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2a \\ a \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -3b \\ b \end{pmatrix}$. This is a solution comprised of two linearly independent solutions, with a, b arbitrary parameters.