1. **Definition:** *n*-vector
   An vector ***a*** with *n*-component is an *n*-tuple of real numbers, ***a*** $=\{a_1, a_2, \ldots, a_n\}$. We call this an *n*-vector. a_i, $i=1, 2, \ldots, n$ are the components of ***a***. It has *n* components.

2. As an special example, for $n=3$, ***a***$=\{a_1, a_2, a_3\}$. ***a*** can be imagined as a point in 3-space, the 3-dimensional space we human resides in. For example, the 3-vector ***a***$=\{0,0,0\}$ represents a point with spatial coordinates $\{0, 0, 0\}$.

3. Imagine the collection of all possible 3-vectors into a set *V* containing all points in the 3-space. We call the set of all 3-vectors, (or in other words, all points in the 3-D space), R^3. Each vector in R^3 is equivalent to a point in the 3-space.

4. Similarly, R^2 is the set of all 2-vectors. R^2 is the set of all points in 2-space.

5. *R*, the set of all real number, is the set of all '1-vector' ('1-vector' is just the real scalar we all familiar with). The collection of all 'points' in the 1-space is equivalent to the set of all points in a 1-dimensional 'real-number line'.

6. For the 2-vectors and 3-vectors, we know that we can add and do scalar multiplication on them according to well-defined rules of vector addition and scalar multiplication. As an illustration, consider this: Given two 3-vectors in R^3, ***a***$=\{a_1, a_2, a_3\}$ and ***b***$=\{b_1, b_2, b_3\}$, the vector addition ***a + b*** is defined as a new 3-vector, ***c*** $=\{a_1+b_1, a_2+b_2, a_3+b_3\}$. Similarly, the scalar multiplication between a scalar *k* and a vector ***a*** is defined as a new vector ***d***$=\{ka_1, ka_2, ka_3\}$.

7. **Definition:** Consider a set *V* containing some elements on which operations of **vector addition** and **scalar multiplication** are defined. The set *V* is called a **vector space** if the following ten properties are satisfied:

DEFINITION 7.5 Vector Space

Let *V* be a set of elements on which two operations called **vector addition** and **scalar multiplication** are defined. Then *V* is said to be a **vector space** if the following ten properties are satisfied.

Axioms for Vector Addition

(*i*) If **x** and **y** are in *V*, then **x** + **y** is in *V*.
(*ii*) For all **x**, **y** in *V*, **x** + **y** = **y** + **x**. (commutative law)
(*iii*) For all **x**, **y**, **z** in *V*, **x** + (**y** + **z**) = (**x** + **y**) + **z**. (associative law)
(*iv*) There is a unique vector **0** in *V* such that **0** + **x** = **x** + **0** = **0**. (zero vector)
(*v*) For each **x** in *V*, there exists a vector −**x** such that **x** + (−**x**) = (−**x**) + **x** = **0**. (negative of a vector)

Axioms for Scalar Multiplication

(*vi*) If *k* is any scalar and **x** is in *V*, then *k***x** is in *V*.
(*vii*) $k(\mathbf{x}+\mathbf{y}) = k\mathbf{x} + k\mathbf{y}$
(*viii*) $(k_1 + k_2)\mathbf{x} = k_1\mathbf{x} + k_2\mathbf{x}$ (distributive laws)
(*ix*) $k_1(k_2\mathbf{x}) = (k_1k_2)\mathbf{x}$
(*x*) $1\mathbf{x} = \mathbf{x}$

8. Consider the 3-space, R^3. As mentioned, this a vector space. Can you justify this claim by referring to the definition as given?

Ans:
This is a vector space because (*i*) vector addition and scalar multiplication are well defined on all of the 3-vectors, the elements in R^3, (*ii*) all of the 3-vectors, the elements in R^3, fulfill the 10 axioms. In particular, all 3-vectors are closed under vector addition and closed under scalar multiplication.

9. Explain what do you understand by (*i*) 'closure under vector addition'. (*ii*) 'closure under scalar multiplication'.

Ans:
(*i*) Closure under vector addition means: any two vectors in R^3, when added vectorially, will result in a vector which is also an element of R^3.
(*ii*) Similarly, closure under scalar multiplication means: Any vector in R^3, when multiply by a scalar *k* will result in a vector which is also an element of R^3.

10. Consider R^2. Is it also a vector space? How about the set of all real number, the 1-space, *R*? How do you convince yourself that they are indeed also vector space?

Ans:
Both are also vector spaces, since both of these set fulfill all criteria of being a vector space as defined.

11. **Definition:** A set of vectors V_s from a vector space *V* is a **subspace** of *V* if V_s is closed under addition and scalar multiplication.

Example: The set containing only the element 0, $V_s=\{0\}$, is a subspace of the vector space *R*, since the $\{0\}$ is *(i)* a element vector from R,

(ii) closed under scalar multiplication:

$k\cdot 0 = 0 \in V_s$,

(iii) closed under vector addition:

$0 + 0 = 0 \in V_s$.

Note that the subspace {0} has only a single element. The criteria of being closed under addition are fulfilled: "if x and y are element is V_s, then $x + y$ is also an element in V_s". Here, x=0, y=0, because there is no any other element in V_s other than 0. In other words, 'any element' in {0} (the x), when vectorially added to 'any element' in {0} (the y) will result in $x+ y = 0$, an element of V_s.

12. Every vector space V has at least two subspaces. One of if is the zero subspace, {**0**}, which is illustrated above. Can you think of what's the other one?

Ans: The vector space V itself.

13. **Definition:** Consider a set S containing vectors $\mathbf{x}_1$, $\mathbf{x}_2$, …$\mathbf{x}_m$ in a vector space V. (To help you visualize better, think of V as the vector space of R^3 that contains an infinite number of 3-vectors. Think of S as a set containing, say, m=3 vectors selected from R^3.) We form linear combinations of these m vectors in the form of $k_1\mathbf{x}_1+ k_2\mathbf{x}_2+ \ldots k_m\mathbf{x}_m$, where k_i are scalars. The set of all linear combinations of the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, …$\mathbf{x}_m$ is called the **span** of the vectors, and is written as **Span(*S*)**.

14. Span(S) is a subspace of V. Span(S) is said to be a subspace spanned by the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, …$\mathbf{x}_n$.

15. If every vector in the vector space V can be written as a linear combination of the vectors in S, then S is called a **spanning set** for V.

Example: Let V be the vector space containing all 3-vectors, R^3. Consider the set S={**i**, **j**, **k**} containing the three rectangular unit vectors. The set of all linear combination $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, where a, b, c are scalar, is the span of the vectors {**i**, **j**, **k**}, Span(S). Span(S) is a subspace in R^3 spanned by **i**, **j**, **k**.

16. We say 'the set S={**i**, **j**, **k**} is a spanning set for R^3'. Think of Span(S) in terms of the set of all possible linear combination in terms of **i**, **j**, **k**, $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$. Can you imagine what does Span(S) represent? *Hint*: Imagine the point at the tip of the 3-vector $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$. Imagine the pervasive cloud form by the tip of $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ when a, b, c vary continuously.

Ans: Span(S) represents all the spatial points in R^3. As is well known, all the vectors in the vector space R^3 can be expressed as linear combination $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$.

17. Can you think of any other spanning set for R^3?

Ans: e.g. {**r**, **θ**, **ϕ**}.

18. Is the set {**i**, **j**, **k**, **i**+**j**, **k**+**i**, **i** + **k**} also spanning set for R^3?

Ans: Both answers are yes

19. Is {**i**, **j**} a spanning set for R^3? Explain your answer.

Ans: No, since not all vectors in R^3 can be expressed as linear combination of $a\mathbf{i}+ b\mathbf{j}$.

20. Consider the set S containing the following 4 3-vectors: $K_1=[1,1,1]^T$, $K_2=[1,3,5]^T$, $K_3=[1,5, 3]^T$, $K_4=[5, 3, 1]^T$; $S=\{K_1, K_1, K_3\ K_4\}$. How would you prove that the S is the spanning set of R^3 (or in other words, S span R^3)? *Hint:* To prove that the set of vectors in S span R^3, one needs to prove the existence of the solution $X=[x_1, x_2, x_3, x_4]^T$ for the non-homogeneous equation system $A = x_1K_1 + x_2K_2 + x_3K_3 + x_4K_4$, given an arbitrary 3-vector $A=(a,b,c)^T$ from R^3**.** If the solution X exists, then S spans R^3, otherwise it doesn't. The reasoning is: If the solution X exists, this means that any arbitrary vector A from R^3 can always be expressed as a unique linear combination in the form of $x_1K_1 + x_2K_2 + x_3K_3 + x_4K_4$. Hence, by definition, if the set of vectors in S is a spanning set of R^3.

Ans:

$$A=\begin{pmatrix}K_1 & K_2 & K_3 & K\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&1&1&5\\1&3&5&3\\1&5&3&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\equiv KX$$

$$K=\begin{pmatrix}1&1&1&5\\1&3&5&3\\1&5&3&1\end{pmatrix}\sim\begin{pmatrix}1&0&0&6\\0&1&0&-1\\0&0&1&0\end{pmatrix}$$

Hence, rank of K, $r = 3$. The rank of $[K \mid A]$ is also 3 irrespective of what the values of a, b, c are. By the fundamental theorem in page 76, Ayers, the system $KX = A$ is consistent. Hence, it is always possible to express any 3-vector A in R^3 as linear combination in K_1, K_2, K_3, K_4 in the form

$$A = x_1K_1 + x_2K_2 + x_3K_3 + x_4K_4,$$

with the solution of x_1, x_2, x_3, x_4 guaranteed to exist. This prove $S=\{K_1, K_2, K_3, K_4\}$ spans R^3.

21. In general, given a set of m n-vectors, $K_i=(k_1, k_2,\ldots, k_n)^T$, $i=1,2,\ldots m$, we can determines whether they span a vector space R^n, the vector space containing the set of all n-vector by looking for the existence proof of solution X to the non-homogeneous system. The procedure is as followed:

22. Let $K=\left(K_1, K_2,\ldots, K_m\right)$, an n by m matrix,

    $X=\left(x_1, x_2,\ldots, x_m\right)^T$, an m by 1 column vector,

    $A=\left(a_1, a_2,\ldots, a_n\right)^T$, an arbitrary n-vector in R^n. Consider the NH system $A = x_1K_1 + x_2K_2 + \cdots + x_mK_m = KX$. If the solution for the NH systems does not exist, i.e. $\text{rank}[K] \neq \text{rank}\,[K \mid A]$, then the set of vectors K_i does not span R^n. Otherwise, they do.

23. In (20), we see that the set $\{K_1, K_2, K_3, K_4\}$ comprises of 4 3-vector spans R^3. Can we span R^3 with less than 4 3-vectors (e.g., say, 3 or even 2 3-vectors)? In general, for a vector space V containing elements made up of n-vectors, we want to know what is the smallest number of linearly independent n-vector that spans the vector space V.

24. In fact, out of the four 3-vectors in the set S in (20), only three are linearly independent (refer DQ 13, Chapter 9), namely K_1, K_2, K_3, whereas K_4 can be expressed as a linear combination of the other three vectors.

    (*i*) Prove the linearly independence of the vector set K_1, K_2, K_3. (*Hint:* Refer to DQ 12, 13, 14 in Chapter 9.)

    (*ii*) Prove, using the procedure mentioned in (22) above, that this set of vectors K_1, K_2, K_3 spans R^3.

Ans:

(*i*) Consider the homogeneous problem $KX = 0$, where $K=\left(K_1, K_2, K_3\right)$, a 3 by 3 matrix, and $X=\left(x_1, x_2, x_3\right)^T$.

$$K=\begin{pmatrix}1&1&3\\1&3&5\\1&5&3\end{pmatrix}\sim\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

, hence rank $[K] = 3$; so is the number of unknown = 3. The HE system also has the same number of unknowns. Hence, the HE system, $KX = 0$, admits only trivial solution, $X=0$. By definition, this proves the linearly independence of the vector set $\{K_1, K_2, K_3\}$.

(*ii*) $KX=A$, $A=(a_1, a_2, a_3)^T$, an arbitrary 3-vector in R^3. Rank $[K|A]$ = Rank$[K]$ =3, hence, solution X exists. This means that an arbitrary 3-vector in R^3 can always be expressed as a linear combination in K_1, K_2, K_3. According to definition, $\{K_1, K_2, K_3\}$ spans R^3.

25. Now, we ask: can any of the 2 vectors (which are necessarily linearly independent) form the set $\{K_1, K_2, K_3\}$ span R^3? The answer can be proven to be negative. (Prove this). So, it appears that the minimum number of linearly independent 3-vectors to span R^3 is 3, not 2.

26. **Definition:** The minimum number of linearly independent vectors that is required to span a vector space is called the **dimension** of the vector space. In the above example, the dimension of the vector space R^3 is 3 since the minimum number of linearly independent vectors in R^3 is 3.

27. **Definition:** Consider a vector space V with dimension r. A set of r linearly independent vectors in V is called the **basis** (or basis set) of the vector space. It happens that given any set of r vectors, which are linearly independent, from V, they (*i*) will form a basis set for V, and (*ii*) any vector in V can be expressed as a unique linear combination in this set of r vectors.

28. Let's consider the vector space R^3. We know that the dimension of it is r=3. (*i*) If I simply pick any three vectors in R^3, say $X_1 = (a, b, c)$, $X_2 = (d, e, f)$, $X_3 = (g, h, i)$, in general, will the set $\{X_1, X_2, X_3\}$ form a basis for R^3? (*ii*) Is the basis set of R^3 unique? (*iii*) How many basis set can R^3 possibly has?

Ans:

(i) Not in general. Only a set of 3 linearly independent vectors in R^3 is the basis of R^3. Those that are not linearly independent cannot form a basis for R^3.

(ii) No, not unique.

(iii) There can be infinitely many basis set in R^3.

29. Consider the set of three vectors in R^3, $S=\{E_1, E_2, E_3\}$, where $E_1= [1, 0, 0]^T$, $E_2= [0, 1, 0]^T$, $E_3= [0, 0, 1]^T$.

(i) Are the vectors in S linearly independent (you should be able to answer this simply question by visual inspection)?

(ii) Do the vectors in the set S form a basis set for R^3?

(iii) Do the vectors in the set S span R^3?

(iv) Can every vector in R^3 be expressed as linear combination of E_1, E_2, E_3?

(v) What's the name of these E-vectors? (*Hint: see page 88 of Ayers).* (Note: we will refer this basis set by the name 'the E-basis').

Ans:

(*i - iv*) Yes. (*v*) Elementary or unit vector over R^3.

30. You may like to refer to Ayers page 88. Say I have an arbitrary vector in R^3, $X=(a, b, c)^T$.

(i) Write X as a linear combination of the unit vectors, E_i, defined in (30).

(ii) What are the components (or referred to as 'coordinates') of X relative to the E-basis? Write these components in the form of a column vector and call it 'the component vector of X relative to the E-basis', denoted by X_E.

Ans:

(i) $X = (a, b, c)^T = aE_1 + bE_2 + cE_3$.

(ii) $X_E = (a, b, c)^T$.

31. In the previous question, we have an arbitrary vector in R^3, X. Let's say that the vector X when expressed in the E-basis is represented by the component vectors X_E=(1, 2, 3)^T. Normally, a vector is by default expressed in the E-basis. In general, other than the E-basis, we can also represent a vector in other basis set. To illustrate this point, let's consider another basis set $Z = \{Z_1, Z_2, Z_3\}$ ('the Z-basis'), where $Z_1= [2, -1, 3]^T$, $Z_2= [1, 2, -1]^T$, $Z_3= [1, -1, -1]^T$. What is the component vector of X relative to the Z-basis, X_Z? [*Hint:* In order to obtain X_Z, you need to express X as a linear combination of $\{Z_1, Z_2, Z_3\}$: $X_E = a_1Z_1+ a_2Z_2+ a_3Z_3$. Then the component vector of X in the Z-basis is simply $X_Z= (a_1, a_2, a_3)^T$.]

Ans:

$$X_E = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = a_1 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + a_1 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Writing the above compactly in matrix form,

$$X_E = \begin{pmatrix} Z_1 & Z_2 & Z_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = Z \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = Z \cdot X_Z, \text{ where}$$

$$Z = \begin{pmatrix} Z_1 & Z_2 & Z_3 \end{pmatrix}, X_Z = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

The solution is then $X_Z = Z^{1} X_E$

$$X_Z = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 3 & -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{5} & 0 & \frac{1}{5} \\ \frac{4}{15} & \frac{1}{3} & -\frac{1}{15} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{11}{15} \\ -\frac{4}{3} \end{pmatrix}.$$

32. Refer to Example 5, page 88 Ayers. Now, see if you can do things another way round: If the component vector of X is given in the Z representation, i.e. $X_Z = (1,2,3)^T$ is known. What is component vector of X in the E-basis? In other words, what is X_E? *Hint*: Follow the procedure as described in (32), then try to find a similar relation that relates X_E to X_Z in the form of

$$X_E = [\text{some matrix}] \cdot X_Z$$

Ans:

From the previous procedure, we have $X_E = Z \cdot X_Z$. Hence, it

is straight forward to obtain X_E :

$$X_E = Z \cdot X_Z = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 3 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix}.$$