1. **Definition:** *n*-vector
   An vector $\boldsymbol{a}$ with n-component is an n-tuple of real numbers, $\boldsymbol{a} = \{a_1, a_2, \ldots, a_n\}$. We call this an n-vector. a_i, $i=1, 2, \ldots, n$ are the components of $\boldsymbol{a}$. It has n components.

2. As an special example, for $n=3$, $\boldsymbol{a}=\{a_1, a_2, a_3\}$. $\boldsymbol{a}$ can be imagined as a point in 3-space, the 3-dimensional space we human resides in. For example, the 3-vector $\boldsymbol{a}=\{0,0,0\}$ represents a point with spatial coordinates {0 , 0, 0}.

3. Imagine the collection of all possible 3-vectors into a set V containing all points in the 3-space. We call the set of all 3-vectors, (or in other words, all points in the 3-D space), R^3. Each vector in R^3 is equivalent to a point in the 3-space.

4. Similarly, R^2 is the set of all 2-vectors. R^2 is the set of all points in 2-space.

5. R, the set of all real number, is the set of all ‘1-vector’ (‘1-vector’ is just the real scalar we all familiar with). The collection of all ‘points’ in the 1-space is equivalent to the set of all points in a 1-dimensional ‘real-number line’.

6. For the 2-vectors and 3-vectors, we know that we can add and do scalar multiplication on them according to well-defined rules of vector addition and scalar multiplication. As an illustration, consider this: Given two 3-vectors in R^3, $\boldsymbol{a}=\{a_1, a_2, a_3\}$ and $\boldsymbol{b}=\{b_1, b_2, b_3\}$, the vector addition $\boldsymbol{a} + \boldsymbol{b}$ is defined as a new 3-vector, $\boldsymbol{c} = \{a_1+b_1, a_2+b_2, a_3+b_3\}$. Similarly, the scalar multiplication between a scalar k and a vector $\boldsymbol{a}$ is defined as a new vector $\boldsymbol{d}=\{ka_1, ka_2, ka_3\}$.

7. **Definition:** Consider a set V containing some elements on which operations of **vector addition** and **scalar multiplication** are defined. The set V is called a **vector space** if the following ten properties are satisfied:

DEFINITION 7.5 Vector Space

Let V be a set of elements on which two operations called **vector addition** and **scalar multiplication** are defined. Then V is said to be a **vector space** if the following ten properties are satisfied.

Axioms for Vector Addition

(*i*) If $\mathbf{x}$ and $\mathbf{y}$ are in V, then $\mathbf{x} + \mathbf{y}$ is in V.
(*ii*) For all $\mathbf{x}, \mathbf{y}$ in V, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. (commutative law)
(*iii*) For all $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in V, $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$. (associative law)
(*iv*) There is a unique vector $\mathbf{0}$ in V such that $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{0}$. (zero vector)
(*v*) For each $\mathbf{x}$ in V, there exists a vector $-\mathbf{x}$ such that $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$. (negative of a vector)

Axioms for Scalar Multiplication

(*vi*) If k is any scalar and $\mathbf{x}$ is in V, then $k\mathbf{x}$ is in V.
(*vii*) $k(\mathbf{x} + \mathbf{y}) = k\mathbf{x} + k\mathbf{y}$
(*viii*) $(k_1 + k_2)\mathbf{x} = k_1\mathbf{x} + k_2\mathbf{x}$ (distributive laws)
(*ix*) $k_1(k_2\mathbf{x}) = (k_1k_2)\mathbf{x}$
(*x*) $1\mathbf{x} = \mathbf{x}$

8. Consider the 3-space, R^3. As mentioned, this a vector space. Can you justify this claim by referring to the definition as given?

Ans:

This is a vector space because (*i*) vector addition and scalar multiplication are well defined on all of the 3-vectors, the elements in R^3, (*ii*) all of the 3-vectors, the elements in R^3, fulfill the 10 axioms. In particular, all 3-vectors are closed under vector addition and closed under scalar multiplication.

9. Explain what do you understand by (*i*) 'closure under vector addition'. (*ii*) 'closure under scalar multiplication'.

10. Consider R^2. Is it also a vector space? How about the set of all real number, the 1-space, R? How do you convince yourself that they are indeed also vector space?

11. **Definition:** A set of vectors V_s from a vector space V is a **subspace** of V if V_s is closed under addition and scalar multiplication.

Example: The set containing only the element 0, V_s={0}, is a subspace of the vector space R, since the {0} is

(i) a element vector from R,

(ii) closed under scalar multiplication:

$k \cdot 0 = 0 \in V_s$,

(iii) closed under vector addition:

$0 + 0 = 0 \in V_s$.

Note that the subspace {0} has only a single element. The criteria of being closed under addition are fulfilled: "if x and y are element is V_s, then $x + y$ is also an element in V_s". Here, x=0, y=0, because there is no any other element in V_s other than 0. In other words, ‘any element’ in {0} (the *x*), when vectorially added to ‘any element’ in {0} (the *y*) will result in $x+y=0$, an element of V_s.

12. Every vector space *V* has at least two subspaces. One of if is the zero subspace, {**0**}, which is illustrated above. Can you think of what’s the other one?

13. **Definition:** Consider a set *S* containing vectors $\mathbf{x}_1$, $\mathbf{x}_2$, …$\mathbf{x}_m$ in a vector space *V*. (To help you visualize better, think of *V* as the vector space of R^3 that contains an infinite number of 3-vectors. Think of *S* as a set containing, say, m=3 vectors selected from R^3.) We form linear combinations of these *m* vectors in the form of $k_1\mathbf{x}_1+k_2\mathbf{x}_2+\ldots k_m\mathbf{x}_m$, where k_i are scalars. The set of all linear combinations of the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, …$\mathbf{x}_m$ is called the **span** of the vectors, and is written as **Span(*S*)**.

14. Span(*S*) is a subspace of *V*. Span(*S*) is said to be a subspace spanned by the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, …$\mathbf{x}_n$.

15. If every vector in the vector space *V* can be written as a linear combination of the vectors in *S*, then *S* is called a **spanning set** for *V*.

Example: Let *V* be the vector space containing all 3-vectors, R^3. Consider the set *S*={**i**, **j**, **k**} containing the three rectangular unit vectors. The set of all linear combination $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, where *a*, *b*, *c* are scalar, is the span of the vectors {**i**, **j**, **k**}, Span(*S*). Span(*S*) is a subspace in R^3 spanned by **i**, **j**, **k**.

16. We say ‘the set *S*={**i**, **j**, **k**} is a spanning set for R^3’. Think of Span(*S*) in terms of the set of all possible linear combination in terms of **i**, **j**, **k**, $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$. Can you imagine what does Span(*S*) represent? *Hint*: Imagine the point at the tip of the 3-vector $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$. Imagine the pervasive cloud form by the tip of $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ when *a*, *b*, *c* vary continuously.

17. Can you think of any other spanning set for R^3?

Ans: e.g. {**r**, θ, ϕ}.

18. Is the set {**i**, **j**, **k**, **i**+**j**, **k**+**i**, **i** + **k**} also spanning set for R^3?

19. Is {**i**, **j**} a spanning set for R^3? Explain your answer.

20. Consider the set S containing the following 4 3-vectors: K_1=[1,1,1]^T, K_2=[1,3,5]^T, K_3=[1,5, 3]^T, K_4=[5, 3, 1]^T; S={K_1, K_1, K_3 K_4}. How would you prove that the S is the spanning set of R^3 (or in other words, S span R^3)? *Hint:* To prove that the set of vectors in S span R^3, one needs to prove the existence of the solution X=[x_1, x_2, x_3, x_4]^T for the non-homogeneous equation system $A = x_1K_1 + x_2K_2 + x_3K_3 + x_4K_4$, given an arbitrary 3-vector A=$(a,b,c)^T$ from R^3**.** If the solution X exists, then S spans R^3, otherwise it doesn't. The reasoning is: If the solution X exists, this means that any arbitrary vector A from R^3 can always be expressed as a unique linear combination in the form of $x_1K_1 + x_2K_2 + x_3K_3 + x_4K_4$. Hence, by definition, if the set of vectors in S is a spanning set of R^3.

21. In general, given a set of m n-vectors, K_i=(k_1, k_2,…, k_n)^T , i=1,2,…m, we can determines whether they span a vector space R^n, the vector space containing the set of all n-vector by looking for the existence proof of solution X to the non-homogeneous system. The procedure is as followed:

22. Let K=$(K_1, K_2,..., K_m)$, an n by m matrix, X=$(x_1, x_2,..., x_m)^T$, an m by 1 column vector, A=$(a_1, a_2,..., a_n)^T$, an arbitrary n-vector in R^n. Consider the NH system $A = x_1K_1 + x_2K_2 + \cdots + x_mK_m = KX$. If the solution for the NH systems does not exist, i.e. rank[K] ≠ rank [K | A], then the set of vectors K_i does not span R^n. Otherwise, they do.

23. In (20), we see that the set $\{K_1, K_2, K_3, K_4\}$ comprises of 4 3-vector spans R^3. Can we span R^3 with less than 4 3-vectors (e.g., say, 3 or even 2 3-vectors)? In general, for a vector space *V* containing elements made up of *n*-vectors, we want to know what is the smallest number of linearly independent *n*-vector that spans the vector space *V*.

24. In fact, out of the four 3-vectors in the set *S* in (20), only three are linearly independent (refer DQ 13, Chapter 9), namely K_1, K_2, K_3, whereas K_4 can be expressed as a linear combination of the other three vectors.

    (*i*) Prove the linearly independence of the vector set K_1, K_2, K_3. (*Hint:* Refer to DQ 12, 13, 14 in Chapter 9.)

    (*ii*) Prove, using the procedure mentioned in (22) above, that this set of vectors K_1, K_2, K_3 spans R^3.

25. Now, we ask: can any of the 2 vectors (which are necessarily linearly independent) form the set $\{K_1, K_2, K_3\}$ span R^3? The answer can be proven to be negative. (Prove this). So, it appears that the minimum number of linearly independent 3-vectors to span R^3 is 3, not 2.

26. **Definition:** The minimum number of linearly independent vectors that is required to span a vector space is called the **dimension** of the vector space. In the above example, the dimension of the vector space R^3 is 3 since the minimum number of linearly independent vectors in R^3 is 3.

27. **Definition:** Consider a vector space *V* with dimension *r*. A set of *r* linearly independent vectors in *V* is called the **basis** (or basis set) of the vector space. It happens that given any set of *r* vectors, which are linearly independent, from *V*, they (*i*) will form a basis set for *V*, and (*ii*) any vector in *V* can be expressed as a unique linear combination in this set of *r* vectors.

28. Let's consider the vector space R^3. We know that the dimension of it is r=3.

(i) If I simply pick any three vectors in R^3, say $X_1 = (a, b, c)$, $X_2 = (d, e, f)$, $X_3 = (g, h, i)$, in general, will the set $\{X_1, X_2, X_3\}$ form a basis for R^3?

(ii) Is the basis set of R^3 unique?

(iii) How many basis set can R^3 possibly has?

29. Consider the set of three vectors in R^3, $S=\{E_1, E_2, E_3\}$, where $E_1= [1, 0, 0]^T$, $E_2= [0, 1, 0]^T$, $E_3= [0, 0, 1]^T$.

(i) Are the vectors in S linearly independent (you should be able to answer this simply question by visual inspection)?

(ii) Do the vectors in the set S form a basis set for R^3?

(iii) Do the vectors in the set S span R^3?

(iv) Can every vector in R^3 be expressed as linear combination of E_1, E_2, E_3?

(v) What's the name of these E-vectors? (*Hint: see page 88 of Ayers).* (Note: we will refer this basis set by the name 'the E-basis').

30. You may like to refer to Ayers page 88. Say I have an arbitrary vector in R^3, $X=(a, b, c)^T$.

(i) Write X as a linear combination of the unit vectors, E_i, defined in (30).

(ii) What are the components (or referred to as 'coordinates') of X relative to the E-basis? Write these components in the form of a column vector and call it 'the component vector of X relative to the E-basis', denoted by X_E.

31. In the previous question, we have an arbitrary vector in R^3, X. Let's say that the vector X when expressed in the E-basis is represented by the component vectors $X_E=(1, 2, 3)^T$. Normally, a vector is by default expressed in the E-basis. In general, other than the E-basis, we can also represent a vector in other basis set. To illustrate this point, let's consider another basis set $Z = \{Z_1, Z_2, Z_3\}$ ('the Z-basis'), where $Z_1= [2, -1, 3]^T$, $Z_2= [1, 2, -1]^T$, $Z_3= [1, -1, -1]^T$. What is the component vector of X relative to the Z-basis, X_Z? [*Hint:* In order to obtain X_Z, you need

to express X as a linear combination of $\{Z_1, Z_2, Z_3\}$: $X_E = a_1Z_1 + a_2Z_2 + a_3Z_3$. Then the component vector of X in the Z-basis is simply $X_Z = (a_1, a_2, a_3)^T$.]

32. Refer to Example 5, page 88 Ayers. Now, see if you can do things another way round: If the component vector of X is given in the Z representation, i.e. $X_Z = (1,2,3)^T$ is known. What is component vector of X in the E-basis? In other words, what is X_E? *Hint*: Follow the procedure as described in (32), then try to find a similar relation that relates X_E to X_Z in the form of

$$X_E = [\text{some matrix}]\cdot X_Z$$