Linearly independence

Definition: Consider a set of n-vectors, $\{K_1, K_2, \ldots, K_n\}$.

If there exist a set of coefficient $\{x_1, x_2, \ldots, x_n\} \neq \{0, 0, \ldots, 0\}$ such that $x_1 K_1 + x_2 K_2 + \ldots + x_n K_n = 0$, the set of vector $\{K_1, K_2, \ldots, K_n\}$ is not linearly independent.

In other words, if the only solution for $x_1 K_1 + x_2 K_2 + \ldots + x_n K_n = 0$ is $\{x_1, x_2, \ldots, x_n\} = \{0, 0, \ldots, 0\}$, then the set of vector $\{K_1, K_2, \ldots, K_n\}$ is linearly independent.

Example:
Consider the 3-vectors pair, $X_1^T = [1,2,3]$, $X_2^T = [-1,-2,-3]$.
Find any possible values of k_1 and k_2, with $\{k_1, k_2\} \neq \{0,0\}$, such that
$k_1 X_1^T + k_2 X_2^T = 0$. (ii) Are the vectors linearly dependent or linearly-independent ?

Ans: (i) Any arbitrary value of $k_1 = k_2 \neq 0$ will do. (ii).
They are linearly dependent, since there exist values of $\{k_1, k_2\} \neq \{0,0\}$ such that $X_1^T = -(k_2 / k_1) X_2^T$. Any pairs of vectors that are parallel or antiparallel is not linearly independent.

Note that when, say for example, the set of 3 vectors $\{K_1, K_2, K_3\}$ are not lienarly independent, any of the vector in the set can be expressed as linear combination of the rest, e.g.,
$x_1 K_1 + x_2 K_2 + x_3 K_3 = 0$
$\Rightarrow K_3 = -(x_1 K_1 + x_2 K_2)/x_3$.
In other words, K_3 is not a vector independent from the rest, K_2 and K_1 (because K_3 can be expressed in terms of K_1 and K_2)

Example of linearly independent set of vectors
$X_1^T = [1, 2, 3]$, $X_2^T = [4, 5, 6]$.
The set of two vectors $\{X_1, X_2\}$ is linearly independent.

For $k_1 X_1 + k_2 X_2 = [0, 0, 0]^T$, we need $k_1 + 4k_2 = 0$, $2k_1 + 5k_2 = 0$, $3k_1 + 6k_2 = 0$. The only possible solution is k_1, k_2 both being zero. This means that for $k_1 X_1 + k_2 X_2 = [0, 0, 0]^T$, the coefficients $\{k_1, k_2\}$ must be all zero. This proves the linearly independence of X_1 and X_1.

Refer (9.5) in page 69, Ayres. Given a set of m vectors, we want to know whether they are linearly independent or otherwise. What is the easiest way (or one of the easier ways) to determine the linear independence of such a set of vectors?

Ans: Use row elementary operations to reduce the matrix A formed by these vectors to RREF. The number of non-zero row in the RREF of A is the rank of the matrix A, r. The rank, r, also tells us how many linearly independent vectors are there in the set of m vectors.

If r = m, then the set of this m vectors is linearly independent. If $r < m$, then the set of m vectors is linearly dependent. See theorem V, page 69, Ayers.

Example 2.3, Prof. Rosy Teh's note on Linear Algebra, page 33.
Consider the set S containing the following 4 3-vectors: $K_1 = [1, 1, 1]^T$, $K_2 = [1, 3, 5]^T$,

$K_3^{□} = [1, 5, 3]^T, K_4^{□} = [5, 3, 1]^T$;
$S=\{K_1^{□}, K_2^{□}, K_3^{□}, K_4^{□}\}$. (i) Form the matrix K whose rows are made up of the vectors $K_i^{□}$, i=1, 2, 3, 4. (ii) Reduce K into RREF. (iii) What is the rank of K? (iv) How many linearly independent vectors are there in the set S? (v) Are the vectors in set S linearly independent?

Answer:

$K=(K_1^{□}\ K_2^{□}\ K_3^{□}\ K_4^{□})=\begin{pmatrix} 1 & 1 & 1 & 5 \\ 1 & 3 & 5 & 3 \\ 1 & 5 & 3 & 1 \end{pmatrix}$.

RREF(K)= $\begin{pmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Obviously, the rank of matrix K is r=3.

Since the number of vectors in S, m = 4 is larger than the number of linearly independent vectors, r = 3, the set of vectors in S is **NOT linearly independent** by the virtue of theorem V, page 69, Ayers.

In the above example, the set of vectors $S=\{K_1^{□}, K_2^{□}, K_3^{□}, K_4^{□}\}$ is proven to be not linearly independent. Now we ask another different question: is the set of vector $R=\{K_1^{□}, K_2^{□}, K_3^{□}\}$ linearly independent? To answer the question, we perform the following calculation:

$K1=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, K2=\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, K3=\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$

$K = (K1\ K2\ K3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 5 & 3 \end{pmatrix}$

m = number of vectors forming the matrix K = 3

$RREF[K] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Hence Rank [K] is r = 3
Since r = m = 3 the set of vectors {K1,K2,K3} is linearly independent, by the virtue of theorem V, page 69, Ayers.

Is the set of vector $R=\{K_1^{□}, K_2^{□}, K_4^{□}\}$ linearly independent? To answer the question:

$K1=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, K2=\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, K4=\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$

$K = (K1\ K2\ K4) = \begin{pmatrix} 1 & 1 & 5 \\ 1 & 3 & 3 \\ 1 & 5 & 1 \end{pmatrix}$

m = number of vectors forming the matrix K = 3

$RREF[K] = \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

Rank [K] is r = 2 ≠ m = 3
Hence the set of vector K is not linearly independent.

Is the set of vector $R=\{K_1^{□}, K_2^{□}\}$ linearly independent?

$K1=\begin{pmatrix}1\\1\\1\end{pmatrix}$, $K2=\begin{pmatrix}1\\3\\5\end{pmatrix}$

$K = (K1\ \ K2) = \begin{pmatrix}1 & 1\\1 & 3\\1 & 5\end{pmatrix}$

m = number of vectors forming the matrix K = 2

$RREF[K] = \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\end{pmatrix}$

Hence Rank [K] is r = 2

Since r = m = 2, the set of vectors in K is linearly independent

The span of the vector space $V_3(R)$ **by a set of 4 3-vectors** (Refer to example 2.2, page 33, Prof. Rosy Teh's lecture note on Linear Algebra)

Consider the same set containing 4 3-vectors: $K_1=[1, 1, 1]^T$, $K_2=[1, 3, 5]^T$, $K_3=[1, 5, 3]^T$, $K_4=[5, 3, 1]^T$. Does the vectors $\{K_1, K_2, K_3, K_4\}$ span the vector space $V_3(R)$?

In other words, can any arbitray vector living in $V_3(R)$ be expressed as a linear combination of $\{K_1, K_2, K_3, K_4\}$? If yes, $\{K_1, K_2, K_3, K_4\}$ is said to span $V_3(R)$, else, it does not.

Let an arbitrary vectors in $V_3(R)$ is denoted by a column vector with components a, b, c which are not all zero, $H=[a, b, c]^T$. Then we form a linear combination of $\{K_1, K_2, K_3, K_4\}$ and equate them to H,

$$x_1 K_1 + x_2 K_2 + x_3 K_3 + x_4 K_4 = H.$$

If putting the column vectors K_i into a matrix K,

$$K=(K_1\ K_2\ K_3\ K_4)=\begin{pmatrix}1 & 1 & 1 & 5\\1 & 3 & 5 & 3\\1 & 5 & 3 & 1\end{pmatrix},$$

then $x_1 K_1 + x_2 K_2 + x_3 K_3 + x_4 K_4 = H$ can be compactly expressed as a non- homogeneous equation in matrix form,

$$KX = H$$

where X is $\begin{pmatrix}x1\\x2\\x3\\x4\end{pmatrix}$.

The rank of the matrix K can be deduced from it RREF,

$$RREF[K] = \begin{pmatrix}1 & 0 & 0 & 6\\0 & 1 & 0 & -1\\0 & 0 & 1 & 0\end{pmatrix}$$, which tells us that Rank[K]=3.

So is the rank of the augmented matrix $[K \mid H]$ can be deduced from it RREF $[K \mid H]$,

$$\text{RREF}[K\mid H]=\begin{pmatrix} 1 & 0 & 0 & 6 & \frac{4a}{3}-\frac{b}{6}-\frac{c}{6} \\ 0 & 1 & 0 & -1 & -\frac{a}{6}-\frac{b}{6}+\frac{c}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{6}(-a+2b-c) \end{pmatrix}$$, which tells us that Rank[$K|H$]=3.

Hence the non-homogeneous equation $KX = H$ is consitent (i.e., non-trivial solutions X exist). This means the set of vectors spans $V_3(R)$. In other words, given any arbitrary vector $H = [a, b, c]^T$, it can always be expressed as a linear combanation based on the vectors $\{K_1, K_2, K_3, K_4\}$.

Note: The exact values of x_1, x_2, x_3, x_4 are not important here. What is important is that the non-trivial solution, X, exist.

Note: In the discussion of whether the set of vector $\{K_1, K_2, K_3, K_4\}$ span $V_3(R)$ we do not care whether the vectors $\{K_1, K_2, K_3, K_4\}$ are linearly independent. This is a different question at all, and has to be considered separately.

In the example above, the set of vectors $S=\{K_1, K_2, K_3, K_4\}$ is proven to span $V_3(R)$. We ask another different question: does the set of vector $R=\{K_1, K_2, K_3\}$ span $V_3(R)$? To answer the question:

$$K1=\begin{pmatrix}1\\1\\1\end{pmatrix},\ K2=\begin{pmatrix}1\\3\\5\end{pmatrix},\ K3=\begin{pmatrix}1\\5\\3\end{pmatrix}$$

$$K = (K1\ K2\ K3)=\begin{pmatrix}1&1&1\\1&3&5\\1&5&3\end{pmatrix}$$

$$H \text{ is } \begin{pmatrix}a\\b\\c\end{pmatrix}, X \text{ is } \begin{pmatrix}x1\\x2\\x3\end{pmatrix}.$$

$$KX=H=\begin{pmatrix}1&1&5\\1&3&3\\1&5&1\end{pmatrix}\begin{pmatrix}x1\\x2\\x3\end{pmatrix}=\begin{pmatrix}x1+x2+5\,x3\\x1+3\,x2+3\,x3\\x1+5\,x2+x3\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}.$$

$$\text{RREF}[K]=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\Rightarrow \text{Rank}[K]=3.$$

$$\text{RREF}[K\mid H]=\begin{pmatrix} 1 & 0 & 0 & \frac{4a}{3}-\frac{b}{6}-\frac{c}{6} \\ 0 & 1 & 0 & -\frac{a}{6}-\frac{b}{6}+\frac{c}{3} \\ 0 & 0 & 1 & \frac{1}{6}(-a+2b-c) \end{pmatrix}\Rightarrow \text{Rank}[K|H]=3.$$

Hence the non-homegeneous system K X = H is consistent.
There is non-trivial solution X=[x1,x2,x3]^T such that any arbitray vector H can be expressed as a linear combination of K1,K2 and K3, H = K.X. By definition, the set of vectors {K1, K2, K3} spans S=V_3(R).

In the example above, the set of vectors $S=\{K_1, K_2, K_3\}$ is proven to span $V_3(R)$. We ask another different question: does the set of vector $R=\{K_1, K_2, K_4\}$ spans $V_3(R)$? To answer the question:

$$K = (K1\ K2\ K4) = \begin{pmatrix} 1 & 1 & 5 \\ 1 & 3 & 3 \\ 1 & 5 & 1 \end{pmatrix}$$

$$H \text{ is } \begin{pmatrix} a \\ b \\ c \end{pmatrix}, X \text{ is } \begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix}$$

$$K\ X = H = KX = \begin{pmatrix} x1 + x2 + 5\,x3 \\ x1 + 3\,x2 + 3\,x3 \\ x1 + 5\,x2 + x3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$[K|H] = \begin{pmatrix} 1 & 1 & 5 & a \\ 1 & 3 & 3 & b \\ 1 & 5 & 1 & c \end{pmatrix},$$

$$RREF[K] = \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ Rank}[K] = 2.$$

$$RREF[K|H] = \begin{pmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{Rank } [K|H] = 3$$

"Hence the non-homegeneous system K X = H is NOT consistent because Rank[K|H] is not equal to Rank [K].

There is NO solution X=[x1,x2,x3]^T such that any arbitray vector H can be expressed as linear combination of K1,K2 and K3, H = K.X. By definition, the set of vectors {K1, K2, K4} does not spans S=V_3(R)."

Does the set of vector $R=\{K_1^{\square},K_2^{\square}\}$ span $V_3^{\square}(R)$? To answer this question:

$$K = \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{pmatrix}, X = \begin{pmatrix} x1 \\ x2 \end{pmatrix}, H = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$[K|H] = \begin{pmatrix} 1 & 1 & a \\ 1 & 3 & b \\ 1 & 5 & c \end{pmatrix}$$

$$RREF[K] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{Rank } [K] = 2$$

$$RREF[K|H] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{Rank } [K|H] = 3.$$

Hence the non-homegeneous system K X = H is NOT consistent because Rank[K|H] is not equal to Rank [K]. \nThere are no non-trivial solution for K X = H

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It is not possible to expressed H in terms of linear combination of K1, K2 in
  S=V_3(R). By definition, the set of vector {K1, K2} does not span V_3(R).
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Conclusion:

$\{K_1, K_2, K_3, K_4\}$ span $V_3(R)$, and is not linearly independent.
$\{K_1, K_2, K_3\}$ span $V_3(R)$, and is linearly independent.
$\{K_1, K_2, K_4\}$ does not span $V_3(R)$, and is not linearly independent.
$\{K_1, K_2\}$ does not span $V_3(R)$, and is linearly independent.

Definition: The minimum number of linearly independent vector to span a space is the dimension of the vector space. In the above example, the vector space $V_3(R)$ has a dimension of 3, because that is the minimum of linearly independent vectors that is required to span it.

Definition: Consider a vector space V with dimension r. A set of r linearly independent vectors in V is called the basis (or basis set) of the vector space. It happens that given any set of r vectors, which are linearly independent, they (i) will form a basis set for V, and (ii) any vector in V can be expressed as a unique linear combination in this set of r vectors.

$\{K_1, K_2, K_3\}$ are basis vectors in $V_3(R)$, as any abitrary vector H in $V_3(R)$ can be expressed as a linear combination of $\{K_1, K_2, K_3\}$. There are many other sets of basis vector in $V_3(R)$, other than $\{K_1, K_2, K_3\}$, as long as the set of vectors is linearly independent and spans $V_3(R)$. For example, the set of vectors $\{[1, 0, 0]^T, [0, 1, 0]^T, [0, 0, 1]^T\}$ is a good example of basis vectors. $\{K_1, K_2, K_4\}$, $\{K_1, K_2\}$, $\{K_1, K_2, K_3, K_4\}$, according to the definition, are not basis vectors in $V_3(R)$.