ZCA 110
Linear Algebra
Chapter 12
Basis and coordinates

Please refer to Chapter 11, page 88-89, Ayres.

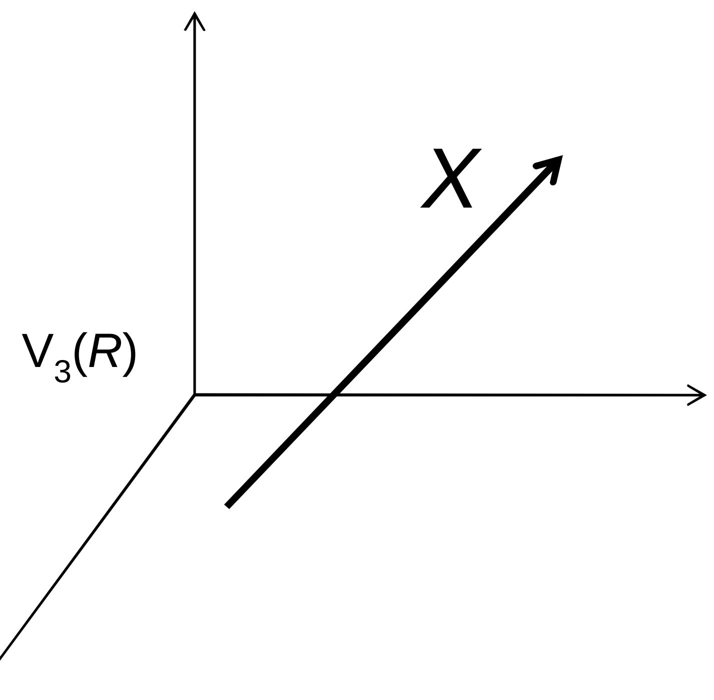
We will discuss how to relate a 3-vector in the vector space $V_3(R)$ in one basis to another basis.

- The description presented in these slides can be trivially generalised to general case of a *n*-vector living in a real $V_n(R)$ space.

A 3-vector living in $V_3(R)$

- Consider a 3-vector X in the real 3-space $V_3(R)$
- This is a mathematical object quantity with three component, expressed in the form of a column vector

$$X = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$



Basis

- Any vector in $V_3(R)$ can be expressed in any basis
- Consider an arbitrary basis, which shall name it as the *W*-basis
- The *W*-basis is comprised of three basis vector
- $\{W_i\} = \{W_1, W_2, W_3\}$, where each W_i is a 3-component column vector

W-basis

$$\{W_i\} = \{W_1, W_2, W_3\}$$

$$W_1 = \begin{pmatrix} w_{11} \\ w_{21} \\ w_{31} \end{pmatrix}, W_2 = \begin{pmatrix} w_{12} \\ w_{22} \\ w_{32} \end{pmatrix}, W_3 = \begin{pmatrix} w_{13} \\ w_{23} \\ w_{33} \end{pmatrix}$$

Vector *X* represented in *W*-basis

- In the W-basis an arbitrary vector *X* is represented as

$$X = a_1 W_1 + a_2 W_2 + a_3 W_3$$

- a_1, a_2, a_3 are the components of the vector *X* in the *W*-basis

- We put all the components a_i in a column vector called

"the coordinate vector of *X* relative to the W-basis"

$$X_W = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Vector *X* represented in *Z*-basis

- Likewise, the same vector *X* can also be represented in other basis, say, a *Z*-basis

$$X = b_1 Z_1 + b_2 Z_2 + b_3 Z_3$$

- b_1, b_2, b_3 are the components of the vector *X* in the *Z*-basis

- We put all the components b_i in a column vector called

"the coordinate vector of *X* relative to the *Z*-basis"

$$X_Z = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

What are the questions we want to answer?

- Given the knowledge of a vector in a specific basis, we want to know what is the representation of the same vector in another basis.

What are the questions we want to answer? (cont.)

- For example, if we know X_Z, and the bases $\{Z_i\}$, we want to know what is X_W.

- Usually, the bases in questions, $\{Z_i\}$ and $\{W_i\}$ are known.

How to relate X_Z to X_w (and vice versa)

$$X = \underbrace{b_1 Z_1 + b_2 Z_2 + b_3 Z_3}_{\text{representation in } Z\text{-basis}} = \underbrace{a_1 W_1 + a_2 W_2 + a_3 W_3}_{\text{representation in } W\text{-basis}}$$

$$= \underbrace{\begin{pmatrix} Z_1 & Z_2 & Z_3 \end{pmatrix}}_{z} \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{X_Z} = \underbrace{\begin{pmatrix} W_1 & W_2 & W_3 \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}}_{X_W}$$

X_Z is related to X_w via the "matrices of the bases", W and Z

$$X = ZX_Z = WX_W$$

$$\underbrace{Z}_{\text{a 3 by 3 matrix}} = \begin{pmatrix} Z_1 & Z_2 & Z_3 \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix};$$

$$\underbrace{W}_{\text{a 3 by 3 matrix}} = \begin{pmatrix} W_1 & W_2 & W_3 \end{pmatrix} = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}$$

$$X_Z = Z^{-1}WX_W \qquad\qquad X_W = W^{-1}ZX_Z$$

Example
(Example 5, Ayres pg 88)

If $Z_1=(2,-1,3)^T$, $Z_2=(1,2,-1)^T$, $Z_3=(1,-1,-1)^T$ is a basis of $V_3(R)$

W is the E-basis (elemental basis), $W_1=(1,0,0)^T$, $W_2=(0,1,0)^T$, $W_3=(0,0,1)^T$

If $X_Z=(1,2,3)^T$, then the component vector of X in the W-basis (E-basis), X_W is

$$X = ZX_Z = WX_W$$

$$\Rightarrow X_W = W^{-1}ZX_Z = IZX_Z = ZX_Z = \begin{pmatrix} Z_1 & Z_2 & Z_3 \end{pmatrix} X_Z$$

$$= \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 3 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix}$$

Conclusion for the previous example

The vector X in the Z-basis representation is given by $X_Z=(1,2,3)^T$

Whereas in the W-basis (which is actually the E-basis) representation the vector X is represented as $X_W=X_E=(7,0,2)^T$

Linear transformation

(Please refer to Chapter 12, Ayres)

Linear transformation

- Roughly speaking, linear transformation is an operation that maps a vector *X* into its image *Y* via a transformation matrix, *A*

$$X \xrightarrow{A} Y$$

- Operationally,

$$Y = AX$$

Y is called "the image of *X* under transformation *A*"

Transformation expressed in different bases

- As discussed in earlier slides, a vector must specified with respect to a basis.
- So is the transformation relation

In W-basis,

$$Y_W = A_W X_W$$

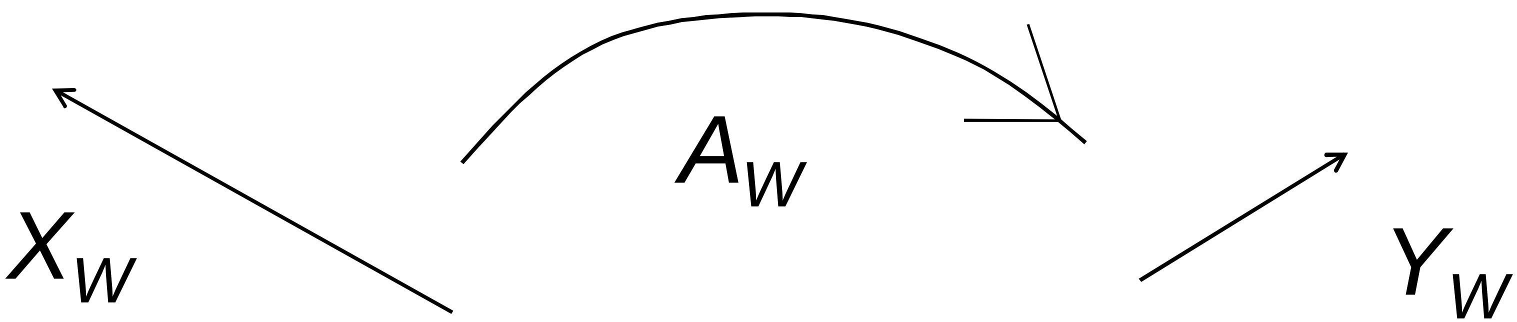
In Z-basis,

$$Y_Z = A_Z X_Z$$

Transformation expressed in *W*-basis

$$Y_W = A_W X_W$$

- X_W is the component vector of X in W-basis
- Y_W is the component vector of Y in W-basis
- X_W is the "original" vector, and Y_W is the image of X_W under the transformation A_W



A note of caution

Note that we are actually talking about the *same* operation A on X **but** NOT two different operations

A_W and A_Z are the same operation expressed in different bases

Both A_W and A_Z are non-identical matrices, but their effect on X is the same: both map X to Y in their respective basis.

A_W and A_Z are related via a similarity transformation

$$A_Z = Q^{-1} A_W Q$$

$$A_W = Q A_Z Q^{-1}$$

where $Q = W^{-1} Z, \quad Q^{-1} = Z^{-1} W$

Proof of $A_W = Q A_Z Q^{-1}, Q \equiv W^{-1} Z$

$$Y_Z = A_Z X_Z$$

$$\text{LHS}: Y_Z = \left(Z^{-1}W\right) Y_W; \qquad \text{RHS}: A_Z X_Z = A_Z\left[\left(Z^{-1}W\right) X_W\right]$$

$$\text{LHS} = \text{RHS:}$$

$$Z^{-1}WY_W = A_Z Z^{-1} W X_W$$

$$Y_W = \left(W^{-1} Z A_Z Z^{-1} W\right) X_W$$

Compare this to $Y_W = A_W X_W$,

$$A_W \equiv \left(W^{-1}Z\right) A_Z \left(Z^{-1}W\right) = Q A_Z Q^{-1} \Leftrightarrow A_Z = Q^{-1} A_W Q,$$

$$Q \equiv W^{-1} Z$$

How to remember the similarity transformation

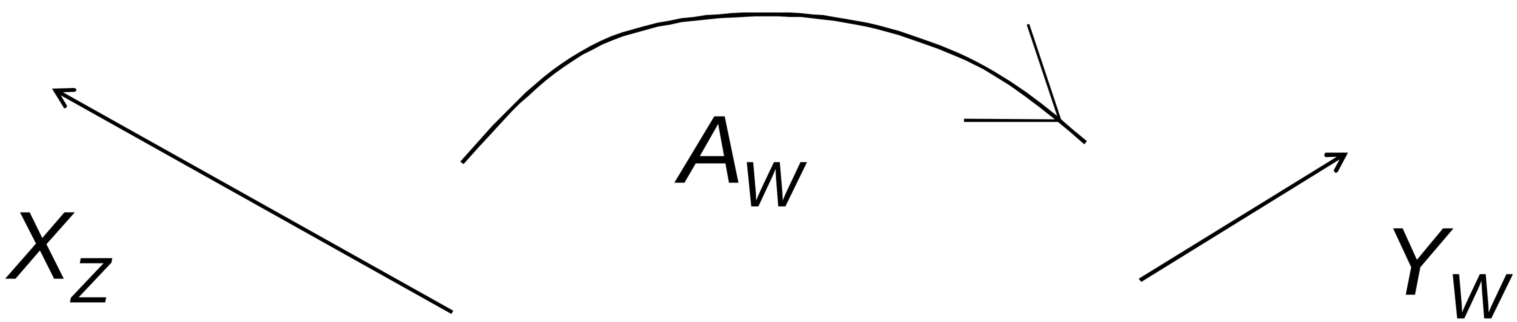
$$A_W = QA_Z Q^{-1}, Q \equiv W^{-1} Z$$

$$A_W = W^{-1} \underbrace{Z A_Z Z^{-1}} W \xrightarrow{X \leftrightarrow Z} A_Z = Z^{-1} \underbrace{W A_W W^{-1}} Z$$

Transformation expressed in Z-basis

$$Y_Z = A_Z X_Z$$

- X_Z is the component vector of X in Z-basis
- Y_Z is the component vector of Y in Z-basis
- X_Z is the “original” vector, and Y_Z is the image of X_Z under the transformation A_Z



What are the questions we want to answer?

- Usually in a given question, the bases $\{Z_i\}$, $\{W_i\}$ are known quantities.
- We are given (*i*) a transformation matrix A in a given basis, say, the Z-basis (i.e., A_Z is know), and (*ii*) an 'original vector' X in a basis, say X_Z.
- The question usually involves the calculation for

(1) Y_Z

(2) X_W

(3) A_W

(4) Y_W .

Ayres, Chapter 12, example 2, page 96

Given $A_Z = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 3 & -1 & -1 \end{pmatrix}$ relative to the Z-basis (which is the E-basis)

$$Z_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, Z_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, Z_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let the W – basis be $W_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, W_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, W_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

(a) Given the vector $X_Z = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ in the Z-basis, find the coordinate

of its image relative to the W-basis

Solutions

$$X_Z = \begin{pmatrix} 3 & 0 & 2 \end{pmatrix}^T$$

$$A_Z = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 3 & -1 & -1 \end{pmatrix}, Z_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, Z_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, Z_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad W_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, W_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, W_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

(a) Find Y_W :

$$Y_Z = A_Z X_Z = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$$

$$Y_W = W^{-1} Z Y_Z = W^{-1} Y_Z$$

$$W = \begin{pmatrix} W_1 & W_2 & W_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{pmatrix} \Rightarrow W^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{pmatrix}^{-1} = \frac{1}{9} \begin{pmatrix} 3 & 3 & 0 \\ 1 & 2 & 3 \\ 5 & -1 & -3 \end{pmatrix}$$

$$Y_W = W^{-1} Y_Z = \frac{1}{9} \begin{pmatrix} 3 & 3 & 0 \\ 1 & 2 & 3 \\ 5 & -1 & -3 \end{pmatrix} \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 14/3 \\ 20/9 \\ 19/9 \end{pmatrix}$$

(b) Find the linear transformation $Y_W = A_W X_W$ corresponding to $Y_Z = A_Z X_Z$.

Solution:

$$A_W = W^{-1} Z A_Z Z^{-1} W = W^{-1} A_Z W = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 3 & 3 & 0 \\ 1 & 2 & 3 \\ 5 & -1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 36 & 21 & -15 \\ 21 & 10 & -11 \\ -3 & 23 & -1 \end{pmatrix}$$

Hence the tranformation relative to the W-basis is

$$Y_W = \frac{1}{9} \begin{pmatrix} 36 & 21 & -15 \\ 21 & 10 & -11 \\ -3 & 23 & -1 \end{pmatrix} X_W .$$

(b) Use the result in (c) to find the image Y_W of $X_W = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$.

Solution:

$$Y_W = A_W X_W = \frac{1}{9}\begin{pmatrix} 36 & 21 & -15 \\ 21 & 10 & -11 \\ -3 & 23 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 7 \end{pmatrix}$$