## Chapter 4 Evaluation of determinants

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

> Consider the operation "add to row *j* by (*k* × row *i*)", $j \neq i$, where *k* is a non-zero scalar. We will conveniently represent the above operation by $R_j^i(k)$, and we have $A \xrightarrow{R_j^i(k)} A' = R_j^i(k)A$.
> Sometimes this operation is symbolized by {*j*} → {*j*} + *k*{*i*}, meaning: "replace row *j* by (row *j* + *k* times row *i*)". In Ayer's notation, this operation is denoted by $H_{ij}(k)$. We will adopt the $R_j^i(k)$ notation for the rest of the course. There is also a similar operation that acts on the columns, denoted by $C_j^i(k)$.

1. Given $A=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 9\\ 8 & 10 & 6\end{pmatrix}$, what is $A' = R_3^2(-2)A$?

Ans:

$$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 9\\ 8 & 10 & 6\end{pmatrix} \xrightarrow{R_3^2(-2)}$$

$$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 9\\ 8-2(4) & 10-2(5) & 6-2(9)\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 9\\ 0 & 0 & -12\end{pmatrix}$$

$= A'$

2. (*i*) What is the determinant of *A* in (1)? (*ii*) That of $A'$? (*iii*) What is your conclusion?

**Ans:**

(*i,ii*) $|A| = |A'| = -12 - 12\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix} = -12(5-8) = 36$;

(*iii*) If two matrices are related by an operation of $R_j^i(k)$, their determinant equals.

3. If you understand the idea intended to be conveyed in (1), what is your strategy if you were asked to evaluate $\begin{vmatrix}1 & 66 & 54\\ 4 & 11 & 9\\ 8 & 10 & 43\end{vmatrix}$? Inspect the values of the elements then think of the best strategy possible.

**Ans:**

One would first try to see if there is any way to reduce a row in the determinant to as many zeroes as possible in $\begin{vmatrix}1 & 66 & 54\\ 4 & 11 & 9\\ 8 & 10 & 43\end{vmatrix}$. One may like to operate $R_1^2(-6)$ so that the second row becomes (-23, 0, 0). Then the determinant becomes easily evaluated.

4. How about evaluating $\begin{vmatrix}1 & 4 & 8\\ 7 & 11 & 10\\ 7 & 8 & 16\end{vmatrix}$?

**Ans:**

Try to transform either column 2 or 3 of the determinant into one containing two zeroes. E.g., carry out the operation $C_3^2(-2)$ so that the third column becomes $(0, -12, 0)^T$. Then the transformed determinant $\begin{vmatrix}1 & 4 & 0\\ 7 & 11 & -12\\ 7 & 8 & 0\end{vmatrix}$ becomes easily evaluated:

$$\begin{vmatrix}1 & 4 & 0\\ 7 & 11 & -12\\ 7 & 8 & 0\end{vmatrix} = (-1)^{2+3}(-12)\begin{vmatrix}1 & 4\\ 7 & 8\end{vmatrix} = 12(-20) = -240$$

**5.** Given $A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$. Carry out successive operations of $R_j^i(k)$ to transform $A$ into a matrix which has as many zero elements as possible in row 3. Call the resultant matrix $B$. (*i*) What is your resultant matrix, $B$? (*ii*) Is $A$ and $B$ equivalent? (*iii*) Does $A$ and $B$ have the determinant? Justify your answer. (*iv*) Evaluate $|A|$.

**Ans:**

**(*i*)**

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\xrightarrow{R_3^1(-7)}$$

$$\begin{pmatrix}1&2&3\\4&5&6\\7-7(1)&8-7(2)&9-7(3)\end{pmatrix}=\begin{pmatrix}1&2&3\\4&5&6\\0&-6&-12\end{pmatrix}$$

$$\xrightarrow{R_2^1(-4)}$$

$$\begin{pmatrix}1&2&3\\4-4(1)&5-4(2)&6-4(3)\\0&-6&-12\end{pmatrix}=\begin{pmatrix}1&2&3\\0&-3&-6\\0&-6&-12\end{pmatrix}$$

$$\xrightarrow{R_3^2(-2)}$$

$$\begin{pmatrix}1&2&3\\0&-3&-6\\0-2(0)&-6-2(-3)&-12-2(-6)\end{pmatrix}=\begin{pmatrix}1&2&3\\0&-3&-6\\0&0&0\end{pmatrix}.$$

(*ii*) Yes. (*iii*) One of the theorems in chapter 3 on the properties of determinant says so. (*iv*) Clearly, $|B|=0$ due to the last row of 3 zeros. Hence, $|A|=|B|=0$.