Chapter 5 Equivalence

Answer the following designed questions. These questions are designed in accordance to the subsections as sequentially presented in Ayers. Try to identify the questions below with the corresponding subsection from which these questions are based on as it will definitely help while answering these questions.

1. Consider an n-square non-zero matrix A. (i) What is the highest possible rank of matrix A? (ii) What is the smallest possible rank of A? (iii) What is the condition on A for it to assume the highest possible rank? (vi) What kind of matrix A is if it fulfils the condition in (iii)?

Ans:
(i) n; (ii) 1. (iii) when $|A| \neq 0$. (iii) non-singular.

2. In general, for an n-square non-zero matrix A, its first minors* are (i) ___-square minors (ii) Can you recognize what the *"n-square minors"* of A is?
(*We will refer "first minors" simply as "minors" in the future unless specify otherwise.)

Ans:
(i) (n-1)-square minors; (ii) the *"n-square minor"* of A is non other than the determinant of A, $|A|$.

3. Consider the 3-square matrix $Y=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{pmatrix}$. ($i$) The minors of Y are ____-square minors. (ii) What is the *"3-square minor"* of Y? (iii) What is the rank of Y? (iv) What is the 4-square minor of Y?

Ans:
(i) The minors of Y are "2-square" minors.
(ii) The *"3-square minor"* of Y is $|Y|=3\neq0$. (iii) The rank of Y is 3 (highest possible rank for a matrix of order 3). (iv) Since the order of A is n=3, the 4-square minor of A is not defined.

4. You may refer to Chapter 3, designed question (12).
Consider the 3-square matrix $X=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. ($i$) What is the value of the *"3-square minor"* of X? (ii) How many "2-square minors" are there in X? (iii) Is ALL of the "2-square minors" zero? (iv) What is the rank of X?

Ans: (i) $|X|$= 0. (ii) There are 9 of them. (iii) The matrix of "2-square minors" of Y is $[M_{ij}]$= $\begin{pmatrix} -3 & -6 & -3 \\ -6 & -12 & -6 \\ -3 & -6 & -3 \end{pmatrix}$. Not all of them are zero. (iv) The rank of X is r = 2.

5. Consider the 3-square matrix $Z=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$. ($i$) What is the value of the *"3-square minor"* of Z? (ii) How many "2-square minors" are there in Z? (iii) Is ALL of the "2-square minors" zero? (iv) How many "1-square minors" are there in Z? (v) Is ALL of the "1-square minors" zero? (vi)What is the rank of X?

Ans:
(i) $|Z|$=0.
(ii) There are 9 of them.
(iii) The matrix of "2-square minors" of Y is $[M_{ij}]$= $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. ALL of the 2-square minors are zero.
(iv) There are 9 "1-square minors" in Z. (v) The matrix of "1-square minors" of Z is $\begin{pmatrix} |1| & |2| & |3| \\ |1| & |2| & |3| \\ |1| & |2| & |3| \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.
(v) Not all of the "1-square minors" of Z is zero. (vi) rank of Z is r = 1.

6. Consider matrix X as defined in (4). What is X', the image of X when it is transformed under the operations

$(i)\,R_2^1$ (ii) C_2^1 (iii) $R_2(3)$ $(iv)\,C_2(3)$

$(v)\,R_2^1(3)$ (vi) $C_2^1(3)$

7. (i) What do you call the operations in (6)?
(ii) What is the determinant of X' in each case in (6)?
(iii) What is the order of X' in each case in (6)?
(iv) What is the rank of X' in each case in (6)?

Ans:

(i) Elementary transformations.
(ii, iii, iv) Determinant, order and rank of a matrix remain unchanged under elementary transformations.

8. Given two examples of equivalent matrices to X as defined in (6).

Ans:
All of the examples in 6(i-vi) are equivalent matrices to X.

9. Let $A=\begin{pmatrix}1&3&0\\0&2&0\\0&0&1\end{pmatrix}$. Find a single row elementary transformation (let's call it $E_{A\to B}$) that transforms A into a 3-square diagonal matrix, B? We will use the notation $B=E_{A\to B}A$.

Ans:

$$A=\begin{pmatrix}1&3&0\\0&2&0\\0&0&1\end{pmatrix}\xrightarrow{R_1^2(-3/2)}\begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}=B$$

10. Find a single row elementary transformation (let's call it $E_{B\to I_3}$) that transforms B in (9) into the unit matrix I_3?

We will denote $I_3=E_{B\to I_3}A$.

Ans:

$$B=\begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}\xrightarrow{R_2(1/2)}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=I_3$$

11. If we define the operation $U=E_{B\to I_3}\,E_{A\to B}$, what will you get if U is operated on A? We will denote the resultant matrix as $V=UA$.

Ans:

$V=UA=E_{B\to I_3}(E_{A\to B}A)=E_{B\to I_3}B=I_3$.

12. Let $A=\begin{pmatrix}1&3&2\\0&2&0\\5&0&1\end{pmatrix}$. Transforms A into a 3-square diagonal matrix, call it D, by successively applying row elementary operations. I can achieve a 3-square diagonal matrix in three steps (I call it $R_{stp}=R_{stp3}\,R_{stp2}\,R_{stp1}$, so that $R_{stp}A$ is diagonal matrix.) How many steps you need, and what is your that R_{stp}?

Ans:

$$A=\begin{pmatrix}1&3&2\\0&2&0\\5&0&1\end{pmatrix}\xrightarrow{R_1^3(-2)}\begin{pmatrix}-9&3&0\\0&2&0\\5&0&1\end{pmatrix}\xrightarrow{R_1^2(-3/2)}$$

$$\begin{pmatrix}-9&0&0\\0&2&0\\5&0&1\end{pmatrix}\xrightarrow{R_3^1(5/9)}\begin{pmatrix}-9&0&0\\0&2&0\\0&0&1\end{pmatrix}=D.$$

Hence $R_{stp}=R_{stp3}\,R_{stp2}\,R_{stp1}=R_3^1(5/9)\,R_1^2(-3/2)\,R_1^3(-2)$.

13. Reduce the resultant diagonal matrix in (12) into a unit matrix by applying a Sequence of row elementary transformations. Call this operation R_d. What is your R_d?

Ans:

$$D=\begin{pmatrix}-9&0&0\\0&2&0\\0&0&1\end{pmatrix}\xrightarrow{R_1(-1/9)}\begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}\xrightarrow{R_2(1/2)}$$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Hence, $R_d = R_2(1/2)\, R_1(-1/9)$

14. Now, I will call the operation that transforms A, as in (12), into a 3-square matrix identity matrix $E_{A\to I_3}$.

    With this notation, we have $E_{A\to I_3} A = I_3$. Write down the form of $E_{A\to I_3}$ in terms of the operations R_{stp} and R_d as obtained in (12) and (13).

Ans:

$E_{A\to I_3} = R_d R_{stp}$

$= R_2(1/2)\, R_1(-1/9)\, R_3^1(5/9)\, R_1^2(-3/2)\, R_1^3(-2)$

15. "Canonical matrix" as mentioned in page 40, Ayers, is a general form of matrix that fulfills the set of properties (*a*)-(*d*) stated in the same page. A special case of canonical matrices are matrices in Row Reduce Echelon form (RREF). These are matrices that have the following properties:

1. Rows of all zeros, if there are any, appear at the bottom of the matrix.
2. The first nonzero entry of a nonzero row is 1. This is called a leading 1.
3. For each nonzero row, the leading 1 appears to the right and below any leading 1's in preceding rows.
4. Any column in which a leading 1 appears has zeros in every other entry.

A matrix in RREF appears as a staircase pattern of leading 1's descending from the upper left corner of the matrix. The columns of the leading 1's are columns of an identity matrix. A matrix is in row echelon form (REF) if properties 1, 2, and 3 above are satisfied.

Now, see if you can demonstrate your understanding on what RREF is by answering the following questions:

Given the following matrices, state whether they are in RREF, REF or neither.

$(i)\ \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (ii) \begin{pmatrix} 0 & 0 & 1 & -6 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$

$(iii) \begin{pmatrix} 1 & 5 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} (iv) \begin{pmatrix} 0 & 0 & 1 & -6 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$

$(v) \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 2 & 5 \end{pmatrix}.$

Ans: (*i*) RREF (*ii*) RREF (*iii*) REF (*iv*)REF (*v*) neither. However, under $R_3(1/2)$, matrix (*v*) will be reduced to REF.

16. Convert the above non-RREF matrices in to RREF via elementary row transformations.

Ans:

$(iii)\ R_1^2(-5) \begin{pmatrix} 1 & 5 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$(iv)\ R_1^2(-2) \begin{pmatrix} 0 & 0 & 1 & -6 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix} =$

$\begin{pmatrix} 0 & 0 & 1 & -6 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$

(*v*)

$\begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 2 & 5 \end{pmatrix} \xrightarrow{R_2^3(-5/2)} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & 0 & -13/2 \\ 0 & 0 & 2 & 5 \end{pmatrix}$

$\xrightarrow{R_1^3(1/2)} \begin{pmatrix} 1 & 2 & 0 & 3/2 \\ 0 & 1 & 0 & -13/2 \\ 0 & 0 & 2 & 5 \end{pmatrix} \xrightarrow{R_1^2(-2)}$

$$\begin{pmatrix} 1 & 0 & 0 & 29/2 \\ 0 & 1 & 0 & -13/2 \\ 0 & 0 & 2 & 5 \end{pmatrix} \xrightarrow{R_3(1/2)} \begin{pmatrix} 1 & 0 & 0 & 29/2 \\ 0 & 1 & 0 & -13/2 \\ 0 & 0 & 1 & 5/2 \end{pmatrix}$$

[See mathematica file for verification].

17. Consider $B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$. (*i*) Transform B into a row echelon form via elementary row transformations. (*ii*) What is the reduced row echelon form of B? (*iii*) Is the REF as obtained in (*ii*) the same as your friend's? (*iv*) Is the RREF as obtained in (*iii*) the same as your friend's? (*v*) What can you conclude from (*iv*) and (*v*)? After this exercise you should have learnt the trick of reducing any generic matrix into RREF form.

Ans:

(*i*) (non-unique)

(*ii*) Using mathematica,

`RowReduce[B]` $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(*iii*) No. (*iv*) Yes. (*v*) REF is not unique but RREF is.