Calculus Lecture Notes for ZCA 110

based on Thomas' Calculus, 11th Edition

by George B. Thomas, Maurice D. Weir, Joel Hass, Frank R. Giordano, Addison Wesley, 11th edition

prepared by

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School of Physics

Universiti Sains Malaysia

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PUSAT PENGAJIAN SAINS FIZIK
UNIVERSITI SAINS MALAYSIA

First Semester, 2016/17 Academic Session

COURSE DETAILS

Course name:	Calculus
Course code:	ZCA 110
Credit hours:	4 (i.e. 4 lectures per week for 14 weeks, plus tutorial sessions)

LECTURERS

- Three separate classes for ZCA 110 (Groups: A, B, and C) handled concurrently by three lecturers:

  A group: Dr. Norhaslinda Mohamed Tahrin (NMT)
  B group: Dr. Yoon Tiem Leong (YTL)
  C group: Dr. Wong Khai Ming (WKM)

COURSE DESCRIPTIONS

- A core course offered by School of Physics
- Course conducted in English, but the students can answer the final exam either in Bahasa Malaysia or English

Duration:	5^{th} September 2016 – 16^{th} December 2016
Semester Break:	24^{th} – 30^{th} October 2016
Meeting times:	Mon 12.00 noon – 12.50 pm Wed 9.00 – 9.50 am Thurs 11.00 – 11.50 am Fri 11.00 – 11.50 am
Pre-requisite:	None, BUT will assume that students are familiar with basic mathematics at STPM or Matrikulasi level (i.e. arithmetic of

addition, subtraction, division and multiplication; basic algebra, geometry, trigonometry, simple differentiation, and integration)

E-learn: For updates, announcements, assignments, etc.

CONTENTS

Preliminaries: Sets, real numbers, rational and complex numbers (read the Appendix section of Thomas' calculus)

The Calculus course offered by School of Physics covers the following topics:

- Functions, limits, and continuity
- Differentiation and its applications
- Integration, techniques of integration, and its applications
- Transcendental functions
- Sequences and series

OBJECTIVES

1. Differentiation: learn the different rules of differentiation, and its applications

2. Integration: learn the different techniques of integration, and its applications

3. To learn the calculus of transcendental functions, and the basic concepts on series

COURSE EXPECTATIONS

After completing this course, students should be:

- Well-versed in the so-called foundation mathematics that will be needed for numerous applications in physics

- Well-prepared for more advanced mathematics courses as well (e.g. ZCT 112/3, ZCT 210/4, ZCT 219/4, etc.)

CONSULTATION HOURS

Consult your respective lecturers for details.

ASSESSMENT

COMPONENTS	DESCRIPTION	WEIGHTAGE
Course work	Three (3) tests – 15% (at 5% each) Quizzes – 5% Assignments – 20%	40%
Final examination	Will cover all topics	60%
Attendance	• will be recorded • students missing tests without valid reasons/M.C. will get zero • students with attendance less than 70% will be barred from sitting for the final examination	
Total		100%

TESTS

	Dates	**Time**	**Venue**
Test 1	21st October 2016 (F)	11.00 – 12.00 noon	E41*
Test 2	28th November 2016 (M)	12.00 – 1.00 pm	E41*
Test 3	9th December 2016 (F)	11.00 – 12.00 noon	E41*

* Basement of PHS II (Adjacent to Eureka building)

Note: All students (A, B, C groups) will sit for the same tests and final examination. Topics covered will be announced later.

ASSIGNMENTS and TUTORIALS

- About eight (8) assignments to be completed by students throughout the course duration
- Students are required to submit them to the respective lecturers
- All assignments will be graded by the tutors
- Assignments received after the respective due date will not be graded (which means that you will get zero for that particular assignment)
- Tutorial sessions – each session is to be held during one of the usual lecture hours. Details of which will be announced later by your respective lecturers

REFERENCES

Main textbook

Thomas' Calculus Early Transcendentals, 11th Edition, G.B. Thomas, as revised by MD Weir, J Hass and F.R. Giordano, Pearson International Edition, 2008

Additional references

1. S.L. Salas, E. Hille, and G.J. Etgen, Calculus, John Wiley & Sons, New York, 9th Edition, 2003, John Wiley & Sons.

2. Edwards and Penny, Calculus, 6th Edition, 2002, Prentice Hall.

3. Gerald L. Bradley and Karl J. Smith, Calculus, 2nd Edition, 1999, Prentice Hall.

Enjoy! ☺

Chapter 1

Preliminaries

1.3

Functions and Their Graphs

Function

- $y = f(x)$
- f represents function (a rule that tell us how to calculate the value of y from the variable x
- x : independent variable (input of f)
- y : dependent variable (the correspoinding output value of f at x)

DEFINITION **Function**

A **function** from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

Definition Domain of the function

The set of D of all possible input values

Definition Range of the function

The set of all values of $f(x)$ as x varies throughout D

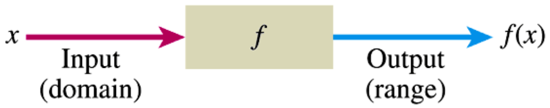


FIGURE 1.22 A diagram showing a function as a kind of machine.

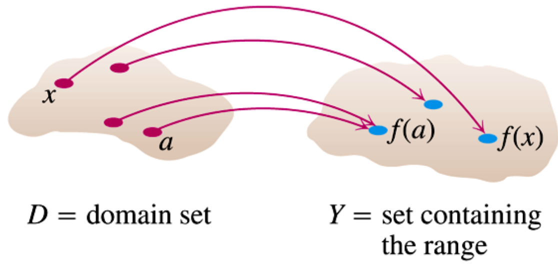


FIGURE 1.23 A function from a set D to a set Y assigns a unique element of Y to each element in D.

Natural Domain

- When a function $y = f(x)$is defined and the domain is not stated explicitly, the domain is assumed to be the largest set of real x-values for the formula gives real y-values.
- e.g. compare "$y = x^2$" c.f. "$y = x^2$, $x \geq 0$"
- Domain may be open, closed, half open, finite, infinite.

Verify the domains and ranges of these functions

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Graphs of functions

- Graphs provide another way to visualise a function
- In set notation, a graph is

$$\{(x,f(x)) \mid x \in D\}$$

- The graph of a function is a useful picture of its behaviour.

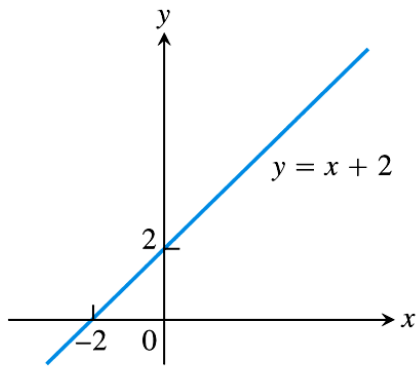


FIGURE 1.24 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

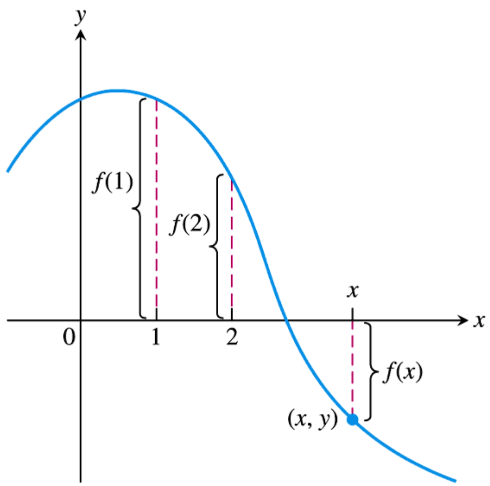


FIGURE 1.25 If (x, y) lies on the graph of f, then the value $y = f(x)$ is the height of the graph above the point x (or below x if $f(x)$ is negative).

Example 2 Sketching a graph

- Graph the function $y = x^2$ over the interval [-2,2]

The vertical line test

- Since a function must be single valued over its domain, no vertical line can intersect the graph of a function more than once.
- If *a* is a point in the domain of a function *f*, the vertical line *x*=*a* can intersect the graph of *f* in a single point (*a*, *f*(*a*)).

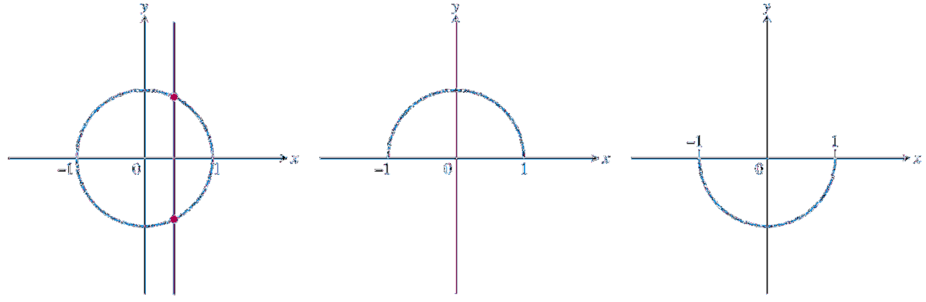


FIGURE 1.28 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of a function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of a function $g(x) = -\sqrt{1 - x^2}$.

Piecewise-defined functions

- The absolute value function

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

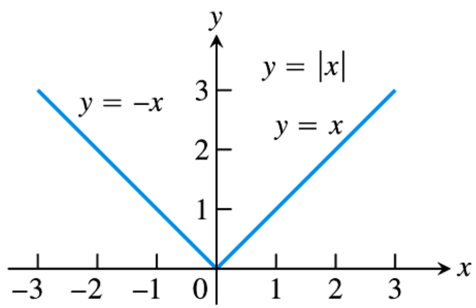


FIGURE 1.29 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

Graphing piecewise-defined functions

- Note: this is *just one function* with a domain covering all real number

$$f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

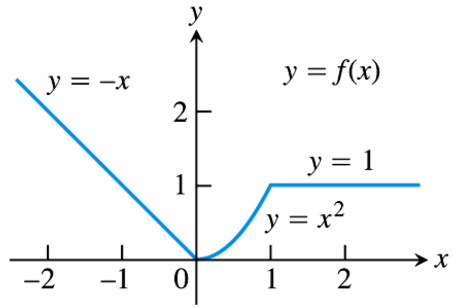


FIGURE 1.30 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 5).

The greatest integer function

- Also called integer floor function
- *f* = [*x*], defined as greatest integer less than or equal to *x*.
- e.g.
- [2.4] = 2
- [2]=2
- [-2] = -2, etc.

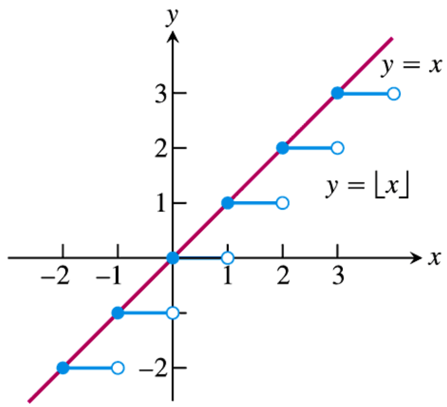


FIGURE 1.31 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x (Example 6).

Note: the graph is the blue colour lines, not the one in red

Writing formulas for piecewise-defined functions

- Write a formula for the function *y*=*f*(*x*) in Figure 1.33

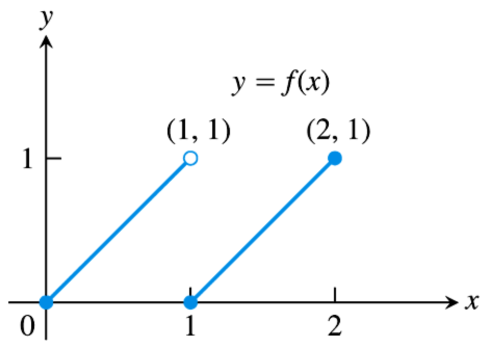


FIGURE 1.33 The segment on the left contains (0, 0) but not (1, 1). The segment on the right contains both of its endpoints (Example 8).

1.4

Identifying Functions;
Mathematical Models

Linear functions

- Linear function takes the form of
- $y=mx+b$
- m, b constants
- m slope of the graph
- b intersection with the y-axis
- The linear function reduces to a constant function $f = c$ when $m = 0$,

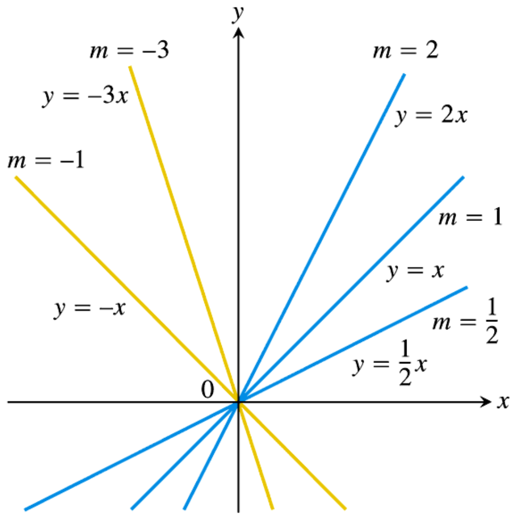


FIGURE 1.34 The collection of lines $y = mx$ has slope m and all lines pass through the origin.

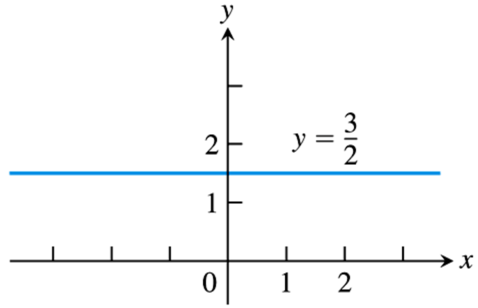


FIGURE 1.35 A constant function has slope $m = 0$.

Power functions

- $f(x) = x^a$
- a constant
- Case (a): $a = n$, a positive integer

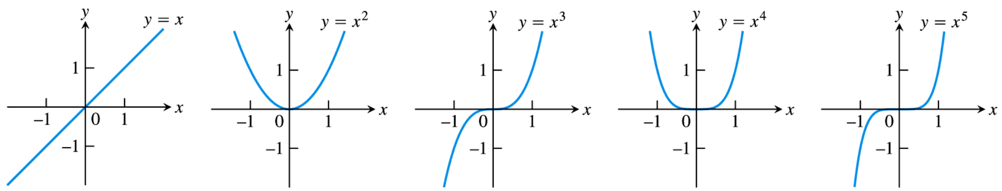


FIGURE 1.36 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$ defined for $-\infty < x < \infty$.

go back

Power functions

- Case (b):
- a = -1 (hyperbola)
- or a=-2

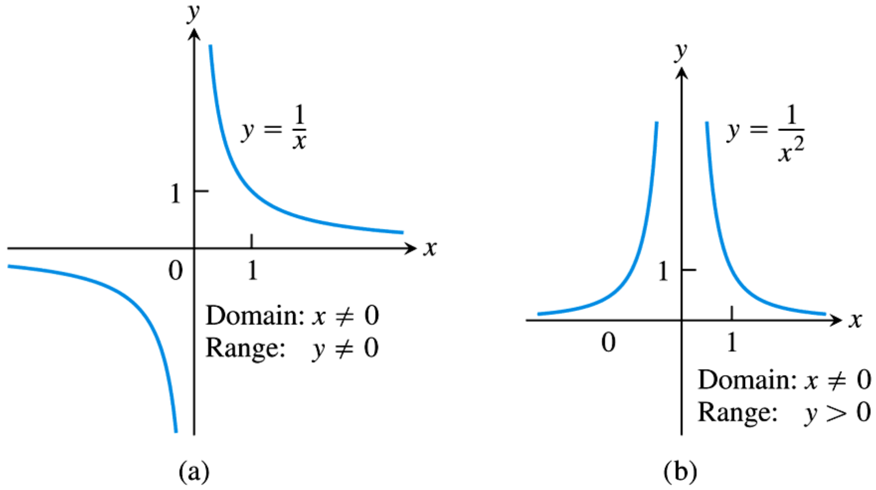


FIGURE 1.37 Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

go back

Power functions

- Case (c):
- a = ½, 1/3, 3/2, and 2/3
- $f(x) = x^{1/2} = \sqrt{x}$ (square root) , domain = $[0 \le x < \infty)$
- $g(x) = x^{1/3} = \sqrt[3]{x}$(cube root), domain = $(-\infty < x < \infty)$

- $p(x) = x^{2/3} = (x^{1/3})^2$, domain = ?
- $q(x) = x^{3/2} = (x^3)^{1/2}$ domain = ?

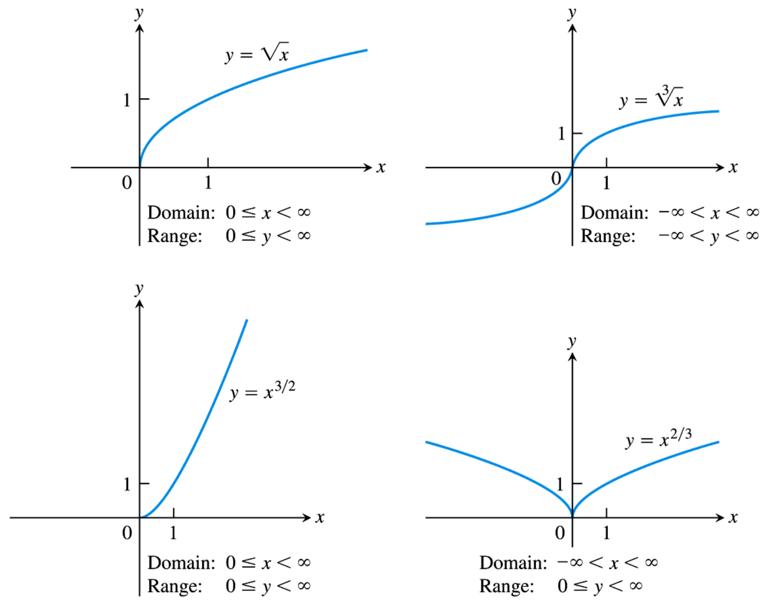


FIGURE 1.38 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

go back

Polynomials

- $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots a_1 x + a_0$
- n nonnegative integer (1,2,3…)
- a's coefficients (real constants)
- If $a_n \neq 0$, n is called the degree of the polynomial

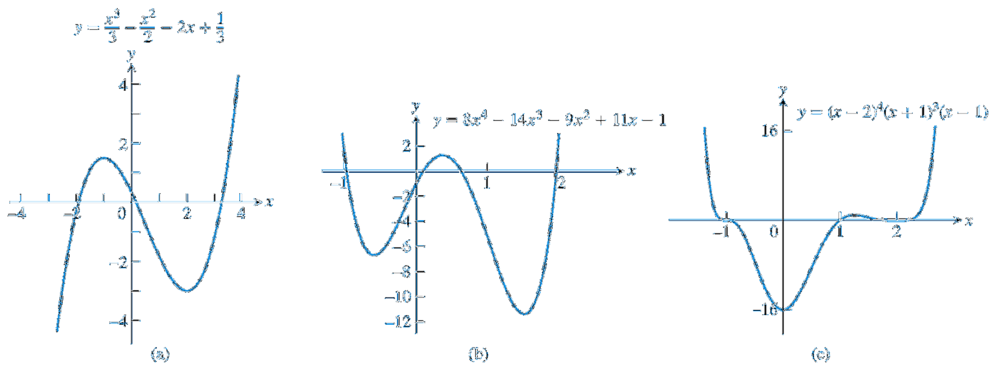


FIGURE 1.39 Graphs of three polynomial functions.

Rational functions

- A rational function is a quotient of two polynomials:
- $f(x) = p(x) / q(x)$
- p,q are polynomials.
- Domain of $f(x)$ is the set of all real number x for which $q(x) \neq 0$.

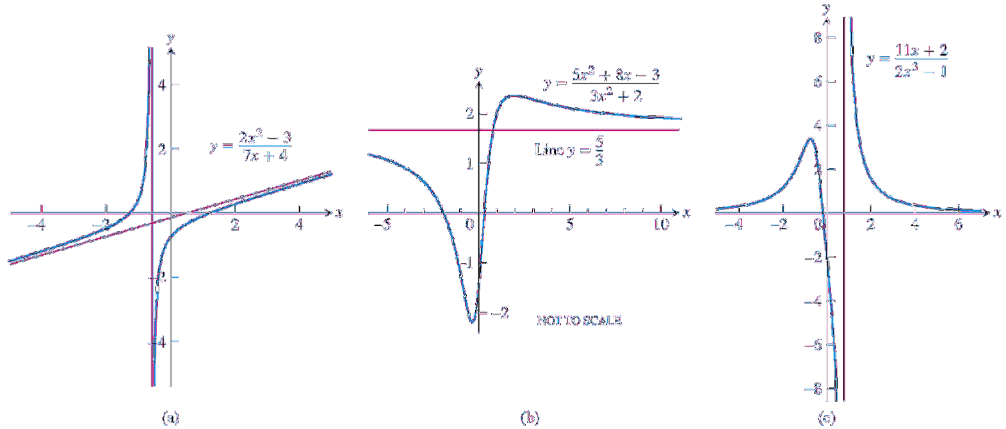


FIGURE 1.40 Graphs of three rational functions.

Algebraic functions

- Functions constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots)

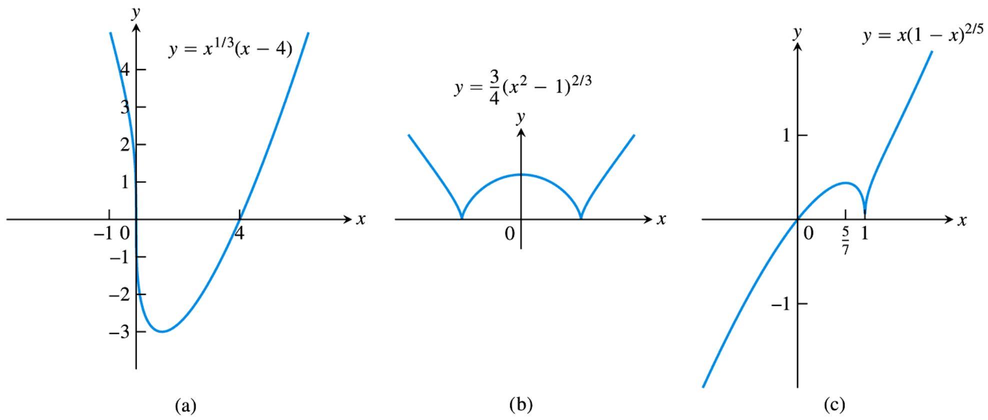


FIGURE 1.41 Graphs of three algebraic functions.

Trigonometric functions

- More details in later chapter

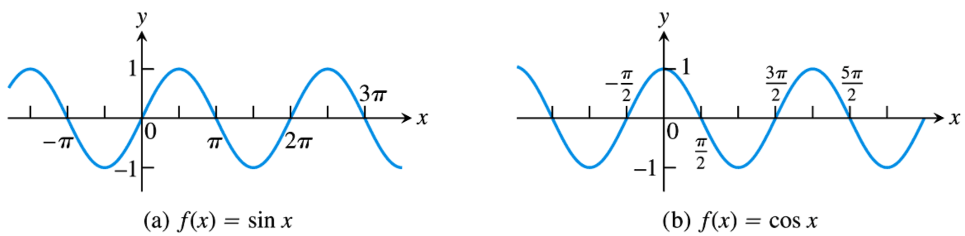


FIGURE 1.42 Graphs of the sine and cosine functions.

Exponential functions

- $f(x) = a^x$
- Where $a > 0$ and $a \neq 0$. a is called the ‘base’.
- Domain (-∞, ∞)
- Range (0, ∞)
- Hence, $f(x) > 0$
- More in later chapter

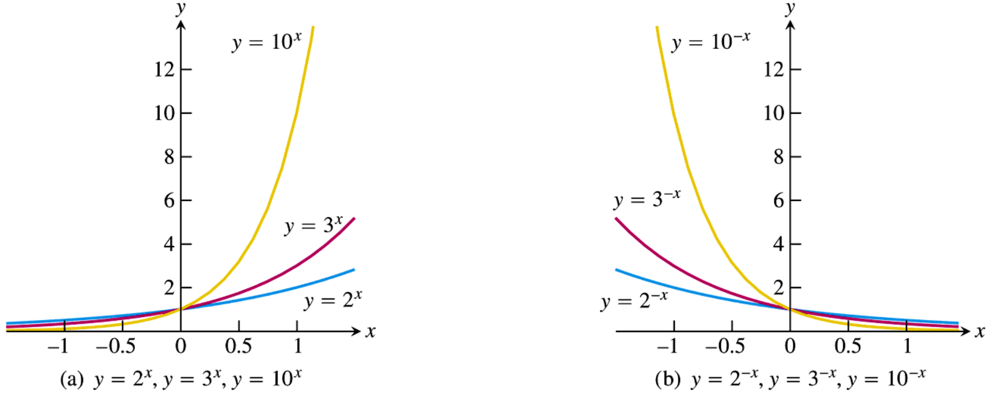


FIGURE 1.43 Graphs of exponential functions.

Note: graphs in (a) are reflections of the corresponding curves in (b) about the y-axis. This amount to the symmetry operation of $x \leftrightarrow -x$.

Logarithmic functions

- $f(x) = \log_a x$
- a is the base
- $a \neq 1$, $a > 0$
- Domain (0, ∞)
- Range (-∞, ∞)
- They are the *inverse functions* of the exponential functions (more in later chapter)

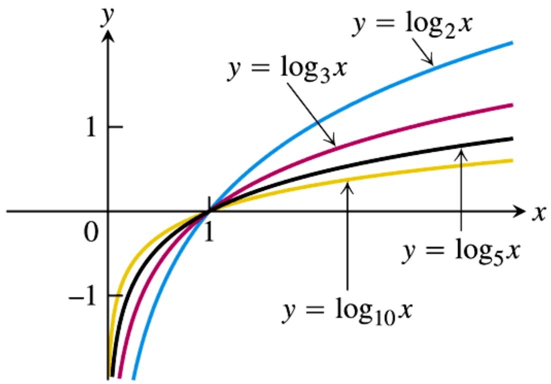


FIGURE 1.44 Graphs of four logarithmic functions.

Transcendental functions

- Functions that are not algebraic
- Include: trigonometric, inverse trigonometric, exponential, logarithmic, hyperbolic and many other functions

Example 1

- Recognizing Functions
- (a) $f(x) = 1 + x - \frac{1}{2}x^5$
- (b) $g(x) = 7^x$
- (c) $h(z) = z^7$
- (d) $y(t) = \sin(t - \pi/4)$

Increasing versus decreasing functions

- A function is said to be increasing if it rises as you move from left to right
- A function is said to be decreasing if it falls as you move from left to right

Function	Where increasing	Where decreasing
$y = x^2$	$0 \leq x < \infty$	$-\infty < x \leq 0$
$y = x^3$	$-\infty < x < \infty$	Nowhere
$y = 1/x$	Nowhere	$-\infty < x < 0$ and $0 < x < \infty$
$y = 1/x^2$	$-\infty < x < 0$	$0 < x < \infty$
$y = \sqrt{x}$	$0 \leq x < \infty$	Nowhere
$y = x^{2/3}$	$0 \leq x < \infty$	$-\infty < x \leq 0$

$y=x^2$, $y=x^3$; $y=1/x$, $y=1/x^2$; $y=x^{1/2}$, $y=x^{2/3}$

DEFINITIONS **Even Function, Odd Function**

A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

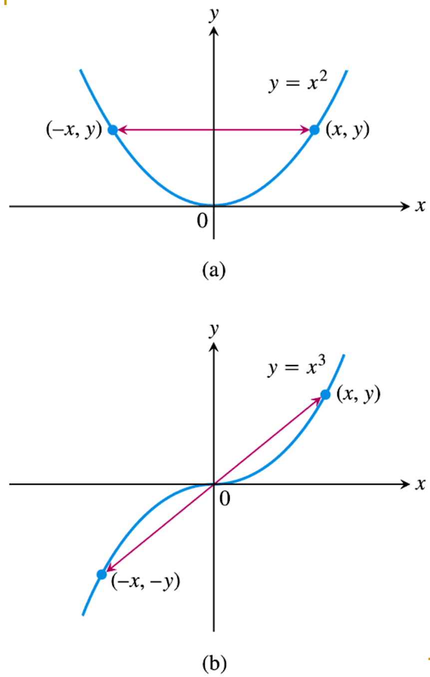


FIGURE 1.46 In part (a) the graph of $y = x^2$ (an even function) is symmetric about the y-axis. The graph of $y = x^3$ (an odd function) in part (b) is symmetric about the origin.

Recognising even and odd functions

- $f(x) = x^2$ Even function as $(-x)^2 = x^2$ for all x, symmetric about the all x, symmetric about the y-axis.
- $f(x) = x^2 + 1$ Even function as $(-x)^2 + 1 = x^2 + 1$ for all x, symmetric about the all x, symmetric about the y-axis.

Recognising even and odd functions

- $f(x) = x$. Odd function as $(-x) = -x$ for all x, symmetric about origin.
- $f(x) = x+1$. Odd function ?

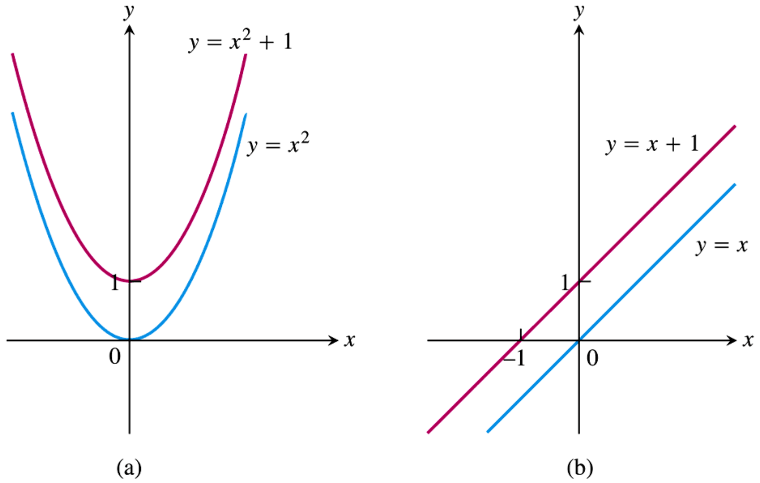


FIGURE 1.47 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd. The symmetry about the origin is lost (Example 2).

1.5

Combining Functions;
Shifting and Scaling Graphs

Sums, differences, products and quotients

- *f*, *g* are functions
- For $x \in D(f) \cap D(g)$, we can define the functions of
- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$,
- $(cf)(x) = cf(x)$, c a real number
- $$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$$

Example 1

- $f(x) = \sqrt{x}$, $g(x) = \sqrt{(1-x)}$,
- The domain common to both *f*,*g* is
- $D(f) \cap D(g) = [0,1]$ (work it out)

Function	Formula	Domain
$f+g$	$(f+g)(x)=\sqrt{x}+\sqrt{1-x}$	$[0, 1] = D(f) \cap D(g)$
$f-g$	$(f-g)(x)=\sqrt{x}-\sqrt{1-x}$	$[0, 1]$
$g-f$	$(g-f)(x)=\sqrt{1-x}-\sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x)=f(x)g(x)=\sqrt{x(1-x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x)=\frac{f(x)}{g(x)}=\sqrt{\frac{x}{1-x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x)=\frac{g(x)}{f(x)}=\sqrt{\frac{1-x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

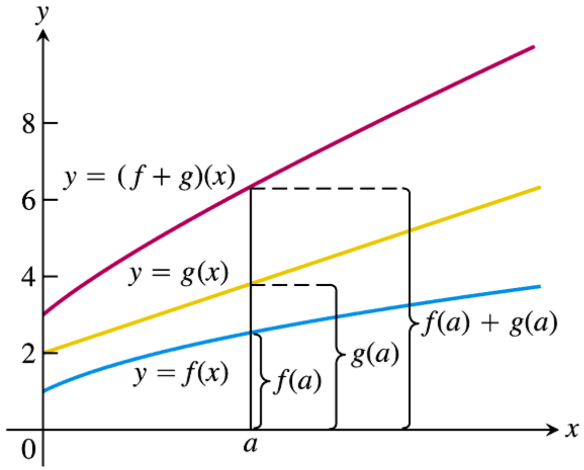


FIGURE 1.50 Graphical addition of two functions.

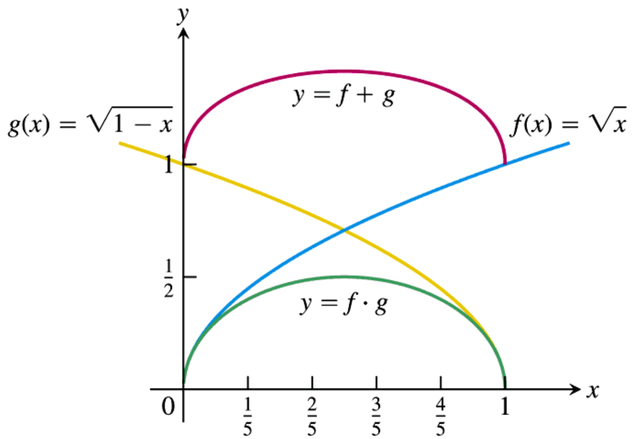


FIGURE 1.51 The domain of the function $f + g$ is the intersection of the domains of f and g, the interval $[0, 1]$ on the x-axis where these domains overlap. This interval is also the domain of the function $f \cdot g$ (Example 1).

Composite functions

- Another way of combining functions

DEFINITION **Composition of Functions**

If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

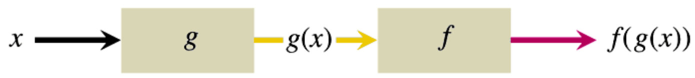


FIGURE 1.52 Two functions can be composed at x whenever the value of one function at x lies in the domain of the other. The composite is denoted by $f \circ g$.

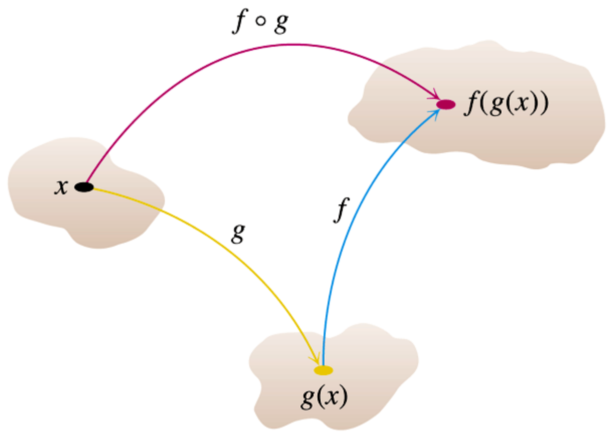


FIGURE 1.53 Arrow diagram for $f \circ g$.

Example 2

- Viewing a function as a composite
- $y(x) = \sqrt{(1 - x^2)}$ is a composite of
- $g(x) = 1 - x^2$ and $f(x) = \sqrt{x}$
- i.e. $y(x) = f[g(x)] = \sqrt{(1 - x^2)}$
- Domain of the composite function is $|x| \leq 1$, or $[-1,1]$
- Is $f[g(x)] = g[f(x)]$?

Example 3

- Read it yourself
- Make sure that you know how to work out the domains and ranges of each composite functions listed

Shifting a graph of a function

Shift Formulas

Vertical Shifts

$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$
	Shifts it *down* $\|k\|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$	Shifts the graph of f *left* h units if $h > 0$
	Shifts it *right* $\|h\|$ units if $h < 0$

Example 4

- (a) $y = x^2$, $y = x^2 + 1$
- (b) $y = x^2$, $y = x^2 - 2$

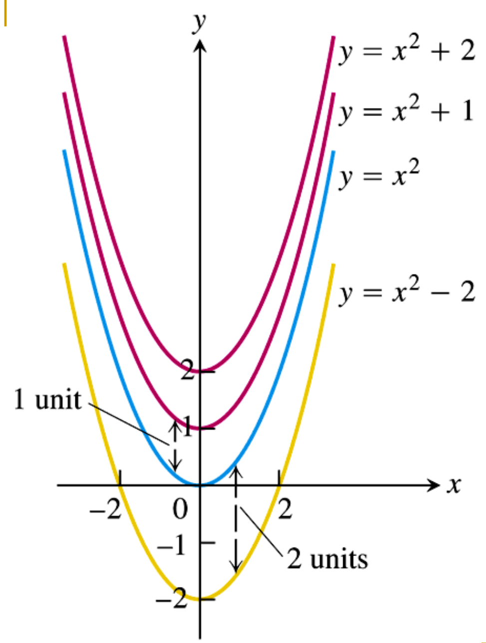


FIGURE 1.54 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Example 4a and b).

Example 4

- (c) $y = x^2$, $y = (x + 3)^2$, $y = (x - 3)^2$

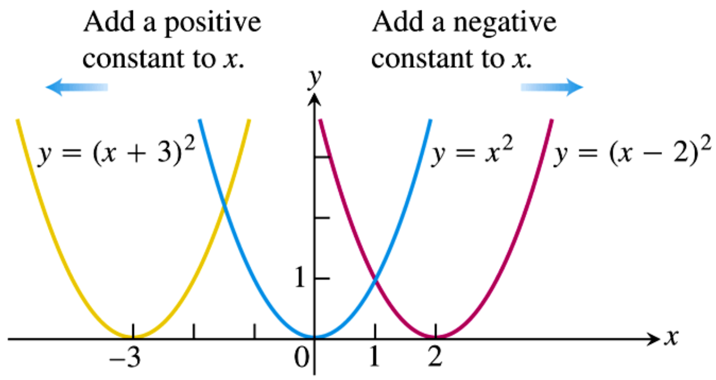


FIGURE 1.55 To shift the graph of $y = x^2$ to the left, we add a positive constant to x. To shift the graph to the right, we add a negative constant to x (Example 4c).

Example 4

- (d) $y = |x|$, $y = |x - 2| - 1$

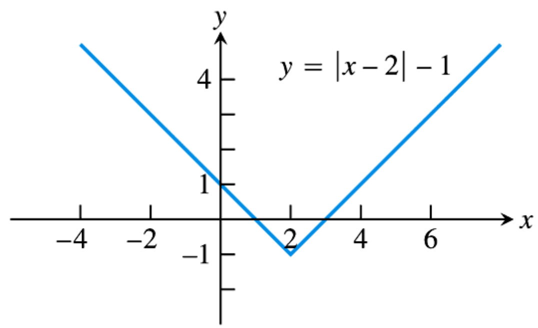


FIGURE 1.56 Shifting the graph of $y = |x|$ 2 units to the right and 1 unit down (Example 4d).

Scaling and reflecting a graph of a function

- To scale a graph of a function is to stretch or compress it, vertically or horizontally.
- This is done by multiplying a constant *c* to the function or the independent variable

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$,

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$,

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

Example 5(a)

- Vertical stretching and compression of the graph $y = \sqrt{x}$ by a factor or 3

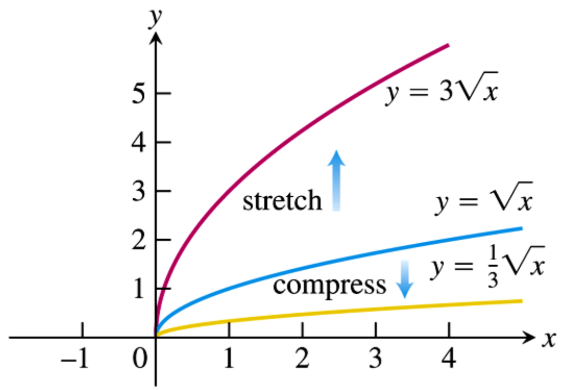


FIGURE 1.57 Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 5a).

Example 5(b)

- Horizontal stretching and compression of the graph $y = \sqrt{x}$ by a factor of 3

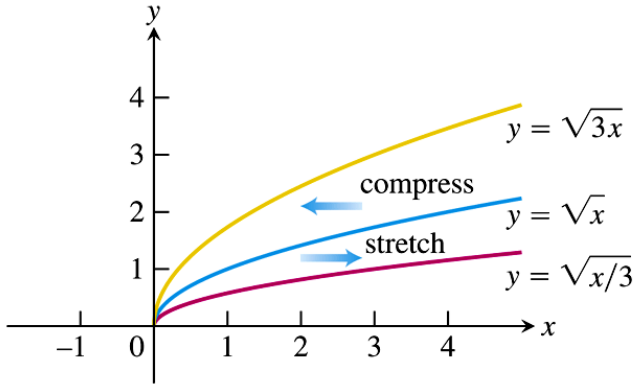


FIGURE 1.58 Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 5b).

Example 5(c)

- Reflection across the *x*- and *y*- axes
- $c = -1$

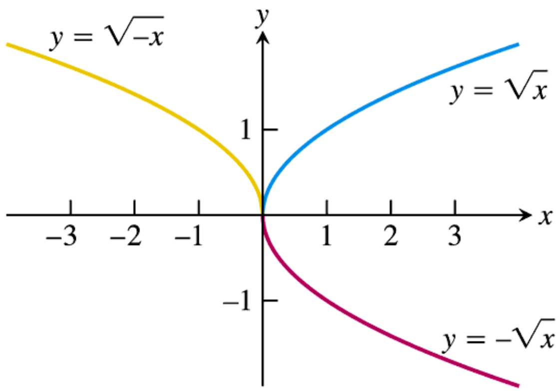


FIGURE 1.59 Reflections of the graph $y = \sqrt{x}$ across the coordinate axes (Example 5c).

EXAMPLE 6

Combining Scalings and Reflections

- Given the function $f(x)=x^4-4x^3+10$ (Figure 1.60a), find formulas to
- (a) compress the graph horizontally by a factor of 2 followed by a reflection across the *y*-axis (Figure 1.60b).
- (b) compress the graph vertically by a factor of 2 followed by a reflection across the *x*-axis (Figure 1.60c).

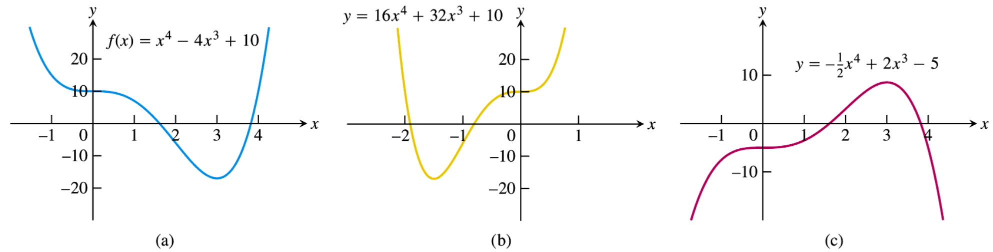


FIGURE 1.60 (a) The original graph of *f*. (b) The horizontal compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the *y*-axis. (c) The vertical compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the *x*-axis (Example 6).

1.6

Trigonometric Functions

Radian measure

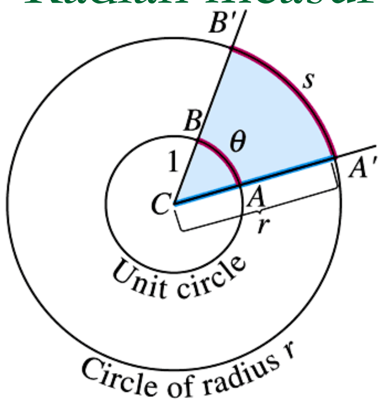


FIGURE 1.63 The radian measure of angle *ACB* is the length θ of arc *AB* on the unit circle centered at *C*. The value of θ can be found from any other circle, however, as the ratio s/r. Thus $s = r\theta$ is the length of arc on a circle of radius r when θ is measured in radians.

Conversion Formulas

$$1 \text{ degree} = \frac{\pi}{180} (\approx 0.02) \text{ radians}$$

$$\text{Degrees to radians: multiply by } \frac{\pi}{180}$$

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57) \text{ degrees}$$

$$\text{Radians to degrees: multiply by } \frac{180}{\pi}$$

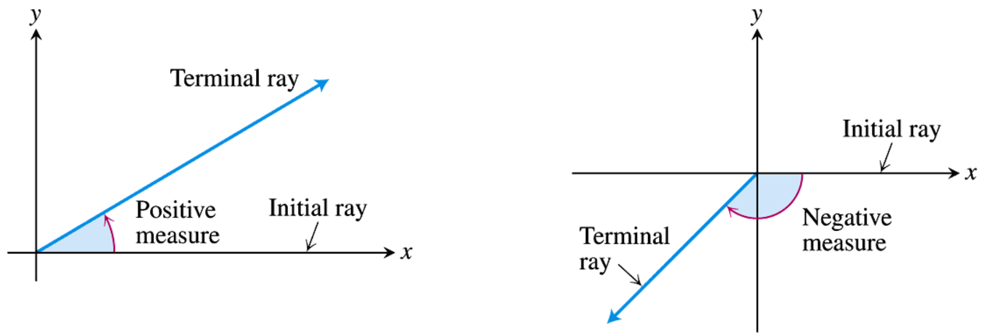


FIGURE 1.65 Angles in standard position in the *xy*-plane.

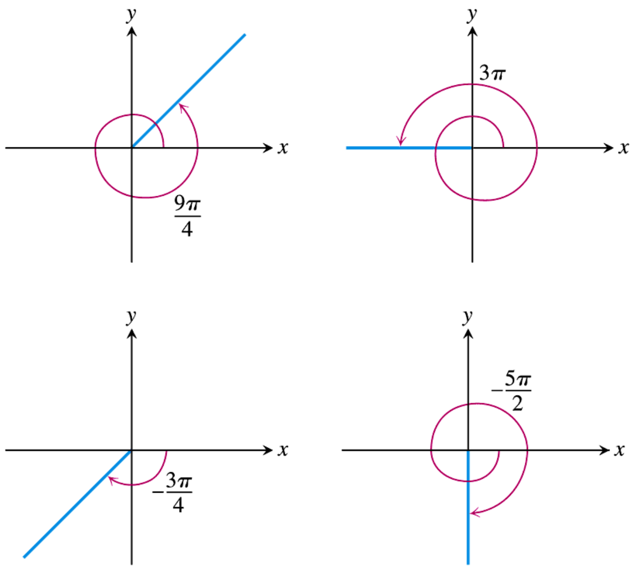
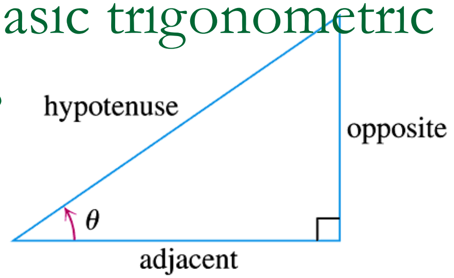


FIGURE 1.66 Nonzero radian measures can be positive or negative and can go beyond 2π.

Angle convention

- Be noted that angle will be expressed in terms of radian unless otherwise specified.
- Get used to the change of the unit

The six basic trigonometric functions



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

FIGURE 1.67 Trigonometric ratios of an acute angle.

Generalised definition of the six trigo functions

- Define the trigo functions in terms of the coordinates of the point *P*(*x*,*y*) on a circle of radius *r*

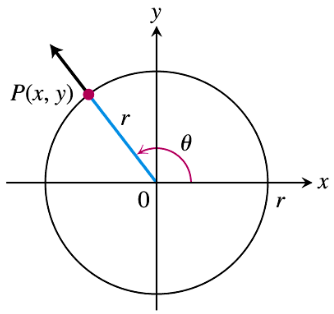


FIGURE 1.68 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.

- **sine:** $\sin\theta = y/r$
- **cosine:** $\cos\theta = x/r$
- **tangent:** $\tan\theta = y/x$
- **cosecant:** $\csc\theta = r/y$
- **secant:** $\sec\theta = r/x$
- **cotangent:** $\cot\theta = x/y$

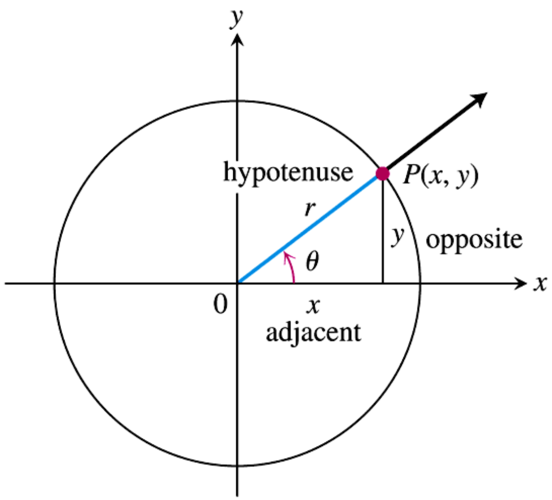


FIGURE 1.69 The new and old definitions agree for acute angles.

Mnemonic to remember when the basic trigo functions are positive or negative

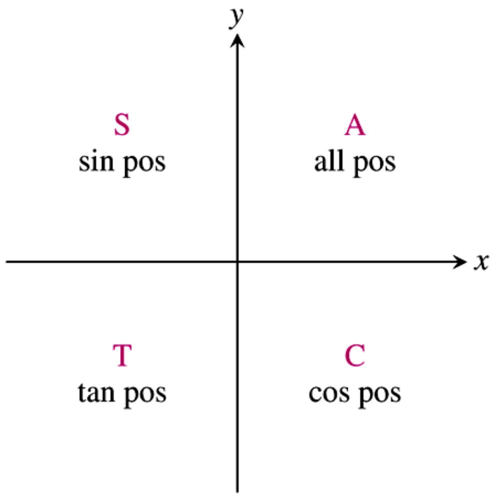


FIGURE 1.70 The CAST rule, remembered by the statement "All Students Take Calculus," tells which trigonometric functions are positive in each quadrant.

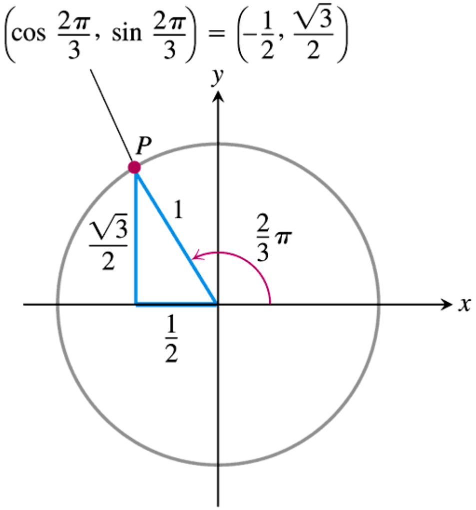


FIGURE 1.71 The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

Periodicity and graphs of the trigo functions

Trigo functions are also periodic.

DEFINITION Periodic Function

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

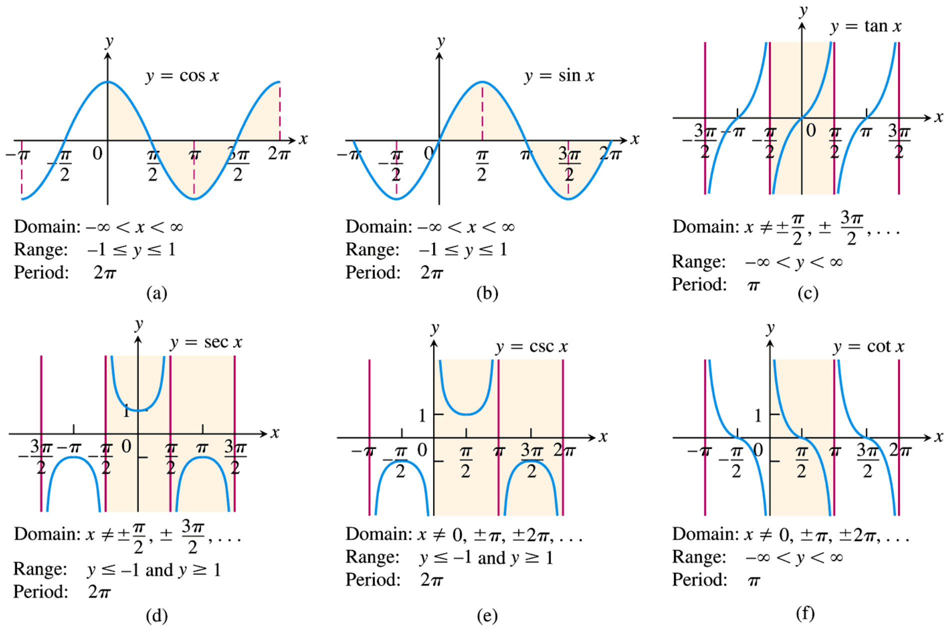


FIGURE 1.73 Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Parity of the trigo functions

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

The parity is easily deduced from the graphs.

Identities

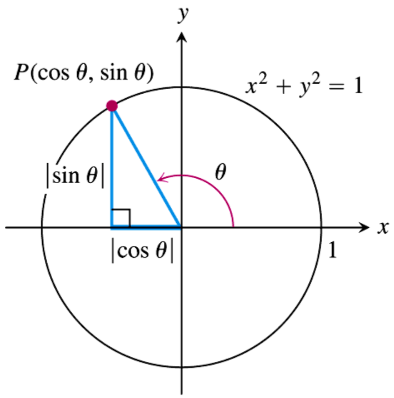


FIGURE 1.74 The reference triangle for a general angle θ.

Applying Pythagorean theorem to the right triangle leads to the identity

$$\cos^2\theta + \sin^2\theta = 1. \qquad (1)$$

Dividing identity (1) by $\cos^2\theta$ and $\sin^2\theta$ in turn gives the next two identities

$$1 + \tan^2\theta = \sec^2\theta.$$
$$1 + \cot^2\theta = \csc^2\theta.$$

Addition Formulas

$$\begin{aligned}\cos(A + B) &= \cos A\cos B - \sin A\sin B\\ \sin(A + B) &= \sin A\cos B + \cos A\sin B\end{aligned} \qquad (2)$$

There are also similar formulas for cos (A-B) and sin (A-B). Do you know how to deduce them?

Double-Angle Formulas

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ \sin 2\theta &= 2\sin\theta\cos\theta\end{aligned} \qquad (3)$$

Identity (3) is derived by setting $A = B$ in (2)

Half-Angle Formulas

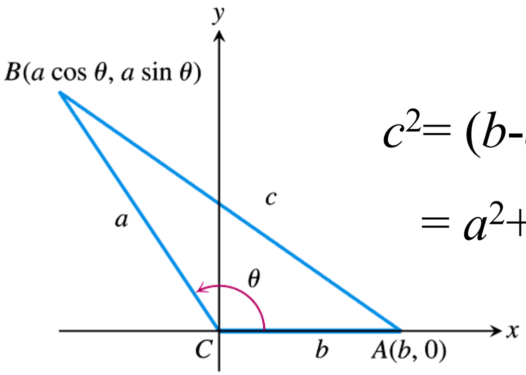
$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad (4)$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \qquad (5)$$

Identities (4,5) are derived by combining (1) and (3(i))

Law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \qquad (6)$$



$$c^2 = (b - a\cos\theta)^2 + (a\sin\theta)^2$$

$$= a^2 + b^2 - 2ab\cos\theta$$

FIGURE 1.75 The square of the distance between A and B gives the law of cosines.

Chapter 2

Limits and Continuity

2.1

Rates of Change and Limits

Average Rates of change and Secant Lines

- Given an arbitrary function $y=f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y, Δy, by the length Δx

DEFINITION Average Rate of Change over an Interval

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

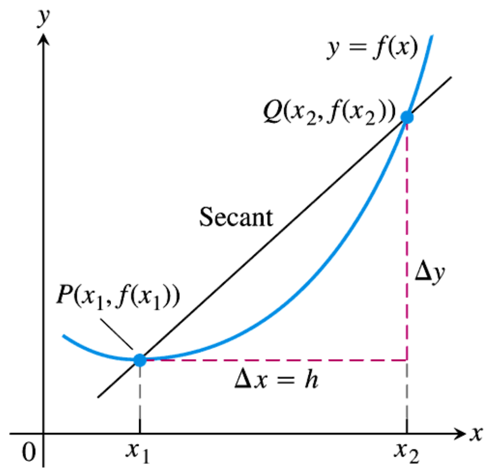


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Example 4

- Figure 2.2 shows how a population of fruit flies grew in a 50-day experiment.
- (a) Find the average growth rate from day 23 to day 45.
- (b) How fast was the number of the flies growing on day 23?

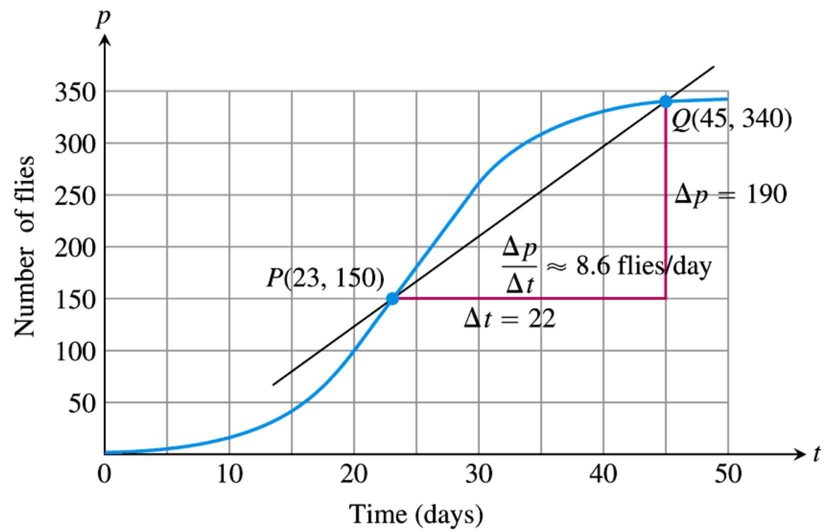


FIGURE 2.2 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line.

Slop at $P \approx (250 - 0)/(35-14)$ = 16.7 flies/day

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\frac{340-150}{45-23} \approx 8.6$
(40, 330)	$\frac{330-150}{40-23} \approx 10.6$
(35, 310)	$\frac{310-150}{35-23} \approx 13.3$
(30, 265)	$\frac{265-150}{30-23} \approx 16.4$

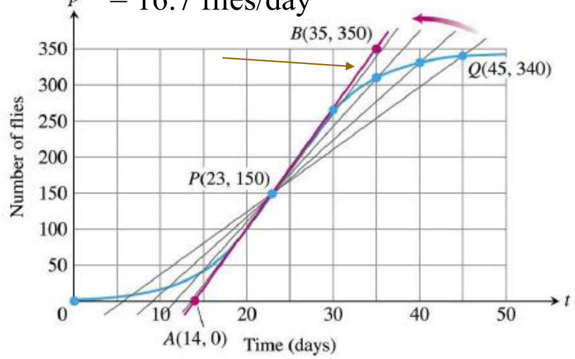


FIGURE 2.3 The positions and slopes of four secants through the point P on the fruit fly graph (Example 4).

The grow rate at day 23 is calculated by examining the average rates of change over increasingly short time intervals starting at day 23. Geometrically, this is equivalent to evaluating the slopes of secants from P to Q with Q approaching P.

Limits of function values

- Informal definition of limit:
- Let f be a function defined on an open interval about x_0, except possibly at x_0 itself.
- If f gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0

$$\lim_{x \to x_0} f(x) = L$$

- "Arbitrarily close" is not yet defined here (hence the definition is informal).

Example 5

- How does the function behave near x=1?

$$f(x) = \frac{x^2 - 1}{x - 1}$$

- Solution:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \quad \text{for } x \neq 1$$

TABLE 2.2 The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

Values of x below and above 1	$f(x) = \frac{x^2-1}{x-1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

We say that $f(x)$ approaches the limit 2 as x approaches 1, $\lim_{x \to 1} f(x) = 2$ or $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$

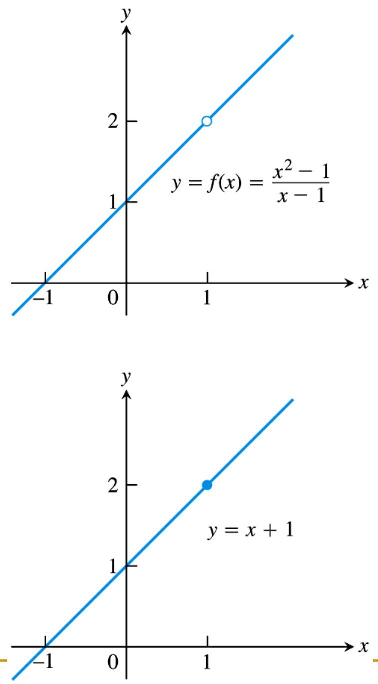


FIGURE 2.4 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 5).

Example 6

- The limit value does not depend on how the function is defined at x_0.

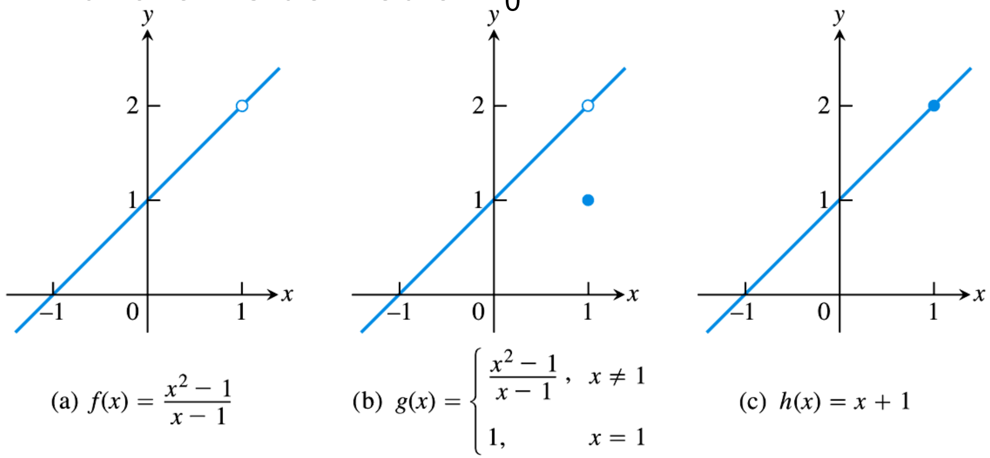


FIGURE 2.5 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 6).

Example 7

- In some special cases $\lim_{x \to x_0} f(x)$ can be evaluated by calculating $f(x_0)$. For example, constant function, rational function and identity function for which $x=x_0$ is defined
- (a) $\lim_{x\to 2} (4) = 4$ (constant function)
- (b) $\lim_{x\to -13} (4) = 4$ (constant function)
- (c) $\lim_{x\to 3} x = 3$ (identity function)
- (d) $\lim_{x\to 2} (5x-3) = 10 - 3 = 7$ (polynomial function of degree 1)
- (e) $\lim_{x\to -2} (3x+4)/(x+5) = (-6+4)/(-2+5) = -2/3$ (rational function)

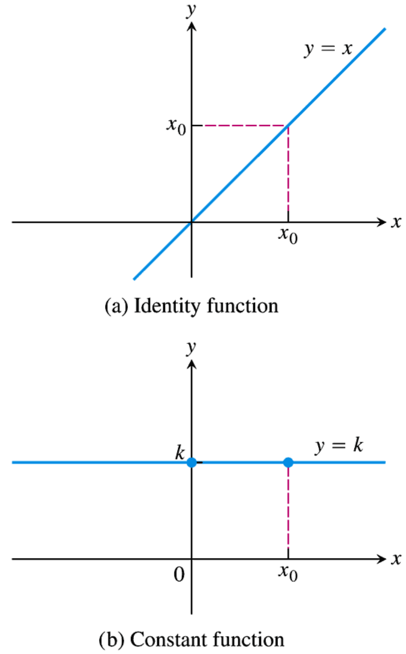


FIGURE 2.6 The functions in Example 8.

Example 9

- A function may fail to have a limit exist at a point in its domain.

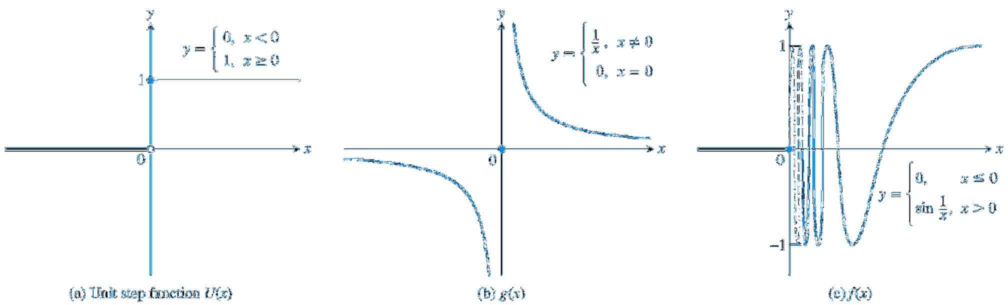


FIGURE 2.7 None of these functions has a limit as x approaches 0 (Example 9).

Jump

Grow to infinities

Oscillate

2.2

Calculating limits using the limits laws

The limit laws

- Theorem 1 tells how to calculate limits of functions that are arithmetic combinations of functions whose limit are already known.

THEOREM 1 Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$

   The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$

   The limit of the difference of two functions is the difference of their limits.

3. *Product Rule:* $\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$

   The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x\to c}(k\cdot f(x)) = k\cdot L$

   The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:* $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$

   The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule:* If r and s are integers with no common factor and $s \neq 0$, then

   $$\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$$

   provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

   The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Example 1 Using the limit laws

- (a) $\lim_{x\to c}(x^3+4x^2-3)$

  $= \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3$

  (sum and difference rule)

  $= c^3 + 4c^2 - 3$

  (product and multiple rules)

Example 1

- (b) $\lim_{x\to c}(x^4+x^2-1)/(x^2+5)$

  $= \lim_{x\to c}(x^4+x^2-1) / \lim_{x\to c}(x^2+5)$

$=(\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1)/(\lim_{x\to c} x^2 + \lim_{x\to c} 5)$

$= (c^4 + c^2 - 1)/(c^2 + 5)$

Example 1

- (c) $\lim_{x\to -2} \sqrt{(4x^2-3)} = \sqrt{\lim_{x\to -2} (4x^2-3)}$

  Power rule with $r/s = \frac{1}{2}$

  $= \sqrt{[\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3]}$

  $= \sqrt{[4(-2)^2 - 3]} = \sqrt{13}$

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example 2

- Limit of a rational function

$$\lim_{x\to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

Eliminating zero denominators algebraically

Identifying Common Factors
It can be shown that if $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.

Example 3 Canceling a common factor

- Evaluate $\lim_{x \to 1} \dfrac{x^2 + x - 2}{x^2 - x}$
- Solution: We can't substitute x=1 since $f(x = 1)$ is not defined. Since $x \neq 1$, we can cancel the common factor of x-1:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{(x-1)(x+2)}{x(x-1)} = \lim_{x \to 1} \frac{(x+2)}{x} = 3$$

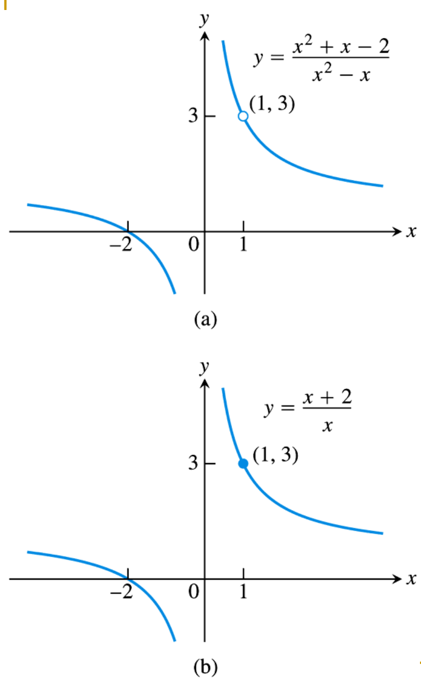


FIGURE 2.8 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$ (Example 3).

The Sandwich theorem

THEOREM 4 The Sandwich Theorem
Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

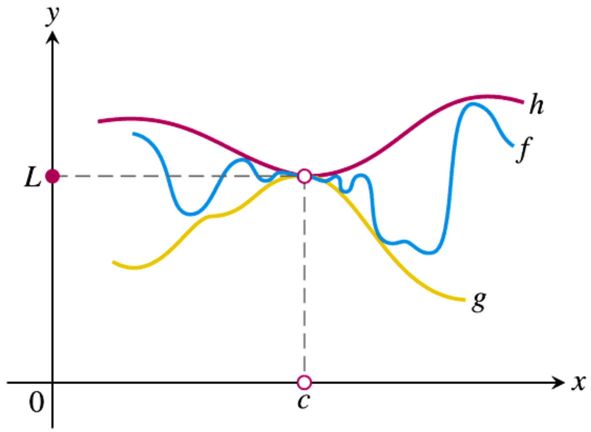


FIGURE 2.9 The graph of f is sandwiched between the graphs of g and h.

Example 6

- (a)
- The function $y = \sin\theta$ is sandwiched between $y = |\theta|$ and $y = -|\theta|$ for all values of θ. Since $\lim_{\theta\to 0} (-|\theta|) = \lim_{\theta\to 0} (|\theta|) = 0$, we have $\lim_{\theta\to 0} \sin\theta = 0$.
- (b)
- From the definition of cos θ,

  $0 \leq 1 - \cos\theta \leq |\theta|$ for all θ, and we have the limit $\lim_{x\to 0} \cos\theta = 1$

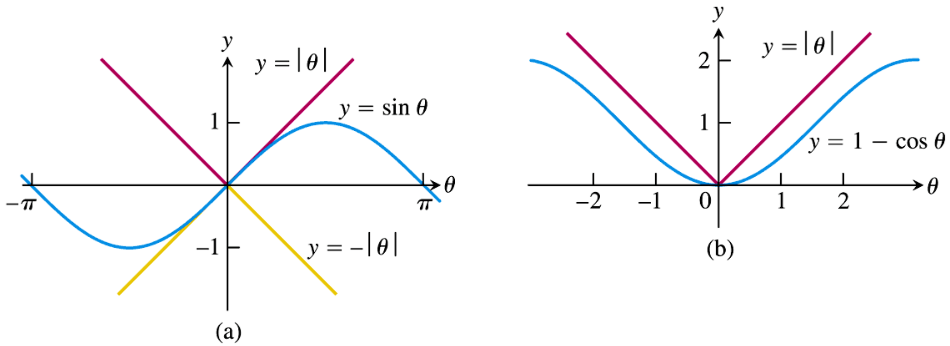


FIGURE 2.11 The Sandwich Theorem confirms that (a) $\lim_{\theta\to 0} \sin\theta = 0$ and (b) $\lim_{\theta\to 0} (1 - \cos\theta) = 0$ (Example 6).

Example 6(c)

- For any function $f(x)$, if $\lim_{x\to 0} (|f(x)|) = 0$, then $\lim_{x\to 0} f(x) = 0$ due to the sandwich theorem.
- Proof:
- $-|f(x)| \leq f(x) \leq |f(x)|$
- Since $\lim_{x\to 0} (|f(x)|) = \lim_{x\to 0} (-|f(x)|) = 0$
- $\Rightarrow \lim_{x\to 0} f(x) = 0$

2.3

The Precise Definition of a Limit

Example 1 A linear function

- Consider the linear function $y = 2x - 1$ near $x_0 = 4$. Intuitively it is close to 7 when x is close to 4, so $\lim_{x \to 0} (2x-1)=7$. How close does x have to be so that $y = 2x - 1$ differs from 7 by less than 2 units?

Solution

- For what value of x is $|y-7| < 2$?
- First, find $|y-7|<2$ in terms of x:

$|y-7|<2 \equiv |2x-8|<2$
$\equiv -2 < 2x-8 < 2$
$\equiv 3 < x < 5$
$\equiv -1 < x - 4 < 1$

Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0=7$.

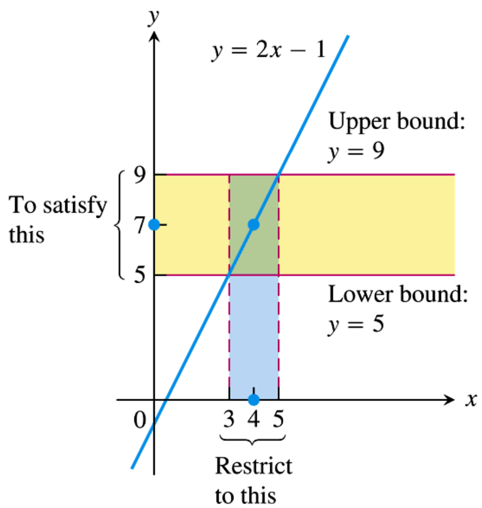


FIGURE 2.12 Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Example 1).

Definition of limit

DEFINITION Limit of a Function
Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Definition of limit

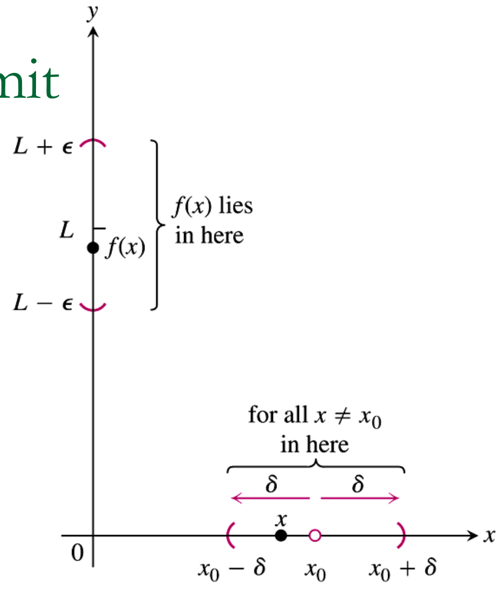


FIGURE 2.14 The relation of δ and ϵ in the definition of limit.

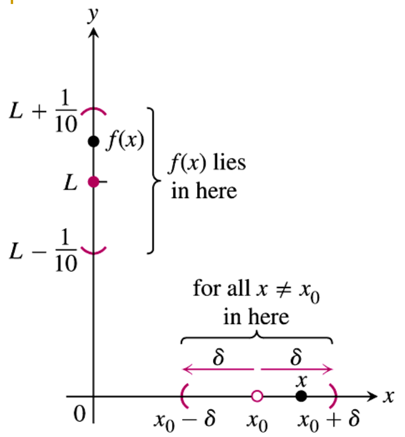
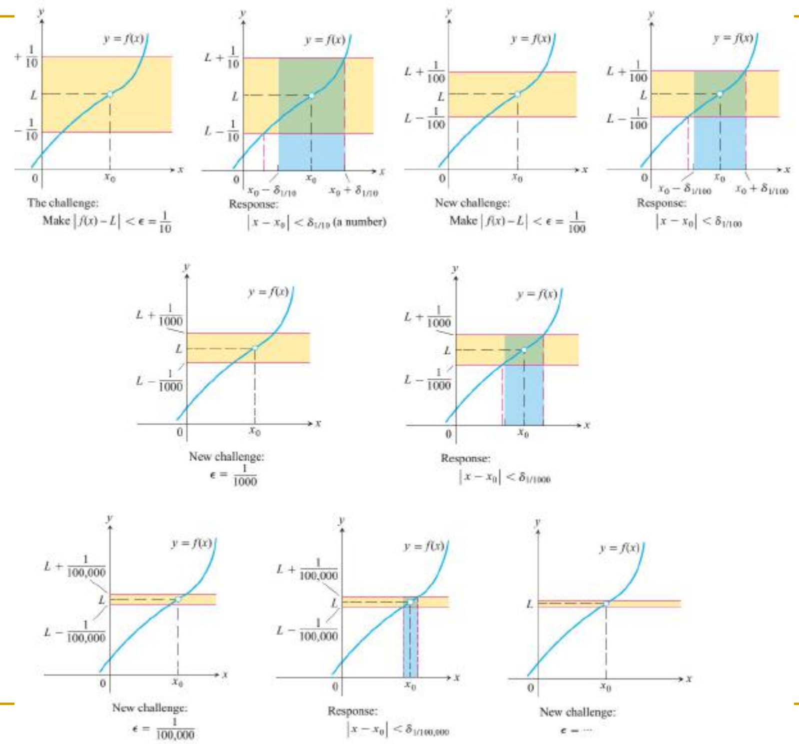


FIGURE 2.13 How should we define $\delta > 0$ so that keeping x within the interval $(x_0 - \delta, x_0 + \delta)$ will keep $f(x)$ within the interval $\left(L - \frac{1}{10}, L + \frac{1}{10}\right)$?

- The problem of proving L as the limit of $f(x)$ as x approaches x_0 is a problem of proving the existence of δ, such that whenever
- $x_0 - \delta < x < x_0 + \delta$,
- $L+\varepsilon < f(x) < L-\varepsilon$ for any arbitrarily small value of ε.
- As an example in Figure 2.13, given $\varepsilon = 1/10$, can we find a corresponding value of δ?
- How about if $\varepsilon = 1/100$? $\varepsilon = 1/1234$?
- If for any arbitrarily small value of ε we can always find a corresponding value of δ, then we has successfully proven that L is the limit of f as x approaches x_0



Example 2 Testing the definition

- Show that

$$\lim_{x \to 1} (5x - 3) = 2$$

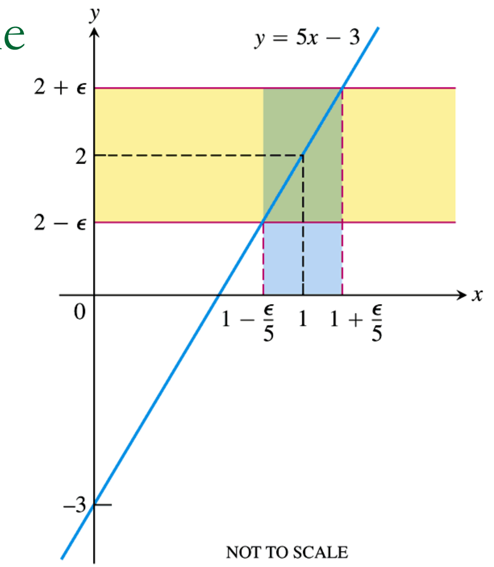


FIGURE 2.15 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

Solution

- Set x_0=1, $f(x)$=5x-3, L=2.
- For any given ε, we have to find a suitable δ > 0 so that whenever

  0<| x – 1|< δ, $x \neq 1$,

  $f(x)$ is within a distance ε from L=2, i.e.

  $|f(x) - 2| < \varepsilon$.

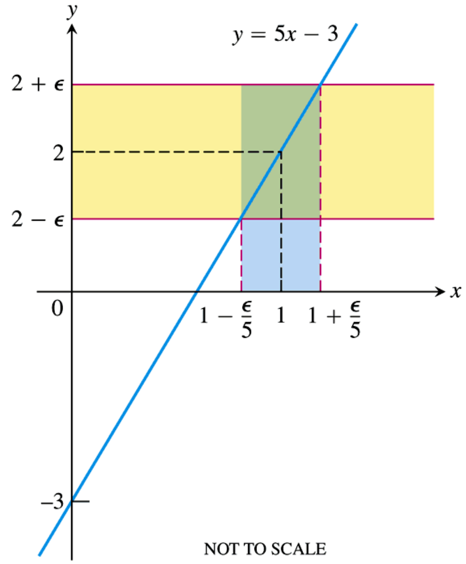
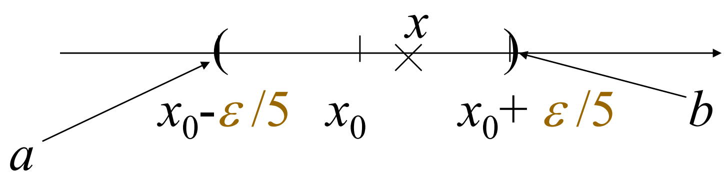


FIGURE 2.15 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

- First, obtain an open interval (a,b) in which $|f(x) - 2| < \varepsilon \equiv |5x - 5| < \varepsilon \equiv$

  $-\varepsilon/5 < x - 1 < \varepsilon/5 \equiv -\varepsilon/5 < x - x_0 < \varepsilon/5$



- choose $\delta < \varepsilon / 5$. This choice will guarantee that $|f(x) - L| < \varepsilon$ whenever $x_0 - \delta < x < x_0 + \delta$.

  We have shown that for any value of ε given, we can always find a corresponding value of δ that meets the "challenge" posed by an ever diminishing ε. This is a proof of existence.

  Thus we have proven that the limit for $f(x)$=5x-3 is L=2 when $x \rightarrow x_0$=1.

Example 3(a)

- Limits of the identity functions
- Prove

$$\lim_{x \to x_0} x = x_0$$

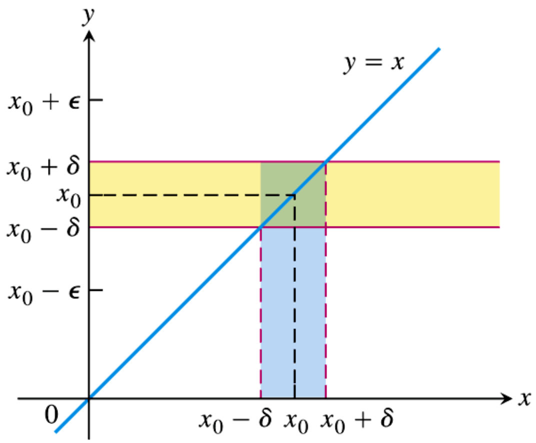


FIGURE 2.16 For the function $f(x) = x$, we find that $0 < |x - x_0| < \delta$ will guarantee $|f(x) - x_0| < \epsilon$ whenever $\delta \leq \epsilon$ (Example 3a).

Solution

- Let ε > 0. We must find δ > 0 such that for all x, 0 < $|x - x_0| < \delta$ implies $|f(x) - x_0| < \varepsilon$., here, $f(x)$=x, the identity function.
- Choose $\delta < \varepsilon$ will do the job.
- The proof of the existence of δ proves

$$\lim_{x \to x_0} x = x_0$$

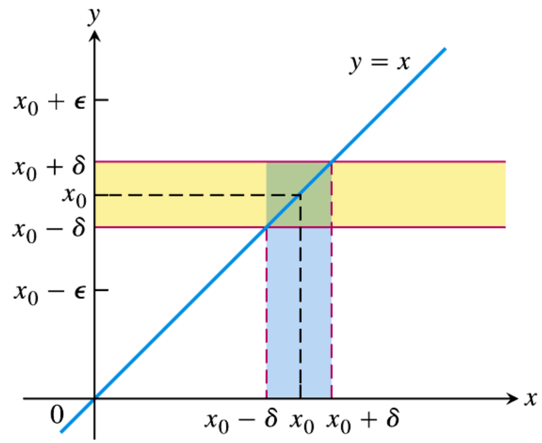


FIGURE 2.16 For the function $f(x) = x$, we find that $0 < |x - x_0| < \delta$ will guarantee $|f(x) - x_0| < \epsilon$ whenever $\delta \leq \epsilon$ (Example 3a).

Example 3(b)

- Limits constant functions
- Prove

$$\lim_{x \to x_0} k = k \quad (k \text{ constant})$$

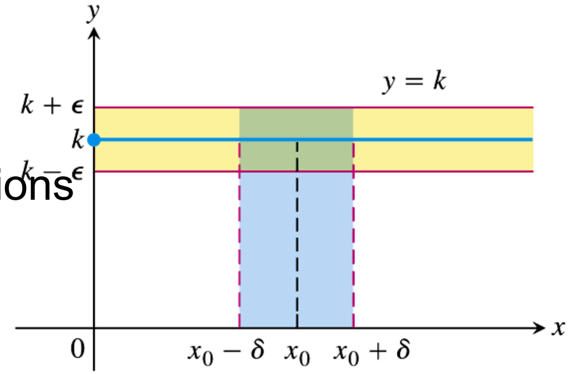


FIGURE 2.17 For the function $f(x) = k$, we find that $|f(x) - k| < \epsilon$ for any positive δ (Example 3b).

Solution

- Let $\varepsilon > 0$. We must find $\delta > 0$ such that for all x, $0 < |x-x_0| < \delta$ implies $|f(x)- k| < \varepsilon$., here, $f(x)=k$, the constant function.
- Choose any δ will do the job.
- The proof of the existence of δ proves

$$\lim_{x \to x_0} k = k$$

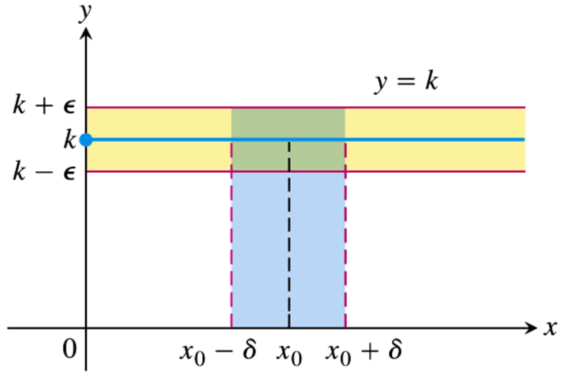


FIGURE 2.17 For the function $f(x) = k$, we find that $|f(x) - k| < \epsilon$ for any positive δ (Example 3b).

Finding delta algebraically for given epsilons

- Example 4: Finding delta algebraically
- For the limit $\lim_{x \to 5} \sqrt{x-1} = 2$

find a $\delta > 0$ that works for $\varepsilon = 1$. That is, find a $\delta > 0$ such that for all x,

$$0 < |x - 5| < \delta \Rightarrow 0 < \left|\sqrt{x-1} - 2\right| < 1$$

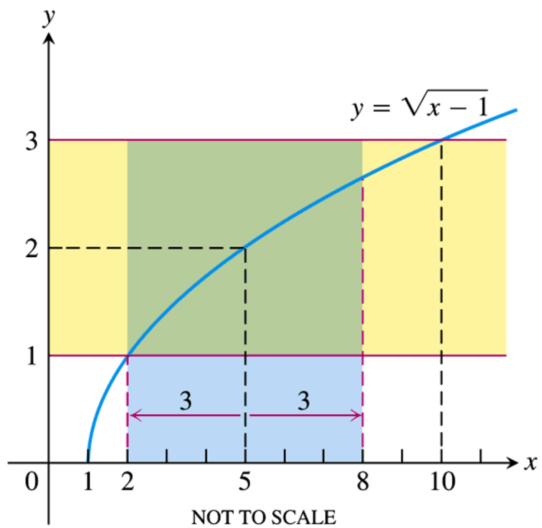


FIGURE 2.19 The function and intervals in Example 4.

Solution

- δ is found by working backward:

How to Find Algebraically a δ for a Given f, L, x_0, and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.
2. *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

Solution

- Step one: Solve the inequality $|f(x)-L|<\varepsilon$

$$0 < \left|\sqrt{x-1} - 2\right| < 1 \Rightarrow 2 < x < 10$$

- Step two: Find a value of $\delta > 0$ that places the open interval $(x_0-\delta, x_0+\delta)$ centered at x_0 inside the open interval found in step one. Hence, we choose $\delta = 3$ or a smaller number

By doing so, the inequality $0<|x - 5| < \delta$ will automatically place x between 2 and 10 to make $0 < \left|f(x) - 2\right| < 1$

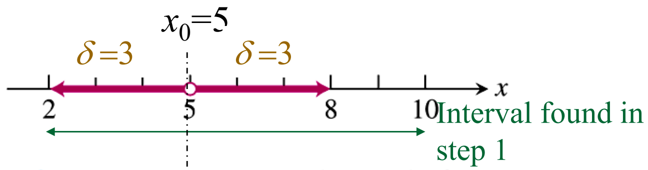


FIGURE 2.18 An open interval of radius 3 about $x_0 = 5$ will lie inside the open interval (2, 10).

Example 5

- Prove that

$$\lim_{x \to 2} f(x) = 4 \text{ if}$$

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$$

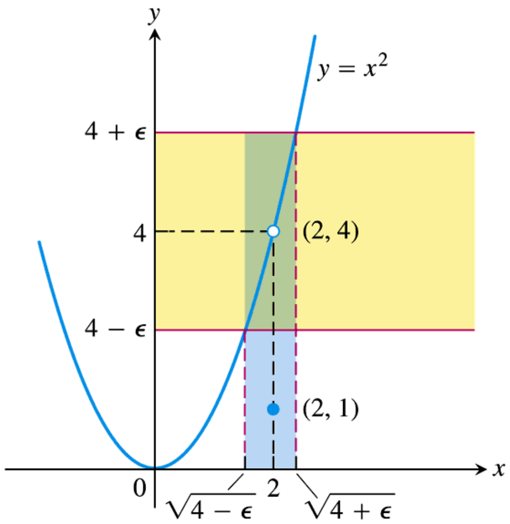


FIGURE 2.20 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \epsilon$.

Solution

- Step one: Solve the inequality $|f(x)-L|<\varepsilon$:

$$0 < \left|x^2 - 2\right| < \varepsilon \Rightarrow \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}, x \neq 2$$

- Step two: Choose
- $\delta <$ min [2-√(4-ε), √(4+ε) – 2]

For all x that obey

- $0 < |x - 2| < \delta$
- $\Rightarrow |f(x)-4|<\varepsilon$
- This completes the proof.

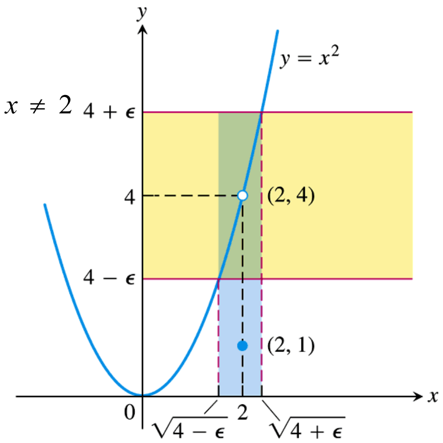


FIGURE 2.20 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \epsilon$.

2.4

One-Sided Limits and
Limits at Infinity

Two sided limit does not exist for *y*;

But

y does has two one-sided limits

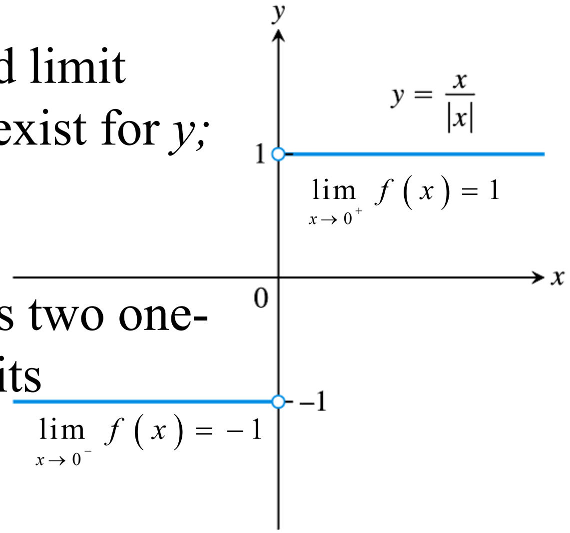
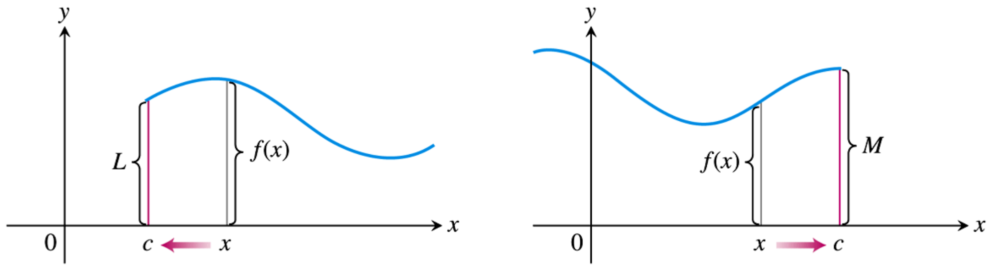


FIGURE 2.21 Different right-hand and left-hand limits at the origin.

One-sided limits



(a) $\lim_{x \to c^+} f(x) = L$ (b) $\lim_{x \to c^-} f(x) = M$

FIGURE 2.22 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

Right-hand limit Left-hand limit

Example 1

- One sided limits of a semicircle

No left hand limit at x= -2;

No two sided limit at x= -2;

No right hand limit at x=2;

No two sided limit at x= 2;

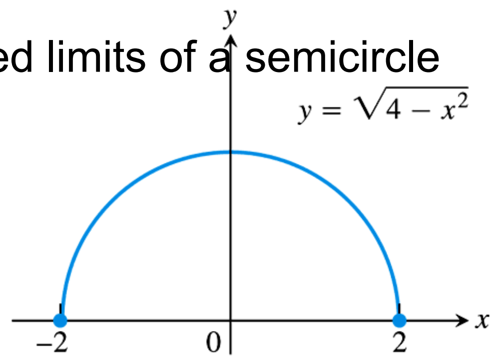


FIGURE 2.23 $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$ and $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$ (Example 1).

THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

Example 2

- Limits of the function graphed in Figure 2.24
- Can you write down all the limits at x=0, x=1, x=2, x=3, x=4?
- What is the limit at other values of x?

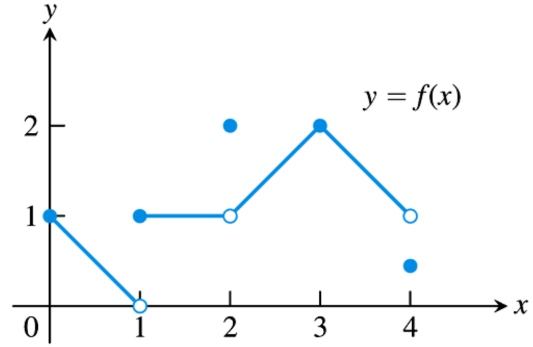


FIGURE 2.24 Graph of the function in Example 2.

Precise definition of one-sided limits

DEFINITIONS Right-Hand, Left-Hand Limits

We say that $f(x)$ has **right-hand limit L at x_0**, and write

$$\lim_{x\to x_0^+} f(x) = L \qquad \text{(See Figure 2.25)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit L at x_0**, and write

$$\lim_{x\to x_0^-} f(x) = L \qquad \text{(See Figure 2.26)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

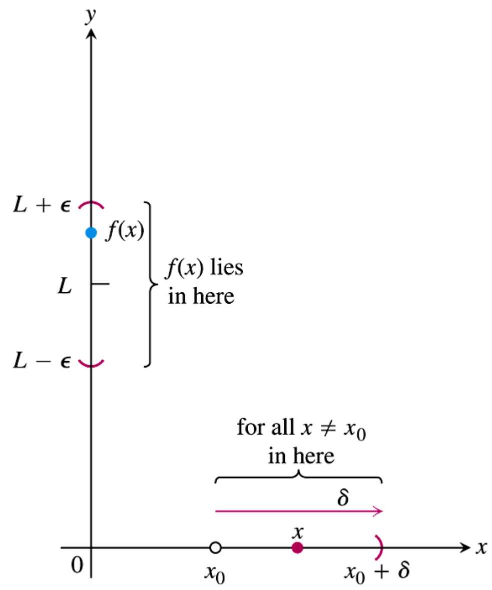


FIGURE 2.25 Intervals associated with the definition of right-hand limit.

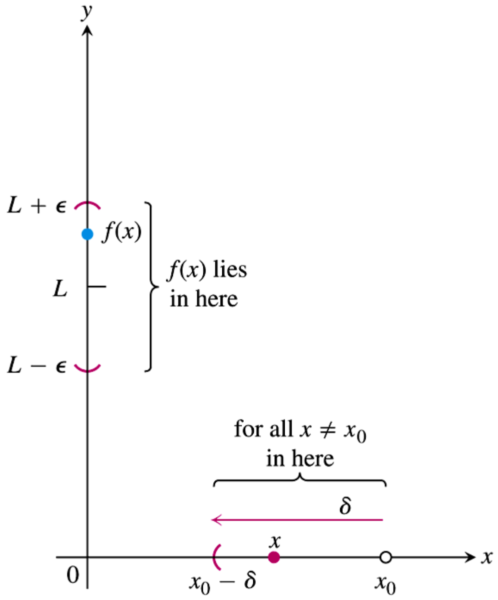


FIGURE 2.26 Intervals associated with the definition of left-hand limit.

Limits involving $(\sin\theta)/\theta$

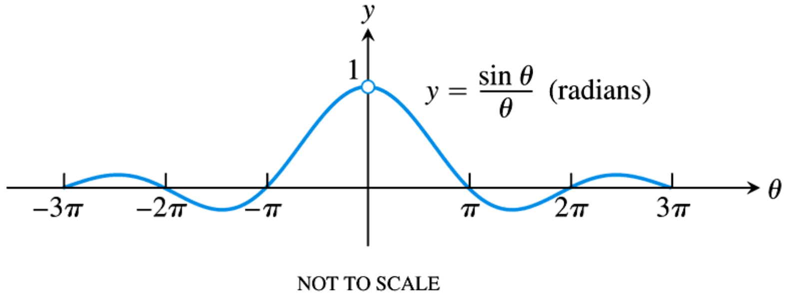


FIGURE 2.29 The graph of $f(\theta) = (\sin\theta)/\theta$.

THEOREM 7

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$

Proof

Area $\Delta OAP = \frac{1}{2} \sin\theta$

Area sector $OAP = \theta/2$

Area $\Delta OAT = \frac{1}{2} \tan\theta$

$\frac{1}{2} \sin\theta < \theta/2 < \frac{1}{2} \tan\theta$

$1 < \theta/\sin\theta < 1/\cos\theta$

$1 > \sin\theta/\theta > \cos\theta$

Taking limit $\theta \to 0^{\pm}$,

$$\lim_{\theta \to 0^{\pm}} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta \to 0} \frac{\sin\theta}{\theta}$$

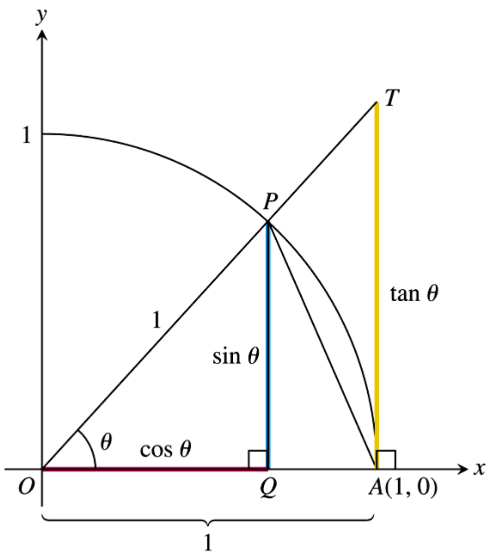


FIGURE 2.30 The figure for the proof of Theorem 7. $TA/OA = \tan\theta$, but $OA = 1$, so $TA = \tan\theta$.

Example 5(a)

- Using theorem 7, show that

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

Example 5(b)

- Using theorem 7, show that

$$\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$$

Finite limits as $x \to \pm\infty$

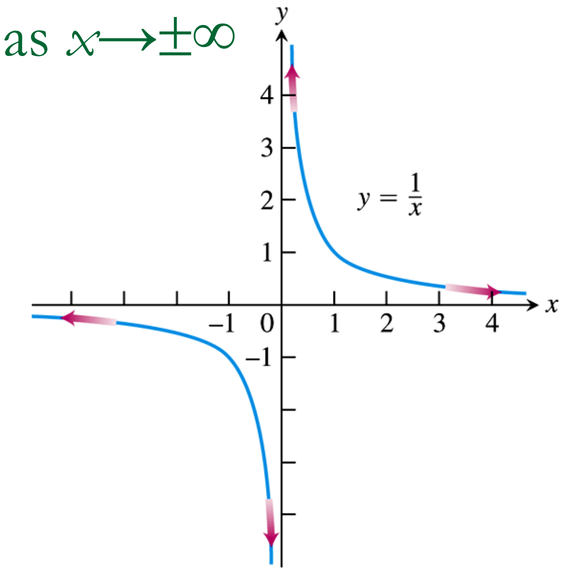


FIGURE 2.31 The graph of $y = 1/x$.

Precise definition

DEFINITIONS Limit as x approaches ∞ or $-\infty$

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

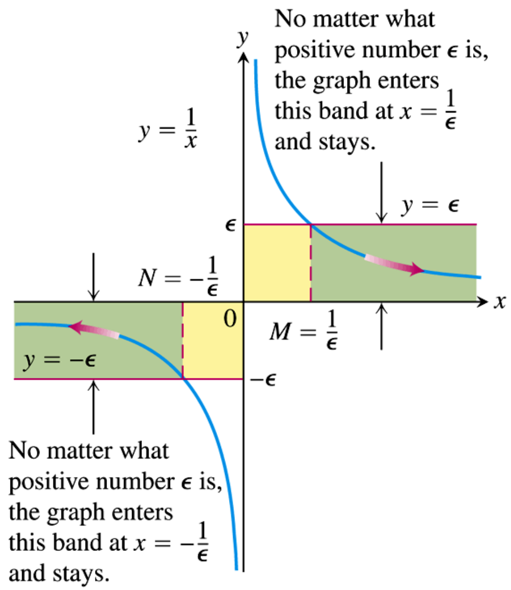


FIGURE 2.32 The geometry behind the argument in Example 6.

Example 6

- Limit at infinity for

$$f(x) = \frac{1}{x}$$

- (a) Show that

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

- (b) Show that

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

THEOREM 8 Limit Laws as $x \to \pm\infty$

If L, M, and k, are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L \quad \text{and} \quad \lim_{x \to \pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to \pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to \pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x \to \pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x \to \pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

Example 7(a)

- Using Theorem 8

$$\lim_{x \to \infty} \left(5 + \frac{1}{x} \right) = \lim_{x \to \infty} 5 + \lim_{x \to \infty} \frac{1}{x} = 5 + 0 = 5$$

Example 7(b)

$$\lim_{x \to \infty} \frac{\pi \sqrt{3}}{x^2} = \pi \sqrt{3} \lim_{x \to \infty} \frac{1}{x^2}$$

$$= \pi \sqrt{3} \lim_{x \to \infty} \frac{1}{x} \cdot \lim_{x \to \infty} \frac{1}{x}$$

$$= \pi \sqrt{3} \cdot 0 \cdot 0 = 0$$

Limits at infinity of rational functions

- Example 8

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to\infty}\frac{5+(8/x)-(3/x^2)}{3+(2/x^2)}=$$

$$=\frac{5+\lim_{x\to\infty}(8/x)-\lim_{x\to\infty}(3/x^2)}{3+\lim_{x\to\infty}(2/x^2)}=\frac{5+0-0}{3+0}=\frac{5}{3}$$

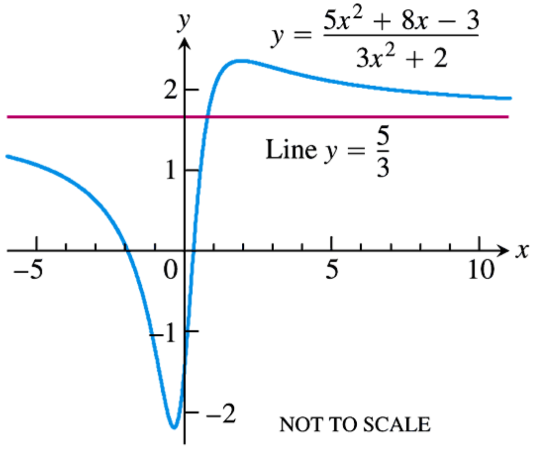


FIGURE 2.33 The graph of the function in Example 8. The graph approaches the line $y = 5/3$ as $|x|$ increases.

go back

Example 9

- Degree of numerator less than degree of denominator

$$\lim_{x\to\infty}\frac{11x+2}{2x^3-1}=\lim_{x\to\infty}\ldots=0$$

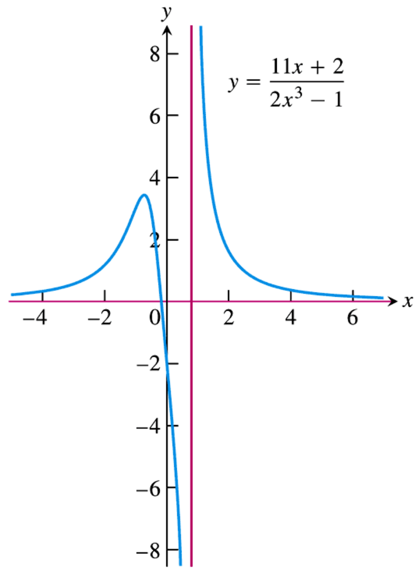


FIGURE 2.34 The graph of the function in Example 9. The graph approaches the x-axis as $|x|$ increases.

Horizontal asymptote

- x-axis is a horizontal asymptote

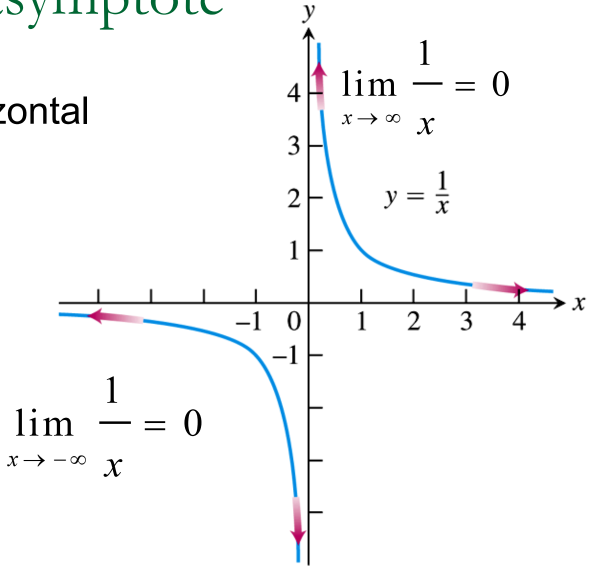


FIGURE 2.31 The graph of $y = 1/x$.

DEFINITION Horizontal Asymptote

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

Figure 2.33 has the line $y=5/3$ as a horizontal asymptote on both the right and left because

$$\lim_{x \to \infty} f(x) = \frac{5}{3} \qquad \lim_{x \to -\infty} f(x) = \frac{5}{3}$$

Oblique asymptote

- Happen when the degree of the numerator polynomial is one greater than the degree of the denominator
- By long division, recast $f(x)$ into a linear function plus a remainder. The remainder shall $\to 0$ as $x \to \pm\infty$. The linear function is the asymptote of the graph.

Example 12

- Find the oblique asymptote

$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

- Solution

$$f(x) = \frac{2x^2 - 3}{7x + 4} = \underbrace{\left(\frac{2}{7}x - \frac{8}{49}\right)}_{\text{linear function}} + \frac{-115}{49(7x + 4)}$$

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \left(\frac{2}{7}x - \frac{8}{49}\right) + \lim_{x \to \pm\infty} \frac{-115}{49(7x + 4)}$$

$$= \lim_{x \to \pm\infty} \left(\frac{2}{7}x - \frac{8}{49}\right) + 0 = \lim_{x \to \pm\infty} \left(\frac{2}{7}x - \frac{8}{49}\right)$$

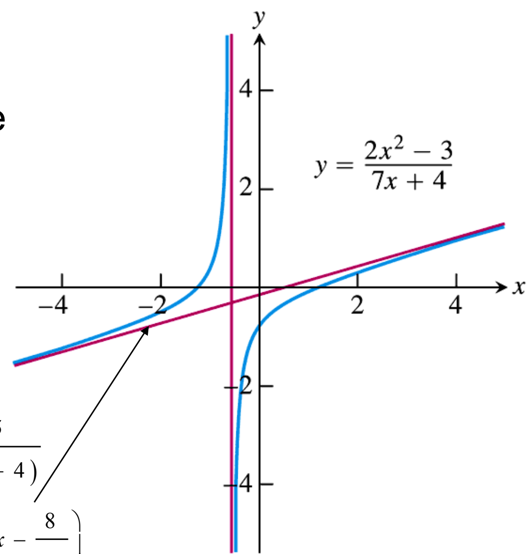


FIGURE 2.36 The function in Example 12 has an oblique asymptote.

2.5

Infinite Limits and Vertical Asymptotes

Infinite limit

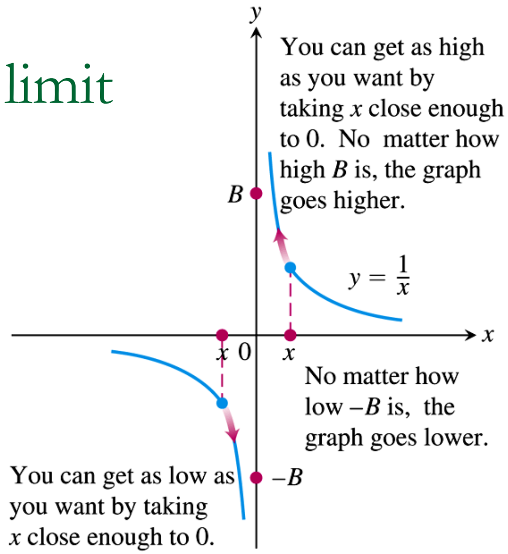


FIGURE 2.37 One-sided infinite limits:

$$\lim_{x\to0^+}\frac{1}{x}=\infty \quad \text{and} \quad \lim_{x\to0^-}\frac{1}{x}=-\infty$$

Example 1

- Find $\lim_{x\to1^+}\frac{1}{x-1}$ and $\lim_{x\to1^-}\frac{1}{x-1}$

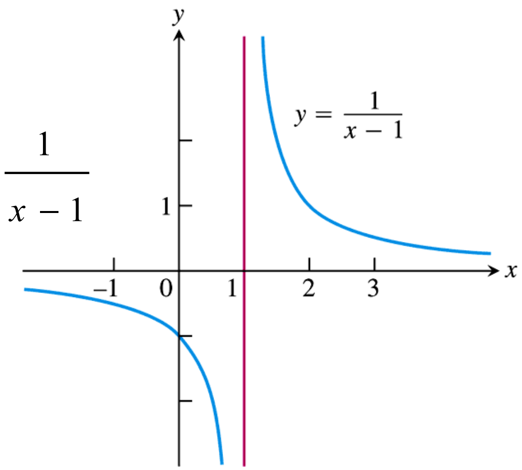


FIGURE 2.38 Near $x = 1$, the function $y = 1/(x-1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 1).

Example 2 Two-sided infinite limit

- Discuss the behavior of

$$(a) \qquad f(x)=\frac{1}{x^2} \text{ near } x=0$$

$$(b) \qquad g(x)=\frac{1}{(x+3)^2} \text{ near } x=-3$$

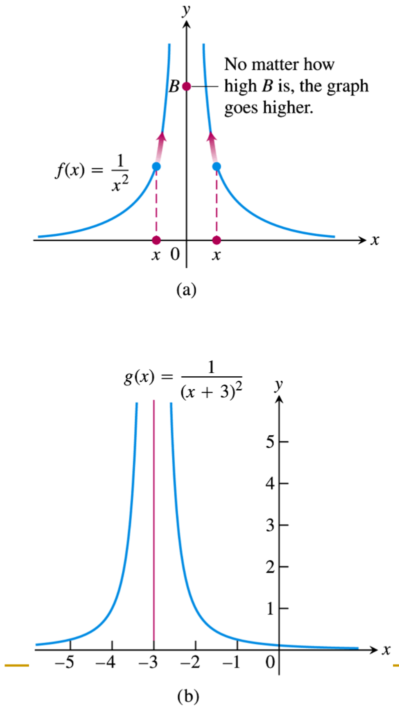


FIGURE 2.39 The graphs of the functions in Example 2. (a) $f(x)$ approaches infinity as $x \to 0$. (b) $g(x)$ approaches infinity as $x \to -3$.

Example 3

- Rational functions can behave in various ways near zeros of their denominators

$$(a)\lim_{x\to 2}\frac{(x-2)^2}{x^2-4}=\lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{(x-2)}{(x+2)}=0$$

$$(b)\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{(x+2)}=\frac{1}{4}$$

$$(c)\lim_{x\to 2^+}\frac{x-3}{x^2-4}=\lim_{x\to 2^+}\frac{x-3}{(x-2)(x+2)}=-\infty \quad (\text{note: } x>2)$$

$$(d)\lim_{x\to 2^-}\frac{x-3}{x^2-4}=\lim_{x\to 2^-}\frac{x-3}{(x-2)(x+2)}=+\infty \quad (\text{note: } x<2)$$

Example 3

$$(e)\lim_{x\to 2}\frac{x-3}{x^2-4}=\lim_{x\to 2}\frac{x-3}{(x-2)(x+2)} \quad \text{limit does not exist}$$

$$(f)\lim_{x\to 2}\frac{2-x}{(x-2)^3}=-\lim_{x\to 2}\frac{x-2}{(x-2)(x-2)^2}=-\lim_{x\to 2}\frac{1}{(x-2)^2}=-\infty$$

Precise definition of infinite limits

DEFINITIONS Infinity, Negative Infinity as Limits

1. We say that **$f(x)$ approaches infinity as x approaches x_0**, and write

$$\lim_{x\to x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > B.$$

2. We say that **$f(x)$ approaches negative infinity as x approaches x_0**, and write

$$\lim_{x\to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) < -B.$$

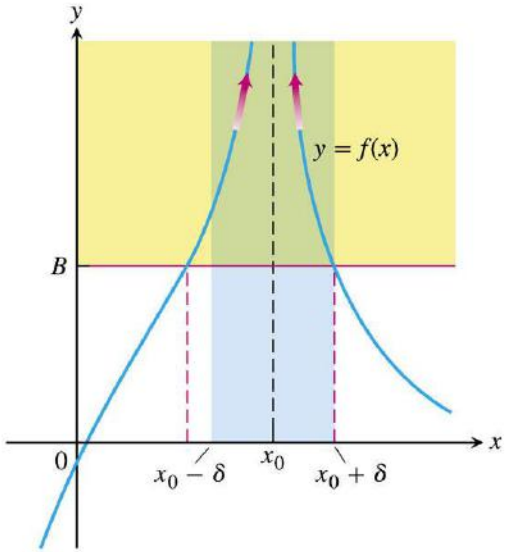


FIGURE 2.40 For $x_0 - \delta < x < x_0 + \delta$, the graph of $f(x)$ lies above the line $y = B$.

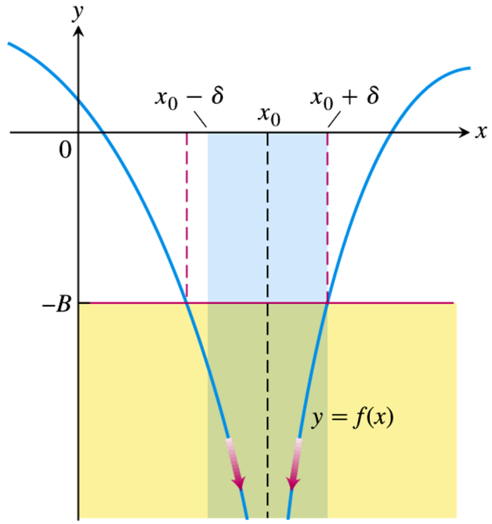


FIGURE 2.41 For $x_0 - \delta < x < x_0 + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

Example 4

- Using definition of infinit limit
- Prove that

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Given $B>0$, we want to find $\delta>0$ such that

$$0 <| x - 0 |< \delta \quad \text{implies} \quad \frac{1}{x^2} > B$$

Example 4

Now

$$\frac{1}{x^2} > B \text{ if and only if } x^2 < 1/B \equiv | x |< 1/\sqrt{B}$$

By choosing $\delta = 1/\sqrt{B}$

(or any smaller positive number), we see that

$$| x |< \delta \quad \text{implies} \quad \frac{1}{x^2} > \frac{1}{\delta^2} \geq B$$

Vertical asymptotes

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

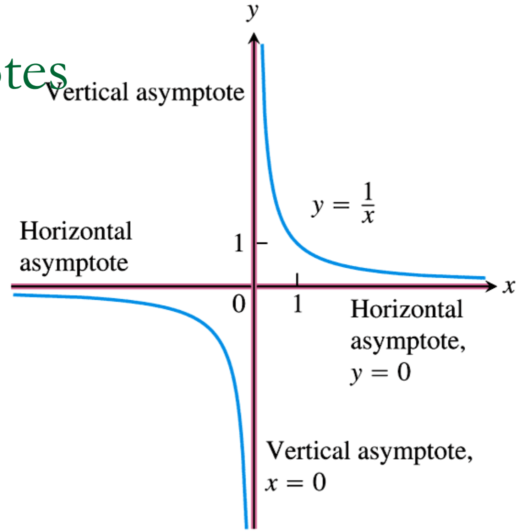


FIGURE 2.42 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

DEFINITION Vertical Asymptote

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty.$$

Example 5 Looking for asymptote

- Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x+3}{x+2}$$

- Solution:

$$y = 1 + \frac{1}{x+2}$$

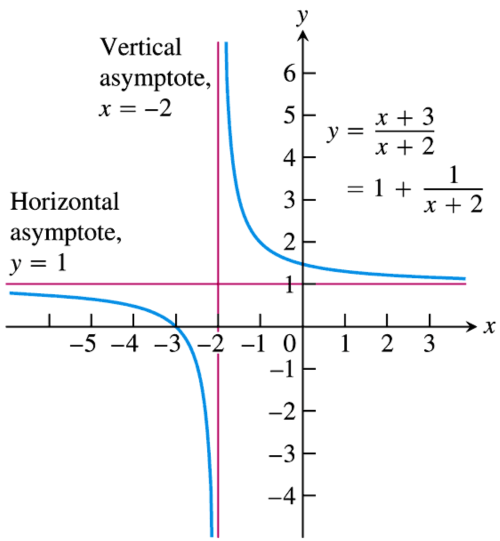


FIGURE 2.43 The lines $y = 1$ and $x = -2$ are asymptotes of the curve $y = (x + 3)/(x + 2)$ (Example 5).

Asymptotes need not be two-sided

- Example 6

$$f(x) = -\frac{8}{x^2 - 2}$$

- Solution:

$$f(x) = -\frac{8}{x^2 - 2} = -\frac{8}{(x-2)(x+2)}$$

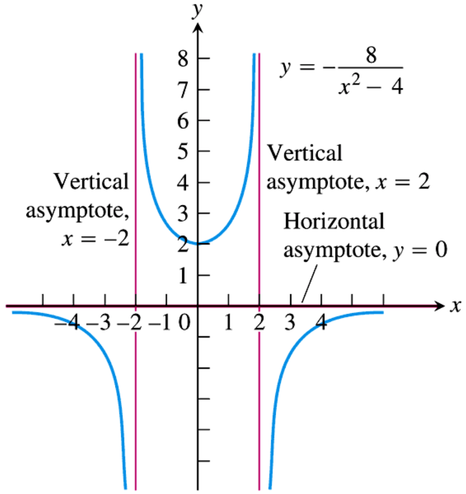


FIGURE 2.44 Graph of $y = -8/(x^2 - 4)$. Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided (Example 6).

Example 8

- A rational function with degree of numerator greater than degree of denominator

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

- Solution:

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\frac{x}{2} + 1}_{\text{linear}} + \underbrace{\frac{1}{2x - 4}}_{\text{remainder}}$$

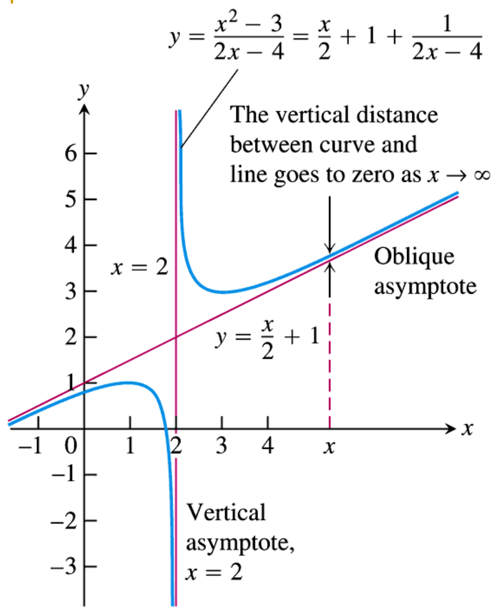


FIGURE 2.47 The graph of $f(x) = (x^2 - 3)/(2x - 4)$ has a vertical asymptote and an oblique asymptote (Example 8).

2.6

Continuity

Continuity at a point

- Example 1
- Find the points at which the function *f* in Figure 2.50 is continuous and the points at which *f* is discontinuous.

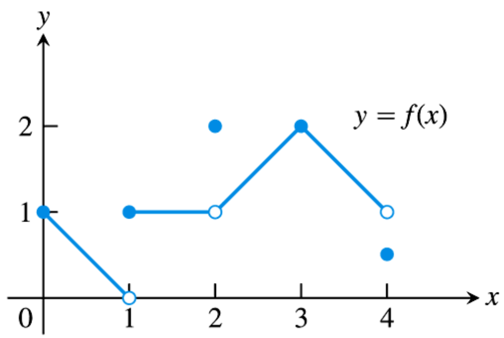


FIGURE 2.50 The function is continuous on [0, 4] except at $x = 1, x = 2$, and $x = 4$ (Example 1).

- *f* continuous:
- At $x = 0$
- At $x = 3$
- At $0 < c < 4$, $c \neq 1,2$

- *f* discontinuous:
- At $x = 1$
- At $x = 2$
- At $x = 4$
- $0 > c$, $c > 4$
- Why?

- To define the continuity at any point in a function's domain, we need to define continuity at an interior point and continuity at an endpoint

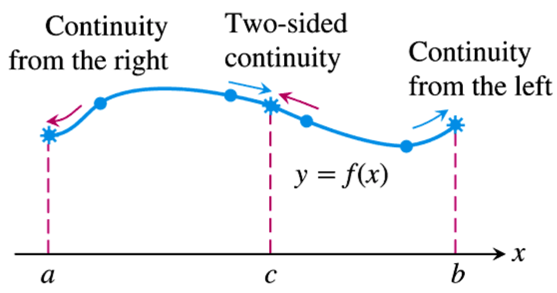


FIGURE 2.51 Continuity at points *a*, *b*, and *c*.

DEFINITION Continuous at a Point

Interior point: A function $y = f(x)$ is **continuous at an interior point *c*** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint *a*** or is **continuous at a right endpoint *b*** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

Example 2

- A function continuous throughout its domain

$$f(x) = \sqrt{4 - x^2}$$

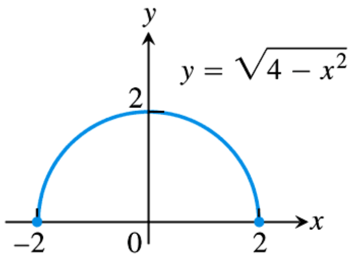


FIGURE 2.52 A function that is continuous at every domain point (Example 2).

Example 3

- The unit step function has a jump discontinuity

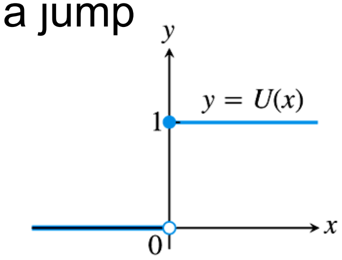


FIGURE 2.53 A function that is right-continuous, but not left-continuous, at the origin. It has a jump discontinuity there (Example 3).

Summarize continuity at a point in the form of a test

Continuity Test
A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value)

For one-sided continuity and continuity at an endpoint, the limits in part 2 and part 3 of the test should be replaced by the appropriate one-sided limits.

Example 4

- The greatest integer function,
- y=int x
- The function is not continuous at the integer points since limit does not exist there (left and right limits not agree)

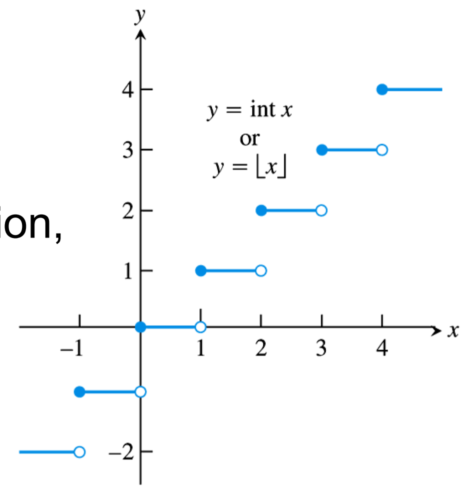


FIGURE 2.54 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

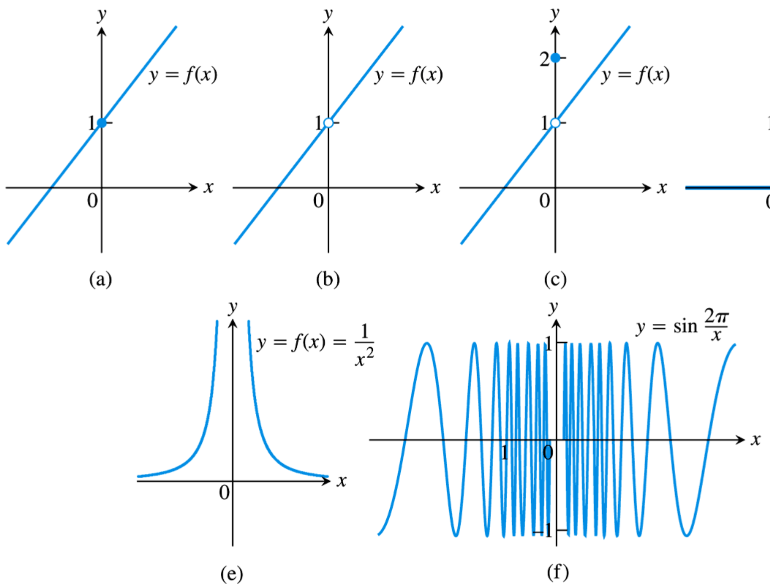


FIGURE 2.55 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

Discontinuity types

- (b), (c) removable discontinuity
- (d) jump discontinuity
- (e) infinite discontinuity
- (f) oscillating discontinuity

Continuous functions

- A function is continuous on an interval if and only if it is continuous at every point of the interval.
- Example: Figure 2.56
- 1/*x* not continuous on [-1,1] but continuous over (-∞,0) ∪ (0, ∞)

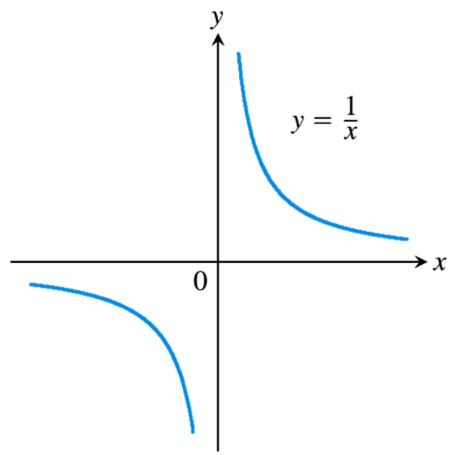


FIGURE 2.56 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$ (Example 5).

Example 5

- Identifying continuous function
- (a) $f(x)=1/x$
- (b) $f(x)= x$
- Ask: is $1/x$ continuous over its domain?

THEOREM 9 Properties of Continuous Functions
If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

Example 6

- Polynomial and rational functions are continuous
- (a) Every polynomial is continuous by
- (i) $\lim_{x \to c} P(x) = P(c)$
- (ii) Theorem 9
- (b) If $P(x)$ and $Q(x)$ are polynomial, the rational function $P(x)/Q(x)$ is continuous whenever it is defined.

Example 7

- Continuity of the absolute function
- $f(x) = |x|$ is everywhere continuous

- Continuity of the sinus and cosinus function
- $f(x) = \cos x$ and $\sin x$ is everywhere continuous

Composites

- All composites of continuous functions are continuous

THEOREM 10 Composite of Continuous Functions
If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

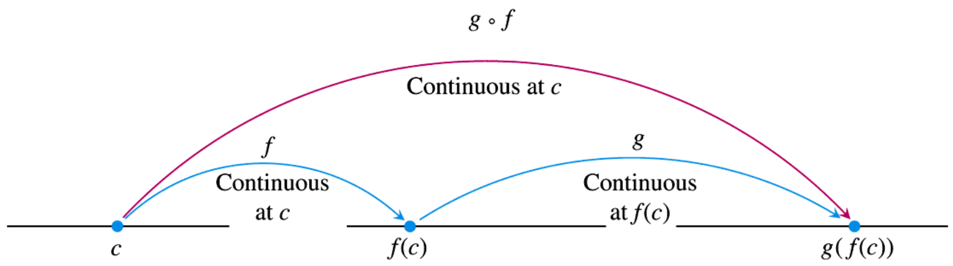


FIGURE 2.57 Composites of continuous functions are continuous.

Example 8

- Applying Theorems 9 and 10
- Show that the following functions are continuous everywhere on their respective domains.

$$(a)\ y = \sqrt{x^2 - 2x - 5} \qquad (b)\ y = \frac{x^{2/3}}{1 + x^4}$$

(a) $y = \sqrt{x^2 - 2x - 5}$

(a) The square root function is continuous on $[0, \infty)$ because it is a rational power of the continuous identity function $f(x) = x$ (Part 6, Theorem 9). The given function is then the composite of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$.

$$y(x) = f \circ g;$$

$$f(t) = \sqrt{t} = t^{1/2};$$

$$g(x) = x^2 + 2x - 5$$

g(*x*) is continuous in all *x* since it is a polynomial, according to Example 6.

f(*t*) is continuous in all *t* due to Part 6 in Theorem 9.

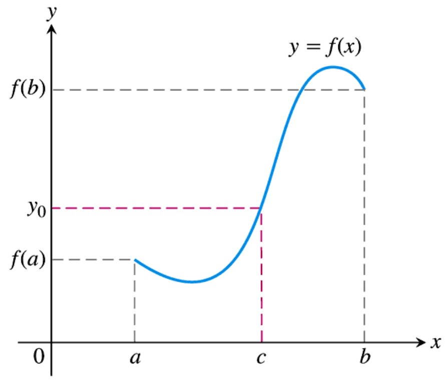
Hence, *f* [*g*(*x*)] = is continuous, according to Theorem 10.

(b) $y = \dfrac{x^{2/3}}{1 + x^4}$

The numerator is a rational power of the identity function; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.

This is the application of theorem 9.

THEOREM 11 The Intermediate Value Theorem for Continuous Functions
A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



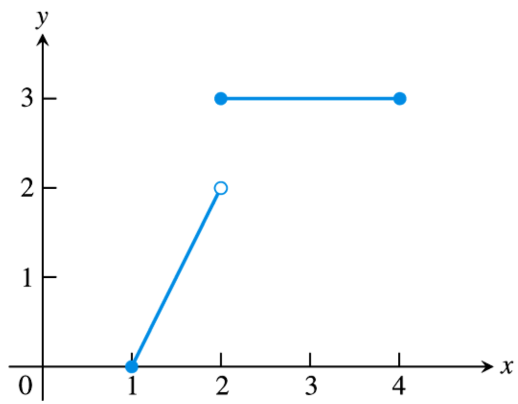


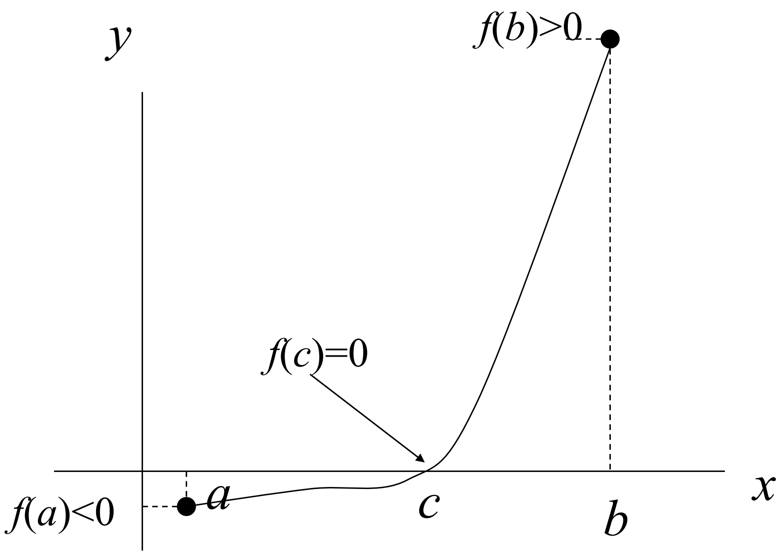
FIGURE 2.61 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

Consequence of root finding

- A solution of the equation $f(x)=0$ is called a root.
- For example, $f(x)= x^2 + x - 6$, the roots are $x=2$, $x=-3$ since $f(-3)=f(2)=0$.
- Say f is continuous over some interval.
- Say a, b (with $a < b$) are in the domain of f, such that $f(a)$ and $f(b)$ have opposite signs.
- This means either $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$
- Then, as a consequence of theorem 11, there must exist at least a point c between a and b, i.e. $a < c < b$ such that $f(c)= 0$. $x=c$ is the root.



Example

- Consider the function *f*(*x*) = *x* - cos *x*
- Prove that there is at least one root for *f*(*x*) in the interval [0, $\pi/2$].

- **Solution**
- *f*(*x*) is continuous on (-∞, ∞).
- Say *a* = 0, *b* = $\pi/2$.
- *f*(*x*=0) = -1; *f*(*x* = $\pi/2$) = $\pi/2$
- *f*(*a*) and *f*(*b*) have opposite signs
- Then, as a consequence of theorem 11, there must exist at least a point *c* between *a* and *b*, i.e. *a*=0 < *c* < *b*= $\pi/2$ such that *f*(*c*)= 0. *x*=*c* is the root.

2.7

Tangents and Derivatives

What is a tangent to a curve?

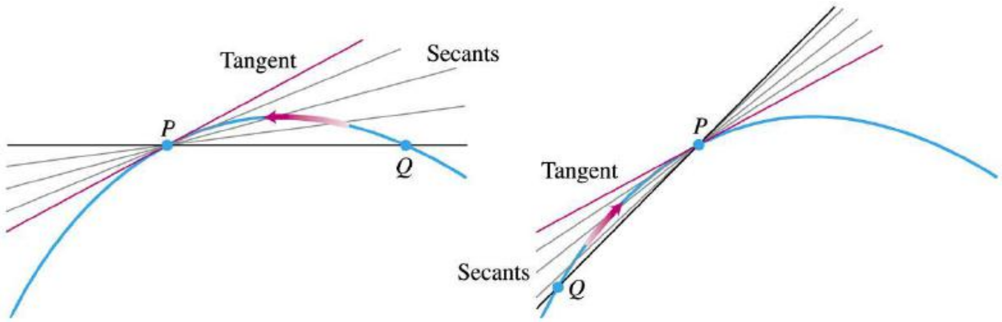


FIGURE 2.65 The dynamic approach to tangency. The tangent to the curve at *P* is the line through *P* whose slope is the limit of the secant slopes as $Q \to P$ from either side.

See Mathematica simulation, 2.7_tangent.nb

DEFINITIONS **Slope, Tangent Line**

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

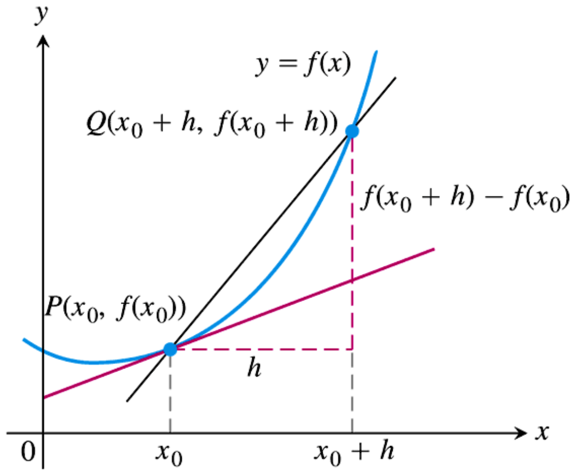


FIGURE 2.67 The slope of the tangent line at P is $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

Example 1: Tangent to a parabola

- Find the slope of the parabola $y=x^2$ at the point $P(2,4)$. Write an equation for the tangent to the parabola at this point.

Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope
$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
3. If the limit exists, find the tangent line as
$$y = y_0 + m(x - x_0).$$

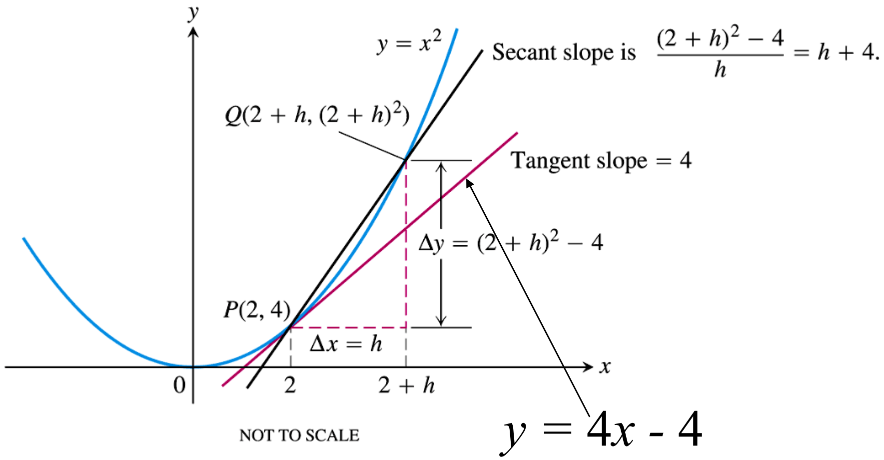


FIGURE 2.66 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ (Example 1).

Example 3

- Slope and tangent to $y=1/x$, $x\neq 0$
- (a) Find the slope of $y=1/x$ at $x = a \neq 0$
- (b) Where does the slope equal -1/4?
- (c) What happens to the tangent of the curve at the point (a, $1/a$) as a changes?
- *(see Mathematica simulation, 2.7_tangent.nb)*

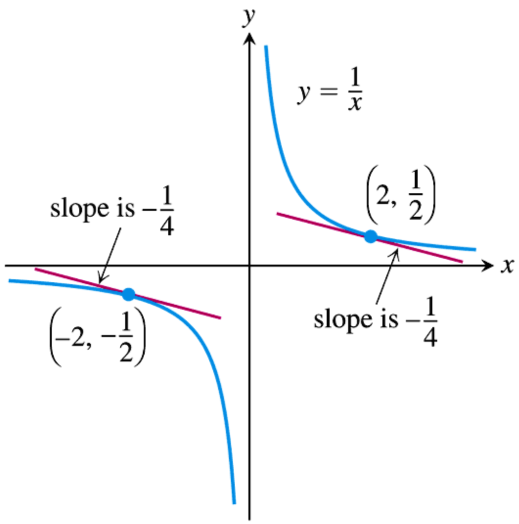


FIGURE 2.68 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 3).

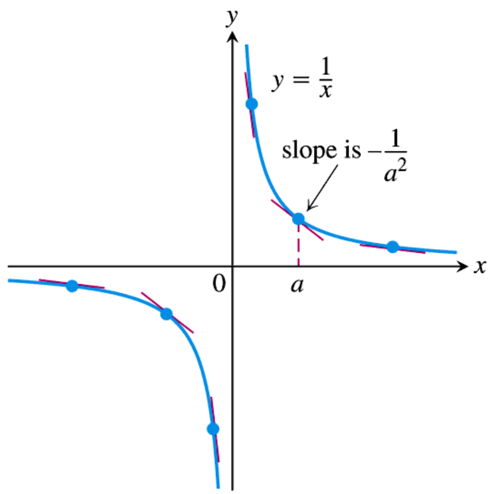


FIGURE 2.69 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away.

difference quotient of f at x_0 with increment h.

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

- If the limit $h \rightarrow 0$ of the quotient exists, it is called

derivative of f at x_0.

Chapter 3

Differentiation

3.1

The Derivative as a Function

DEFINITION **Derivative Function**

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

- The limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

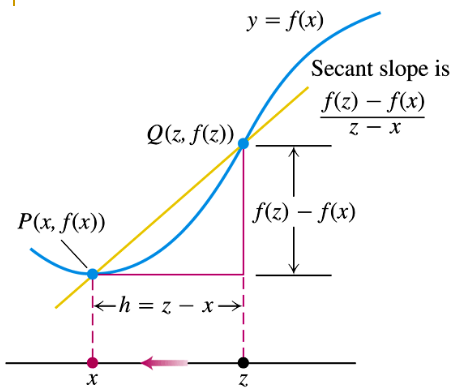
- when it existed, is called the Derivative of f at x_0.
- View derivative as a function derived from f

- If f' exists at x, f is said to be differentiable (has a derivative) at x
- If f' exists at every point in the domain of f, f is said to be differentiable.

If write $z = x + h$, then $h = z - x$

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$



Derivative of f at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

FIGURE 3.1 The way we write the difference quotient for the derivative of a function f depends on how we label the points involved.

Calculating derivatives from the definition

- Differentiation: an operation performed on a function $y = f(x)$
- d/dx operates on $f(x)$
- Write as
$$\frac{d}{dx} f(x)$$
- f' is taken as a shorthand notation for
$$\frac{d}{dx} f(x)$$

Example 1: Applying the definition

- Differentiate
$$f(x) = \frac{x}{x-1}$$
- Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{x+h}{x+h-1}\right) - \left(\frac{x}{x-1}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}$$

Example 2: Derivative of the square root function

- (a) Find the derivative of $y = \sqrt{x}$ for $x>0$
- (b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$

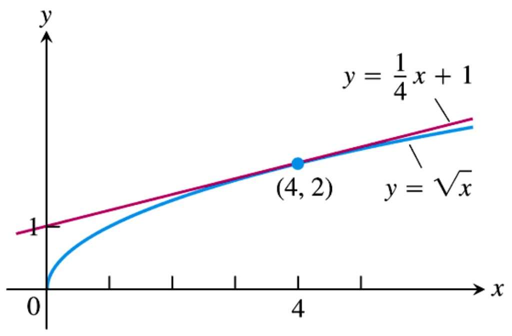


FIGURE 3.2 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

Notations

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx} f(x)\right|_{x=a}$$

Differentiable on an interval; One sided derivatives

- A function $y = f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval.
- It is differentiable on a closed interval $[a,b]$ if it is differentiable on the interior (a,b) and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h}$$

$$\lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h}$$

exist at the endpoints

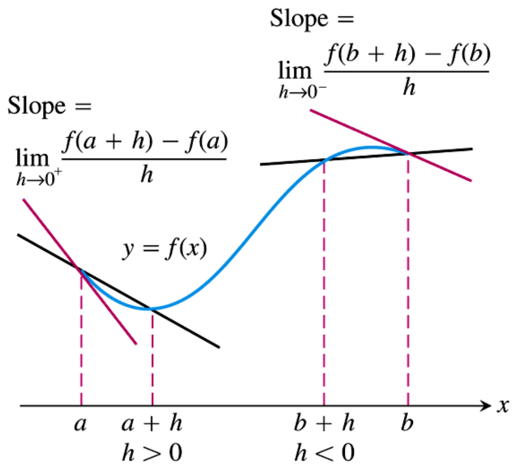


FIGURE 3.5 Derivatives at endpoints are one-sided limits.

- A function has a derivative at a point if an only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

Example 5

- $y = |x|$ is not differentiable at $x = 0$.
- Solution:
- For $x > 0$,

$$\frac{d\,|x|}{dx} = \frac{d}{dx}(x) = 1$$

- For $x < 0$,

$$\frac{d\,|x|}{dx} = \frac{d}{dx}(-x) = -1$$

- At $x = 0$, the right hand derivative and left hand derivative differ there. Hence $f(x)$ not differentiable at $x = 0$ but else where.

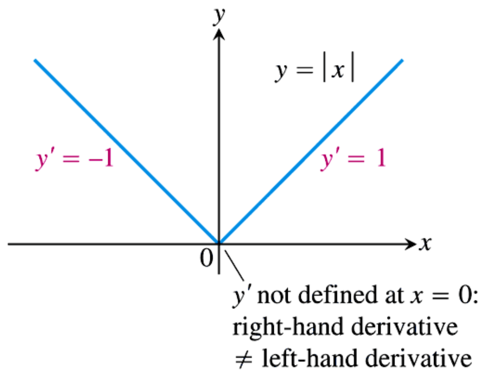


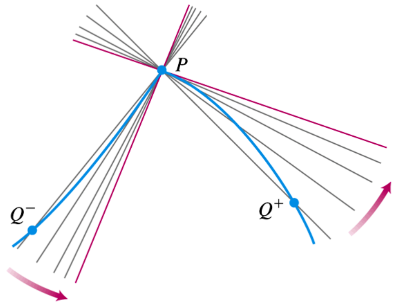
FIGURE 3.6 The function $y = |x|$ is not differentiable at the origin where the graph has a "corner."

Example 6

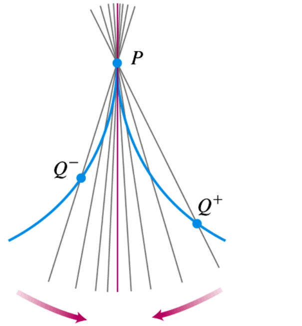
- $y = \sqrt{x}$ is not differentiable at $x = 0$
- The graph has a vertical tangent at $x = 0$

When Does a function not have a derivative at a point?

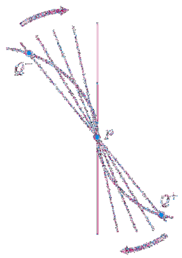
1. a *corner*, where the one-sided derivatives differ.



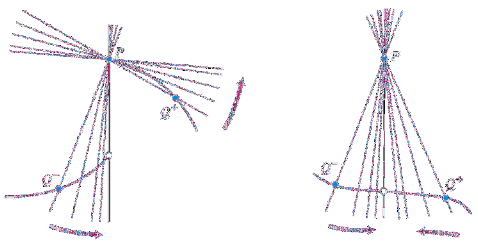
2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.



3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).



4. a *discontinuity*.



Differentiable functions are continuous

THEOREM 1 Differentiability Implies Continuity
If f has a derivative at $x = c$, then f is continuous at $x = c$.

The converse is false: continuity does not necessarily implies differentiability

Example

- $y = |x|$ is continuous everywhere, including $x = 0$, but it is not differentiable there.

The equivalent form of Theorem 1

- If f is not continuous at $x = c$, then f is not differentiable at $x = c$.
- Example: the step function is discontinuous at $x = 0$, hence not differentiable at $x = 0$.

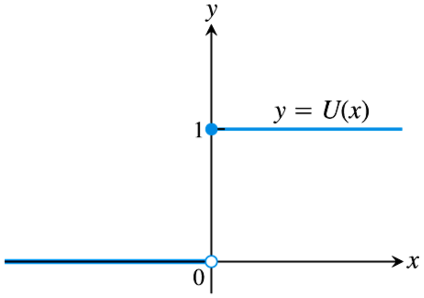


FIGURE 3.7 The unit step function does not have the Intermediate Value Property and cannot be the derivative of a function on the real line.

The intermediate value property of derivatives

THEOREM 2 Darboux's Theorem
If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

- See section 4.4

3.2

Differentiation Rules

Powers, multiples, sums and differences

RULE 1 Derivative of a Constant Function
If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Example 1

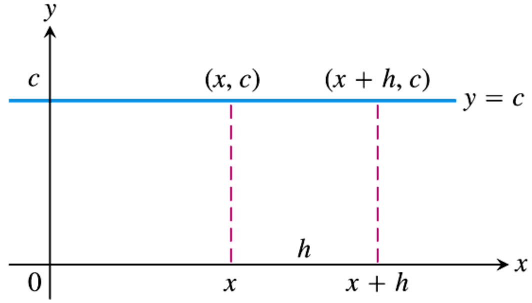


FIGURE 3.8 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

RULE 2 Power Rule for Positive Integers
If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

RULE 3 Constant Multiple Rule
If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

In particular, if $u = x^n$, $\frac{d}{dx}(cx^n) = cx^{n-1}$

Example 3

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x^{2-1} = 6x$$

$$\frac{d}{dx}(x^2) = 2x^{2-1} = 2x$$

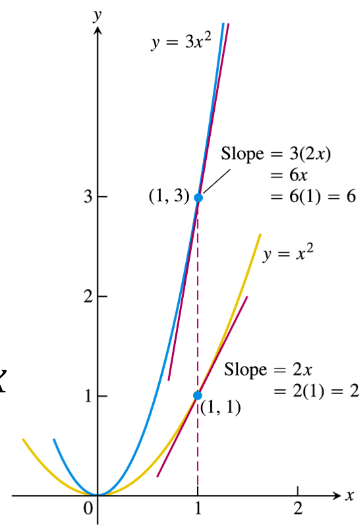


FIGURE 3.9 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinates triples the slope (Example 3).

RULE 4 Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Example 5

$$y = x^3 + \frac{4}{3}x^2 - 5x + 1$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(\frac{4}{3}x^2) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$= 3x^2 + \frac{8}{3}x - 5$$

Example 6

- Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

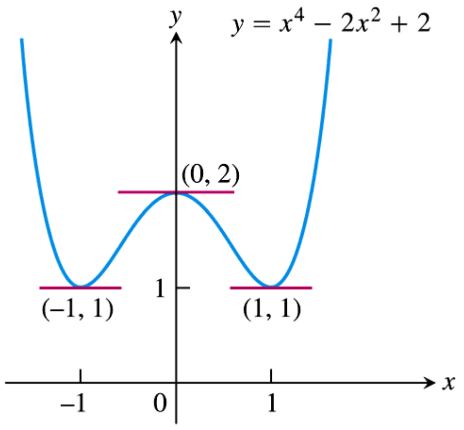


FIGURE 3.10 The curve $y = x^4 - 2x^2 + 2$ and its horizontal tangents (Example 6).

Products and quotients

- Note that

$$\frac{d}{dx}(x\cdot x)=\frac{d}{dx}(x^2)=2x$$
$$\frac{d}{dx}(x\cdot x)\neq\frac{d}{dx}(x)\cdot\frac{d}{dx}(x)=1$$

RULE 5 Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}.$$

Example 7

- Find the derivative of $y=\frac{1}{x}\left(x^2+\frac{1}{x}\right)$

Example 8: Derivative from numerical values

- Let $y = uv$. Find $y'(2)$ if $u(2)=3$, $u'(2)=-4$, $v(2) = 1$, $v'(2) = 2$

Example 9

- Find the derivative of $y=(x^2+1)(x^3+3)$

RULE 6 Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Negative integer powers of x

- The power rule for negative integers is the same as the rule for positive integers

RULE 7 Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Example 11

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right) = (-1)\,x^{-1-1} = -x^{-2}$$

$$\frac{d}{dx}\left(\frac{4}{x^3}\right) = \frac{d}{dx}\left(4x^{-3}\right) = 4\cdot(-3)\,x^{-3-1} = 12x^{-4}$$

Example 12: Tangent to a curve

- Find the tangent to the curve $y = x + \frac{2}{x}$ at the point (1,3)

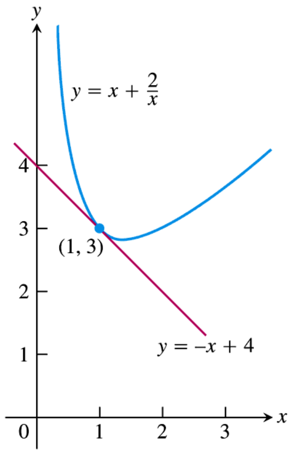


FIGURE 3.11 The tangent to the curve $y = x + (2/x)$ at (1, 3) in Example 12. The curve has a third-quadrant portion not shown here. We see how to graph functions like this one in Chapter 4.

Example 13

- Find the derivative of $y = \dfrac{(x-1)(x^2-2x)}{x^4}$

Second- and higher-order derivative

- Second derivative

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(y')$$
$$= y'' = D^2(f)(x) = D_x^2 f(x)$$

- *n*th derivative

$$y^{(n)} = \frac{d}{dx} y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y$$

Example 14

$$y = x^3 - 3x^2 + 2$$
$$y' = 3x^2 - 6x$$
$$y'' = 6x - 6$$
$$y''' = 6$$
$$y^{(4)} = 0$$

3.3

The Derivative as a Rate of Change

Instantaneous Rates of Change

DEFINITION **Instantaneous Rate of Change**

The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Example 1: How a circle's area changes with its diameter

- $A = \pi D^2/4$
- How fast does the area change with respect to the diameter when the diameter is 10 m?

Motion along a line

- Position $s = f(t)$
- Displacement, $\Delta s = f(t+ \Delta t) - f(t)$
- Average velocity
- $v_{av} = \Delta s/\Delta t = [f(t+ \Delta t) - f(t)] /\Delta t$
- The instantaneous velocity is the limit of v_{av} when $\Delta t \to 0$

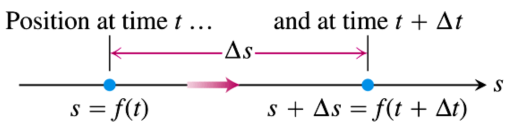


FIGURE 3.12 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$.

DEFINITION **Velocity**

Velocity (instantaneous velocity) is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

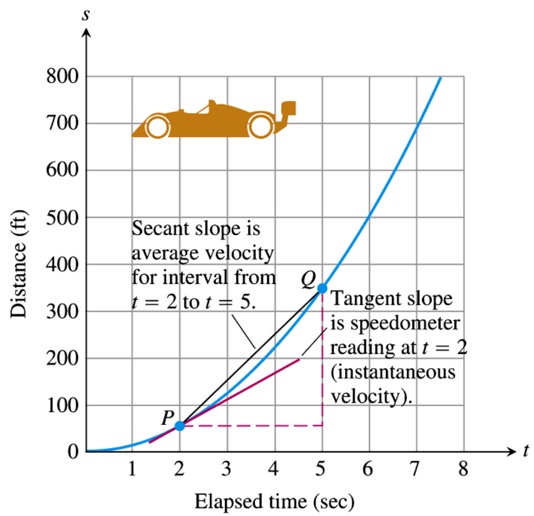


FIGURE 3.13 The time-to-distance graph for Example 2. The slope of the tangent line at P is the instantaneous velocity at $t = 2$ sec.

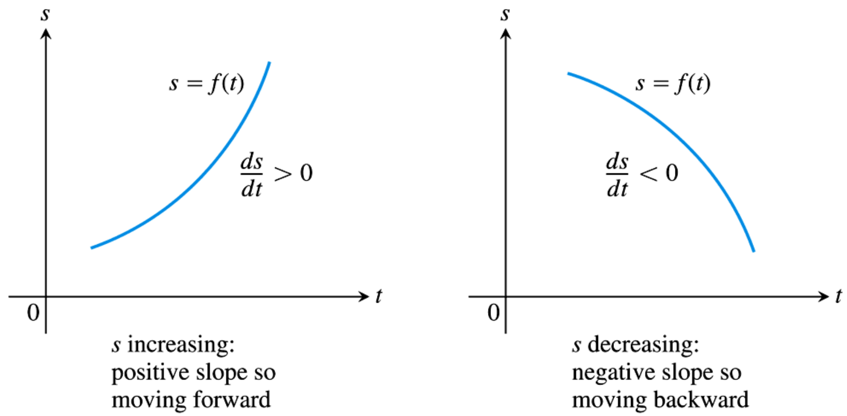


FIGURE 3.14 For motion $s = f(t)$ along a straight line, $v = ds/dt$ is positive when s increases and negative when s decreases.

DEFINITION Speed

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

Example 3

- Horizontal motion

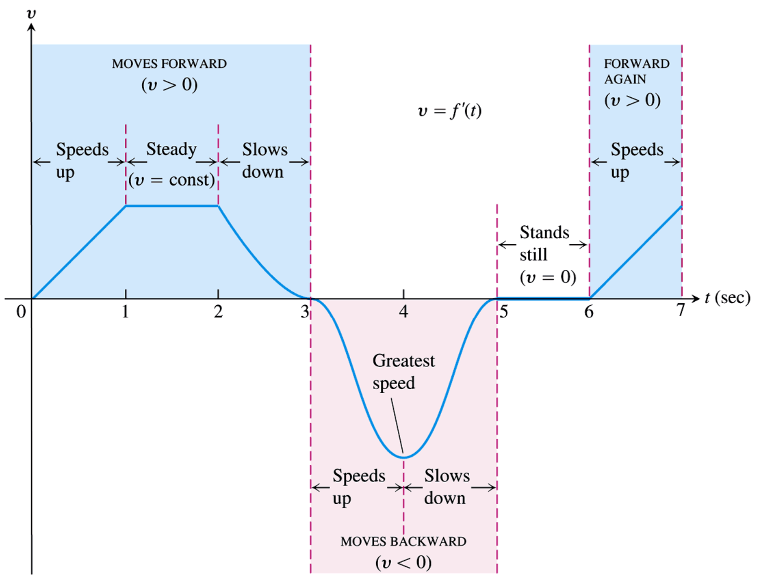


FIGURE 3.15 The velocity graph for Example 3.

DEFINITIONS Acceleration, Jerk

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Example 4

- Modeling free fall $s = \frac{1}{2}gt^2$
- Consider the free fall of a heavy ball released from rest at t = 0 sec.
- (a) How many meters does the ball fall in the first 2 sec?
- (b) What is the velocity, speed and acceleration then?

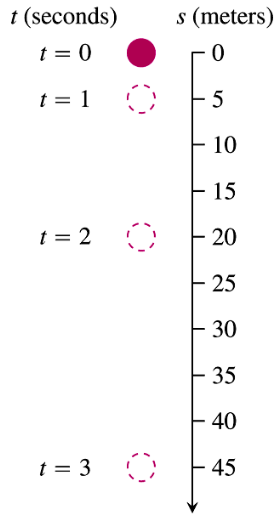


FIGURE 3.16 A ball bearing falling from rest (Example 4).

Modeling vertical motion

- A dynamite blast blows a heavy rock straight up with a launch velocity of 160 m/sec. It reaches a height of $s = 160t - 16t^2$ ft after t sec.
- (a) How high does the rock go?
- (b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?
- (c) What is the acceleration of the rock at any time t during its flight?
- (d) When does the rock hit the ground again?

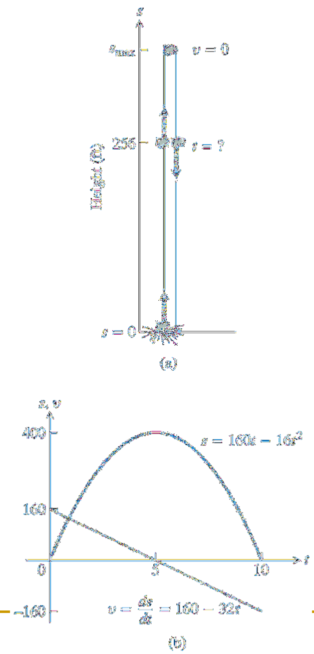


FIGURE 3.17 (a) The rock in Example 5. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

3.4

Derivatives of Trigonometric Functions

Derivative of the sine function

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \cdots$$

Derivative of the cosine function

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \cdots$$

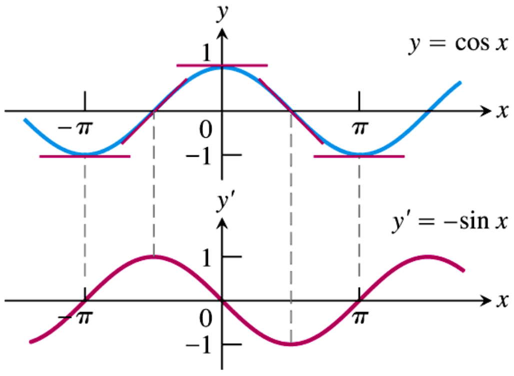


FIGURE 3.23 The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

Example 2

$$(a)\, y = 5x + \cos x$$

$$(b)\, y = \sin x \cos x$$

$$(c)\, y = \frac{\cos x}{1 - \sin x}$$

Derivative of the other basic trigonometric functions

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Example 5

- Find *d*(tan *x*)/*dx*

Example 6

- Find *y''* if *y* = sec *x*

Example 7: Finding a trigonometric limit

- Trigonometric functions are differentiable, hence are continuous throughout their domains.
- So we can calculate limits of algebraic combinations and composites of trigonometric functions by direct substitution.

$$\lim_{x\to 0}\frac{\sqrt{2+\sec x}}{\cos(\pi-\tan x)}=\frac{\sqrt{2+\sec 0}}{\cos(\pi-\tan 0)}$$
$$=\frac{\sqrt{2+1}}{\cos(\pi-0)}=\frac{\sqrt{3}}{-1}=-\sqrt{3}$$

- Note that you can only evaluate the limit of the form

$$\lim_{x\to x_0}\frac{P(x)}{Q(x)}$$

by direct substitution, i.e.,

$$\lim_{x\to x_0}\frac{P(x)}{Q(x)}=\frac{P(x_0)}{Q(x_0)}$$

only when $P(x)$ and $Q(x)$ are both continuous at x_0

3.5

The Chain Rule and Parametric Equations

Differentiating composite functions

- Example:
- $y = f(u) = \sin u$
- $u = g(x) = x^2 - 4$
- How to differentiate $F(x) = f \circ g = f[g(x)]$?
- Use chain rule

Derivative of a composite function

- Example 1: Relating derivatives
- *y* = (3/2)*x* = (1/2)(3*x*)
-    = *y*[*u*(*x*)]
- *y*(*u*) = *u*/2; *u*(*x*) = 3*x*
- *dy/dx* = 3/2;
- *dy/du* = ½; *du/dx* = 3;
- *dy/dx* = (*dy/du*)·(*du/dx*) (Not an accident)

Example 2

$$y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$$

$$y = u^2; u = 3x^2 + 1$$

$$\frac{dy}{du}\cdot\frac{du}{dx} = 2u \cdot 6$$

$$= 2(3x^2 + 1)\cdot 6x = 36x^3 + 12x$$

c.f.

$$\frac{dy}{dx} = \frac{d}{dx}\left(9x^4 + 6x^2 + 1\right) = 36x^3 + 12x$$

THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

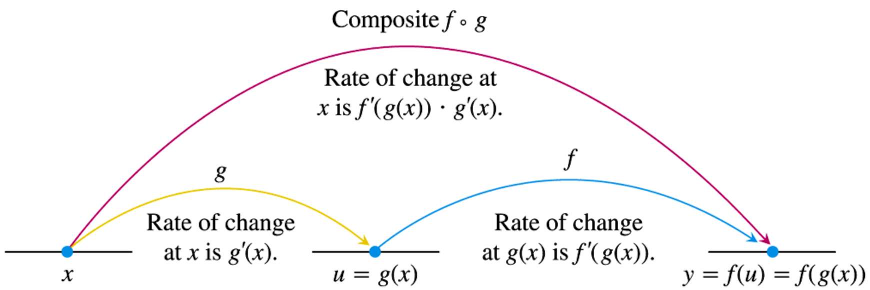


FIGURE 3.27 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x.

Example 3

- Applying the chain rule
- $x(t)= \cos(t^2 + 1)$. Find dx/dt.
- Solution:
- $x(u)= \cos(u)$; $u(t)= t^2 + 1$;
- $dx/dt = (dx/du)\cdot(du/dt) = \ldots$

Alternative form of chain rule

- If $y = f[g(x)]$, then
- $dy/dx = f'[g(x)]\cdot g'(x)$
- Think of f as 'outside function', g as 'inside-function', then
- dy/dx = differentiate the outside function and evaluate it at the inside function let alone; then multiply by the derivative of the inside function.

Example 4

- Differentiating from the outside in:

$$y = \sin(x^2 + x) = f(u) = f[g(x)]$$

$$\underbrace{f(u) = \sin u}_{\text{outside function}}; \quad \underbrace{g(x) = x^2 + x}_{\text{inside function}}$$

$$\frac{dy}{dx} = f'[g(x)]\cdot g'(x)$$

$$\frac{dy}{dx} = \cos \underbrace{(x^2 + x)}_{\substack{\text{inside function}\\ \text{left alone}}} \cdot \underbrace{(2x+1)}_{\substack{\text{derivative of}\\ \text{the inside function}}}$$

Example 5

- A three-link 'chain'
- Find the derivative of $g(t) = \tan(5 - \sin 2t)$

$$\begin{aligned} g'(t) &= \frac{d}{dt}\left(\tan\left(5 - \sin 2t\right)\right) \\ &= \sec^2(5 - \sin 2t)\cdot\frac{d}{dt}\left(5 - \sin 2t\right) \\ &= \sec^2(5 - \sin 2t)\cdot\left(0 - \cos 2t\cdot\frac{d}{dt}(2t)\right) \\ &= \sec^2(5 - \sin 2t)\cdot(-\cos 2t)\cdot 2 \end{aligned}$$

Example 6

- Applying the power chain rule

$$(a)\frac{d}{dx}(5x^3 - x^4)^7$$

$$(b)\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$$

Example 7

- (a) Find the slope of tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$
- (b) Show that the slope of every line tangent to the curve $y = 1/(1-2x)^3$ is positive

Parametric equations

- A way of expressing both the coordinates of a point on a curve, (*x*,*y*) as a function of a third variable, *t*.
- The path or locus traced by a point particle on a curve is then well described by a set of two equations:
- $x = f(t)$, $y = g(t)$

DEFINITION Parametric Curve
If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

The variable t is a parameter for the curve

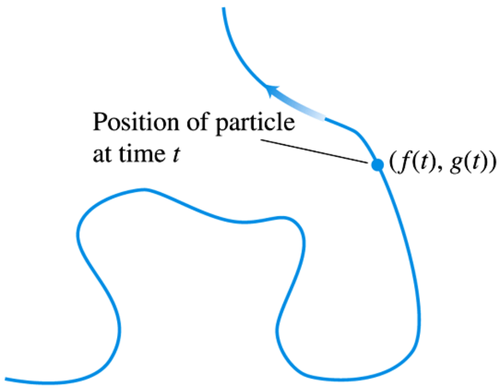


FIGURE 3.29 The path traced by a particle moving in the xy-plane is not always the graph of a function of x or a function of y.

Example 9

- Moving counterclockwise on a circle
- Graph the parametric curves
- $x = \cos t$, $y = \sin t$,
  $0 \le t \le 2\pi$

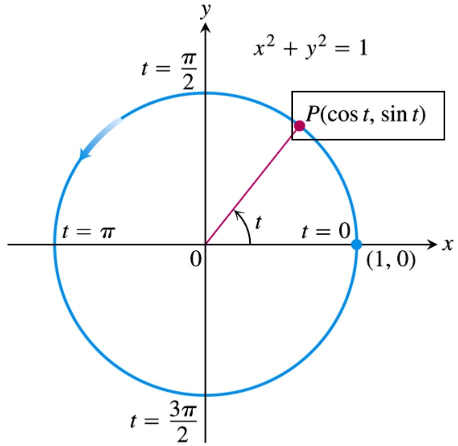


FIGURE 3.30 The equations $x = \cos t$ and $y = \sin t$ describe motion on the circle $x^2 + y^2 = 1$. The arrow shows the direction of increasing t (Example 9).

Example 10

- Moving along a parabola
- $x = \sqrt{t}$, $y = t$, $0 \le t$
- Determine the relation between x and y by eliminating t.
- $y = t = (\sqrt{t})^2 = x^2$
- The path traced out by P (the locus) is only half the parabola, $x \ge 0$

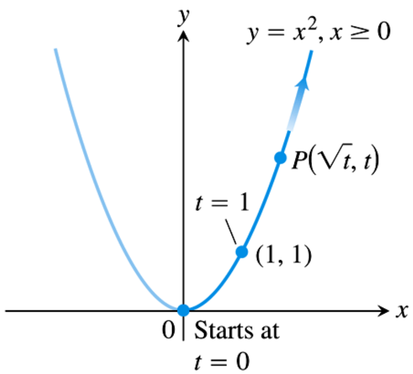


FIGURE 3.31 The equations $x = \sqrt{t}$ and $y = t$ and the interval $t \ge 0$ describe the motion of a particle that traces the right-hand half of the parabola $y = x^2$ (Example 10).

Slopes of parametrized curves

- A parametrized curved $x = f(t)$, $y = g(t)$ is differentiable at t if f and g are differentiable at t.
- At a point on a differentiable parametrised curve where y is also a differentiable function of x, i.e. $y = y(x) = y[x(t)]$,
- chain rule relates dx/dt, dy/dt, dy/dx via

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \qquad (2)$$

Example 12

- Differentiating with a parameter
- If $x = 2t + 3$ and $y = t^2 - 1$, find the value of dy/dx at $t = 6$.

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \qquad (3)$$

(3) is just the parametric formula (2) by

$y \longrightarrow y' = dy/dx$

Example 14 Finding d^2y/dx^2 for a parametrised curve

- Find d^2y/dx^2 as a function of t if $x = t - t^2$, $y = t - t^3$.

3.6

Implicit Differentiation

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

Example 1: Differentiating implicitly

- Find *dy/dx* if $y^2 = x$

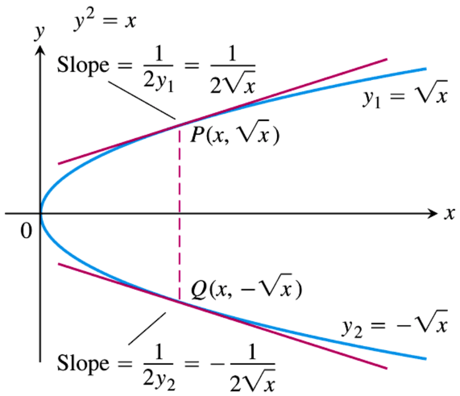


FIGURE 3.37 The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of x on the interval $x \geq 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for y.

Example 2

- Slope of a circle at a point
- Find the slope of circle $x^2 + y^2 = 25$ at (3, -4)

Example 3

- Differentiating implicitly
- Find *dy/dx* if $y^2 = x^2 + \sin xy$

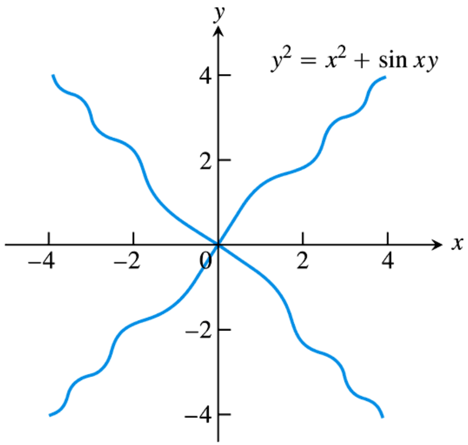


FIGURE 3.39 The graph of $y^2 = x^2 + \sin xy$ in Example 3. The example shows how to find slopes on this implicitly defined curve.

Lenses, tangents, and normal lines

If slop of tangent is m_t, the slope of normal, m_n, is given by the relation

$m_n m_t = -1$, or

$m_n = -1/m_t$

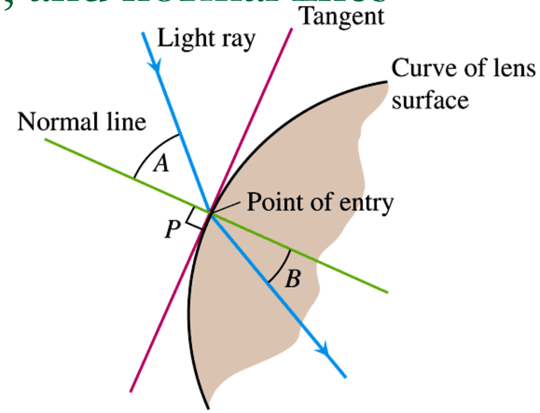


FIGURE 3.40 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

Example 4: Tangent and normal to the folium of Descartes

- Show that the point (2,4) lies on the curve $x^2 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there.

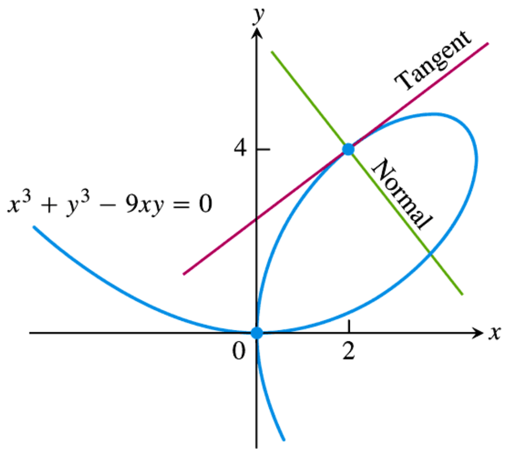


FIGURE 3.41 Example 4 shows how to find equations for the tangent and normal to the folium of Descartes at (2, 4).

Derivative of higher order

- Example 5
- Finding a second derivative implicitly
- Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Rational powers of differentiable functions

THEOREM 4 Power Rule for Rational Powers
If p/q is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$, and

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

Theorem 4 is proved based on $d/dx(x^n) = nx^{n-1}$ (where n is an integer) using implicit differentiation

- Theorem 4 provide a extension of the power chain rule to rational power:

$$\frac{d}{dx}u^{p/q} = \frac{p}{q}u^{(p/q)-1}\frac{du}{dx}$$

- $u \neq 0$ if $(p/q) < 1$, (p/q) rational number, u a differential function of x

Example 6

- Using the rational power rule
- (a) $d/dx\ (x^{1/2}) = 1/2x^{-1/2}$ for $x > 0$
- (b) $d/dx\ (x^{2/3}) = 2/3\ x^{-1/3}$ for $x \neq 0$
- (c) $d/dx\ (x^{-4/3}) = -4/3\ x^{-7/3}$ for $x \neq 0$

Proof of Theorem 4

- Let *p* and *q* be integers with *q* > 0 and

$$y = x^{p/q} \equiv y^q = x^p$$

- Explicitly differentiating both sides with respect to *x*…

Example 7

- Using the rational power and chain rules
- (a) Differentiate $(1-x^2)^{1/4}$
- (b) Differentiate $(\cos x)^{-1/5}$

3.7

Related Rates

EXAMPLE 1 Pumping Out a Tank

How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L / min?

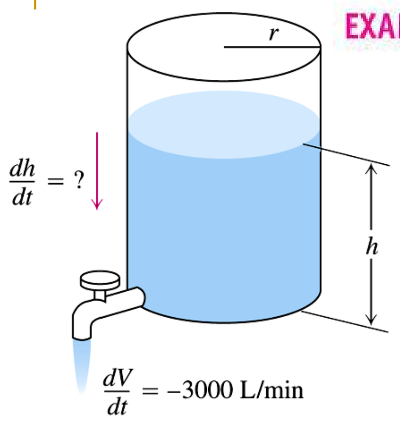


FIGURE 3.42 The rate of change of fluid volume in a cylindrical tank is related to the rate of change of fluid level in the tank (Example 1).

Solution

Geometrically, Volume *V*, is a function of height *h*, *V*=*V*(*h*)
Height, *h*, is a function of time, *h*=*h*(*t*). *r*, radius, is fixed.

Combining both, *V*=*V*[*r*(*t*)]

By chain rule, the derivative of *V* with respect to *t* is

$$\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt} \longrightarrow \frac{dh}{dt}=\frac{dV}{dt}/\frac{dV}{dh}$$

We are asked to find $\frac{dh}{dt}$, given $\frac{dV}{dt}=-3000\,L/min$

Solution

$$V=\pi r^2 h \Rightarrow \frac{dV}{dh}=\pi r^2 \qquad \frac{dV}{dt}=-3000\,L/min$$

$$\frac{dh}{dt}=\frac{dV}{dt}/\frac{dV}{dh} \longrightarrow \frac{dh}{dt}=\frac{-3000L}{min}\cdot\frac{1}{\pi r^2}$$

In this example, conversion of unit must be taken care of properly

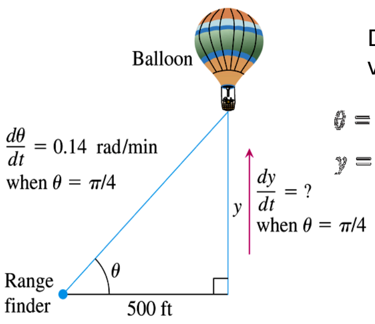
$$1\,m^3=1000\,L$$

If *r* = 1 m, $\frac{dh}{dt}=\frac{-3000\left(10^{-3}m^3\right)}{min}\cdot\frac{1}{\pi(1m)^2}=-\frac{3}{\pi}\frac{m}{min}$

If *r* = 10 m, $\frac{dh}{dt}=\frac{-3000\left(10^{-3}m^3\right)}{min}\cdot\frac{1}{\pi(10m)^2}=-\frac{3}{100\pi}\frac{m}{min}$

EXAMPLE 2 A Rising Balloon

A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?



Draw the scenario and label the relevant variables (and name them)

θ = the angle in radians the range finder makes with the ground.

y = the height in feet of the balloon.

FIGURE 3.43 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

EXAMPLE 2 A Rising Balloon

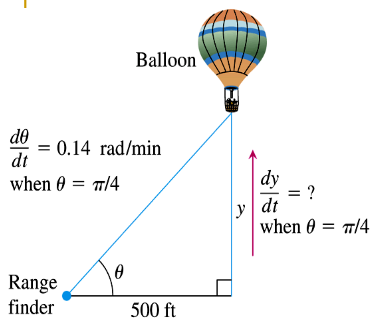


FIGURE 3.43 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

Geometrically, *y* is a function of angle θ.

$$y=x\tan\theta$$

θ is a function of time, $\theta=\theta(t)$.

x, the horizontal distance,is fixed.

Combining both, *y* = *y* [$\theta(t)$]

By chain rule, the derivative of *y* with respect to *t* is

$$\frac{dy}{dt}=\frac{dy}{d\theta}\frac{d\theta}{dt}$$

EXAMPLE 2 A Rising Balloon

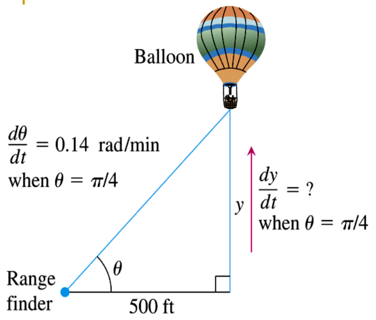


FIGURE 3.43 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

$$y = x\tan\theta \qquad \frac{dy}{d\theta} = x\sec^2\theta$$

$$\frac{dy}{dt} = x\sec^2\theta \cdot \frac{d\theta}{dt}$$

$$\text{Given } \frac{d\theta}{dt} = 0.14(\text{rad/min})$$

$$\text{at } \theta = \pi/4,$$

$$\frac{dy}{dt} = \frac{500\text{ft}\cdot\sec^2\frac{\pi}{4}}{(0.14\text{ rad/min})}$$

$$= 500\text{ ft}\cdot\left(\sqrt{2}\right)^2\cdot(0.14\text{ rad/min}) = 140\text{ft/min}$$

Note: radian is dimensionless (hence unit-less)

3.8

Linearization and differentials

Linearization

- Say you have a very complicated function, $f(x)$=sin (cot $^2 x$), and you want to calculate the value of $f(x)$ at $x = \pi/2 + \delta$, where δ is a very tiny number. The value sought can be estimated within some accuracy using linearization.

Refer to graph Figure 3.47.

- The point-slope equation of the tangent line passing through the point $(a, f(a))$ on a differentiable function f at $x=a$ is
- $y = mx + c$, where c is $c = f(a) - f'(a)\, a$
- Hence the tangent line is the graph of the linear function
- $L(x) = m \quad x + c$

  $= f'(a)\, x + [f(a) - a\, f'(a)]$

  $= f(a) + f'(a)\,(x - a)$

Definitions

- The tangent line $L(x) = f(a) + f'(a)(x - a)$ gives a good approximation to $f(x)$ as long as x is not to be too far away from $x=a$.
- Or in other words, we say that $L(x)$ is the linearization of f at a.
- The approximation $f(x) \approx L(x)$ of f by L is the standard linear approximation of f at a.
- The point $x = a$ is the center of the approximation.

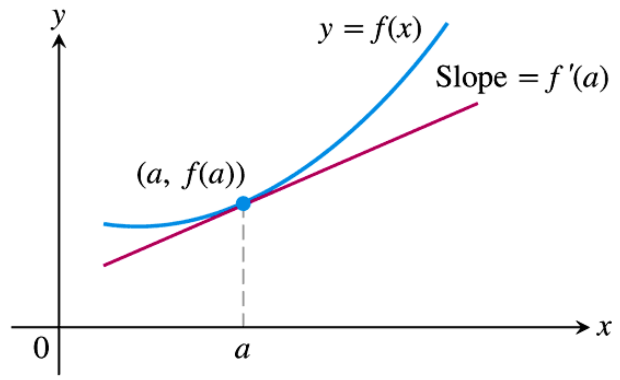


FIGURE 3.47 The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

Example 1 Finding Linearization

- Find the linearization of

$$f(x) = \sqrt{1+x}$$

- at $x = 0$.

$$f(x) = \sqrt{1+x}$$

$$f'(x) = \frac{1}{2(1+x)^{1/2}}$$

The linearization of $f(x)$ at $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) = (1+a)^{1/2} + \frac{1}{2(1+a)^{1/2}}(x-a)$$

$a = 0,$

$f'(0) = \frac{1}{2}; f(0) = 1;$

The linearization of $f(x)$ at $x = a = 0$ is $L(x) = 1 + x/2$

We write $f(x) \approx L(x) = 1 + x/2$

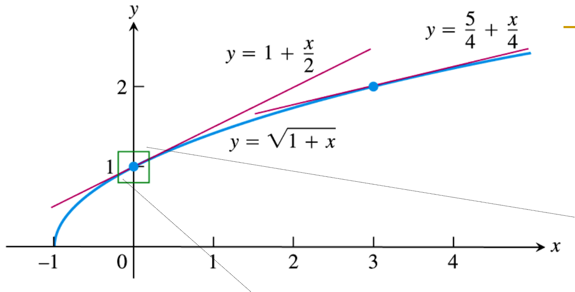


FIGURE 3.48 The graph of $y = \sqrt{1 + x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.49 shows a magnified view of the small window about 1 on the y-axis.

Accuracy of the linearized approximation

- We find that the approximation of *f*(*x*) by *L*(*x*) gets worsened as |*x* – *a*| increases (or in other words, *x* gets further away from *a*).

Approximation	True value	|True value − approximation|
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$<10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$<10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$<10^{-5}$

What $\frac{dy}{dx}$ is not

- Note that the derivative notation $\frac{dy}{dx}$ is not a ratio
- i.e. the derivative of the function *y* = *y*(*x*) with respect to *x*, is not to be understood as the ratio of two values, namely, *dy* and *dx*.
- d*y*/d*x* here denotes the a new quantity derived from *y* when the operation D = d/d*x* is performed on the function *y*, (d/d*x*)[*y*] = D [*y*]

Differential

- Definition:
- Let *y* = *f*(*x*) be a differentiable function. The differential *dy* is
  - *dy* = *f* ′(*x*)*dx*
- *dy* is an dependent variable, i.e., the value of *dy* depends on *f* ′(*x*) and *dx* where *dx* is viewed as an independent variable.
- Once *f* ′(*x*) and *dx* is fixed, then the value of differential *dy* can be calculated.

Example 4 Finding the differential *dy*

- (a) Find dy if $y=x^5 + 37x$.
- (b) Find the value of dy when $x=1$ and $dx = 0.2$

$dy \div dx = f'(x)$

- Referring to the definition of the differentials dy and dx, if we take the ratio of dy and dx, i.e. $dy \div dx$, we get
  $dy \div dx = f'(x)\, dx \,/\, dx = f'(x) \equiv \frac{dy}{dx}$
- In other words, the ratio of the differential dy and dx is equal to the derivative by definition.

Differential of *f*, *df*

- We sometimes use the notation df in place of dy, so that

$$dy = f'(x)\, dx$$

is now written in terms of

- $df = f'(x)\, dx$

- df is called the differential of f

Example of differential of *f*

- If $y = f(x) = 3x^2 - 6$, then the differential of f is

$$df = f'(x)\, dx = 6x\, dx$$

- Note that in the above expression, if we take the ratio $df \,/\, dx$, we obtain
  - $df \,/\, dx = f'(x) = 6x$

The differential form of a function

- For every differentiable function y=*f*(*x*), we can obtain its derivative, $\frac{\mathrm{d}y}{\mathrm{d}x}$
- Corresponds to every derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ there is a differential *df* such that

$$df = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cdot dx$$

In addition, if $f = u + v$, then $df = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\cdot dx + \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\cdot dx$

Example 5

- If *y* = *f*(*x*) = tan 2*x*, the derivative is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sec^2 2x$$

- Correspond to the derivative, the differential of the function, *df*, is given by the product of the derivative d*y*/d*x* and the independent differential *dx*:

$$df = d(\tan 2x) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cdot dx = \left(2\sec^2 2x\right)\cdot dx$$

Example 5

- If *y* = *f*(*x*) = *x*/(*x*+1), then the differential form of the function,

$$y = f(x) = \left(\frac{x}{x+1}\right)$$

$$df = d\left(\frac{x}{x+1}\right) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cdot dx = \frac{(x+1)\frac{\mathrm{d}}{\mathrm{d}x}(x) - x\frac{\mathrm{d}}{\mathrm{d}x}(x+1)}{(x+1)^2}\cdot dx$$

$$= \frac{(x+1)\left[\frac{\mathrm{d}}{\mathrm{d}x}(x)\cdot dx\right] - x\left[\frac{\mathrm{d}}{\mathrm{d}x}(x+1)\cdot dx\right]}{(x+1)^2} = \frac{(x+1)[dx] - x\left[d(x+1)\right]}{(x+1)^2}$$

$$= \frac{(x+1)dx - xd(x+1)}{(x+1)^2} = \frac{(x+1)dx - x\left[\frac{\mathrm{d}}{\mathrm{d}x}(x+1)\cdot dx\right]}{(x+1)^2} = \frac{(x+1)dx - x[1\cdot dx]}{(x+1)^2} = \frac{dx}{(x+1)^2}$$

dy, *df* : any difference?

- Sometimes for a given function, *y* = *f*(*x*), the notation *dy* is used in place of the notation *df*.
- Operationally speaking, it does not matter whether one uses *dy* or *df*.

The derivative dy/dx is not dy divided by dx

- Due to the definition of the differentials dy, dx that their ratio, dy / dx equals to the derivative of the differentiable function $y = f(x)$, i.e. $\frac{dy}{dx} = \frac{\mathrm{d}}{\mathrm{d}x}(y) \equiv f'(x)$
- we can then move the differential dy or dx around, such as $dy = f'(x)dx$
- When we do so, we need to be reminded that dy and dx are differentials, a pair of variables, instead of thinking that the derivative $\frac{\mathrm{d}}{\mathrm{d}x}y$ is made up of a numerator "dy" and a denominator "dx" that are separable

Estimation with differential

- Referring to figure 3.51, geometrically, one can sees that if x, originally at $x=a$, changes by dx (where dx is an independent variable, the differential of x), $f(a)$ will change by

$$\Delta y = f(a+dx) - f(a)$$

- Δy can be approximated by the change of the linearization of f at $x=a$, $L(x)=f(a)+f'(a)(x-a)$,

$$\Delta y \approx \Delta L = L(a+dx)-L(a)= f'(a)dx = df(a)$$

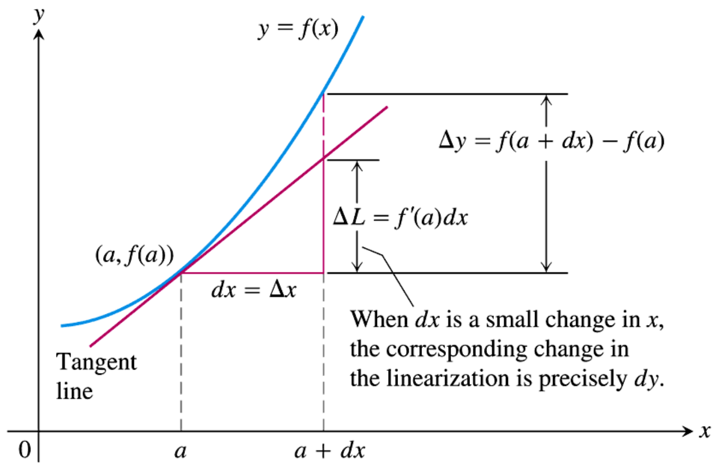


FIGURE 3.51 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

$\Delta y \approx dy$ allows estimation of $f(a+dx)$

- In other words, Δy centered around $x=a$ is approximated by $df(a)$ ($\equiv dy$, where the differential is evaluated at $x=a$):

$$\Delta y \approx dy$$

- or equivalently,

$$\Delta y = f(a+dx) - f(a) \approx dy = f'(a)dx$$

- This also allows us to estimate the value of $f(a+dx)$ if $f'(a)$, $f(a)$ are known, and dx is not too large, via

$$f(a+dx) \approx f(a)+ f'(a)dx$$

Example 6

- Figure 3.52
- The radius *r* of a circle increases from *a*=10 m to 10.1 m. Use *dA* to estimate the increase in circle's area *A.* Estimate the area of the enlarged circle and compare your estimate to your true value.

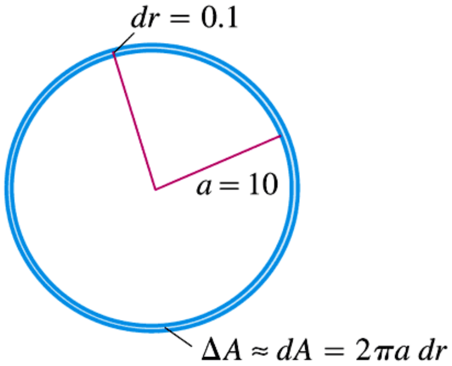
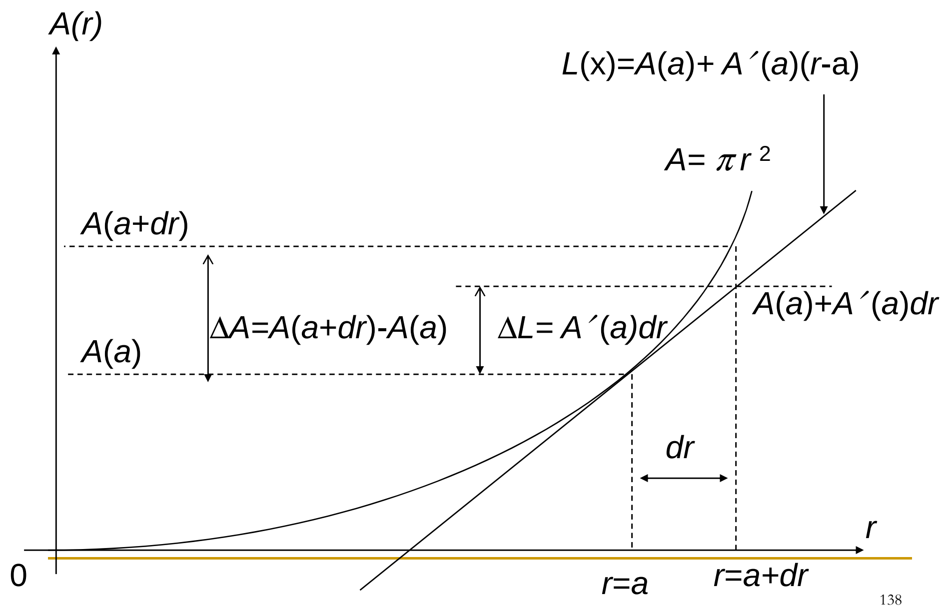


FIGURE 3.52 When dr is small compared with a, as it is when $dr = 0.1$ and $a = 10$, the differential $dA = 2\pi a\, dr$ gives a way to estimate the area of the circle with radius $r = a + dr$ (Example 6).



Solution to example 6

- Let a = 10 m, $a+dr$ = 10.1 m $\Rightarrow$ dr = 0.1m
- $A(r) = \pi r^2 \Rightarrow A(a) = \pi(10\text{ m})^2 = 100\pi\text{ cm}^2$
- $\Delta A \approx A'(a)dr = 2\pi(a)dr = 2\pi(10\text{ m})(0.1\text{ m}) =$

  $= 2\pi\text{ m}^2.$
- $A(a+dr) = A(a) + \Delta A \approx A(a) + A'(a)dr$

  $= 102\pi\text{ m}^2$ (this is an estimation)
- c.f the true area is $\pi(a+dr)^2 = \pi(10.1)^2 = 102.01\pi\text{ m}^2$

Chapter 4

Applications of Derivatives

4.1

Extreme Values of Functions

DEFINITIONS Absolute Maximum, Absolute Minimum

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

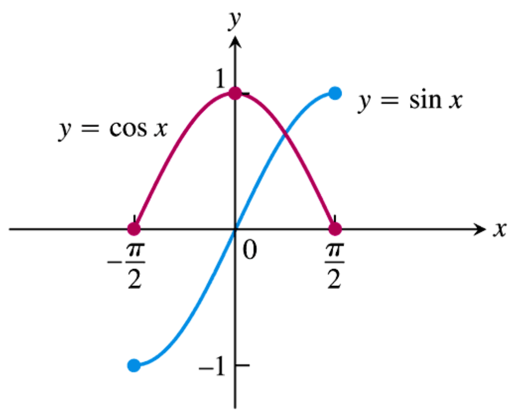


FIGURE 4.1 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

Example 1

- Exploring absolute extrema
- The absolute extrema of the following functions on their domains can be seen in Figure 4.2

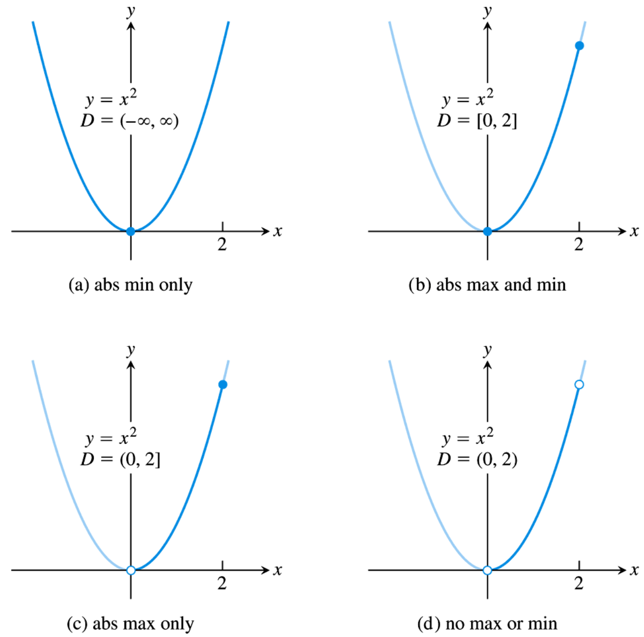


FIGURE 4.2 Graphs for Example 1.

THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a, b]$ (Figure 4.3).

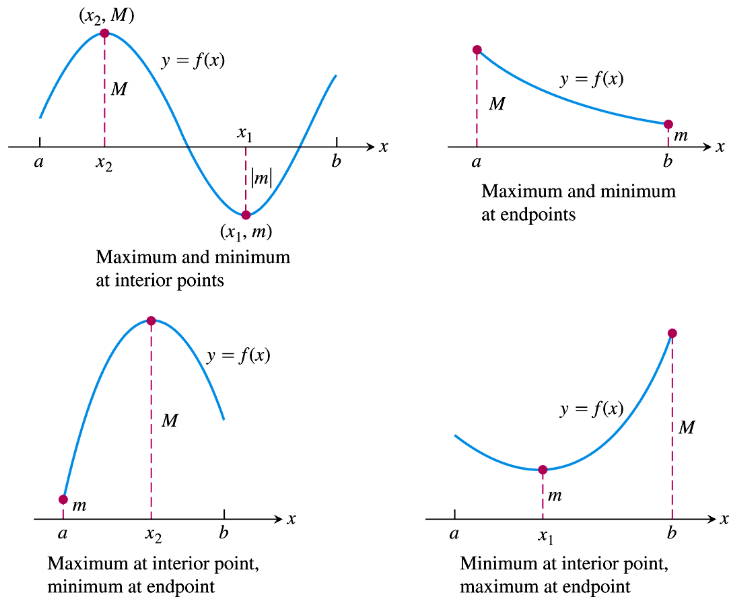


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

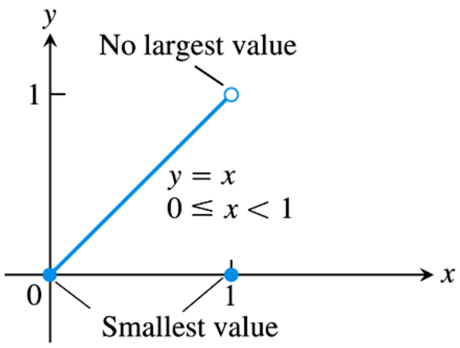


FIGURE 4.4 Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of $[0, 1]$ except $x = 1$, yet its graph over $[0, 1]$ does not have a highest point.

Local (relative) extreme values

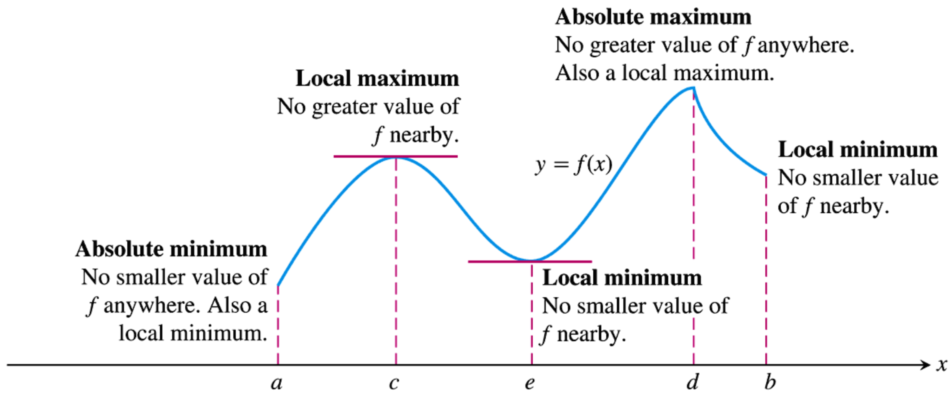


FIGURE 4.5 How to classify maxima and minima.

DEFINITIONS Local Maximum, Local Minimum

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \le f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \ge f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

Finding Extrema…with a not-always-effective method.

THEOREM 2 The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

Be careful not to misinterpret theorem 2 because its converse is false. A differentiable function may have a critical point at $x = c$ without having a local extreme value there. E.g. at point $x = 0$ of function $y = x^3$.

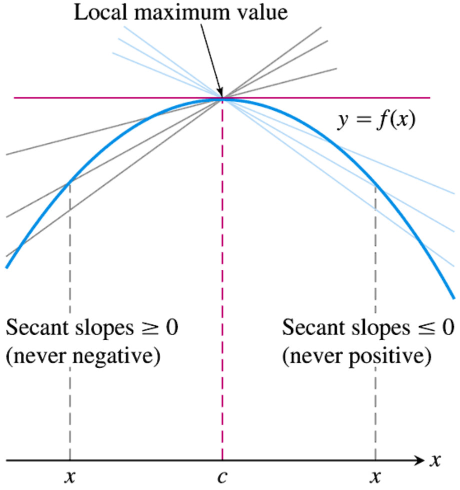


FIGURE 4.6 A curve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

DEFINITION Critical Point
An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

How to find the absolute extrema of a continuous function *f* on a finite closed interval

1. Evaluate *f* at all critical point and endpoints
2. Take the largest and smallest of these values.

Example 2: Finding absolute extrema

- Find the absolute maximum and minimum of $f(x) = x^2$ on [-2,1].

Example 3: Absolute extrema at endpoints

- Find the absolute extrema values of $g(t) = 8t - t^4$ on [-2,1].

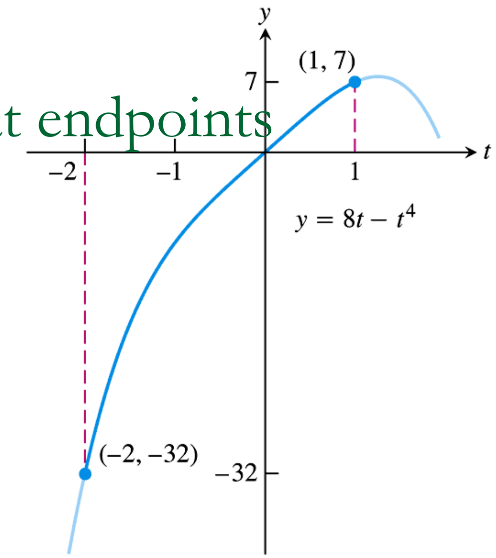


FIGURE 4.7 The extreme values of $g(t) = 8t - t^4$ on $[-2, 1]$ (Example 3).

Example 4: Finding absolute extrema on a closed interval

- Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval [-2,3].

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$

The point (0,f(0)) is a critical point by definition

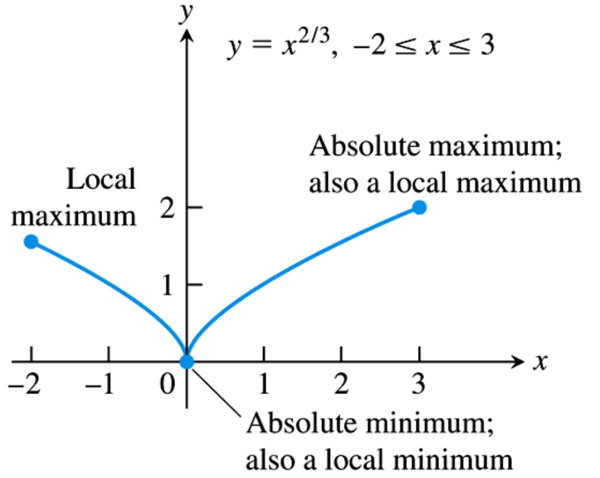
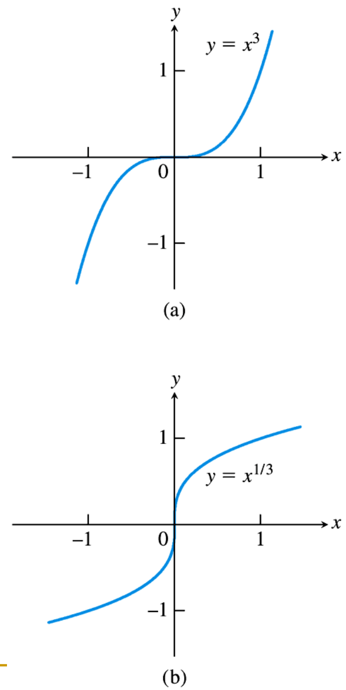


FIGURE 4.8 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).



- Not every critical point or endpoints signals the presence of an extreme value.

FIGURE 4.9 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

4.2

The Mean Value Theorem

THEOREM 3 Rolle's Theorem
Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If

$$f(a) = f(b),$$

then there is at least one number c in (a, b) at which

$$f'(c) = 0.$$

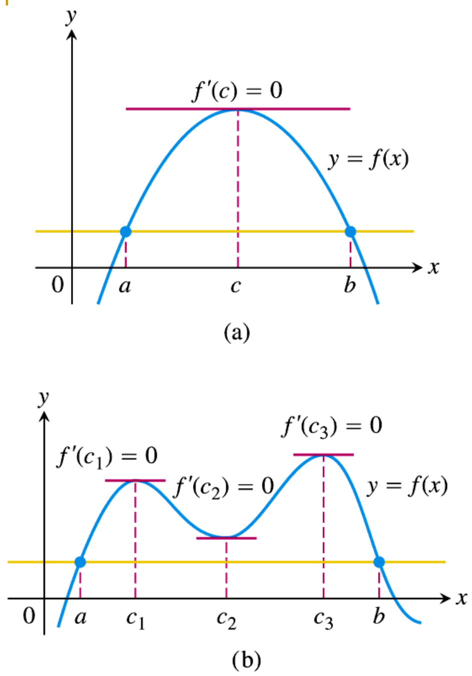
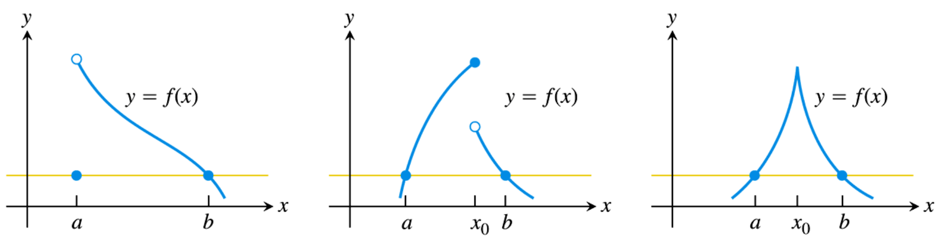


FIGURE 4.10 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

FIGURE 4.11 There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.

Example 1

- Horizontal tangents of a cubic polynomial

$$f(x) = \frac{x^3}{3} - 3x$$

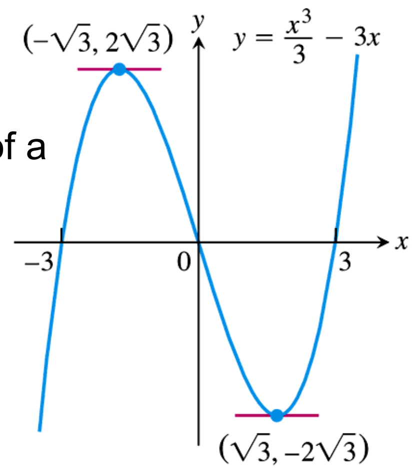


FIGURE 4.12 As predicted by Rolle's Theorem, this curve has horizontal tangents between the points where it crosses the x-axis (Example 1).

Example 2 Solution of an equation $f(x)=0$

- Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

Solution

1. Apply Intermediate value theorem to show that there exist at least one root
2. Apply Rolle's theotem to prove the uniqueness of the root.

$f'(x) = 3x^2 + 3$ is never zero

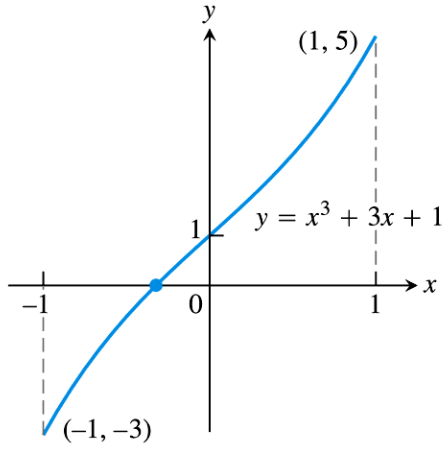


FIGURE 4.13 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the x-axis between -1 and 0 (Example 2).

The mean value theorem

THEOREM 4 The Mean Value Theorem

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

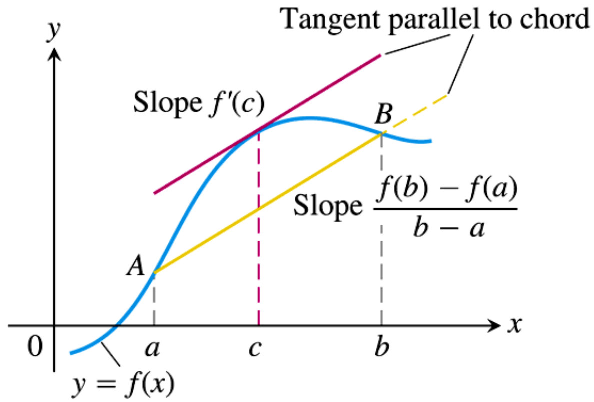


FIGURE 4.14 Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

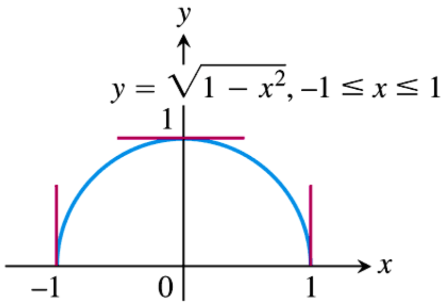


FIGURE 4.17 The function $f(x) = \sqrt{1 - x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though f is not differentiable at -1 and 1.

Example 3

- The function $f(x) = x^2$

is continuous for 0 ≤ *x*≤2 and differentiable for 0 < *x* < 2.

At some point *c* in the interval 0 < *x* < 2 the derivative *f '*(*x*)=2*x* must have the value (4-0)/(2-0)=2.

In this case, *f '*(*c*)=2*c* = 2.

That is, at *x*=*c*=1, *f '*(*c*) = the slope of the chord AB (see Figure 4.18)

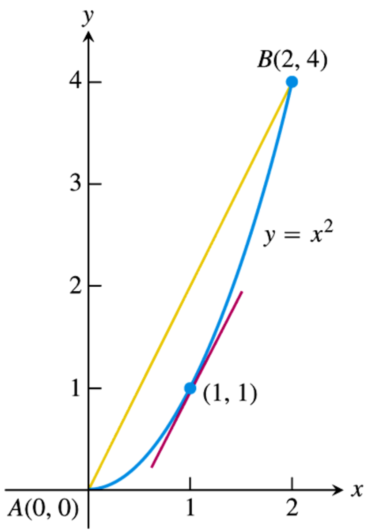


FIGURE 4.18 As we find in Example 3, $c = 1$ is where the tangent is parallel to the chord.

Mathematical consequences

COROLLARY 1 Functions with Zero Derivatives Are Constant

If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

COROLLARY 2 Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant on (a, b).

Corollary 1 can be proven using the Mean Value Theorem

- Say $x_1, x_2 \in (a,b)$ such that $x_1 < x_2$
- By the MVT on $[x_1,x_2]$ there exist some point c between x_1 and x_2 such that
  $f'(c) = [f(x_2) - f(x_1)] / (x_2 - x_1)$
- Since $f'(c) = 0$ for all c lying in (a,b),
  $f(x_2) - f(x_1) = 0$, hence $f(x_2) = f(x_1)$ for $x_1, x_2 \in (a,b)$.
- This is equivalent to $f(x)$ = a constant for $x \in (a,b)$.

Proof of Corollary 2

- At each point $x \in (a,b)$ the derivative of the difference between function $h = f - g$ is
  $h'(x) = f'(x) - g'(x) = 0$ (because $f'(x) = g'(x)$)
- Thus $h(x) = C$ on (a,b) by Corollary 1. That is $f(x) - g(x) = C$ on (a,b), so

$$f(x) = C + g(x).$$

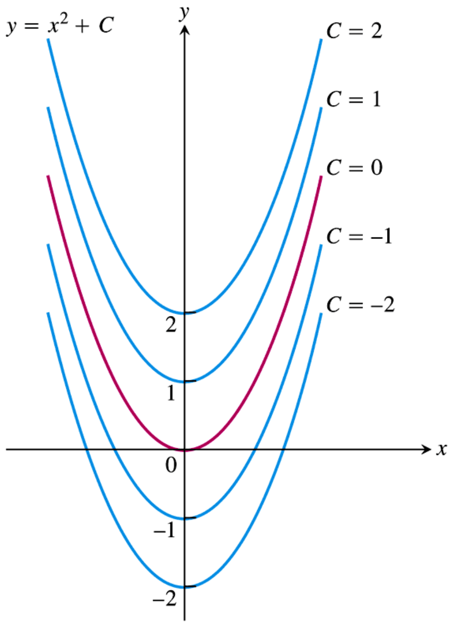


FIGURE 4.20 From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives on an interval can differ only by a vertical shift there. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for selected values of C.

Example 5

- Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point (0,2).

Solution Since $f(x)$ has the same derivative as $g(x) = -\cos x$, we know that $f(x) = -\cos x + C$ for some constant C. The value of C can be determined from the condition that $f(0) = 2$ (the graph of f passes through (0, 2)):

$$f(0) = -\cos(0) + C = 2, \qquad \text{so} \qquad C = 3.$$

The function is $f(x) = -\cos x + 3$. ■

4.3

Monotonic Functions and The First Derivative Test

Increasing functions and decreasing functions

DEFINITIONS **Increasing, Decreasing Function**

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

A function that is increasing or decreasing on I is called **monotonic** on I.

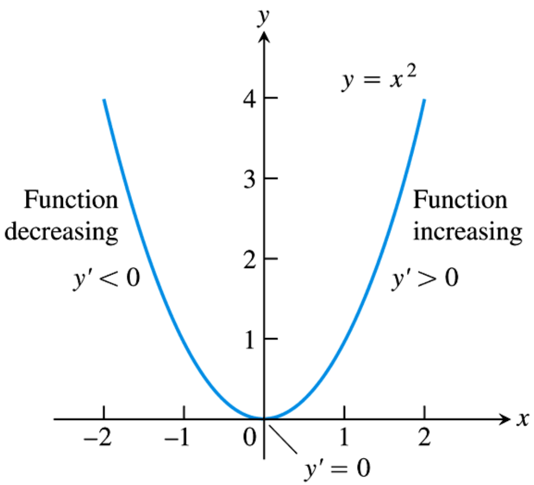


FIGURE 4.21 The function $f(x) = x^2$ is monotonic on the intervals $(-\infty, 0]$ and $[0, \infty)$, but it is not monotonic on $(-\infty, \infty)$.

COROLLARY 3 **First Derivative Test for Monotonic Functions**

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Mean value theorem is used to prove Corollary 3

Example 1

- Using the first derivative test for monotonic functions $f(x) = x^3 - 12x - 5$
- Find the critical point of $f(x) = x^3 - 12x - 5$ and identify the intervals on which *f* is increasing and decreasing.

Solution

$$f'(x) = 3(x+2)(x-2)$$

$$f' + \quad \text{for } -\infty < x < -2$$

$$f' - 12 \quad \text{for } -2 < x < 2$$

$$f' + \quad \text{for } 2 < x < \infty$$

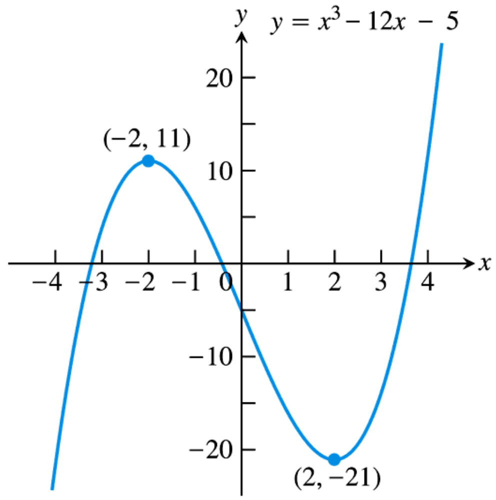


FIGURE 4.22 The function $f(x) = x^3 - 12x - 5$ is monotonic on three separate intervals (Example 1).

First derivative test for local extrema

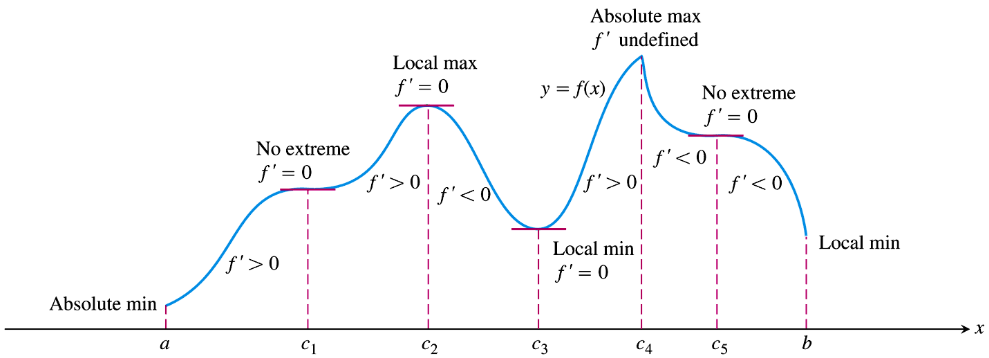


FIGURE 4.23 A function's first derivative tells how the graph rises and falls.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

Example 2: Using the first derivative test for local extrema

- Find the critical point of

$$f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$$

- Identify the intervals on which *f* is increasing and decreasing. Find the function's local and absolute extreme values.

$$f' = \frac{4(x-1)}{3x^{2/3}};\ f' - \text{ve} \quad \text{for } x < 0;$$

$$f' - \text{ve} \quad \text{for } 0 < x < 1;\ f' + \text{ve} \quad \text{for } x > 1$$

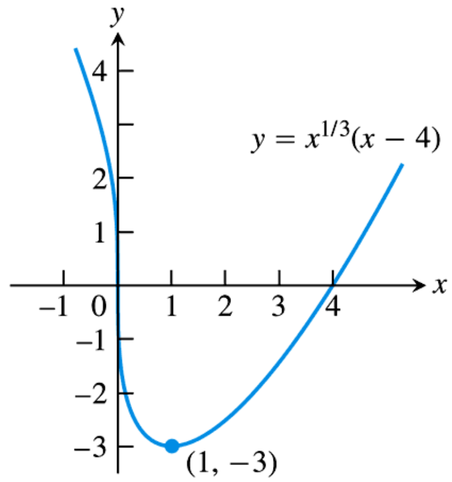


FIGURE 4.24 The function $f(x) = x^{1/3}(x-4)$ decreases when $x < 1$ and increases when $x > 1$ (Example 2).

4.4

Concavity and Curve Sketching

Concavity

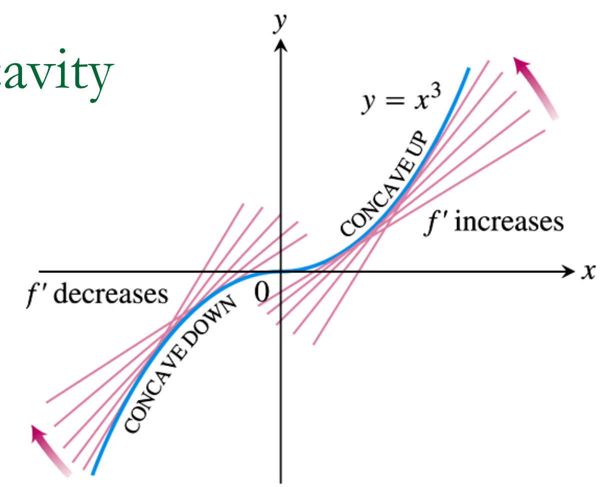


FIGURE 4.25 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

go back

DEFINITION Concave Up, Concave Down

The graph of a differentiable function $y = f(x)$ is

(a) concave up on an open interval I if f' is increasing on I

(b) concave down on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.

2. If $f'' < 0$ on I, the graph of f over I is concave down.

Example 1(a): Applying the concavity test

- Check the concavity of the curve $y = x^3$
- Solution: $y'' = 6x$
- $y'' < 0$ for $x < 0$; $y'' > 0$ for $x > 0$;

Link to Figure 4.25

Example 1(b): Applying the concavity test

- Check the concavity of the curve $y = x^2$
- Solution: $y'' = 2 > 0$

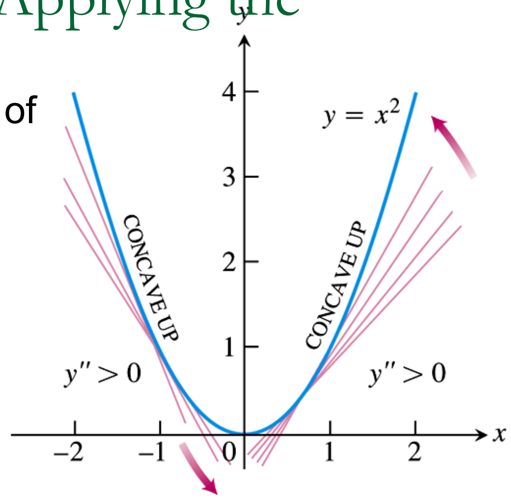


FIGURE 4.26 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

Example 2

- Determining concavity
- Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

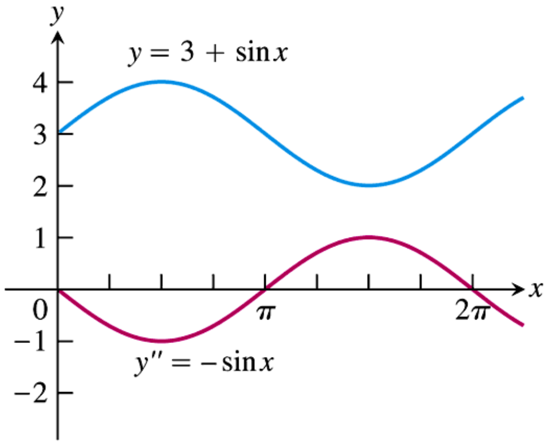


FIGURE 4.27 Using the graph of y'' to determine the concavity of y (Example 2).

Point of inflection

DEFINITION Point of Inflection
A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

Example 3: $y'' = 0$ not necessarily means existence of inflection point

- An inflection point may not exist where $y'' = 0$
- The curve $y = x^4$ has no inflection point at x=0. Even though $y'' = 12x^2$ is zero there, it does not change sign.

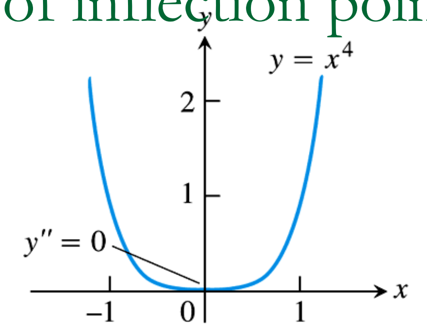


FIGURE 4.28 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 3).

Example 4: Existence of inflection does not necessarily needs $y'' = 0$ means

- An inflection point may occur where y'' = 0 does not exist
- The curve $y = x^{1/3}$ has a point of inflection at x=0 but y'' does not exist there.
- $y'' = -(2/9)x^{-5/3}$

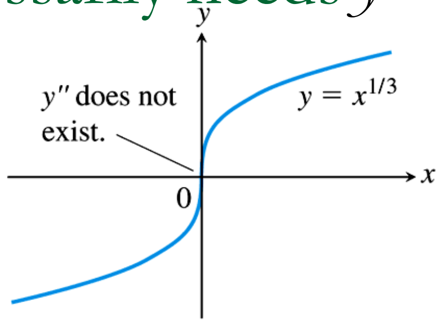
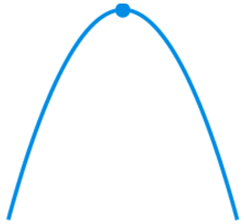


FIGURE 4.29 A point where y'' fails to exist can be a point of inflection (Example 4).

Second derivative test for local extrema

THEOREM 5 Second Derivative Test for Local Extrema
Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



$f' = 0, f'' < 0$
$\Rightarrow$ local max

$f' = 0, f'' > 0$
$\Rightarrow$ local min

Example 6: Using f' and f'' to graph f

- Sketch a graph of the function
  $f(x) = x^4 - 4x^3 + 10$
  using the following steps.

(a) Identify where the extrema of *f* occur
(b) Find the intervals on which *f* is increasing and the intervals on which *f* is decreasing
(c) Find where the graph of *f* is concave up and where it is concave down.
(d) Identify the slanted/vertical/horizontal asymtots, if there is any
(e) Sketch the general shape of the graph for *f*.
(f) Plot the specific points. Then sketch the graph.

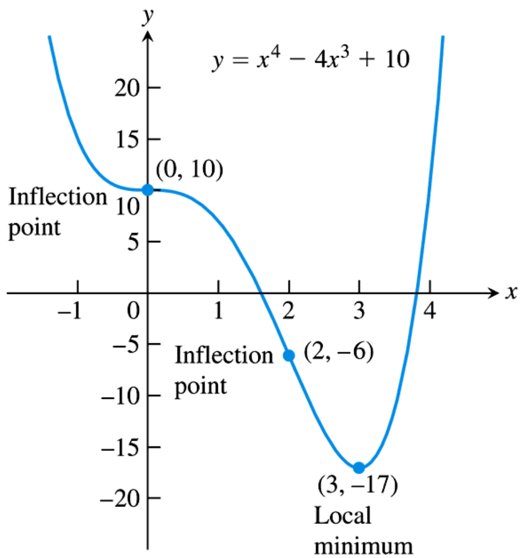


FIGURE 4.30 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 6).

Example

- Using the graphing strategy
- Sketch the graph of
- $f(x) = (x + 1)^2 / (x^2 + 1)$.

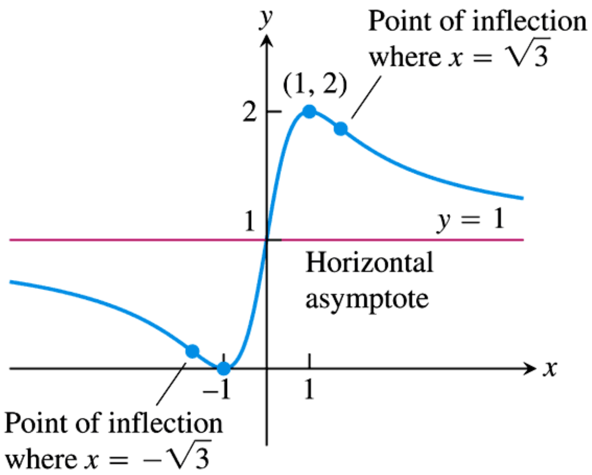
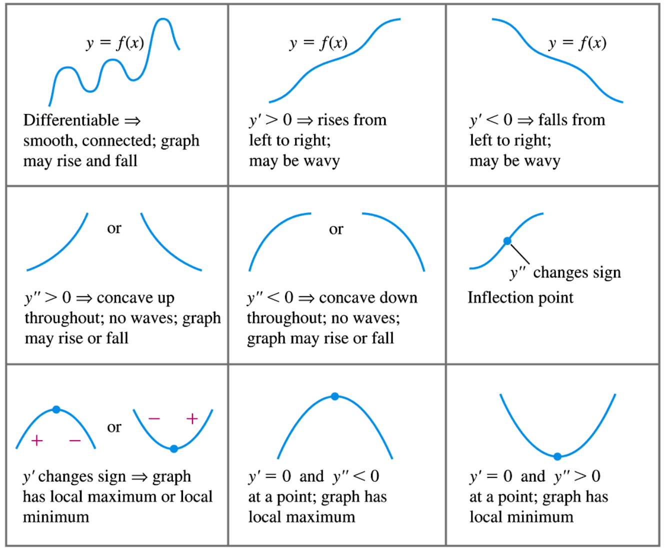


FIGURE 4.31 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$ (Example 7).

Learning about functions from derivatives



4.5

Applied Optimization Problems

Example 1

- An open-top box is to be cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

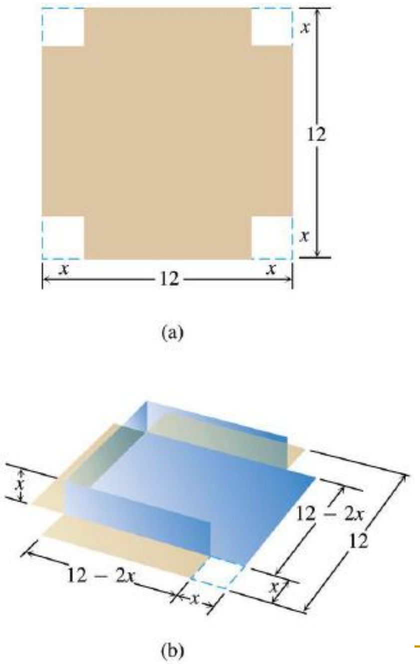


FIGURE 4.32 An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume (Example 1)?

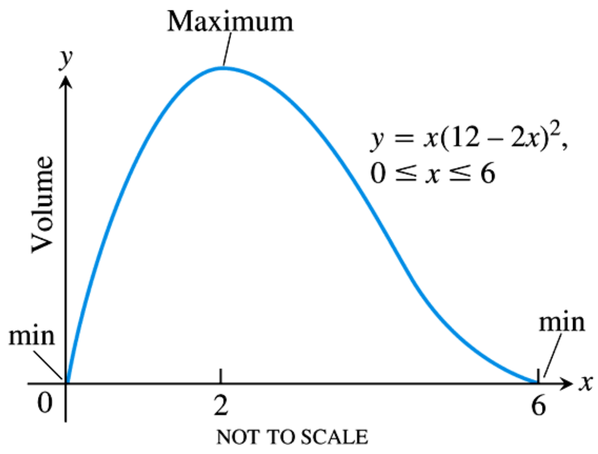


FIGURE 4.33 The volume of the box in Figure 4.32 graphed as a function of x.

Example 2

- Designing an efficient cylindrical can
- Design a 1-liter can shaped like a right circular cylinder. What dimensions will use the least material?

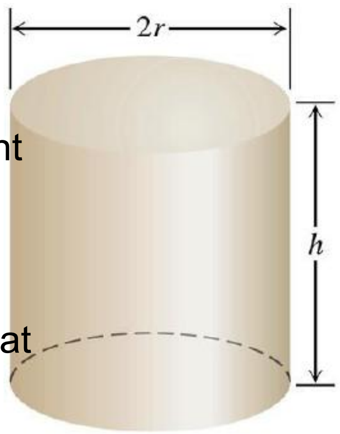


FIGURE 4.34 This 1-L can uses the least material when $h = 2r$ (Example 2).

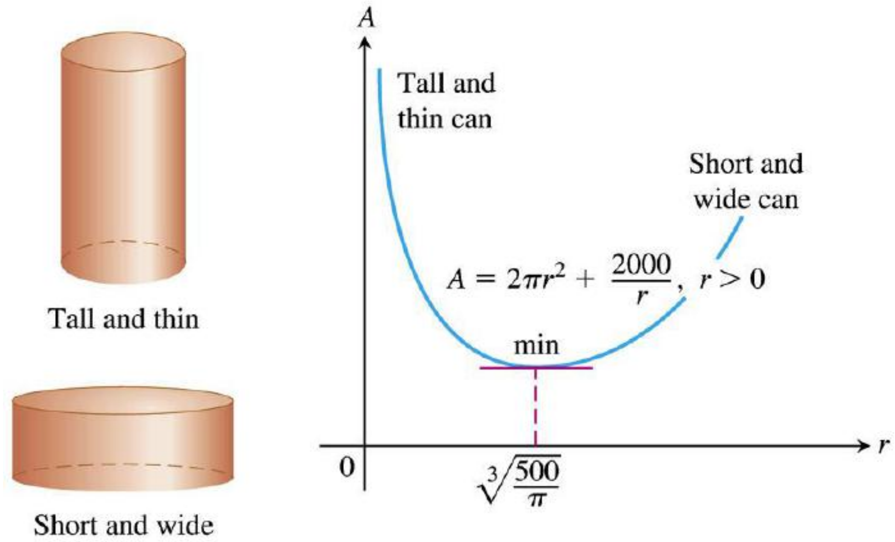


FIGURE 4.35 The graph of $A = 2\pi r^2 + 2000/r$ is concave up.

Example 3

- Inscribing rectangles
- A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what is its dimension?

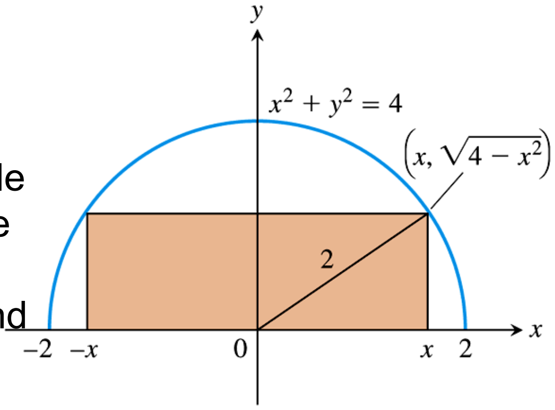


FIGURE 4.36 The rectangle inscribed in the semicircle in Example 3.

Solution

- Form the function of the area *A* as a function of *x*: $A=A(x)=x(4-x^2)^{1/2}$; $x > 0$.
- Seek the maximum of *A*:

4.6

Indeterminate Forms and L' Hopital's Rule

Indeterminate forms 0/0

THEOREM 6 L'Hôpital's Rule (First Form)
Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Example 1

- Using L′ Hopital's Rule
- (a)

$$\lim_{x \to 0} \frac{3x - \sin x}{x} = \left. \frac{3 - \cos x}{1} \right|_{x=0} = 2$$

- (b)

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \left. \frac{\frac{1}{2\sqrt{1+x}}}{1} \right|_{x=0} = \frac{1}{2}$$

THEOREM 7 L'Hôpital's Rule (Stronger Form)
Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

Example 2(a)

- Applying the stronger form of L′ Hopital's rule
- (a)

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2} = \lim_{x \to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x}$$

$$= \lim_{x \to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = \frac{-1}{8}$$

Example 2(b)

- Applying the stronger form of L′ Hopital's rule
- (b)

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$

THEOREM 8 Cauchy's Mean Value Theorem
Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Using L'Hôpital's Rule
To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Example 3

- Incorrect application of the stronger form of L′ Hopital's

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2}$$

Example 4

- Using L′ Hopital's rule with one-sided limits

$$(a)\ \lim_{x \to 0^+} \frac{\sin x}{x^2} = \lim_{x \to 0^+} \frac{\cos x}{2x} = \ldots$$

$$(b)\ \lim_{x \to 0^-} \frac{\sin x}{x^2} = \lim_{x \to 0^-} \frac{\cos x}{2x} = \ldots$$

Indeterminate forms ∞/∞, ∞·0, ∞- ∞

- If $f \to \pm\infty$ and $g \to \pm\infty$ as $x \to a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

- *a* may be finite or infinite

Example 5
Working with the indeterminate form ∞/∞

$$(a)\ \lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x}$$

$$\lim_{x \to (\pi/2)^-} \frac{\sec x}{1 + \tan x} = \lim_{x \to (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to (\pi/2)^-} \sin x = 1$$

$$\lim_{x \to (\pi/2)^+} \frac{\sec x}{1 + \tan x}. = \ldots$$

Example 5(b)

$$(b) \lim_{x \to \infty} \frac{x - 2x^2}{3x^2 + 5x} = \ldots$$

Example 6

- Working with the indeterminate form ∞·0

$$\lim_{x \to \infty} \left(x \sin \frac{1}{x} \right)$$

Example 7

- Working with the indeterminate form ∞ - ∞

$$\lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x \to 0}\left(\frac{x - \sin x}{x \sin x}\right) = \ldots$$

4.8

Antiderivatives

Finding antiderivatives

DEFINITION Antiderivative
A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example 1

- Finding antiderivatives
- Find an antiderivative for each of the following functions
- (a) $f(x) = 2x$
- (b) $f(x) = \cos x$
- (c) $h(x) = 2x + \cos x$

The most general antiderivative

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Example 2 Finding a particular antiderivative

- Find an antiderivative of $f(x) = \sin x$ that satisfies $F(0) = 3$
- Solution: $F(x)=\cos x + C$ is the most general form of the antiderivative of $f(x)$.
- We require $F(x)$ to fulfill the condition that when x=3 (in unit of radian), $F(x)$=0. This will fix the value of C, as per
- $F(3)= 3 = \cos 3 + C \Rightarrow 3 - \cos 3$
- Hence, $F(x)= \cos x + (3 - \cos 3)$ is the antiderivative sought

TABLE 4.2 Antiderivative formulas

	Function	General antiderivative
1.	x^n	$\frac{x^{n+1}}{n+1} + C, \quad n \neq -1, n$ rational
2.	$\sin kx$	$-\frac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\frac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

Example 3 Finding antiderivatives using table 4.2

- Find the general antiderivative of each of the following functions.
- (a) $f(x) = x^5$
- (b) $g(x) = 1/x^{1/2}$
- (c) $h(x) = \sin 2x$
- (d) $i(x) = \cos (x/2)$

Example 4 Using the linearity rules for antiderivatives

- Find the general antiderivative of
- $f(x) = 3/x^{1/2} + \sin 2x$

DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\, dx.$$

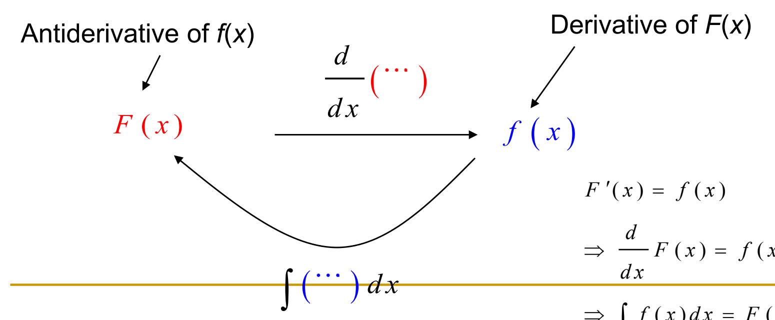
The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

- In other words, given a function $f(x)$, the most general form of its antiderivative, previously represented by the symbol $F(x) + C$, where C denotes an arbitrary constant, is now being represented in the form of an indefinite integral, namely,

$$\int f(x)\,dx \equiv F(x) + C$$

Operationally, the indefinite integral of *f*(*x*) means …

The indefinite integral of *f*(*x*) is the inverse of the operation of derivative taking of *f*(*x*)



$$F'(x) = f(x)$$

$$\Rightarrow \frac{d}{dx}F(x) = f(x)$$

$$\Rightarrow \int f(x)dx = F(x) + C$$

Example of indefinite integral notation

$$\int 2x\; dx = x^2 + C$$

$$\int \cos x\; dx = \sin x + C$$

$$\int (2x + \cos x)\; dx = x^2 + \sin x + C$$

Example 7 Indefinite integration done term-by term and rewriting the constant of integration

- Evaluate

$$\int \left(x^2 - 2x + 5\right) dx = \int x^2 dx - \int 2x dx + \int 5 dx = \dots$$

Chapter 5

Integration

5.1

Estimating with Finite Sums

Riemann Sums

Approximating area bounded by the graph between $[a,b]$

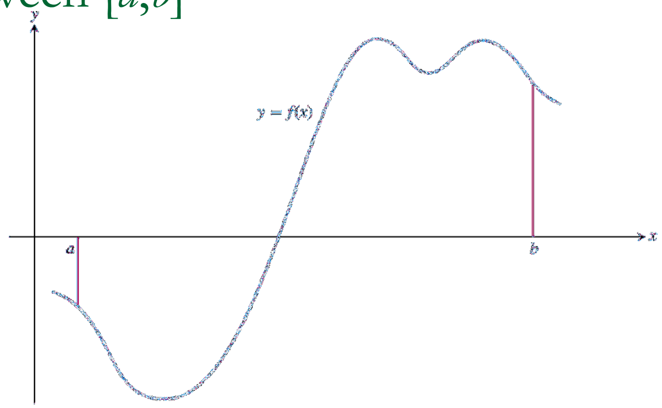


FIGURE 5.8 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

Area is approximately given by

$$f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \dots + f(c_n)\Delta x_n$$

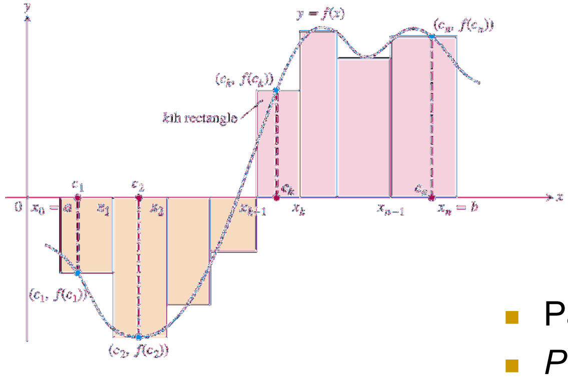
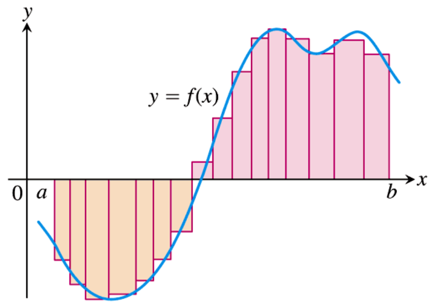
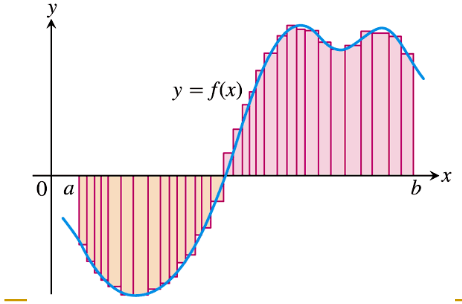


FIGURE 5.9 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis.

- Partition of $[a,b]$ is the set of
- $P = \{x_0, x_1, x_2, \dots x_{n-1}, x_n\}$
- $a = x_0 < x_1 < x_2 \dots < x_{n-1} < x_n = b$
- $c_n \in [x_{n-1}, x_n]$
- $||P||$ = norm of P = the largest of all subinterval width



(a)



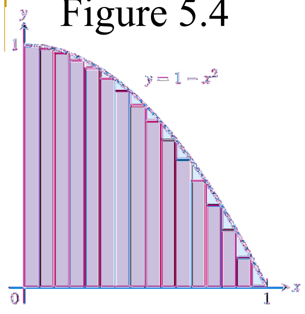
(b)

Riemann sum for f on $[a,b]$

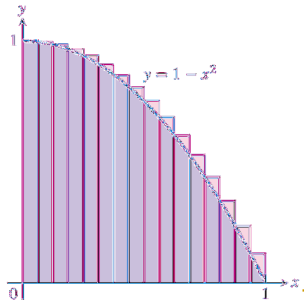
$$R_n = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \ldots + f(c_n)\Delta x_n$$

FIGURE 5.10 The curve of Figure 5.9 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x-axis with increasing accuracy.

Figure 5.4



(a)



(b)

- Let the true value of the area is R
- Two approximations to R:
- $c_n = x_n$ corresponds to case (a). This under estimates the true value of the area R if n is finite.
- $c_n = x_{n-1}$ corresponds to case (b). This over estimates the true value of the area S if n is finite.

go back

Limits of finite sums

- Example 5 The limit of finite approximation to an area
- Find the limiting value of lower sum approximation to the area of the region R below the graphs $f(x) = 1 - x^2$ on the interval [0,1] based on Figure 5.4(a)

Solution

- $\Delta x_k = (1 - 0)/n = 1/n \equiv \Delta x$; $k = 1,2,\ldots n$
- Partition on the x-axis: $[0,1/n], [1/n, 2/n],\ldots, [(n-1)/n,1]$.
- $c_k = x_k = k\Delta x = k/n$
- The sum of the stripes is

$R_n = \Delta x_1\, f(c_1) + \Delta x_2\, f(c_2) + \Delta x_3\, f(c_3) + \ldots + \Delta x_n\, f(c_n)$
$= \Delta x\, f(1/n) + \Delta x\, f(2/n) + \Delta x\, f(3/n) + \ldots + \Delta x_n\, f(1)$
$= \sum_{k=1}^{n} \Delta x\, f(k\Delta x) = \Delta x \sum_{k=1}^{n} f(k/n)$
$= (1/n) \sum_{k=1}^{n} [1 - (k/n)^2]$
$= \sum_{k=1}^{n} 1/n - k^2/n^3 = 1 - (\sum_{k=1}^{n} k^2)/n^3$
$= 1 - [(n)(n+1)(2n+1)/6]/n^3 = 1 - [2n^3 + 3n^2 + n]/(6n^3)$

$\sum_{k=1}^{n} k^2 = (n)(n+1)(2n+1)/6$

- Taking the limit of $n \to \infty$

$$\lim_{n \to \infty} R_n = R = \left(1 - \frac{2n^3 + 3n^2 + n}{6n^3} \right) = 1 - 2/6 = 2/3$$

- The same limit is also obtained if $c_n = x_{n-1}$ is chosen instead.
- For all choice of $c_n \in [x_{n-1}, x_n]$ and partition of P, the same limit for S is obtained when $n \to \infty$

5.3

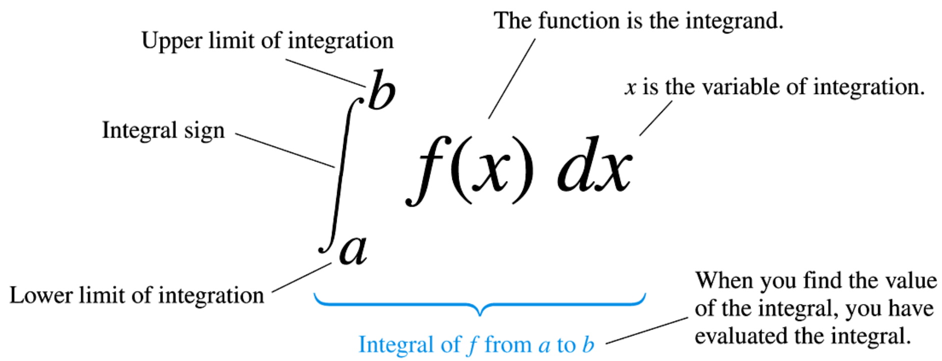
The Definite Integral

DEFINITION The Definite Integral as a Limit of Riemann Sums

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number I is the **definite integral of f over $[a, b]$** and that I is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - I \right| < \epsilon .$$



"The integral from a to b of f of x with respect to x"

- The limit of the Riemann sums of *f* on [*a*,*b*] converges to the finite integral *I*

$$\lim_{||P||\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = I = \int_a^b f(x)dx$$

- We say *f* is integrable over [*a*,*b*]
- Can also write the definite integral as

$$I = \int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du$$

$$= \int_a^b f(\text{what ever}) \quad d(\text{what ever})$$

- The variable of integration is what we call a 'dummy variable'

THEOREM 1 The Existence of Definite Integrals
A continuous function is integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

Question: is a non continuous function integrable?

Integral and nonintegrable functions

- Example 1
- A nonintegrable function on [0,1]

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

- Not integrable

Properties of definite integrals

THEOREM 2
When f and g are integrable, the definite integral satisfies Rules 1 to 7 in Table 5.3.

TABLE 5.3 Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — Also a Definition
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any Number k

   $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ — $k = -1$
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. *Max-Min Inequality:* If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

   $f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)

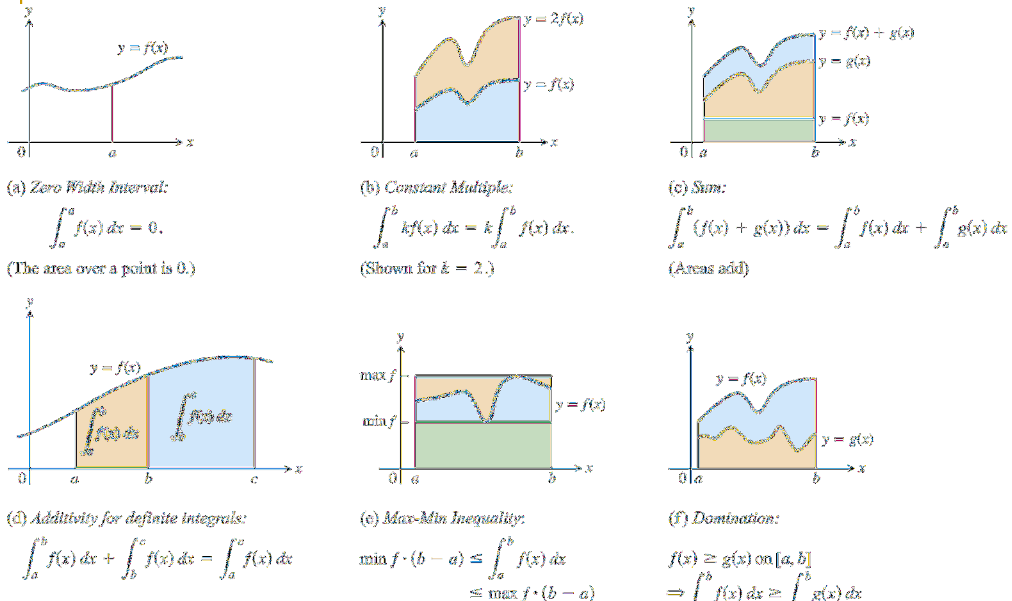


FIGURE 5.11

Example 3 Finding bounds for an integral

- Show that the value of $\int_0^1 \sqrt{1+\cos x}\,dx$ is less than 3/2

- **Solution**
- Use rule 6 Max-Min Inequality

Area under the graphs of a nonnegative function

DEFINITION Area Under a Curve as a Definite Integral

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

Example 4 Area under the line $y = x$

- Compute $\int_0^b x\,dx$ (the Riemann sum)

  and find the area A under $y = x$ over the interval $[0,b]$, $b>0$

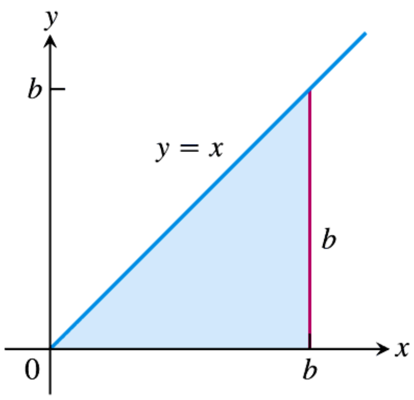


FIGURE 5.12 The region in Example 4 is a triangle.

Solution

By geometrical consideration:

A=(1/2)×high×width= (1/2)×b×b= $b^2/2$

Choose partition of n subinterval with equal width:

$$\{0 = x_0, x_1, x_2, \cdots x_n = b\}, \Delta x_k = x_k - x_{k-1} = \Delta x = b/n$$

Riemann sum:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\Delta x f(c_k) = \lim_{n\to\infty}\Delta x\sum_{k=1}^{n} f(x_k)$$

$$= \lim_{n\to\infty}\Delta x\sum_{k=1}^{n} x_k = \lim_{n\to\infty}\Delta x\sum_{k=1}^{n} k\Delta x$$

$$= \lim_{n\to\infty}(\Delta x)^2\sum_{k=1}^{n} k = \lim_{n\to\infty}\left(\frac{b}{n}\right)^2\sum_{k=1}^{n} k$$

$$= \lim_{n\to\infty}\left(\frac{b}{n}\right)^2\frac{n(n+1)}{2} = \lim_{n\to\infty}\left(\frac{b}{n}\right)^2\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty}\frac{b^2}{2}\left(1+\frac{1}{n}\right) = \frac{b^2}{2}$$

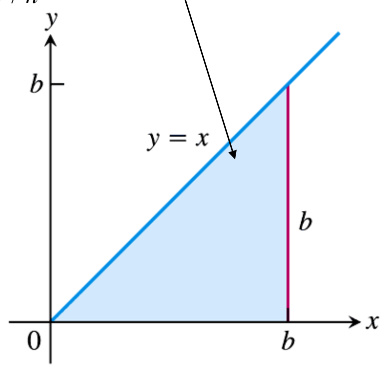


FIGURE 5.12 The region in Example 4 is a triangle.

Using the additivity rule for definite integration:

$$\int_0^b x\,dx = \int_0^a x\,dx + \int_a^b x\,dx$$

$$\to \int_a^b x\,dx = \int_0^b x\,dx - \int_0^a x\,dx = \frac{b^2}{2} - \frac{a^2}{2}, a < b$$

Using geometry, the area is the area of a trapezium

A= (1/2)(b-a)(b+a)

= $b^2/2$ - $a^2/2$

Both approaches to evaluate the area agree

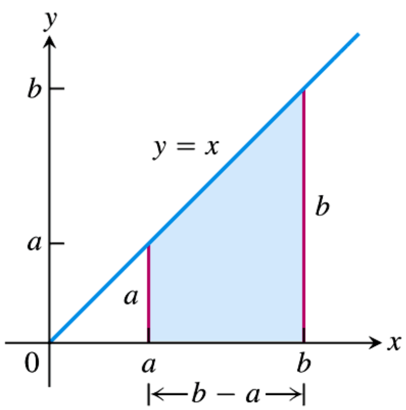


FIGURE 5.13 The area of this trapezoidal region is $A = (b^2 - a^2)/2$.

- One can prove the following Riemannian sum of the functions $f(x)=c$ and $f(x)= x^2$:

$$\int_a^b c\,dx = c(b-a), \quad c \text{ any constant} \qquad (2)$$

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}, \quad a < b \qquad (3)$$

Average value of a continuous function revisited

- Average value of nonnegative continuous function *f* over an interval [*a*,*b*] is

$$\frac{f(c_1)+f(c_2)+\cdots f(c_n)}{n}=\frac{1}{n}\sum_{k=1}^{n} f(c_k)$$

$$=\frac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k)=\frac{1}{b-a}\sum_{k=1}^{n}\Delta x f(c_k)$$

- In the limit of $n \to \infty$, the average =

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

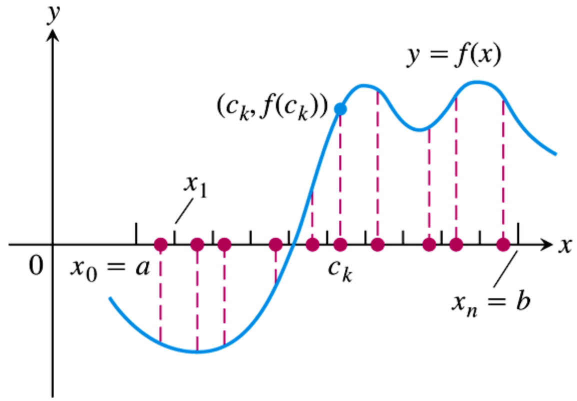


FIGURE 5.14 A sample of values of a function on an interval [*a*, *b*].

DEFINITION The Average or Mean Value of a Function

If f is integrable on $[a, b]$, then its **average value on** $[a, b]$, also called its **mean value**, is

$$\text{av}(f)=\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example 5 Finding average value

- Find the average value of $f(x)=\sqrt{4-x^2}$ over [-2,2]

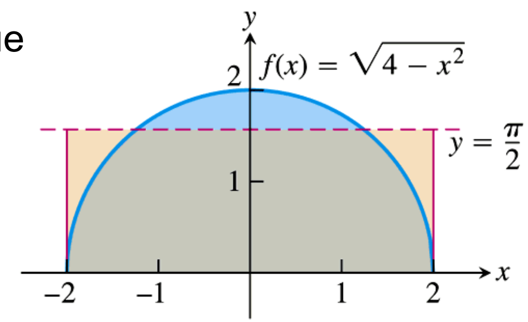


FIGURE 5.15 The average value of $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 5).

5.4

The Fundamental Theorem of Calculus

Mean value theorem for definite integrals

THEOREM 3 The Mean Value Theorem for Definite Integrals
If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

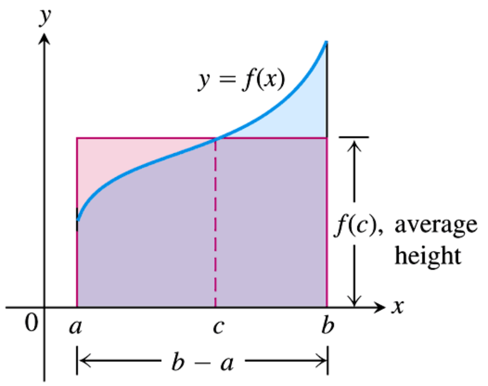


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

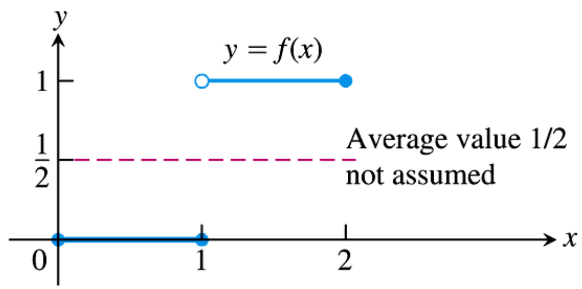


FIGURE 5.17 A discontinuous function need not assume its average value.

Example 1 Applying the mean value theorem for integrals

- Find the average value of $f(x)=4-x$ on [0,3] and where f actually takes on this value as some point in the given domain.

- **Solution**
- Average = 5/2
- Happens at $x=3/2$

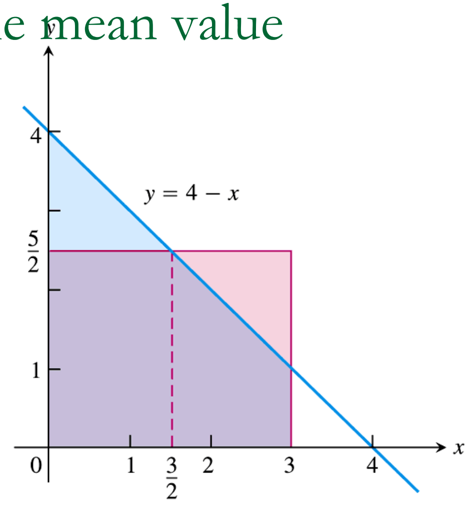


FIGURE 5.18 The area of the rectangle with base [0, 3] and height 5/2 (the average value of the function $f(x) = 4 - x$) is equal to the area between the graph of f and the x-axis from 0 to 3 (Example 1).

Fundamental theorem Part 1

- Define a function $F(x)$: $F(x) = \int_a^x f(t)\,dt$
- $x, a \in I$, an interval over which $f(t) > 0$ is integrable.
- The function $F(x)$ is the area under the graph of $f(t)$ over $[a,x]$, $x > a$

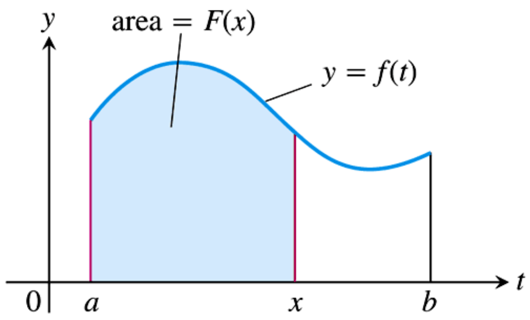


FIGURE 5.19 The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.

Fundamental theorem Part 1 (cont.)

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt$$

$$\frac{F(x+h) - F(x)}{h} = \underbrace{\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c)}_{\text{mean value theorem}};\ x \le c \le x+h$$

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = F'(x) = f(x)$$

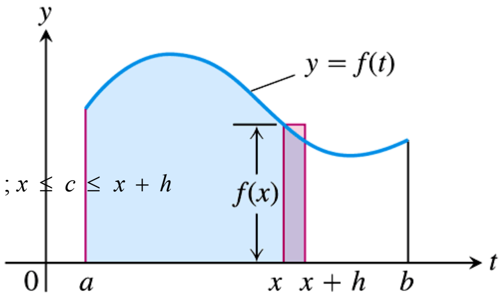


FIGURE 5.20 In Equation (1), $F(x)$ is the area to the left of x. Also, $F(x + h)$ is the area to the left of $x + h$. The difference quotient $[F(x + h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

The above result holds true even if f is not positive definite over $[a,b]$

THEOREM 4 The Fundamental Theorem of Calculus Part 1

If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

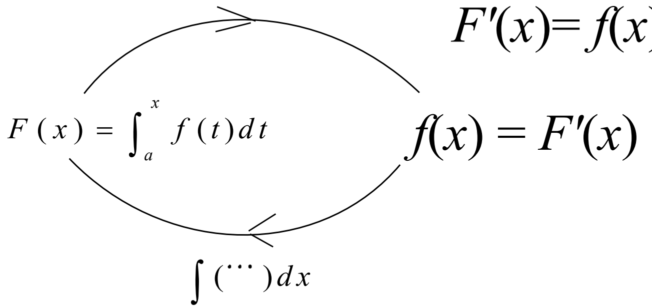
$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \qquad (2)$$

Note: Convince yourself that

(i) $F(x)$ is an antiderivative of $f(x)$

(ii) $f(x)$ is a derivative of $F(x)$

$f(x)$ is a derivative of $F(x)$ because $F'(x) = f(x)$



$F(x)$ is an antiderivative of $f(x)$ because $F(x) = \int_a^x f(t)dt$

The main use of theorem 4 is …

- It tells us that $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$
- In pragmatic terms, if a function is expressed in terms of an integral of the form

$$F(x) = \int_a^x f(t)dt$$

then the derivative of $F(x)$, $\frac{d}{dx}F(x)$, is simply $f(x)$

Example 3 Applying the fundamental theorem

- Use the fundamental theorem to find

$(a)\ \frac{d}{dx}\int_a^x \cos t\,dt$ $\qquad (b)\ \frac{d}{dx}\int_a^x \frac{1}{1+t^2}dt$

$(c)\ \frac{dy}{dx}$ if $y = \int_x^5 3t\sin t\,dt$ $\qquad (d)\ \frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t\,dt$

Solution for (d): you have to invoke chain rule

$$\frac{d}{dx}F(x), \text{ where } F(x) \equiv \int_1^{x^2} \cos t\,dt$$

- Chain rule says if $F(x)=(f \circ u)(x)= f[u(x)]$,

$$\frac{d}{dx}F(x) = \frac{d}{dx}(f \circ u)(x) = \frac{d}{dx}f[u(x)] = \frac{d}{du}f(u)\cdot\frac{d}{dx}u(x)$$

Solution for (d): you have to invoke chain rule

$F(x) = \int_1^{x^2} \cos t\,dt$ is a composite function of the form $F(x)=f[u(x)]$

$$F(x) = f[u(x)], \text{ where}$$

$$f(u) = \int_1^{u} \cos t\,dt, \quad u(x) = x^2$$

so that

$$\frac{d}{dx}F(x) = \frac{d}{du}f(u)\cdot\frac{d}{dx}u(x) = \frac{d}{du}\left(\int_1^{u}\cos t\,dt\right)\cdot\frac{d}{dx}(x^2)$$

$$= \frac{d}{du}\left(\int_1^{u}\cos t\,dt\right)\cdot 2x = \cos u \cdot 2x = 2x\cos\left(x^2\right)$$

Example 4 Constructing a function with a given derivative and value

- Find a function $y = f(x)$ on the domain $(-\pi/2, \pi/2)$ with derivative $dy/dx = \tan x$ that satisfies $f(3)=5$.

- The strategy:
- Use the fundamental theorem of calculus.
- Think along this line: find a function $F(x)$ of the form

$$F(x) = \int_a^x q(t)dt$$

such that $\frac{d}{dx}F(x) = q(x), \text{ with } q(x) = \tan x$

Example 4 (Cont. 1)

Solution

- Stage 1: If $F(x) = \int_a^x \tan t\,dt$, then $\frac{dF}{dx} = \tan x.$

- Stage 2: construct the function $f(x)$ using $F(x)$, and then try to make $f(x)$ so constructed fulfills the condition of $f(3)=5$.
- The way to construct $f(x)$ from $F(x)$ is obviously

$$y = f(x) = F(x) + \text{constant (so that } \frac{dy}{dx} = \tan x)$$

$$= \int_a^x \tan t\,dt + \text{constant}$$

Example 4 (Cont. 2)

$$f(x) = \int_a^x \tan t\,dt + \text{constant}$$

- Find the values of a and constant so that $f(3)=5$
- This can be done by choosing $a = 3$, constant $=5$.
- Verify this:

$$f(3) = \int_{a=3}^{x=3} \tan t\,dt + 5 = 0 + 5 = 5$$

- So, finally, the function we are seeking is

$$f(x) = \int_3^x \tan t\,dt + 5$$

Fundamental theorem, part 2 (The evaluation theorem)

THEOREM 4 (Continued) The Fundamental Theorem of Calculus Part 2

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

To calculate the definite integral of f over $[a,b]$, do the following

- 1. Find an antiderivative F of f, and
- 2. Calculate the number

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big]_a^b \quad \text{or} \quad \Big[F(x)\Big]_a^b$$

To summarise

$$\frac{d}{dx}\int_a^x f(t)dt = \frac{dF(x)}{dx} = f(x)$$

$$\int_a^x \left(\frac{dF(t)}{dt}\right)dt = \int_a^x f(t)dt = F(x) - F(a)$$

Example 5 Evaluating integrals

$$(a)\int_0^{\pi} \cos x\, dx$$

$$(b)\int_{-\pi/4}^{0} \sec x \tan x\, dx$$

$$(c)\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx$$

Example 7 Canceling areas

- Compute
- (a) the definite integral of $f(x)$ over $[0,2\pi]$
- (b) the area between the graph of $f(x)$ and the x-axis over $[0,2\pi]$

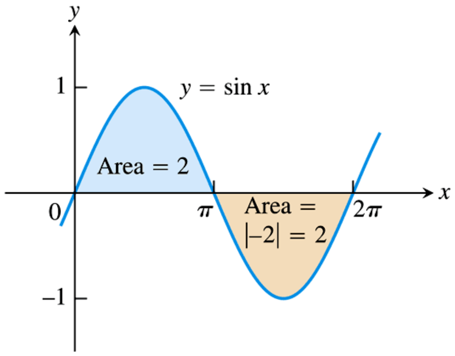


FIGURE 5.22 The total area between $y = \sin x$ and the x-axis for $0 \le x \le 2\pi$ is the sum of the absolute values of two integrals (Example 7).

Example 8 Finding area using antiderivative

- Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.
- **Solution**
- First find the zeros of f.
- $f(x) = x(x+1)(x-2)$

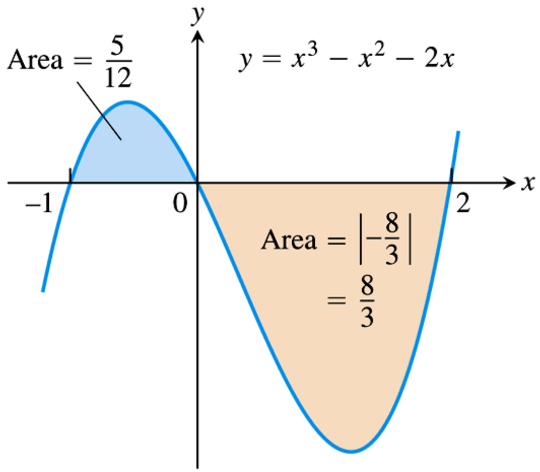


FIGURE 5.23 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 8).

5.5

Indefinite Integrals and the Substitution Rule

Note

- The indefinite integral of *f* with respect to *x*,

$$\int f(x)\,dx$$

is a function plus an arbitrary constant

- A definite integral $\int_a^b f(x)\,dx$ is a number.

Antiderivative and indefinite integral in terms of variable *x*

- If *F*(*x*) is an antiderivative of *f*(*x*),

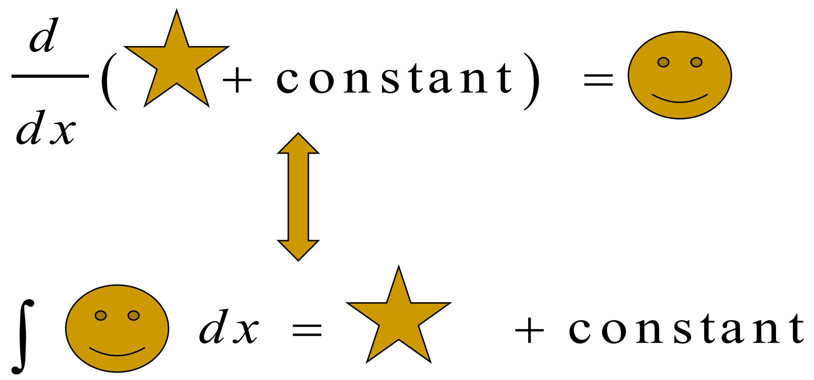
$$\frac{d}{dx}F(x) = f(x)$$

$$\Rightarrow$$

- the indefinite integral of *f*(*x*) is

$$\int f(x)dx = F(x) + C$$

A useful mnemonic



Example:

$$\frac{d}{dx}(\tan x + \text{constant}) = \sec^2 x$$

$$\int \sec^2 x\,dx = \tan x + \text{constant}$$

Antiderivative and indefinite integral with chain rule

$\frac{d}{dx}F(x) = f(x)$, i.e., $F(x)$ antiderivative of $f(x)$,

$\Rightarrow \frac{d}{du}F[u] = f(u)$, where $u = u(x)$.

Applying chain rule to $\frac{d}{dx}F[u]$:

$$\frac{d}{dx}F[u] = \frac{du(x)}{dx}\cdot\frac{dF(u)}{du} = \frac{du}{dx}\cdot f(u) \Rightarrow \frac{d}{dx}F[u] = \frac{du}{dx}\cdot f(u)$$

In other words, $F[u]$ is an antiderivative of $\frac{du}{dx}\cdot f(u)$, so that we can write

$$\int\left(\frac{du}{dx}\cdot f(u)\right)dx = F[u] + C$$

The power rule in integral form

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n\frac{du}{dx} \rightarrow \int\left(u^n\frac{du}{dx}\right)dx = \left(\frac{u^{n+1}}{n+1}\right) + C$$

$$\int\left(u^n\frac{du}{dx}\right)dx = \int u^n\left(\frac{du}{dx}dx\right) = \int u^n du$$

differential of $u(x)$, du is $du = \frac{du}{dx}dx$

If u is any differentiable function, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1, n \text{ rational}). \qquad (1)$$

Example 1 Using the power rule

The strategy is to convert the integral into the form $\int u^n\,du = \frac{u^{n+1}}{n+1} + C$

$$\int\sqrt{1+y^2}\cdot 2y\;dy = ?$$

Let $u = 1 + y^2, du = \frac{du}{dy}dy = 2y\,dy$.

$$\int\sqrt{1+y^2}\cdot 2y\;dy = \int\sqrt{u}\cdot du = \ldots$$

Example 2 Adjusting the integrand by a constant

$$\int\sqrt{4t-1}\;dt = ?$$

Let $u = 4t - 1, du = 4dt$,

$$\int\sqrt{4t-1}\;dt = \int u\;dt = \frac{1}{4}\int u\;4dt = \frac{1}{4}\int\sqrt{u}\;du = \ldots$$

Substitution: Running the chain rule backwards

THEOREM 5 The Substitution Rule
If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

let $u = g(x); \int f[g(x)]\cdot g'(x)dx = \int f(u)\cdot \frac{du}{dx}dx = \int f(u)du$

Used to find the integration with the integrand in the form of the product of $f[g(x)]\cdot g'(x)$

$$\int \underbrace{f[\underbrace{g(x)}]}_{f(u)}\cdot \underbrace{g'(x)dx}_{du} = \int f(u)du$$

Example 3 Using substitution

$$\int \cos\underbrace{(7x+5)}_{u}\ \underbrace{dx}_{\frac{1}{7}du} = \int \cos u\cdot\frac{du}{7} = \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7x+5) + C$$

Example 4 Using substitution

$$\int x^2 \sin x^3\ dx = ?$$

$$u = x^3; du = 3x^2 dx$$

$$\int \sin \underbrace{x^3}_{u}\ \underbrace{x^2 dx}_{\frac{1}{3}du} = \int \sin u \cdot \frac{1}{3}du = -\frac{1}{3}\cos u + C$$

$$= \frac{1}{3}\cos x^3 + C$$

Example 5 Using Identities and substitution

$$\int \frac{1}{\cos^2 2x}\ dx = \int \sec^2 2x\ dx = \int \sec^2 \underbrace{2x}_{u}\ \underbrace{dx}_{\frac{1}{2}du} =$$

$$\frac{1}{2}\int \underbrace{\sec^2 u\, du}_{\frac{d}{du}\tan u} = \frac{1}{2}\tan u + C = \frac{1}{2}\tan 2x + C$$

Example 6 Using different substitutions

$$\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz = \int \underbrace{\left(z^2+1\right)^{-1/3}}_{u^{-1/3}} \underbrace{2z\,dz}_{du} = \int u^{-1/3}\,du = \ldots$$

The integrals of $\sin^2 x$ and $\cos^2 x$

- Example 7

$$\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx$$

$$= \frac{x}{2} - \frac{1}{2}\int \cos \underbrace{2x}_{u}\,\underbrace{dx}_{\frac{1}{2}du}$$

$$= \frac{x}{2} - \frac{1}{4}\int \cos u\,du = \ldots$$

The integrals of $\sin^2 x$ and $\cos^2 x$

- Example 7(b)

$$\int \cos^2 x\,dx = \frac{1}{2}\int (\cos 2x + 1)\,dx = \ldots$$

Example 8 Area beneath the curve $y=\sin^2 x$

- For Figure 5.24, find
- (a) the definite integral of $y(x)$ over $[0,2\pi]$.
- (b) the area between the graph and the x-axis over $[0,2\pi]$.

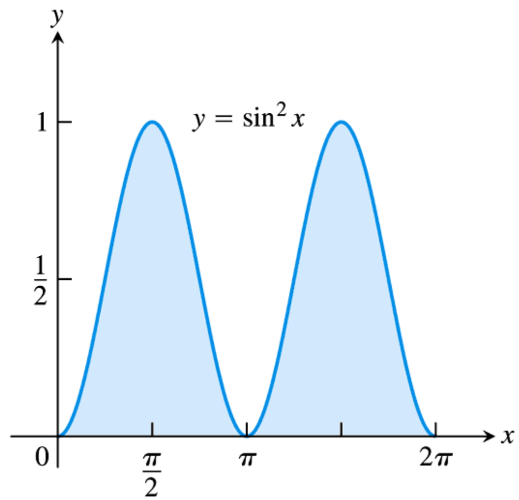


FIGURE 5.24 The area beneath the curve $y = \sin^2 x$ over $[0, 2\pi]$ equals π square units (Example 8).

5.6

Substitution and
Area Between Curves

Substitution formula

THEOREM 6 Substitution in Definite Integrals
If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$$\text{let } u = g(x);\ \int_{x=a}^{x=b} f[g(x)]\cdot g'(x)dx = \int_{x=a}^{x=b} f[u]\cdot\frac{du}{dx}dx = \int_{u=g(a)}^{u=g(b)} f(u)du$$

Example 1 Substitution

- Evaluate

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

$$\int_{x=-1}^{x=1} \underbrace{\sqrt{x^3+1}}_{u^{1/2}}\cdot \underbrace{3x^2dx}_{du} = \int_{u(x=-1)}^{u(x=1)} u^{1/2}\cdot du = \ldots$$

Example 2 Using the substitution formula

$$\int_{x=\pi/4}^{x=\pi/2} \cot x\csc^2 x dx = ?$$

$$\int \cot x\csc^2 x dx = \int \underbrace{\cot x}_{u}\cdot\underbrace{\csc^2 x dx}_{-du} = -\int u du = -\frac{u^2}{2}+c$$

$$= -\frac{\cot^2 x}{2}+c$$

$$\int_{\pi/4}^{\pi/2} \cot x\csc^2 x dx = -\frac{\cot^2 x}{2}\Bigg|_{\pi/4}^{\pi/2} = \frac{\cot^2 x}{2}\Bigg|_{\pi/2}^{\pi/4} = \frac{1}{2}\left[\underbrace{\cot^2(\pi/4)}_{1} - \underbrace{\cot^2(\pi/2)}_{0}\right] = \frac{1}{2}$$

Definite integrals of symmetric functions

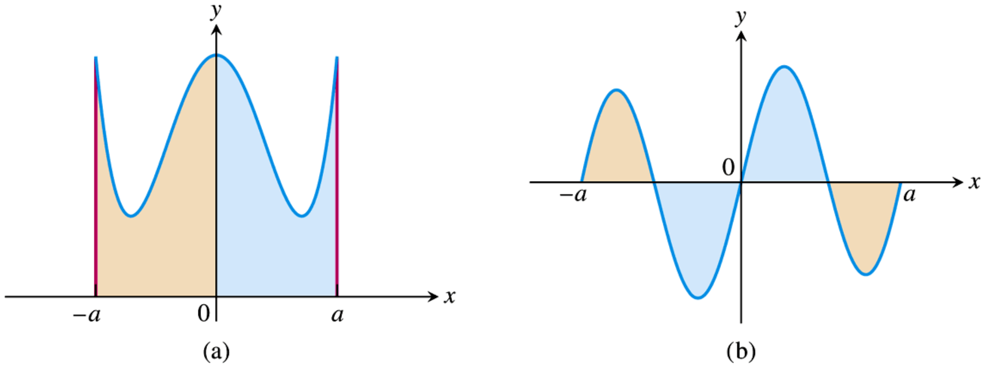


FIGURE 5.26 (a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ (b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

Theorem 7

Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Example 3 Integral of an even function

Evaluate $\int_{-2}^{2} \left(x^4 - 4x^2 + 6\right) dx$

Solution:

$f(x) = x^4 - 4x^2 + 6;$

$f(-x) = (-x)^4 - 4(-x)^2 + 6 = x^4 - 4x^2 + 6 = f(x)$

even function

How about integration of the same function from x=-1 to x=2

Area between curves

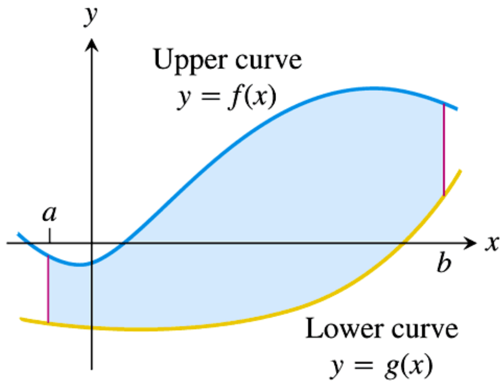


FIGURE 5.27 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

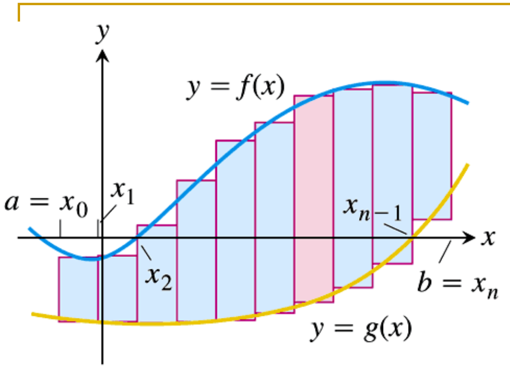


FIGURE 5.28 We approximate the region with rectangles perpendicular to the x-axis.

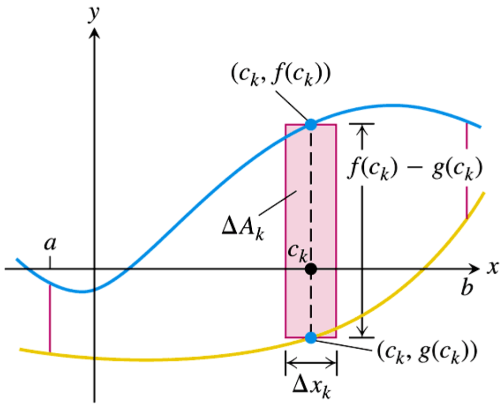


FIGURE 5.29 The area ΔA_k of the kth rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k.

$$A \approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} \Delta x_k \left[\left(f(c_k) - g(c_k) \right) \right]$$

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \Delta x_k \left[\left(f(c_k) - g(c_k) \right) \right] = \int_a^b \left[f(x) - g(x) \right] dx$$

DEFINITION **Area Between Curves**

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Example 4 Area between intersecting curves

- Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

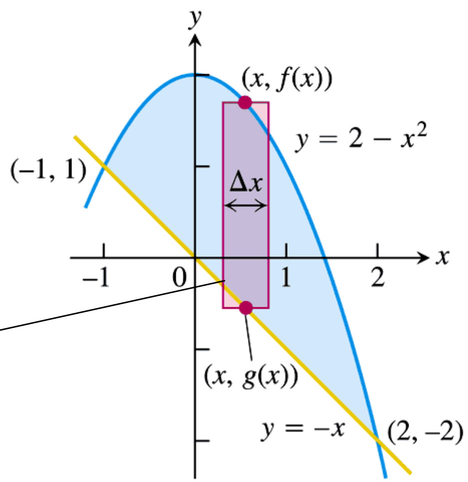


FIGURE 5.30 The region in Example 4 with a typical approximating rectangle.

$$\Delta A = \left(f(x) - g(x) \right) \cdot \Delta x$$

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} \Delta A_k = \int_0^A dA;$$

$$A = \int_{a=-1}^{b=2} [\underbrace{f(x)}_{2-x^2} - \underbrace{g(x)}_{x}]dx$$

$$= \int_{-1}^{2} \left(2 - x^2 - x \right) dx = \ldots$$

Example 5 Changing the integral to match a boundary change

- Find the area of the shaded region

$$Area = A + B$$

$$A = \int_0^2 \sqrt{x}\, dx;$$

$$B = \int_2^4 \sqrt{x} - (x - 2)\, dx$$

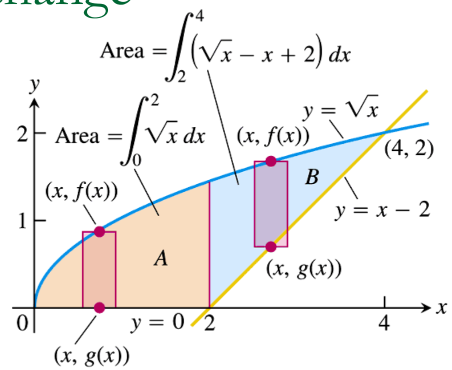


FIGURE 5.31 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 5.

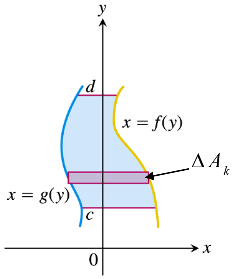
Integration with Respect to y

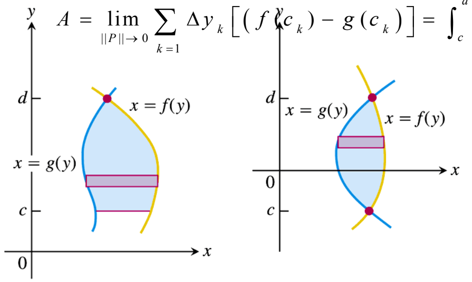
If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these

$$A \approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} \Delta y_k \left[\left(f(c_k) - g(c_k) \right) \right]$$

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \Delta y_k \left[\left(f(c_k) - g(c_k) \right) \right] = \int_c^d [f(y) - g(y)]\,dy$$





use the formula

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

Example 6 Find the area of the region in Example 5 by integrating with respect to y

$$\Delta A = (f(y) - g(y)) \cdot \Delta y$$

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} \Delta A_k = \int_{y=0}^{y=4} [f(y) - g(y)]\,dy$$

$$= \int_0^2 (y + 2) - (y^2)\,dy = \ldots$$

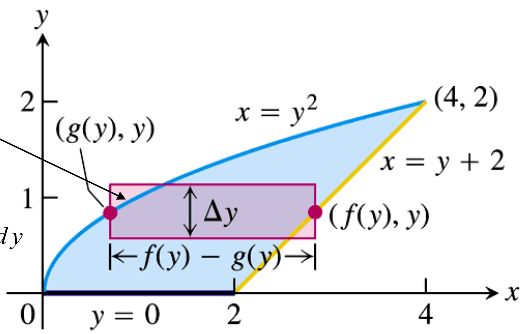
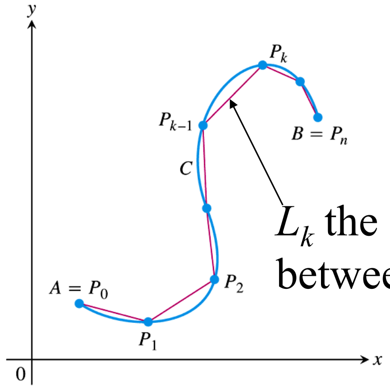


FIGURE 5.32 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y (Example 6).

6.3

Lengths of Plane Curves

Length of a parametrically defined curve



L_k the line segment between P_k and P_{k-1}

$$L = \lim_{\|P\| \to 0} \sum_{k}^{n} L_k$$

FIGURE 6.24 The curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1, and so on, ending at $B = P_n$.

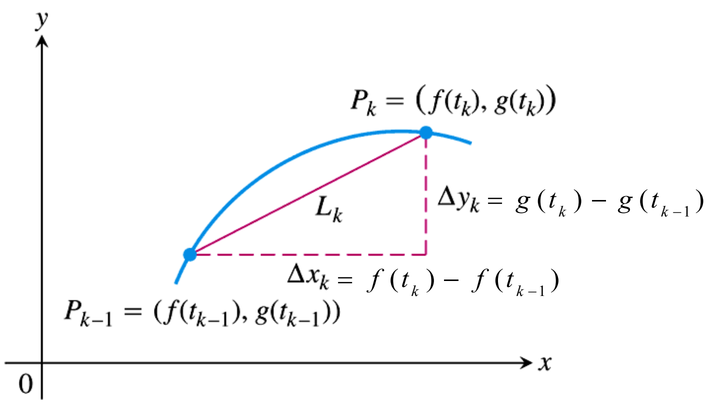


FIGURE 6.25 The arc $P_{k-1}P_k$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

y is parametried by t via $y = g(t)$;

x is parametried by t via $x = f(t)$.

$$\Delta y_k = g(t_k) - g(t_{k-1}) = g'(t_k^*) \cdot (t_k - t_{k-1}) = g'(t_k^*) \cdot \Delta t;$$

$$\Delta x_k = f(t_k) - f(t_{k-1}) = f'(t_k^{**}) \cdot (t_k - t_{k-1}) = f'(t_k^{**}) \cdot \Delta t$$

due to mean value theorem

$$L_k = \sqrt{(\Delta y_k)^2 + (\Delta x_k)^2} = \Delta t \sqrt{(g'(t_k^*))^2 + (f'(t_k^{**}))^2}$$

$$L = \lim_{n \to \infty} \sum_{k}^{n} L_k = \lim_{\|P\| \to 0} \sum_{k}^{n} L_k$$

$$= \lim_{\|P\| \to 0} \sum_{k}^{n} \Delta t \sqrt{(g'(t_k^*))^2 + (f'(t_k^{**}))^2}$$

$$= \int_a^b \sqrt{(g'(t))^2 + (f'(t))^2}\, dt = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

DEFINITION Length of a Parametric Curve

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Example 1 The circumference of a circle

- Find the length of the circle of radius r defined parametrically by
- $x=r\cos t$ and $y=r\sin t$, $0 \le t \le 2\pi$

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt \equiv \int_0^{2\pi} \sqrt{(r\cos t)^2 + (r\sin t)^2}\, dt$$

$$= r\int_0^{2\pi} dt = 2\pi r$$

Length of a curve $y = f(x)$

Assign the parameter $x = t$, the length of the curve $y = f(x)$ is then given by

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$$y = y[x(t)] \Rightarrow \frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} = \frac{dy}{dx} \qquad \left(\because \frac{dx}{dt} = 1\right)$$

$$L = \int_a^b dt \sqrt{\left(\frac{dy}{dx}\cdot\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \int_a^b dx \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}$$

$$= \int_a^b dx \sqrt{[f'(x)]^2 + 1}$$

Formula for the Length of $y = f(x)$, $a \le x \le b$

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \qquad (2)$$

Example 3 Applying the arc length formula for a graph

- Find the length of the curve

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1, \quad 0 \leq x \leq 1$$

Dealing with discontinuity in *dy*/*dx*

- At a point on a curve where *dy*/*dx* fails to exist and we may be able to find the curve's length by expressing *x* as a function of *y* and applying the following

Formula for the Length of $x = g(y)$, $c \leq y \leq d$
If g is continuously differentiable on $[c, d]$, the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \quad (3)$$

Example 4 Length of a graph which has a discontinuity in *dy*/*dx*

- Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.
- **Solution**
- $dy/dx = (1/3)\,(2/x)^{1/3}$ is not defined at $x=0$.
- $dx/dy = 3y^{1/2}$ is continuous on $[0,1]$.

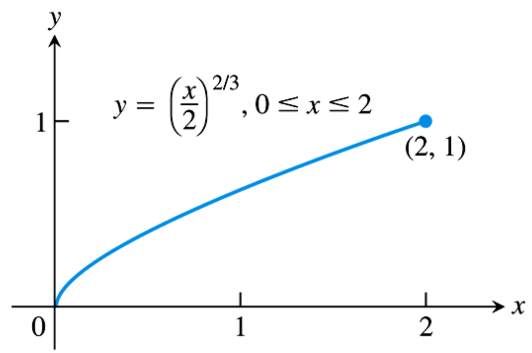


FIGURE 6.27 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Example 4).

Chapter 7

Transcendental Functions

7.1

Inverse Functions and Their Derivatives

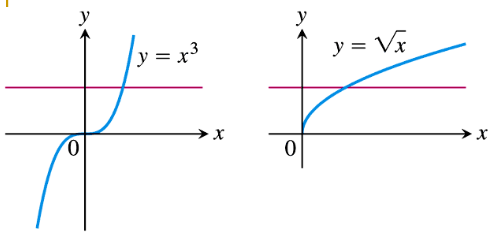
DEFINITION One-to-One Function

A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

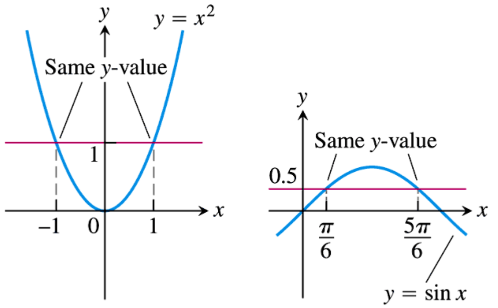
Example 1 Domains of one-to-one functions

- (a) $f(x) = x^{1/2}$ is one-to-one on any domain of nonnegative numbers
- (b) $g(x) = \sin x$ is NOT one-to-one on $[0,\pi]$ but one-to-one on $[0,\pi/2]$.

The Horizontal Line Test for One-to-One Functions
A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.



One-to-one: Graph meets each horizontal line at most once.



Not one-to-one: Graph meets one or more horizontal lines more than once.

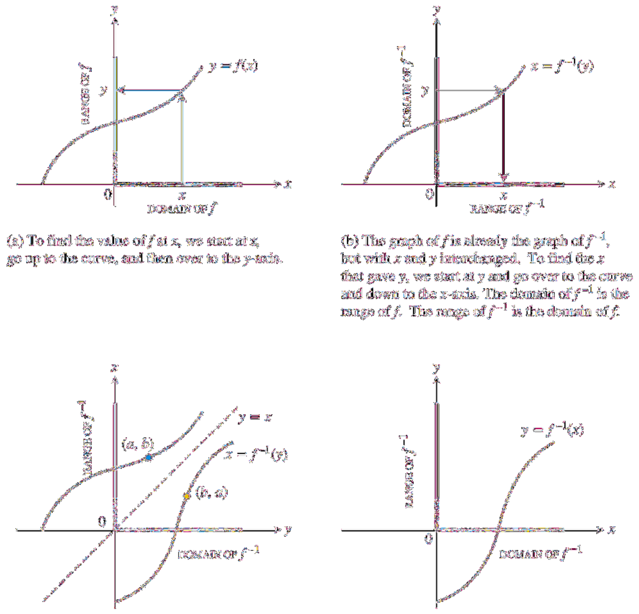
FIGURE 7.1 Using the horizontal line test, we see that $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$, but $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

DEFINITION Inverse Function

Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a.$$

The domain of f^{-1} is R and the range of f^{-1} is D.



(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.

(b) The graph of f is already the graph of f^{-1}, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

(c) To draw the graph of f^{-1} in the more usual way, we reflect the system in the line $y = x$.

(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

FIGURE 7.2 Determining the graph of $y = f^{-1}(x)$ from the graph of $y = f(x)$.

Finding inverses

- 1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.

2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where $f^{-1}(x)$ is expressed in the conventional format with x as the independent variable and y as the dependent variables.

Example 2 Finding an inverse function

- Find the inverse of $y = x/2 + 1$, expressed as a function of x.

- **Solution**
- 1. solve for x in terms of y: $x = 2(y - 1)$
- 2. interchange x and y: $y = 2(x - 1)$
- The inverse function $f^{-1}(x) = 2(x - 1)$
- Check:
- $f^{-1}[f(x)] = 2[f(x) - 1] = 2[(x/2 + 1) - 1] = x = f[f^{-1}(x)]$

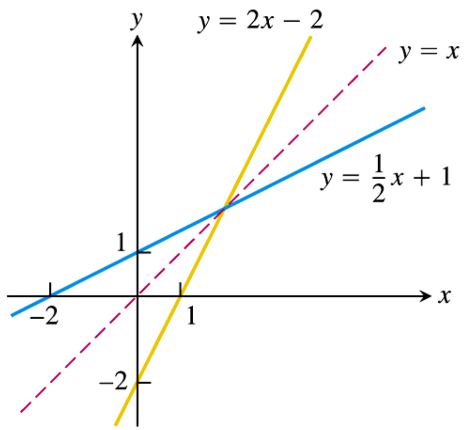


FIGURE 7.3 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$. The slopes are reciprocals of each other (Example 2).

Example 3 Finding an inverse function

- Find the inverse of $y = x^2$, $x \geq 0$, expressed as a function of x.
- **Solution**
- 1. solve for x in terms of y: $x = \sqrt{y}$
- 2. interchange x and y: $y = \sqrt{x}$
- The inverse function $f^{-1}(x) = \sqrt{x}$

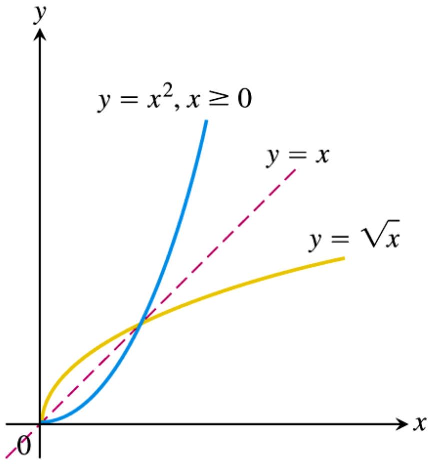


FIGURE 7.4 The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 3).

Derivatives of inverses of differentiable functions

- From example 2 (a linear function)
- $f(x) = x/2 + 1$; $f^{-1}(x) = 2(x + 1)$;
- $df(x)/dx = 1/2$; $df^{-1}(x)/dx = 2$,
- i.e. $df(x)/dx = 1/df^{-1}(x)/dx$
- Such a result is obvious because their graphs are obtained by reflecting on the $y = x$ line.
- Does the reciprocal relationship between the slopes of f and f^{-1} holds for other functions as well?

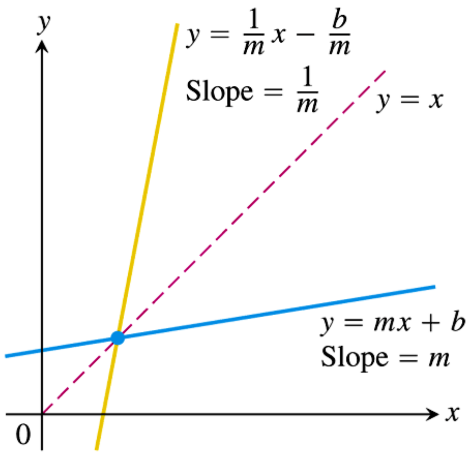
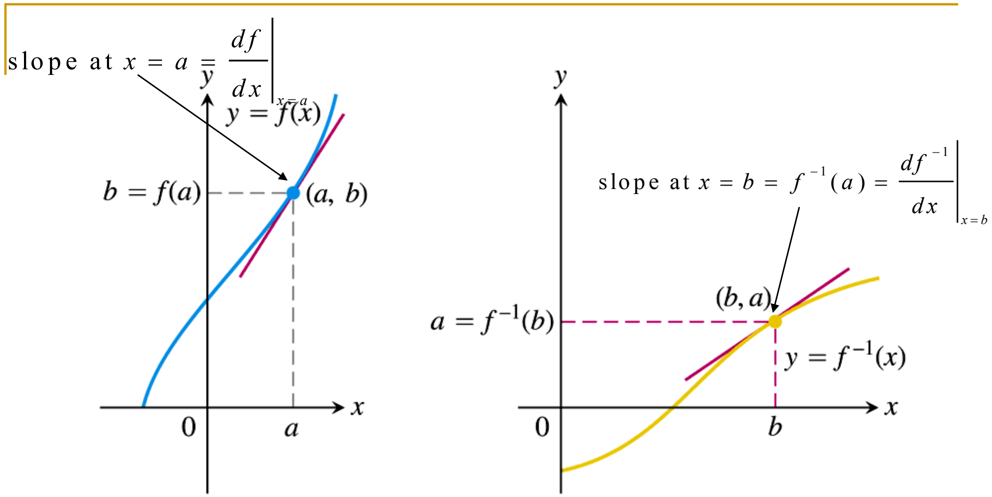


FIGURE 7.5 The slopes of nonvertical lines reflected through the line $y = x$ are reciprocals of each other.



The slopes are reciprocal: $(f^{-1})'(b) = \frac{1}{f'(a)}$ or $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$

FIGURE 7.6 The graphs of inverse functions have reciprocal slopes at corresponding points.

THEOREM 1 The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(b)}} \qquad (1)$$

Example 4 Applying theorem 1

- The function $f(x) = x^2$, $x \geq 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$, and $(f^{-1})'(x) = 1/(2\sqrt{x})$.
- Theorem 1 predicts that the derivative of $f^{-1}(x)$ is

$(f^{-1})'(x) = 1/f'[f^{-1}(x)] = 1/f'[\sqrt{x}]$

$= 1/(2\sqrt{x})$

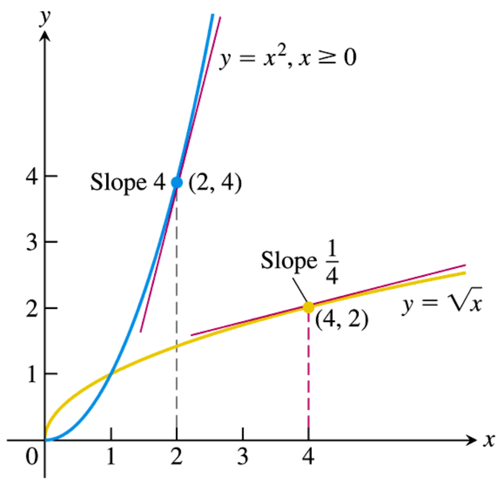


FIGURE 7.7 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 4).

Example 5 Finding a value of the inverse derivative

- Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without a formula for f^{-1}.
- The point for f is (2,6); The corresponding point for f^{-1} is (6,2).
- **Solution**
- $df/dx = 3x^2$
- $df^{-1}/dx|_{x=6} = 1/(df/dx|_{x=2})$

$= 1/3x^2|_{x=2} = 1/12$

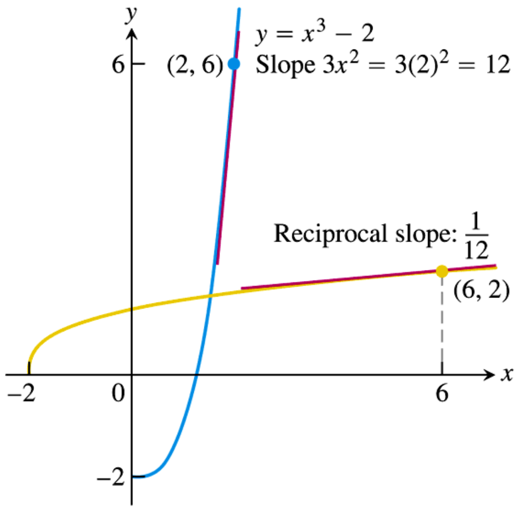


FIGURE 7.8 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 5).

7.2

Natural Logarithms

Definition of natural logarithmic fuction

DEFINITION **The Natural Logarithm Function**

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$

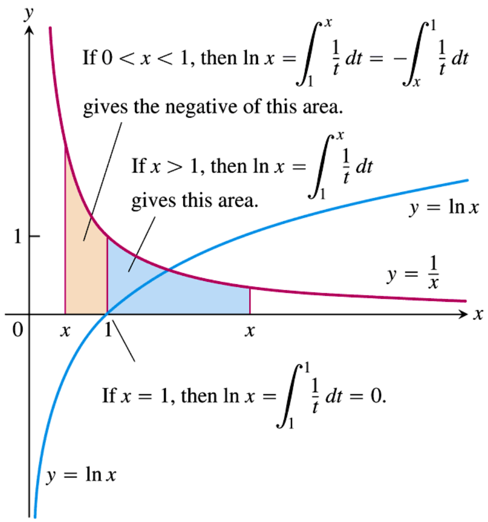


FIGURE 7.9 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left.

- Domain of ln x = $(0,\infty)$
- Range of ln x = $(-\infty,\infty)$
- ln x is an increasing function since $dy/dx = 1/x > 0$

$$\ln x = \int_1^x \frac{1}{x}\,dx$$

TABLE 7.1 Typical 2-place values of ln x

x	ln x
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

e lies between 2 and 3

$\ln x = 1$

DEFINITION The Number *e*

The number e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1$$

By definition, the antiderivative of ln x is just $1/x$

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0 \qquad (1)$$

Let $u = u(x)$. By chain rule,

$d/dx\,[\ln u(x)] = d/du(\ln u)\cdot du(x)/dx$

$=(1/u)\cdot du(x)/dx$

Example 1 Derivatives of natural logarithms

$$(a)\ \frac{d}{dx}\ln 2x =$$

$$(b)\quad u = x^2 + 3;\ \frac{d}{dx}\ln u = \frac{du}{dx}\frac{1}{u} =$$

Properties of logarithms

THEOREM 2 Properties of Logarithms

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln ax = \ln a + \ln x$
2. *Quotient Rule*: $\ln \frac{a}{x} = \ln a - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ — Rule 2 with $a = 1$
4. *Power Rule*: $\ln x^r = r \ln x$ — r rational

Example 2 Interpreting the properties of logarithms

$$(a)\ln 6 = \ln(2 \cdot 3) = \ln 2 + \ln 3;$$

$$(b)\ln 4 - \ln 5 = \ln(4/5) = \ln 0.8$$

$$(c)\ln(1/8) = \ln 1 - \ln 2^3 = -3\ln 2$$

Example 3 Applying the properties to function formulas

$$(a)\ln 4 + \ln \sin x = \ln(4 \sin x);$$

$$(b)\ln \frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$$

$$(c)\ln(\sec x) = \ln \frac{1}{\cos x} = -\ln \cos x$$

$$(d)\ln \sqrt[3]{x+1} = \ln(x+1)^{1/3} = (1/3)\ln(x+1)$$

Proof of ln ax = ln a + ln x

- ln ax and ln x have the same derivative:

$$\frac{d}{dx}\ln ax = \frac{d(ax)}{dx}\frac{1}{ax} = a\frac{1}{ax} = \frac{1}{x} = \frac{d}{dx}\ln x$$

- Hence, by the corollary 2 of the mean value theorem, they differs by a constant C

$$\ln ax = \ln x + C$$

- We will prove that C = ln a by applying the definition ln x at x = 1.

Estimate the value of ln 2

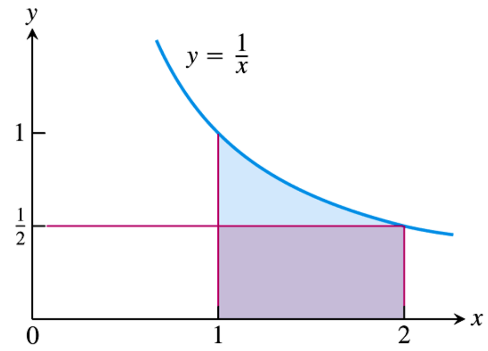


FIGURE 7.10 The rectangle of height $y = 1/2$ fits beneath the graph of $y = 1/x$ for the interval $1 \le x \le 2$.

$$\ln 2 = \int_1^2 \frac{1}{x}dx$$

$$\frac{1}{2}\cdot(2-1) < \int_1^2 \frac{1}{x}dx < 1\cdot(2-1) = 1$$

$$\frac{1}{2} < \ln 2 < 1$$

The integral $\int (1/u)\, du$

From $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

For $u > 0$

Taking the integration on both sides gives

$$\int \frac{d}{dx}\ln u\, dx = \int \frac{1}{u}\frac{du}{dx}dx.$$

Let $y = \ln u \rightarrow \frac{d}{dx}\ln u\, dx = \frac{dy}{dx}dx = dy \rightarrow \int \frac{d}{dx}\ln u\, dx = \int dy = \int d\ln u$

$$\int d\ln u = \int \frac{du}{u} \rightarrow \ln u + C' = \int \frac{du}{u};$$

For $u < 0$:

$-u > 0$,

$$\int \frac{d}{dx}\ln(-u)dx = \int \frac{1}{(-u)}\frac{d(-u)}{dx}dx$$

$$\int d\ln(-u) = \int \frac{du}{u} \rightarrow \ln(-u) + C'' = \int \frac{du}{u}$$

Combining both cases of $u > 0, u < 0$,

$$\int \frac{du}{u} = \ln|u| + C$$

recall: $\int u^n du = \frac{u^{n+1}}{n+1} + C, n$ rational, $\neq -1$

If u is a differentiable function that is never zero,

$$\int \frac{1}{u}du = \ln|u| + C. \qquad (5)$$

From $\int u^{-1}du = \ln|u| + C$.

let $u = f(x)$.

$$\int u^{-1}du = \int \frac{du}{u} = \int \frac{df(x)}{f(x)} = \int \frac{\frac{df(x)}{dx}dx}{f(x)}$$

$$\Rightarrow \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

Example 4 Applying equation (5)

$$(a)\int \frac{2x\,dx}{x^2-5} = \int \frac{d(x^2-5)}{x^2-5} = \ln|x^2-5| + C$$

$$(b)\int_{-\pi/2}^{\pi/2} \frac{4\cos x}{3+2\sin x}\,dx = \dots$$

The integrals of tan *x* and cot *x*

$$\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc x| + C$$

Example 5

$$\int \tan 2x\,dx = \int \frac{\sin 2x}{\cos 2x}\,dx = \int \frac{-\frac{1}{2}\frac{d}{dx}\cos 2x}{\cos 2x}\,dx$$

$$= -\frac{1}{2}\int \frac{d\cos 2x}{\cos 2x} = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C$$

$$= -\frac{1}{2}\ln|\cos 2x| + C$$

$$= \frac{1}{2}\ln|\sec 2x| + C$$

Example 6 Using logarithmic differentiation

- Find *dy/dx* if

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1},\ x > 1$$

$$\ln y = \ln(x^2+1) + (1/2)\ln(x+3) - \ln(x-1)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\ln(x^2+1) + \frac{1}{2}\frac{d}{dx}\ln(x+3) - \frac{d}{dx}\ln(x-1)$$

$$\frac{1}{y}\frac{dy}{dx} = \dots$$

7.3

The Exponential Function

The inverse of ln x and the number e

- ln x is one-to-one, hence it has an inverse. We name the inverse of ln x, $\ln^{-1} x$ as exp (x)

$$\lim_{x\to\infty} \ln^{-1} x = \infty, \quad \lim_{x\to-\infty} \ln^{-1} x = 0$$

The graph of the inverse of ln x

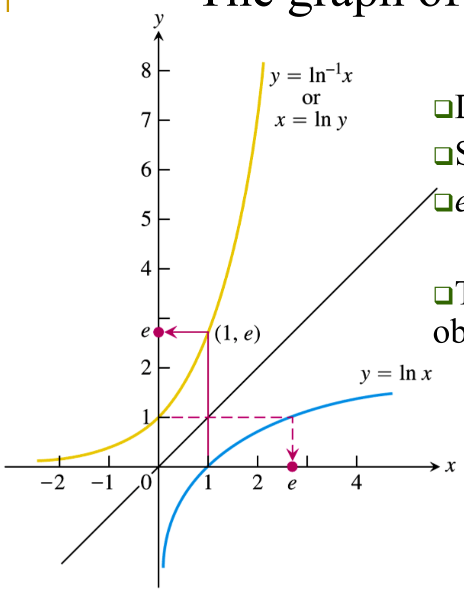


FIGURE 7.11 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

- Definition of e as $\ln e = 1$.
- So, $e = \ln^{-1}(1) = \exp(1)$
- $e = 2.718281828459045\ldots$ (an irrational number)
- The approximate value for e is obtained numerically (later).

The function $y = e^x$ in terms of the exponential function exp

- We can raise the number e to a rational power r, e^r
- e^r is positive since e is positive, hence e^r has a logarithm (recall that logarithm is defied only for positive number).
- From the power rule of theorem 2 on the properties of natural logarithm, $\ln x^r = r \ln x$, where r is rational, we have

$$\ln e^r = r$$

- We take the inverse to obtain

$$\ln^{-1} (\ln e^r) = \ln^{-1} (r)$$

$$e^r = \ln^{-1} (r) \equiv \exp r, \text{ for } r \text{ rational.}$$

The number e to a real (possibly irrational) power x

- How do we define e^x where x is irrational?
- This can be defined by assigning e^x as exp x since $\ln^{-1}(x)$ is defined (because the inverse function of ln x is defined for all real x).

DEFINITION The Natural Exponential Function
For every real number x, $e^x = \ln^{-1} x = \exp x$.

Note: please do make a distinction between e^x and exp x. They have different definitions.

e^x is the number e raised to the power of real number x.

exp x is defined as the inverse of the logarithmic function, $\exp x = \ln^{-1} x$

Typical values of e^x

x	e^x (rounded)
−1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

Inverse Equations for e^x and ln x

$$e^{\ln x} = x \qquad (\text{all } x > 0) \tag{2}$$

$$\ln(e^x) = x \qquad (\text{all } x) \tag{3}$$

- (2) follows from the definition of the exponent function:
- From e^x = exp x, let $x \to \ln x$
- $e^{\ln x} = \exp[\ln x] = x$ (by definition).
- For (3): From e^x = exp x, take logarithm both sides, $\to \ln e^x = \ln[\exp x] = x$ (by definition)

Example 1 Using inverse equations

$$(a)\ \ln e^{2} = \dots$$

$$(b)\ \ln e^{-1} = \dots$$

$$(c)\ \ln \sqrt{e} = \ln e^{1/2} = \dots$$

$$(d)\ \ln e^{\sin x} = \dots$$

$$(f)\ e^{\ln 2} = \dots$$

$$(g)\ e^{\ln\left(x^{2}+1\right)} = \dots$$

$$(h)\ e^{3\ln 2} = e^{\ln 2^{3}} = \dots$$

$$(i)\ e^{3\ln 2} = e^{3\cdot\ln 2} = \left(e^{\ln 2}\right)^{3} = \dots$$

Example 2 Solving for an exponent

- Find k if $e^{2k}=10$.

The general exponential function a^x

- Since $a = e^{\ln a}$ for any positive number a
- $a^x = (e^{\ln a})^x = e^{x\ln a}$

DEFINITION General Exponential Functions

For any numbers $a > 0$ and x, the exponential function with base a is

$$a^{x} = e^{x \ln a}.$$

For the first time we have a precise meaning for an irrational exponent. (previously a^x is defined for only rational x and a)

Example 3 Evaluating exponential functions

$$(a)\ 2^{\sqrt{3}} = \left(e^{\ln 2}\right)^{\sqrt{3}} = e^{\sqrt{3}\ln 2} \approx e^{1.20} \approx 3.32$$

$$(b)\ 2^{\pi} = \left(e^{\ln 2}\right)^{\pi} = e^{\pi \ln 2} \approx e^{2.18} \approx 8.8$$

Laws of exponents

THEOREM 3 Laws of Exponents for e^x

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1x_2} = (e^{x_2})^{x_1}$

Theorem 3 also valid for a^x

Proof of law 1

$$y_1 = e^{x_1}, y_2 = e^{x_2}$$

$$\Rightarrow x_1 = \ln y_1, x_2 = \ln y_2$$

$$\Rightarrow x_1 + x_2 = \ln y_1 + \ln y_2 = \ln y_1 y_2$$

$$\Rightarrow \exp(x_1 + x_2) = \exp(\ln y_1 y_2)$$

$$e^{x_1+x_2} = y_1 y_2 = e^{x_1} e^{x2}$$

Example 4 Applying the exponent laws

$$(a) e^{x+\ln 2} =$$

$$(b) e^{-\ln x} =$$

$$(c) \frac{e^{2x}}{e} =$$

$$(d) \left(e^3\right)^x =$$

The derivative and integral of e^x

$$f(x) = \ln x, y = e^x = \ln^{-1} x = f^{-1}(x)$$

$$\frac{dy}{dx} = \frac{d}{dx} e^x = \frac{d}{dx} f^{-1}(x) = \frac{1}{\left.\dfrac{df(x)}{dx}\right|_{x=f^{-1}(x)}}$$

$$= \frac{1}{\left.(1/x)\right|_{x=f^{-1}(x)}} = \frac{1}{\left.(1/x)\right|_{x=y}} = y = e^x$$

$$\frac{d}{dx} e^x = e^x \qquad (5)$$

Example 5 Differentiating an exponential

$$\frac{d}{dx}\left(5e^{x}\right) =$$

By the virtue of the chain rule, we obtain

If u is any differentiable function of x, then

$$\frac{d}{dx}e^{u} = e^{u}\frac{du}{dx}. \qquad (6)$$

$$f(u) = e^{u}; u = u(x);$$

$$\frac{d}{dx}\left(e^{u(x)}\right) = \frac{d}{dx}f(u) = \frac{df(u)}{du}\frac{du(x)}{dx} = e^{u}\frac{du}{dx}$$

$$\int e^{u}\,du = e^{u} + C.$$

This is the integral equivalent of (6)

Example 7 Integrating exponentials

$$(a)\int_{0}^{\ln 2} e^{3x}dx =$$

$$(b)\int_{0}^{\pi/2} e^{\sin x}\cos x\,dx = \int_{0}^{\pi/2} \underbrace{e^{\sin x}}_{e^{u}}\underbrace{\cos x\,dx}_{du}$$

$$= \int_{u(0)}^{u(\pi/2)} e^{u}du$$

$$= e^{u}\Big|_{u(0)}^{u(\pi/2)} = e^{u(\pi/2)} - e^{u(0)} = e^{\sin(\pi/2)} - e^{\sin(0)} = e - 1$$

The number *e* expressed as a limit

THEOREM 4 The Number *e* as a Limit

The number e can be calculated as the limit

$$e = \lim_{x\to 0}(1 + x)^{1/x}.$$

Proof

- If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$. But by definition of derivative,
- $$f'(y) = \lim_{h \to 0} \frac{f(y+h) - f(y)}{h}$$
$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(x)}{x}$$
$$= \lim_{x \to 0} \frac{\ln(1+x) - \ln(1)}{x} = \lim_{x \to 0} \frac{\ln(1+x)}{x}$$
$$= \lim_{x \to 0} \left[\ln(1+x)\right]^{\frac{1}{x}} = \ln\left[\lim_{x \to 0} (1+x)^{\frac{1}{x}}\right] = 1 \quad (\text{since } f'(1) = 1)$$
$$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = \lim_{y \to \infty} (1 + \frac{1}{y})^{y} = e$$

Define x^n for any real $x > 0$ as $x^n = e^{n \ln x}$.

Here n need not be rational but can be any real number as long as x is positive.

Then we can take the logarithm of x^n:

$$\ln x^n = \ln\left(e^{n \ln x}\right) = n \ln x.$$

Note: c.f the power rule in theorem 2.

Can you tell the difference?

Once x^n is defined via $x^n = e^{n \ln x}$, we can take its differentiation:

$$\frac{d}{dx} x^n = \frac{d}{dx}\left(\overbrace{e^{n \ln x}}^{u(x)}\right) = \frac{du}{dx}\frac{de^u}{du} = \frac{n}{x} e^{n \ln x} = \frac{n}{x} x^n = n x^{n-1}$$

$$\Rightarrow \frac{d}{dx} x^n = n x^{n-1}$$

Note: Can you tell the difference between this formula and the one we discussed in earlier chapters (Theorem 4, Chapter 3)?

- By virtue of chain rule,

$u = u(x)$;

$$\frac{d}{dx} u^n = \frac{du(x)}{dx}\frac{du^n}{du} = \frac{du(x)}{dx} n u^{n-1}$$

Power Rule (General Form)

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx}.$$

Example 9 using the power rule with irrational powers

$$(a)\ \frac{d}{dx}x^{\sqrt{2}} \equiv \frac{du^{n}}{dx} = \frac{du}{dx}nu^{n-1}$$

$$\frac{du}{dx}nu^{n-1} \equiv \frac{dx}{dx}\sqrt{2}x^{\sqrt{2}-1} = \sqrt{2}x^{\sqrt{2}-1}$$

$$(b)\ \frac{d}{dx}(2+\sin 3x)^{\pi} \equiv \frac{du^{n}}{dx} = \frac{du}{dx}nu^{n-1}$$

$$\frac{du}{dx}nu^{n-1} \equiv \frac{d(2+\sin 3x)}{dx}\pi u^{\pi-1} = 3\pi(2+\sin 3x)^{\pi-1}\cos 3x$$

7.4

a^x and $\log_a x$

The derivative of a^x

$$a^x = e^{x\ln a}$$

$$\frac{d}{dx}a^x = \frac{d}{dx}\left(e^{\overset{u}{x\ln a}}\right) = \frac{d}{dx}(x\ln a)\frac{d}{du}\left(e^u\right)$$

$$= e^u\ln a = e^{x\ln a}\ln a = a^x\ln a$$

By virtue of the chain rule,

$$\frac{d}{dx}a^{u(x)} = \frac{du}{dx}\frac{d}{du}\left(a^u\right) = a^u\ln a\frac{du}{dx}$$

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u\ln a\,\frac{du}{dx}. \qquad (1)$$

Example 1: Differentiating general exponential functions

$$(a)\ \frac{d}{dx}3^x = \frac{d}{dx}\left(e^{\overset{u}{x\ln 3}}\right) = \frac{d}{dx}(x\ln 3)\frac{d}{du}\left(e^u\right)$$

$$= \ln 3\cdot e^{x\ln 3} = 3^x\ln 3$$

$$(b)\ \frac{d}{dx}3^{-x} = -\frac{d}{d(-x)}3^{\overset{u}{(-x)}} = -\frac{d}{du}3^u = -\frac{d}{du}3^u$$

$$= -3^u\ln 3 = -3^{(-x)}\ln 3 = -\ln 3/3^x$$

$$(c)\ \frac{d}{dx}3^{\overset{u}{\sin x}} = \frac{du}{dx}\frac{d}{du}3^u = \frac{d(\sin x)}{dx}3^u\ln 3 = 3^{\sin x}\ln 3\cdot\cos x$$

Other power functions

- Example 2 Differentiating a general power function
- Find dy/dx if $y = x^x$, $x > 0$.
- **Solution:** Write x^x as a power of e
- $x^x = e^{x\ln x}$

$$\frac{d}{dx}\left(e^{\overset{u}{x\ln x}}\right) = \frac{du}{dx}\frac{d}{du}\left(e^u\right) = \frac{d}{dx}(x\ln x)\cdot\left(e^u\right) = \ldots$$

Integral of a^u

From $\frac{d}{dx}a^{u(x)} = a^u \ln a \frac{du}{dx}$, devide by $\ln a$:

$$\Rightarrow \frac{1}{\ln a}\frac{d}{dx}a^{u(x)} = a^u \frac{du}{dx}$$

$$\Rightarrow \frac{d}{dx}a^{u(x)} = a^u \ln a \frac{du}{dx}, \text{ integrate both sides wrp to } dx:$$

$$\Rightarrow \int\left(\frac{d}{dx}a^u\right)dx = \int\left(a^u \ln a \frac{du}{dx}\right)dx:$$

$$\Rightarrow \int da^u = \ln a \int a^u du + C$$

$$\Rightarrow \int a^u du = \frac{1}{\ln a}\int da^u = \frac{a^u}{\ln a}$$

$$\int a^u\, du = \frac{a^u}{\ln a} + C. \qquad (2)$$

Example 3 Integrating general exponential functions

$$(a)\int 2^x dx = \frac{2^x}{\ln 2} + C$$

$$(b)\int 2^{\overset{u}{\sin x}} \overset{du}{\cos dx} = \int 2^u du = \ldots$$

Logarithm with base a

DEFINITION $\log_a x$

For any positive number $a \neq 1$,

$\log_a x$ is the inverse function of a^x.

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0) \qquad (3)$$

$$\log_a(a^x) = x \qquad (\text{all } x) \qquad (4)$$

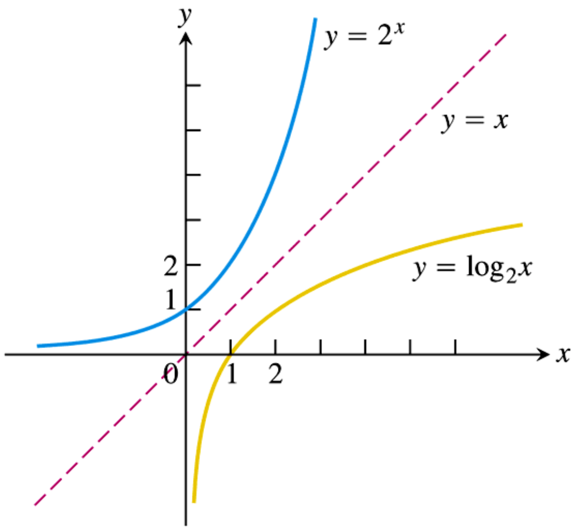


FIGURE 7.13 The graph of 2^x and its inverse, $\log_2 x$.

Example 4 Applying the inverse equations

$(a)\ \log_2 2^5 = 5$

$(b)\ 2^{\log_2 3} = 3$

$(c)\ \log_{10} 10^{(-7)} = -7$

$(d)\ 10^{\log_{10} 4} = 4$

Evaluation of $\log_a x$

Taking ln on both sides of $a^{\log_a x} = x$ gives

$\ln(a^{\log_a x}) = \ln x$

LHS, $\ln(a^{\log_a x}) = \log_a x \ln a.$

Equating LHS to RHS yields

$\log_a x \ln a = \ln x$

$$\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a} \qquad (5)$$

- Example: $\log_{10} 2 = \ln 2 / \ln 10$

TABLE 7.2 Rules for base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:* $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule:* $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Reciprocal Rule:* $\log_a \frac{1}{y} = -\log_a y$
4. *Power Rule:* $\log_a x^y = y \log_a x$

- Proof of rule 1:

$\ln xy = \ln x + \ln y$

divide both sides by $\ln a$

$$\frac{\ln(xy)}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a}$$

$\log_a(xy) = \log_a x + \log_a y$

Derivatives and integrals involving $\log_a x$

$$\frac{d}{dx}(\log_a u) = \frac{du}{dx}\frac{d(\log_a u)}{du}$$

$$\frac{d}{du}(\log_a u) = \frac{d}{du}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{du}(\ln u) = \frac{1}{\ln a}\frac{1}{u}$$

$$\frac{d}{dx}(\log_a u) = \frac{du}{dx}\cdot\left(\frac{1}{\ln a}\frac{1}{u}\right) = \frac{1}{\ln a}\left(\frac{1}{u}\right)\cdot\frac{du}{dx}$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}$$

Example 5

$$(a)\ \frac{d}{dx}\left(\log_{10}\overbrace{(3x+1)}^{u}\right) = \frac{du}{dx}\frac{d(\log_{10} u)}{du}$$

$$= \frac{d}{dx}(3x+1)\frac{1}{\ln 10}\frac{d(\ln u)}{du} = \frac{3}{\ln 10}\frac{1}{(3x+1)}$$

$$(b)\int\frac{\log_2 x}{x}dx = \frac{1}{\ln 2}\int\underbrace{\ln x}_{u}\ \underbrace{\frac{dx}{x}}_{d(\ln x)\equiv du} = \frac{1}{\ln 2}\int u\,du = \ldots$$

7.5

Exponential Growth and Decay

The law of exponential change

- For a quantity y increases or decreases at a rate proportional to it size at a give time t follows the law of exponential change, as per

$$\frac{dy}{dt} \propto y(t) \Rightarrow \frac{dy}{dt} = ky(t).$$

k is the proportional constant.

Very often we have to specify the value of y at some specified time, for example the initial condition $y(t=0) = y_0$

Rearrange the equation $\frac{dy}{dt} = ky$:

$$\frac{1}{y}\frac{dy}{dt} = k \rightarrow \int \frac{1}{y}\frac{dy}{dt}dt = \int k\,dt$$

$$\rightarrow \int \frac{1}{y}dy = k\int dt = kt \rightarrow \ln|y| = kt + \ln C$$

$$\rightarrow y = \pm Ce^{kt} = Ae^{kt}, A = \pm C.$$

Put in the initial value of y at $t = 0$ is y_0:

$$\rightarrow y(0) = y_0 = Ae^{k\cdot 0} = A \rightarrow y = y_0 e^{kt}$$

The Law of Exponential Change

$$y = y_0 e^{kt} \qquad (2)$$

Growth: $k > 0$ Decay: $k < 0$

The number k is the **rate constant** of the equation.

Example 1 Reducing the cases of infectious disease

- Suppose that in the course of any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1000? Assume the law of exponential change applies.

Example 3 Half-life of a radioactive element

- The effective radioactive lifetime of polonium-210 is very short (in days). The number of radioactive atoms remaining after t days in a sample that starts with y_0 radioactive atoms is $y = y_0 \exp(-5\times10^{-3}t)$. Find the element's half life.

Solution

- Radioactive elements decay according to the exponential law of change. The half life of a given radioactive element can be expressed in term of the rate constant k that is specific to a given radioactive species. Here $k=-5\times10^{-3}$.
- At the half-life, $t= t_{1/2}$,
  $y(t_{1/2})= y_0/2 = y_0 \exp(-5\times10^{-3}\, t_{1/2})$
  $\exp(-5\times10^{-3}\, t_{1/2}) = 1/2$

→ $\ln(1/2) = -5\times10^{-3}\, t_{1/2}$

→ $t_{1/2} = -\ln(1/2)/5\times10^{-3} = \ln(2)/5\times10^{-3} = \ldots$

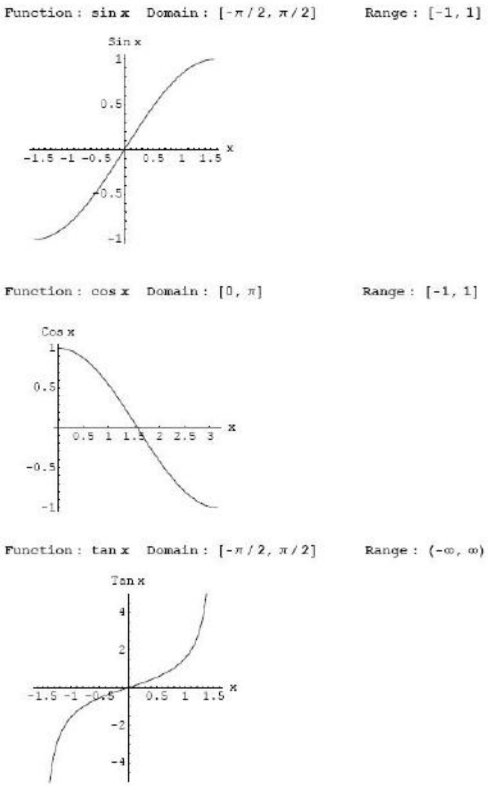
7.7

Inverse Trigonometric Functions

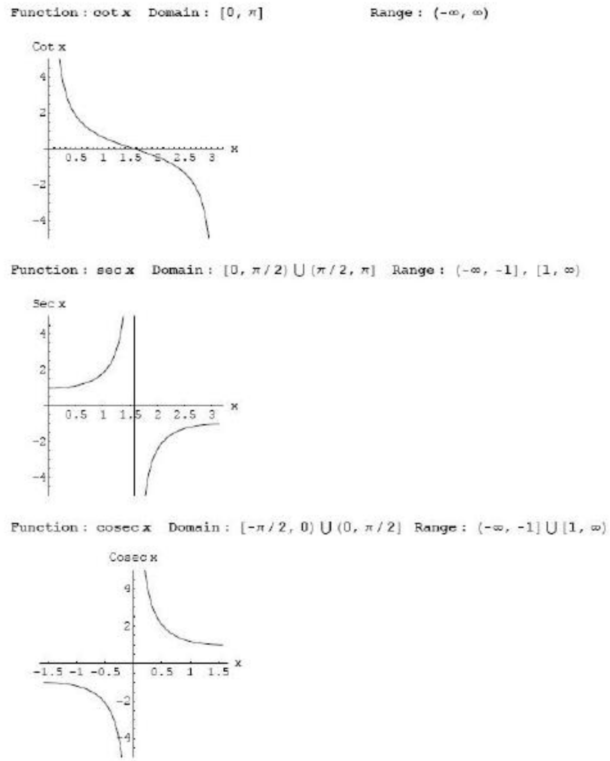
Defining the inverses

- Trigo functions are periodic, hence not one-to-one in the their domains.
- If we restrict the trigonometric functions to intervals on which they are one-to-one, then we can define their inverses.

Domain restriction that makes the trigonometric functions one-to-one



Domain restriction that makes the trigonometric functions one-to-one



Inverses for the restricted trigo functions

$$y = \sin^{-1} x = \arcsin x$$

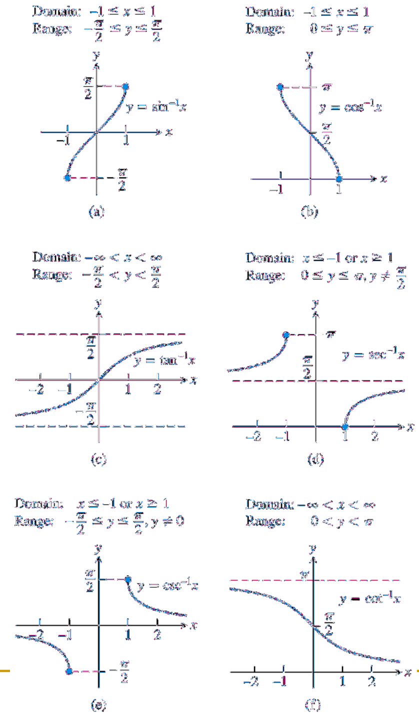
$$y = \cos^{-1} x = \arccos x$$

$$y = \tan^{-1} x = \arctan x$$

$$y = \cot^{-1} x = \operatorname{arccot} x$$

$$y = \sec^{-1} x = \operatorname{arcsec} x$$

$$y = \csc^{-1} x = \operatorname{arccsc} x$$



- The graphs of the inverse trigonometric functions can be obtained by reflecting the graphs of the restricted trigo functions through the line *y = x.*

FIGURE 7.17 Graphs of the six basic inverse trigonometric functions.

DEFINITION Arcsine and Arccosine Functions

$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.

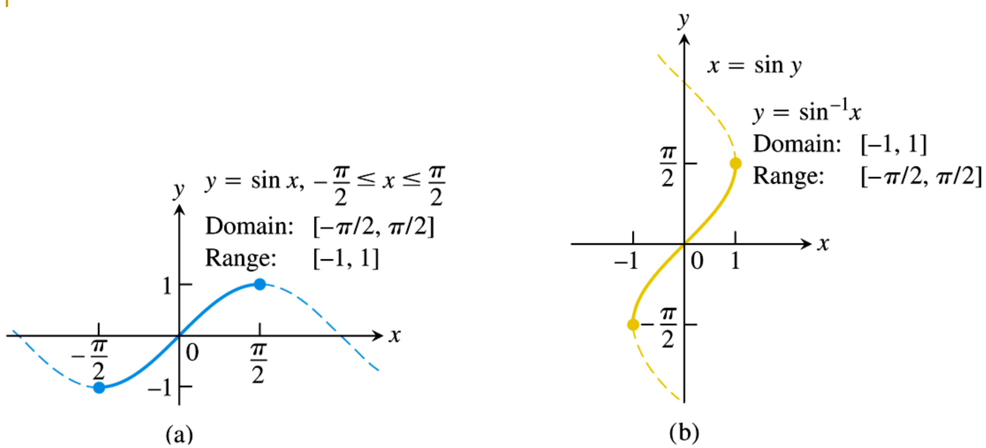


FIGURE 7.18 The graphs of (a) $y = \sin x$, $-\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

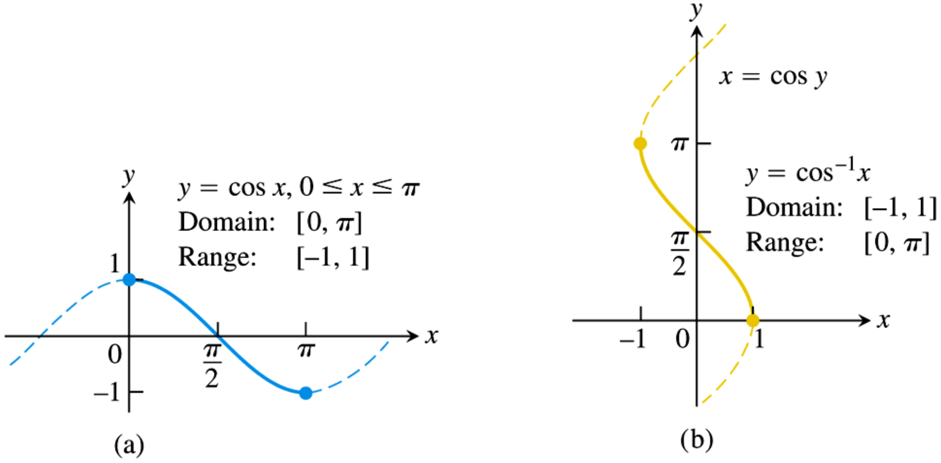
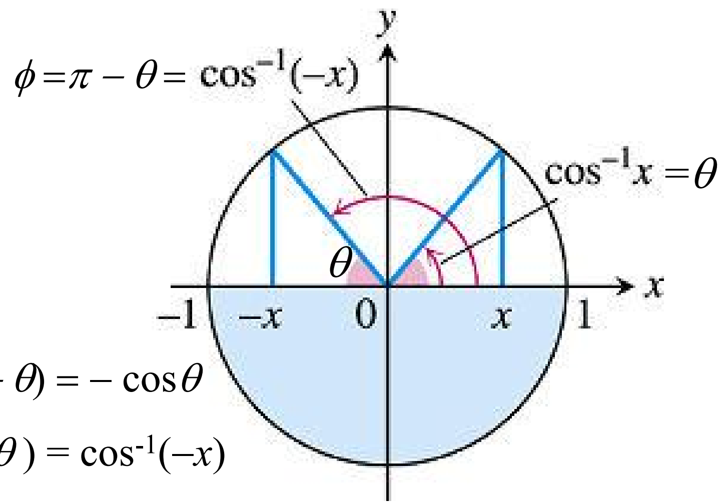


FIGURE 7.19 The graphs of (a) $y = \cos x$, $0 \le x \le \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

Some specific values of $\sin^{-1} x$ and $\cos^{-1} x$

x	$\sin^{-1} x$
$\sqrt{3}/2$	$\pi/3$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/6$
$-1/2$	$-\pi/6$
$-\sqrt{2}/2$	$-\pi/4$
$-\sqrt{3}/2$	$-\pi/3$

x	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/3$
$-1/2$	$2\pi/3$
$-\sqrt{2}/2$	$3\pi/4$
$-\sqrt{3}/2$	$5\pi/6$



$\theta = \cos^{-1} x;$

$\cos\phi = \cos(\pi - \theta) = -\cos\theta$

$\phi = \cos^{-1}(-\cos\theta) = \cos^{-1}(-x)$

Add up θ and ϕ:

$\theta + \phi = \cos^{-1} x + \cos^{-1}(-x)$

FIGURE 7.20 $\cos^{-1} x$ and $\cos^{-1}(-x)$ are supplementary angles (so their sum is π).

$\pi = \cos^{-1} x + \cos^{-1}(-x)$

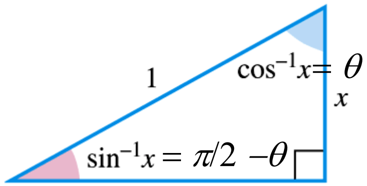


FIGURE 7.21 $\sin^{-1} x$ and $\cos^{-1} x$ are complementary angles (so their sum is $\pi/2$).

$$\cos^{-1} x = \theta; \sin^{-1} x = \left(\frac{\pi}{2} - \theta\right);$$

$$\cos^{-1} x + \sin^{-1} x + = \theta + \left(\frac{\pi}{2} - \theta\right) = \frac{\pi}{2}$$

link to slide derivatives of the other three

DEFINITION **Arctangent and Arccotangent Functions**

$y = \tan^{-1} x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$.

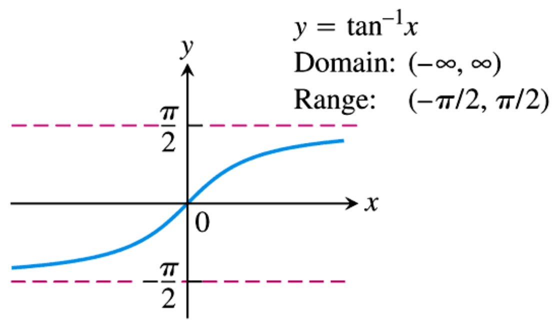


FIGURE 7.22 The graph of $y = \tan^{-1} x$.

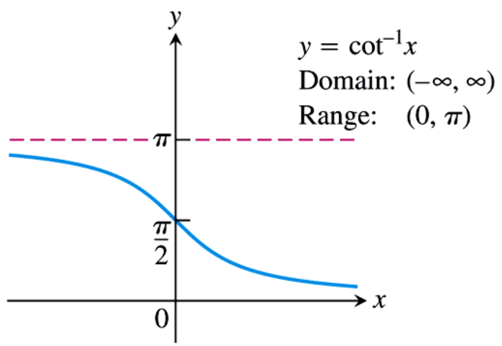


FIGURE 7.23 The graph of $y = \cot^{-1} x$.

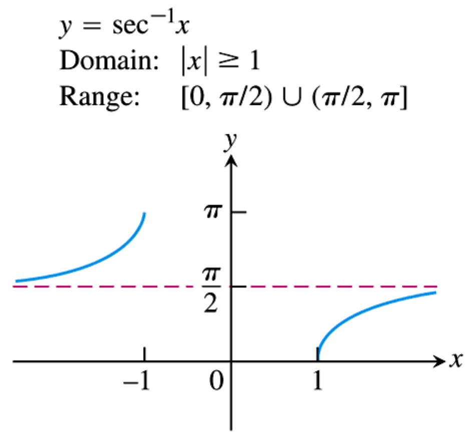


FIGURE 7.24 The graph of $y = \sec^{-1} x$.

Domain: $|x| \geq 1$
Range: $0 \leq y \leq \pi,\ y \neq \frac{\pi}{2}$

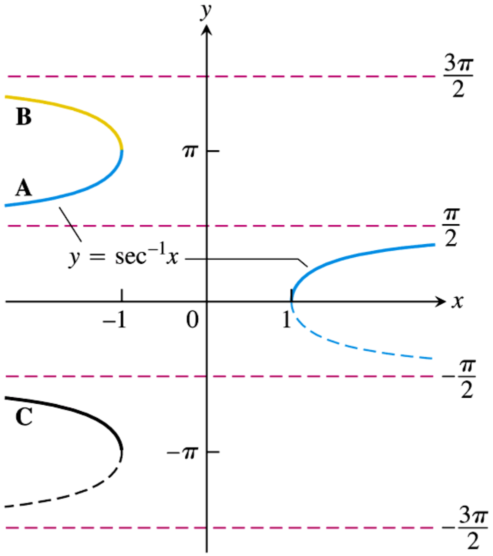


FIGURE 7.26 There are several logical choices for the left-hand branch of $y = \sec^{-1} x$. With choice **A**, $\sec^{-1} x = \cos^{-1}(1/x)$, a useful identity employed by many calculators.

$y = \csc^{-1}x$
Domain: $|x| \geq 1$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$

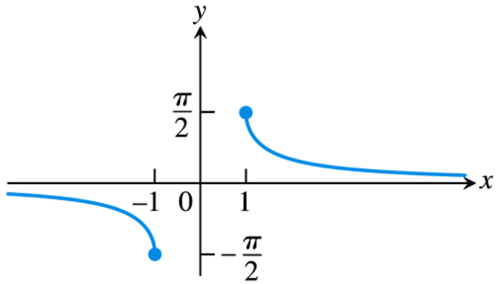


FIGURE 7.25 The graph of $y = \csc^{-1} x$.

Some specific values of $\tan^{-1} x$

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

Example 4

- Find cos α, tan α, sec α, csc α if $\alpha = \sin^{-1}(2/3)$.

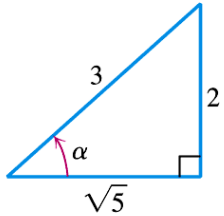


FIGURE 7.27 If $\alpha = \sin^{-1}(2/3)$, then the values of the other basic trigonometric functions of α can be read from this triangle (Example 4).

- sin $\alpha = 2/3$
- ...

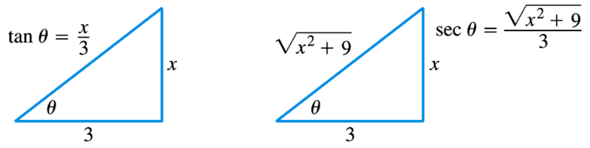
EXAMPLE 5 Find $\sec\left(\tan^{-1}\frac{x}{3}\right)$.

Solution We let $\theta = \tan^{-1}(x/3)$ (to give the angle a name) and picture θ in a right triangle with

$$\tan\theta = \text{opposite}/\text{adjacent} = x/3.$$

The length of the triangle's hypotenuse is

$$\sqrt{x^2 + 3^2} = \sqrt{x^2 + 9}.$$



Thus,

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \sec\theta$$

$$= \frac{\sqrt{x^2+9}}{3}. \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

The derivative of $y = \sin^{-1} x$

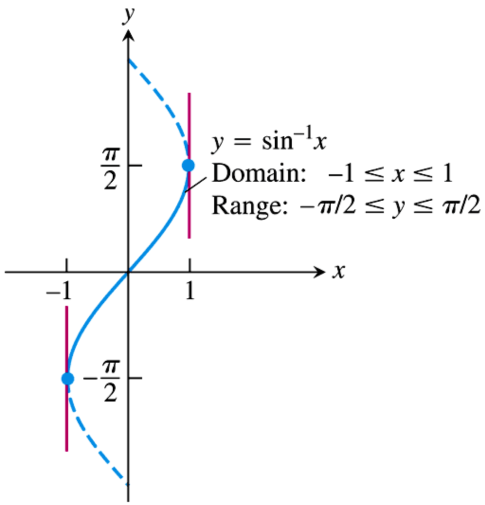
$f(x) = \sin^{-1} x \Rightarrow f^{-1}(x) = \sin x;$

$$\frac{df(x)}{dx} = \frac{1}{\left.\frac{df^{-1}(x)}{dx}\right|_{x=f(x)}} = \frac{1}{\cos x|_{x=f(x)}} = \frac{1}{\cos f(x)}$$

Let $y = f(x) = \sin^{-1} x \to x = \sin y \Rightarrow \cos y = \sqrt{1 - x^2}$

$$\frac{1}{\cos(f(x))} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$



$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

Note that the graph is not differentiable at the end points of $x=\pm 1$ because the tangents at these points are vertical.

FIGURE 7.29 The graph of $y = \sin^{-1} x$.

The derivative of $y = \sin^{-1} u$

If $u = u(x)$ is an diffrentiable function of x,

$$\frac{d}{dx}\sin^{-1} u = ?$$

Use chain rule: Let $y = \sin^{-1} u$

$$\frac{d}{dx}\sin^{-1} u = \frac{du}{dx}\frac{d}{du}\left(\sin^{-1} u\right) = \frac{du}{dx}\frac{1}{\sqrt{1-u^2}}$$

Note that $|u| < 1$ for the formula to apply

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1.$$

Example 7 Applying the derivative formula

$$\frac{d}{dx}\sin^{-1}x^2 = \ldots$$

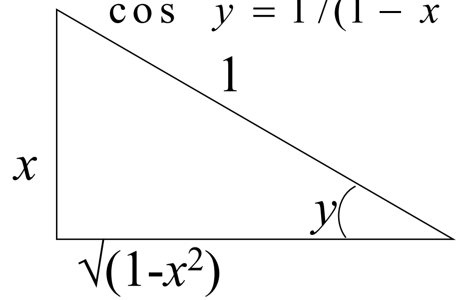
The derivative of $y = \tan^{-1} u$

$$y = \tan^{-1}x \Rightarrow x = \tan y$$

$$1 = \frac{d}{dx}(\tan y) = \frac{dy}{dx}\sec^2 y$$

$$\frac{dy}{dx} = \cos^2 y = 1/(1-x^2)$$

$$\cos^2 y = 1/(1-x^2)$$



By virtue of chain rule, we obtain

$$\frac{d}{dx}(\tan^{-1}u) = \frac{1}{1+u^2}\frac{du}{dx}.$$

Example 8

$$x(t) = \tan^{-1}\sqrt{t}.$$

$$\left.\frac{dx}{dt}\right|_{t=16} = ?$$

The derivative of $y = \sec^{-1} x$

$$y = \sec^{-1}x \Rightarrow x = \sec y$$

$$1 = \frac{d}{dx}(\sec y) = \frac{dy}{dx}\sec y \tan y$$

$$\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$$

$$\frac{d}{dx}\sec^{-1}x = \cos y \cot y = \pm\frac{1}{x}\frac{1}{\sqrt{(x^2-1)}}$$

$$\because \frac{dy}{dx} > 0 \text{ (from Figure 7.30)},$$

$$\frac{dy}{dx} = \frac{1}{|x|}\frac{1}{\sqrt{(x^2-1)}}$$

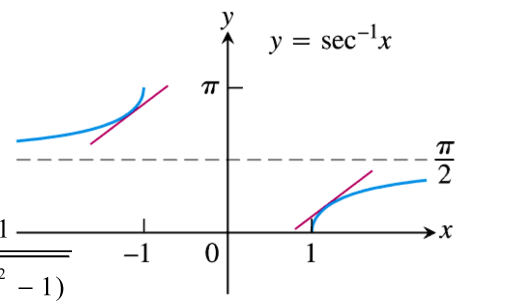


FIGURE 7.30 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

The derivative of $y = \sec^{-1} u$

By virtue of chain rule, we obtain

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad |u| > 1.$$

Example 5 Using the formula

$$\frac{d}{dx}\sec^{-1}(5x^4) = \ldots$$

Derivatives of the other three

- The derivative of $\cos^{-1}x$, $\cot^{-1}x$, $\csc^{-1}x$ can be easily obtained thanks to the following identities:

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

Link to fig. 7.21

TABLE 7.3 Derivatives of the inverse trigonometric functions

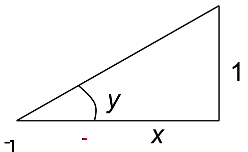
1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{du/dx}{\sqrt{1 - u^2}}, \quad |u| < 1$
2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{du/dx}{\sqrt{1 - u^2}}, \quad |u| < 1$
3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{du/dx}{1 + u^2}$
4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{du/dx}{1 + u^2}$
5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{du/dx}{|u|\sqrt{u^2 - 1}}, \quad |u| > 1$
6. $\dfrac{d(\csc^{-1} u)}{dx} = \dfrac{-du/dx}{|u|\sqrt{u^2 - 1}}, \quad |u| > 1$

Example 10 A tangent line to the arccotangent curve

- Find an equation for the tangent to the graph of $y = \cot^{-1} x$ at $x = -1$.

- Use either $\dfrac{df^{-1}(x)}{dx} = \dfrac{1}{\left.\dfrac{df(x)}{dx}\right|_{x=f^{-1}(x)}}$

- Or



Ans = $-\dfrac{1}{1+x^2}$

Integration formula

- By integrating both sides of the derivative formulas in Table 7.3, we obtain three useful integration formulas in Table 7.4.

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1.	$\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$	(Valid for $u^2 < a^2$)
2.	$\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$	(Valid for all u)
3.	$\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$	(Valid for $\lvert u\rvert > a > 0$)

Example 11 Using the integral formulas

$$(a)\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} =$$

$$(b)\int_0^1 \frac{dx}{1+x^2} =$$

$$(c)\int_{2/\sqrt{3}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} =$$

Example 13 Completing the square

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-4x)}} = \int \frac{dx}{\sqrt{-[(x-2)^2-4]}}$$

$$= \int \frac{dx}{\sqrt{4-(x-2)^2}} = \int \frac{du}{\sqrt{2^2-u^2}} = \dots$$

Example 15 Using substitution

$$\int \frac{dx}{\sqrt{e^{2x}-6}} = \int \frac{dx}{\sqrt{\left(e^{x}\right)^{2}-\left(\sqrt{6}\right)^{2}}} =$$

$$\int \frac{1}{\sqrt{\left(e^{x}\right)^{2}-\left(\sqrt{6}\right)^{2}}}\frac{de^{x}}{e^{x}} = \int \frac{1}{\sqrt{u^{2}-\left(\sqrt{6}\right)^{2}}}\frac{du}{u} = \ldots$$

7.8

Hyperbolic Functions

Even and odd parts of the exponential function

- In general:
- $f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$
- $\frac{1}{2}[f(x) + f(-x)]$ is the even part
- $\frac{1}{2}[f(x) - f(-x)]$ is the odd part

- Specifically:
- $f(x) = e^x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})$
- The odd part $\frac{1}{2}(e^x - e^{-x}) \equiv \cosh x$ (hyperbolic cosine of x)
- The even part $\frac{1}{2}(e^x + e^{-x}) \equiv \sinh x$ (hyperbolic sine of x)

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Proof of

$$\sinh 2x = 2\cosh x \sinh x$$

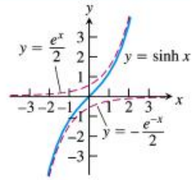
$$\sinh 2x = \frac{1}{2}(e^{2x} - e^{-2x}) = \frac{1}{2}\frac{(e^{4x}-1)}{e^{2x}}$$

$$= \frac{1}{2}\frac{(e^{2x}-1)}{e^{x}}\frac{(e^{2x}+1)}{e^{x}} = \frac{2}{2}\cdot\frac{1}{2}(e^{x}-e^{-x})(e^{x}+e^{-x})$$

$$= 2\cdot\frac{1}{2}(e^{x}-e^{-x})\cdot\frac{1}{2}(e^{x}+e^{-x}) = 2\sinh x \cosh x$$

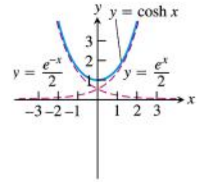
TABLE 7.5 The six basic hyperbolic functions **FIGURE 7.31**

Hyperbolic sine of x: $\sinh x = \dfrac{e^x - e^{-x}}{2}$



(a)

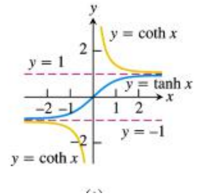
Hyperbolic cosine of x: $\cosh x = \dfrac{e^x + e^{-x}}{2}$



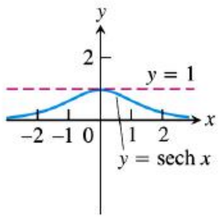
(b)

Hyperbolic tangent: $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent: $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

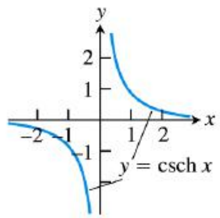


Hyperbolic secant: $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$



(d)

Hyperbolic cosecant: $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$



(e)

Derivatives and integrals

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.8 Integral formulas for hyperbolic functions

$$\int \sinh u\, du = \cosh u + C$$

$$\int \cosh u\, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

$$\frac{d}{dx}\sinh u = \frac{du}{dx}\frac{d}{dx}\sinh x$$

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{1}{2}(e^{x} - e^{-x}) = \frac{1}{2}(e^{x} + e^{-x}) = \cosh x$$

$$\therefore \frac{d}{dx}\sinh u = \frac{du}{dx}\cosh x$$

Example 1 Finding derivatives and integrals

$$(a)\ \frac{d}{dx}\tanh \overbrace{\sqrt{1+t^{2}}}^{u} = \frac{du}{dx}\frac{d}{du}\tanh u$$

$$(b)\int \coth \underbrace{5x}_{u}\, dx = \frac{1}{5}\int \coth u\, du = \frac{1}{5}\int \frac{\cosh u\ du}{\sinh u}$$

$$= \frac{1}{5}\int \frac{\overbrace{d(\sinh u)}^{dv}}{\underbrace{\sinh u}_{v}} = \frac{1}{5}\int \frac{dv}{v} = \frac{1}{5}\ln|v| + C = \frac{1}{5}\ln|\sinh 5x| + C$$

$$(c)\int \sinh^{2} x\ dx = \frac{1}{2}\int(\cosh 2x - 1)\ dx = \ldots$$

$$(d)\int 4e^{x}\sinh x\ dx = 4\int \frac{e^{x} - e^{-x}}{2}\ d\overbrace{e^{x}}^{u} = 2\int u - u^{-1}\ du$$

$$= 2\left(\frac{u^{2}}{2} - \ln|u|\right) + C = (e^{x})^{2} - \ln e^{2x} + C = e^{2x} - 2x + C$$

Inverse hyperbolic functions

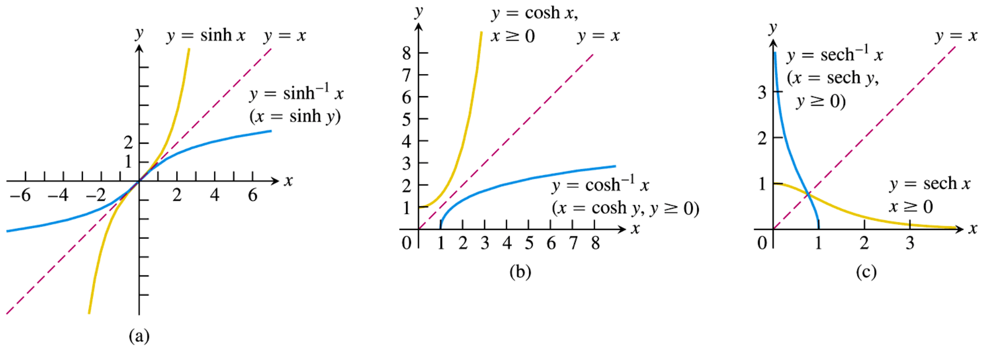


FIGURE 7.32 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

The inverse is useful in integration.

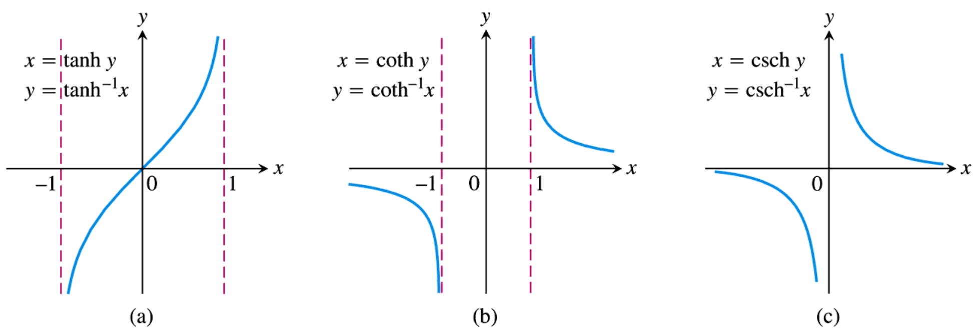


FIGURE 7.33 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

Useful Identities

TABLE 7.9 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1}\frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1}\frac{1}{x}$$

Proof

$$\operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x}.$$

Take sech of $\cosh^{-1}\frac{1}{x}$.

$$\operatorname{sech}\left(\cosh^{-1}\frac{1}{x}\right) = \frac{1}{\cosh\left(\cosh^{-1}\frac{1}{x}\right)} = \frac{1}{\frac{1}{x}} = x$$

$$\operatorname{sech}\left(\cosh^{-1}\frac{1}{x}\right) = x$$

Take sech^{-1} on both sides:

$$\operatorname{sech}^{-1}\left(\operatorname{sech}\left(\cosh^{-1}\frac{1}{x}\right)\right) = \operatorname{sech}^{-1} x \Rightarrow \left(\cosh^{-1}\frac{1}{x}\right) = \operatorname{sech}^{-1} x$$

TABLE 7.10 Derivatives of inverse hyperbolic functions

$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$	
$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx},$	$u > 1$
$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx},$	$\lvert u\rvert < 1$
$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx},$	$\lvert u\rvert > 1$
$\frac{d(\operatorname{sech}^{-1} u)}{dx} = \frac{-du/dx}{u\sqrt{1-u^2}},$	$0 < u < 1$
$\frac{d(\operatorname{csch}^{-1} u)}{dx} = \frac{-du/dx}{\lvert u\rvert\sqrt{1+u^2}},$	$u \neq 0$

Integrating these formulas will allows us to obtain a list of useful integration formula involving hyperbolic functions

e.g.

$$\frac{1}{\sqrt{1+x^2}} = \frac{d}{dx}\sinh^{-1} x$$

$$\rightarrow \int \frac{1}{\sqrt{1+x^2}}dx = \int \frac{d}{dx}\sinh^{-1} x \, dx$$

$$\int \frac{1}{\sqrt{1+x^2}}dx = \sinh^{-1} x + C$$

Proof

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}.$$

let $y = \sinh^{-1} x$

$$x = \sinh y \rightarrow \frac{d}{dx}x = \frac{d}{dx}\sinh y = \frac{dy}{dx}\cosh y$$

$$\rightarrow \frac{dy}{dx} = \operatorname{sech} y = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

$\Rightarrow$ By virtue of chain rule,

$$\frac{d}{dx}\sinh^{-1} u = \frac{du}{dx}\frac{1}{\sqrt{1+u^2}}$$

Example 2 Derivative of the inverse hyperbolic cosine

- Show that

$$\frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}}.$$

Let $y = \cosh^{-1} x\ldots$

Example 3 Using table 7.11

$$\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}}$$

Let $y = 2x$

$$\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}} = \int_0^2 \frac{dy}{\sqrt{3 + y^2}}$$

Scale it again to normalise the constant 3 to 1

$$\text{Let } z = \frac{y}{\sqrt{3}} \rightarrow \int_0^2 \frac{dy}{\sqrt{3 + y^2}} = \int_0^{2/\sqrt{3}} \frac{\sqrt{3}\,dz}{\sqrt{3 + 3z^2}} = \int_0^{2/\sqrt{3}} \frac{dz}{\sqrt{1 + z^2}}$$

$$= \sinh^{-1} z\Big|_0^{2/\sqrt{3}} = \sinh^{-1}(2/\sqrt{3}) - \sinh^{-1}(0) = \sinh^{-1}(2/\sqrt{3}) - 0$$

$$= \sinh^{-1}(2/\sqrt{3})$$

$$\sinh^{-1}(2/\sqrt{3}) = ?$$

Let $q = \sinh^{-1}(2/\sqrt{3})$

$$\sinh q = 2/\sqrt{3} \rightarrow \frac{1}{2}\left(e^q - e^{-q}\right) = \frac{2}{\sqrt{3}}$$

$$e^{2q} - \frac{4}{\sqrt{3}}e^q - 1 = 0$$

$$e^q = \frac{\frac{4}{\sqrt{3}} + \sqrt{\left(-\frac{4}{\sqrt{3}}\right)^2 - 4(-1)}}{2} = \frac{\frac{4}{\sqrt{3}} + \sqrt{\frac{296}{9}}}{2} = 2.682$$

$$\sinh^{-1}(2/\sqrt{3}) = q = \ln 2.682 = 0.9866$$

TABLE 7.11 Integrals leading to inverse hyperbolic functions

1.	$\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C,$	$a > 0$
2.	$\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C,$	$u > a > 0$
3.	$\int \frac{du}{a^2 - u^2} = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C$	if $u^2 < a^2$
	$\int \frac{du}{a^2 - u^2} = \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C,$	if $u^2 > a^2$
4.	$\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C,$	$0 < u < a$
5.	$\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left\|\frac{u}{a}\right\| + C,$	$u \neq 0$ and $a > 0$

Chapter 8

Techniques of Integration

8.1

Basic Integration Formulas

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C$ (any number k)
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \frac{a^u}{\ln a} + C \quad (a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$
21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \quad (a > 0)$
22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \quad (u > a > 0)$

Example 1 Making a simplifying substitution

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\,dx = \int \frac{d(\overbrace{x^2 - 9x}^{u})}{\sqrt{x^2 - 9x + 1}}$$

$$= \int \frac{du}{\sqrt{u + 1}} = \int \frac{d(u+1)}{\sqrt{u + 1}} = \int \frac{dv}{\sqrt{v}} = 2v^{1/2} + C$$

$$= 2(u+1)^{1/2} + C = 2\left(x^2 - 9x + 1\right)^{1/2} + C$$

Example 2 Completing the square

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x-4)^2}} =$$

$$\int \frac{d(x-4)}{\sqrt{16 - (x-4)^2}} = \int \frac{du}{\sqrt{4^2 - u^2}}$$

$$= \sin^{-1}\frac{u}{4} + C = \sin^{-1}\left(\frac{x-4}{4}\right) + C$$

Example 3 Expanding a power and using a trigonometric identity

$$\int (\sec x + \tan x)^2 dx$$

$$= \int (\sec^2 x + \tan^2 x + 2\sec x \tan x) dx.$$

$$\text{Racall:} \tan^2 x = \sec^2 x - 1; \frac{d}{dx}\tan x = \sec^2 x; \frac{d}{dx}\sec x = \tan x \sec x;$$

$$= \int (2\sec^2 x - 1 + 2\sec x \tan x) dx$$

$$= 2\tan x + -x + 2\sec x + C$$

Example 4 Eliminating a square root

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx =$$

$$\cos 4x = \cos 2(2x) = 2\cos^2(2x) - 1$$

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \sqrt{2} \int_0^{\pi/4} |\cos 2x|\, x$$

$$= \sqrt{2} \int_0^{\pi/4} \cos 2x\, dx = \ldots$$

Example 5 Reducing an improper fraction

$$\int \frac{3x^2 - 7x}{3x + 2} dx$$

$$= \int x - 3 + \frac{6}{3x + 2} dx$$

$$= \int x - 3 + \frac{2}{x + 2/3} dx$$

$$= \frac{1}{2}x^2 - 3x + 2\ln|x + \frac{2}{3}| + C$$

Example 6 Separating a fraction

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx$$

$$= 3\int \frac{x}{\sqrt{1-x^2}}\,dx + \int \frac{2}{\sqrt{1-x^2}}\,dx$$

$$= 3\int \frac{\frac{1}{2}d(x^2)}{\sqrt{1-x^2}} + 2\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{3}{2}\int \frac{du}{\sqrt{1-u}} + 2\sin^{-1}x + C \qquad \int \frac{du}{(1-u)^{1/2}} = -2(1-u)^{1/2} + C'$$

$$= \frac{3}{2}[-2(1-u)^{1/2}] + 2\sin^{-1}x + C''$$

$$= -3\sqrt{(1-x^2)} + 2\sin^{-1}x + C''$$

Example 7 Integral of $y = \sec x$

$$\int \sec x\,dx = ?$$

$$d\sec x = \sec x\tan x\,dx$$

$$d\tan x = \sec^2 x\,dx = \sec x\sec x\,dx$$

$$d(\sec x + \tan x) = \sec x(\sec x + \tan x)\,dx$$

$$\sec x\,dx = \frac{d(\sec x + \tan x)}{\sec x + \tan x}$$

$$\int \sec x\,dx = \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} = \ln|\sec x + \tan x| + C$$

TABLE 8.2 The secant and cosecant integrals

1. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$
2. $\displaystyle\int \csc u\,du = -\ln|\csc u + \cot u| + C$

Procedures for Matching Integrals to Basic Formulas

Procedure	Example
Making a simplifying substitution	$\frac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \frac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$
Using a trigonometric identity	$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x\tan x + \tan^2 x$ $= \sec^2 x + 2\sec x\tan x + (\sec^2 x - 1)$ $= 2\sec^2 x + 2\sec x\tan x - 1$
Eliminating a square root	$\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$
Reducing an improper fraction	$\frac{3x^2-7x}{3x+2} = x - 3 + \frac{6}{3x+2}$
Separating a fraction	$\frac{3x+2}{\sqrt{1-x^2}} = \frac{3x}{\sqrt{1-x^2}} + \frac{2}{\sqrt{1-x^2}}$
Multiplying by a form of 1	$\sec x = \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}$ $= \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}$

8.2

Integration by Parts

Product rule in integral form

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$$

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int g(x)\frac{d}{dx}[f(x)]dx + \int f(x)\frac{d}{dx}[g(x)]dx$$

$$f(x)g(x) = \int g(x)f'(x)dx + \int f(x)g'(x)dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Integration by parts formula

Alternative form of Eq. (1)

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

$$g'(x) \equiv \frac{dg(x)}{dx} = h(x) \rightarrow \int g'(x)dx = \int h(x)dx$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

$$\rightarrow \int f(x)h(x)dx = f(x)\left[\int h(x)dx\right] - \int\left\{f'(x)\left[\int h(x)dx\right]\right\}dx$$

Alternative form of the integration by parts formula

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$$

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int g(x)\frac{d}{dx}[f(x)]dx + \int f(x)\frac{d}{dx}[g(x)]dx$$

$$f(x)g(x) = \int g(x)df(x) + \int f(x)dg(x)$$

Let $u = f(x); v = g(x)$.The above formular is recast into the form

$$uv = \int v du + \int u dv$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

Example 4 Repeated use of integration by parts

$$\int x^2 e^x dx = ?$$

Example 5 Solving for the unknown integral

$$\int e^x \cos x dx = ?$$

Evaluating by parts for definite integrals

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \qquad (3)$$

or, equivalently

$$\int_a^b f(x)h(x)dx = f'(x)h(x)\Big]_a^b - \int_a^b \left\{ f'(x)\left[\int h(x)dx\right]\right\} dx$$

Example 6 Finding area

- Find the area of the region in Figure 8.1

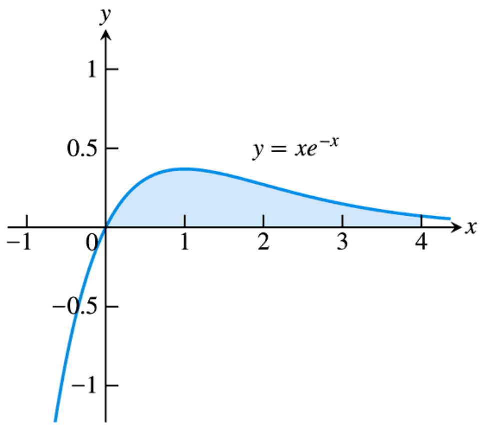


FIGURE 8.1 The region in Example 6.

Solution

$$\int_0^4 xe^{-x}\,dx = \ldots$$

Example 9 Using a reduction formula

- Evaluate $$\int \cos^3 x\,dx$$

- Use

$$\int \cos^n x\,dx = \int \overbrace{\cos^{n-1} x}^{u} \cdot \overbrace{\cos x\,dx}^{dv}$$

$$= \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

8.3

Integration of Rational Functions by Partial Fractions

General description of the method

- A rational function $f(x)/g(x)$ can be written as a sum of partial fractions. To do so:
- (a) The degree of $f(x)$ must be less than the degree of $g(x)$. That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term.
- We must know the factors of $g(x)$. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors.

Reducibility of a polynomial

- A polynomial is said to be **reducible** if it is the product of two polynomials of lower degree.
- A polynomial is **irreducible** if it is not the product of two polynomials of lower degree.

- THEOREM (Ayers, Schaum's series, pg. 305)
- Consider a polynomial $g(x)$ of order $n \geq 2$ (with leading coefficient 1). Two possibilities:

1. $g(x) = (x-r)\, h_1(x)$, where $h_1(x)$ is a polynomial of degree n-1, or
2. $g(x) = (x^2+px+q)\, h_2(x)$, where $h_2(x)$ is a polynomial of degree n-2, and (x^2+px+q) is the irreducible quadratic factor.

Example

$$g(x) = x^3 - 4x = \underbrace{(x-2)}_{\text{linear factor}} \cdot \underbrace{x(x+2)}_{\text{poly. of degree 2}}$$

$$g(x) = x^3 + 4x = \underbrace{(x^2+4)}_{\text{irreducible quadratic factor}} \cdot \underbrace{x}_{\text{poly. of degree 1}}$$

$$g(x) = x^4 - 9 = \underbrace{(x^2+3)}_{\text{irreducible quadratic factor}} \cdot \underbrace{(x+\sqrt{3})(x-\sqrt{3})}_{\text{poly. or degree 2}}$$

$$g(x) = x^3 - 3x^2 - x + 3 = \underbrace{(x+1)}_{\text{linear factor}} \underbrace{(x-2)^2}_{\text{poly. or degree 2}}$$

Quadratic polynomial

- A quadratic polynomial (polynomial or order n = 2) is either reducible or not reducible.
- Consider: $g(x)= x^2+px+q$.
- If $(p^2-4q) \geq 0$, $g(x)$ is reducible, i.e. $g(x) = (x+r_1)(x+r_2)$.
- If $(p^2-4q) < 0$, $g(x)$ is irreducible.

- In general, a polynomial of degree n can always be expressed as the product of linear factors and irreducible quadratic factors:

$$P_n(x) = (x-r_1)^{n_1}(x-r_2)^{n_2}\dots(x-r_l)^{n_l} \times$$
$$(x^2+p_1x+q_1)^{m_1}(x^2+p_2x+q_2)^{m_2}\dots(x^2+p_kx+q_k)^{m_k}$$

$$n = (n_1+n_2+\dots+n_l) + 2(m_1+m_2+\dots+m_l)$$

Integration of rational functions by partial fractions

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example 1 Distinct linear factors

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\,dx = \ldots$$

$$\frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)} = \ldots$$

Example 2 A repeated linear factor

$$\int \frac{6x + 7}{(x+2)^2}\,dx = \ldots$$

$$\frac{6x + 7}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

Example 3 Integrating an improper fraction

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx = \ldots$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{5x - 3}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)} = \ldots$$

Example 4 Integrating with an irreducible quadratic factor in the denominator

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \ldots$$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2} = \ldots$$

Example 5 A repeated irreducible quadratic factor

$$\int \frac{1}{x(x^2+1)^2}\,dx = ?$$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2} = \ldots$$

Other ways to determine the coefficients

- Example 8 Using differentiation
- Find *A*, *B* and *C* in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$\frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3} = \frac{x-1}{(x+1)^3}$$

$$\Rightarrow A(x+1)^2 + B(x+1) + C = x-1$$

$$x = -1 \to C = -2$$

$$\Rightarrow A(x+1)^2 + B(x+1) = x+1$$

$$\Rightarrow A(x+1) + B = 1$$

$$\frac{d}{dx}[A(x+1)+B] = \frac{d}{dx}(1) = 0$$

$$A = 0$$

$$B = 1$$

Example 9 Assigning numerical values to x

- Find *A*, *B* and *C* in

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}$$

$$= \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \equiv f(x)$$

$$= x^2 + 1$$

$$f(1) = 2A + = 1^2 + 1 = 2 \Rightarrow A = 1$$

$$f(2) = -B = 2^2 + 1 = 5; \Rightarrow B = -5$$

$$f(3) = 2C = 3^2 + 1 = 10; \Rightarrow C = 5$$

8.4

Trigonometric Integrals

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the work into three cases.

Case 1 If m is odd, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \quad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \quad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Example 1 *m* is odd

$$\int \sin^3 x \cos^2 x \quad dx = ?$$

$$\int \sin^3 x \cos^2 x \quad dx = -\int \sin^2 x \cos^2 x \quad d(\cos x)$$

$$= \int (\cos^2 x - 1) \cos^2 x \quad d(\cos x)$$

$$= \int (u^2 - 1) u^2 du = \ldots$$

Example 2 *m* is even and *n* is odd

$$\int \cos^5 x \quad dx = ?$$

$$\int \cos^5 x \ dx = \int \cos^4 x \cos x \ dx = \int \left(\cos^2 x\right)^2 \ d(\sin x) =$$

$$= \int (1\text{-}\sin^2 x)^2 \ d \underbrace{\sin x}_{u}$$

$$= \int (1\text{-}u^2)^2 \ du = \int 1 + u^4 - 2u^2 \ du = \ldots$$

Example 3 m and n are both even

$$\int \cos^2 x \sin^4 x \quad dx = ?$$

$$\int \cos^2 x \sin^4 x \, dx =$$

$$\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)^2 \, dx$$

$$= \frac{1}{4}\int \left(1 + \cos^2 2x - \cos 2x - \cos^3 2x\right) dx = \ldots$$

Example 6 Integrals of powers of tan x and sec x

$$\int \sec^3 x dx = ?$$

Use integration by parts.

$$\int \sec^3 x dx = \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x dx}_{dv};$$

$$dv = \sec^2 x dx \to v = \int \sec^2 x dx = \tan x$$

$$u = \sec x \to du = \sec x \tan x dx$$

$$\int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x dx}_{dv}$$

$$= \sec x \tan x - \int \tan x \cdot \underbrace{\sec x \tan x dx}_{du}$$

$$= \sec x \tan x - \int \tan^2 x \sec x dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx$$

$$\int \sec^3 x dx = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \ldots$$

$$\int \sec x dx = \int \sec x \frac{(\tan x + \sec x)}{\tan x + \sec x} dx$$

$$= \int \frac{(\sec x \tan x + \sec^2 x)}{\tan x + \sec x} dx$$

$$= \int \frac{d(\sec x + \tan x)}{\tan x + \sec x}$$

$$= \ln|\sec x + \tan x| + C$$

Example 7 Products of sines and cosines

$$\int \cos 5x \sin 3x dx = ?$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x];$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x];$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

$$\int \cos 5x \sin 3x dx$$

$$= \frac{1}{2}\int [\sin(-2x) + \sin 8x] dx$$

$$= \ldots$$

8.5

Trigonometric Substitutions

Three basic substitutions

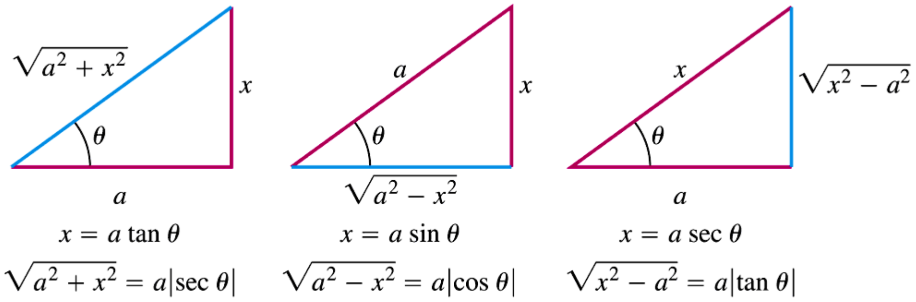


FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

Useful for integrals involving $\sqrt{a^2-x^2}, \sqrt{a^2+x^2}, \sqrt{x^2-a^2}$ in the denominator of the integrand.

Example 1 Using the substitution $x=a\tan\theta$

$$\int \frac{dx}{\sqrt{4+x^2}} = ?$$

$$x = 2\tan y \to dx = 2\sec^2 y dy = 2(\tan^2 y + 1)dy$$

$$\int \frac{dx}{\sqrt{4+4\tan^2 y}} = \int \frac{2(\tan^2 y+1)}{\sqrt{4+4\tan^2 y}} dy$$

$$= \int \frac{(\tan^2 y+1)}{\sqrt{1+\tan^2 y}} dy = \int \sqrt{\sec^2 y} dy = \int |\sec y| dy$$

$$= \ln|\sec y + \tan y| + C$$

Example 2 Using the substitution $x = a\sin\theta$

$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = ?$$

$$x = 3\sin y \to dx = 3\cos y \; dy$$

$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2 y \cdot 3\cos y \; dy}{\sqrt{9-9\sin^2 y}} =$$

$$= 9\int \frac{\sin^2 y \cdot \cos y \; dy}{\sqrt{1-\sin^2 y}}$$

$$= 9\int \sin^2 y dy = \ldots$$

Example 3 Using the substitution $x = a\sec\theta$

$$\int \frac{dx}{\sqrt{25x^2-4}} = ?$$

$$x = \frac{2}{5}\sec y \to dx = \frac{2}{5}\sec y \tan y \; dy$$

$$\int \frac{dx}{\sqrt{25x^2-4}} = \frac{2}{5}\int \frac{\sec y \tan y \; dy}{\sqrt{4\sec^2 y - 4}} = \frac{1}{5}\int \frac{\sec y \tan y \; dy}{\sqrt{\sec^2 y - 1}}$$

$$= \frac{1}{5}\int \frac{\sec y \tan y \; dy}{\sqrt{\sec^2 y - 1}} = \frac{1}{5}\int \sec y \; dy$$

$$= \frac{1}{5}\ln|\sec y + \tan y| + C = \ldots$$

Example 4 Finding the volume of a solid of revolution

$$V = 16\pi \int_0^2 \frac{dx}{(x^2+4)^2} = ?$$

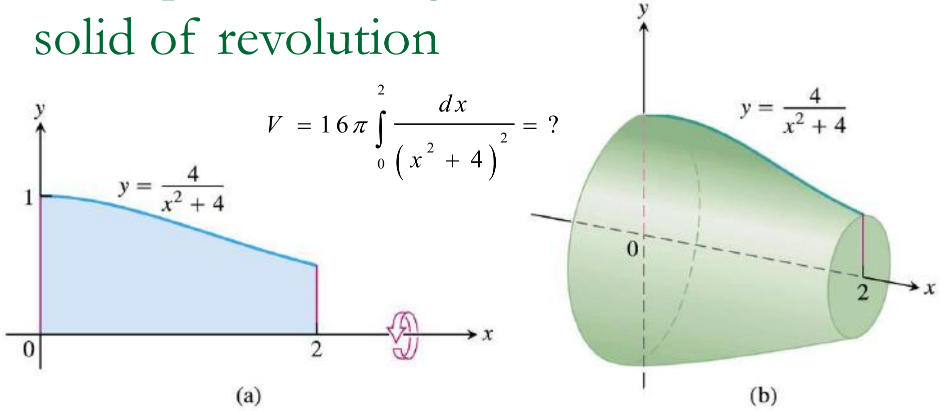


FIGURE 8.7 The region (a) and solid (b) in Example 4.

Solution

$$V = 16\pi \int_0^2 \frac{dx}{(x^2+4)^2} = ?$$

Let $x = 2\tan y \rightarrow dx = 2\sec^2 y dy$

$$V = \pi \int_0^{\pi/4} \frac{2\sec^2 y dy}{(\tan^2 y + 1)^2} = \pi \int_0^{\pi/4} \frac{2\sec^2 y dy}{(\sec^2 y)^2}$$

$$= 2\pi \int_0^{\pi/4} \cos^2 y dy = \ldots$$

8.6

Integral Tables

Integral tables is provided at the back of Thomas'

- T-4 A brief tables of integrals
- Integration can be evaluated using the tables of integral.

EXAMPLE 1 Find

$$\int x(2x+5)^{-1}\,dx.$$

Solution We use Formula 8 (not 7, which requires $n \neq -1$):

$$\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C.$$

With $a = 2$ and $b = 5$, we have

$$\int x(2x+5)^{-1}\,dx = \frac{x}{2} - \frac{5}{4}\ln|2x+5| + C.$$

EXAMPLE 2 Find

$$\int \frac{dx}{x\sqrt{2x+4}}.$$

Solution We use Formula 13(b):

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C, \qquad \text{if } b > 0.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x+4}} = \frac{1}{\sqrt{4}}\ln\left|\frac{\sqrt{2x+4}-\sqrt{4}}{\sqrt{2x+4}+\sqrt{4}}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{2x+4}-2}{\sqrt{2x+4}+2}\right| + C.$$

EXAMPLE 3 Find

$$\int \frac{dx}{x\sqrt{2x-4}}.$$

Solution We use Formula 13(a):

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}}\tan^{-1}\sqrt{\frac{ax-b}{b}} + C.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x-4}} = \frac{2}{\sqrt{4}}\tan^{-1}\sqrt{\frac{2x-4}{4}} + C = \tan^{-1}\sqrt{\frac{x-2}{2}} + C.$$

EXAMPLE 4 Find

$$\int \frac{dx}{x^2\sqrt{2x-4}}.$$

Solution We begin with Formula 15:

$$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}} + C.$$

With $a = 2$ and $b = -4$, we have

$$\int \frac{dx}{x^2\sqrt{2x-4}} = -\frac{\sqrt{2x-4}}{-4x} + \frac{2}{2\cdot 4}\int \frac{dx}{x\sqrt{2x-4}} + C.$$

We then use Formula 13(a) to evaluate the integral on the right (Example 3) to obtain

$$\int \frac{dx}{x^2\sqrt{2x-4}} = \frac{\sqrt{2x-4}}{4x} + \frac{1}{4}\tan^{-1}\sqrt{\frac{x-2}{2}} + C.$$

EXAMPLE 5 Find

$$\int x \sin^{-1} x \, dx.$$

Solution We use Formula 99:

$$\int x^n \sin^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \sin^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \qquad n \neq -1.$$

With $n = 1$ and $a = 1$, we have

$$\int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2\,dx}{\sqrt{1-x^2}}.$$

The integral on the right is found in the table as Formula 33:

$$\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2}x\sqrt{a^2-x^2} + C.$$

With $a = 1$,

$$\int \frac{x^2\,dx}{\sqrt{1-x^2}} = \frac{1}{2} \sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2} + C.$$

The combined result is

$$\int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2}\left(\frac{1}{2} \sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2} + C\right)$$

$$= \left(\frac{x^2}{2} - \frac{1}{4}\right)\sin^{-1} x + \frac{1}{4}x\sqrt{1-x^2} + C'.$$

8.8

Improper Integrals

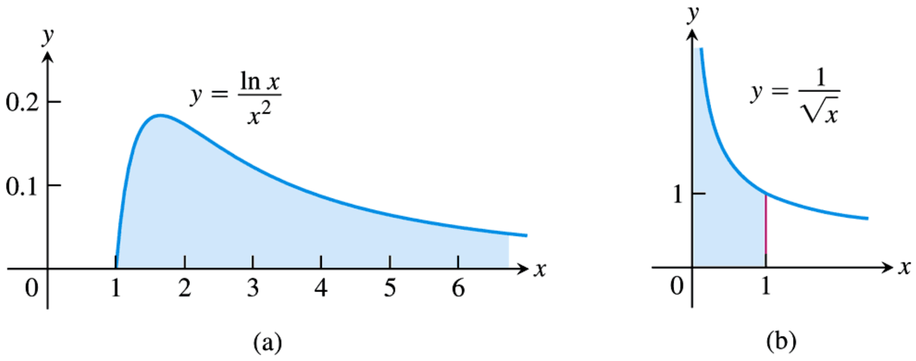


FIGURE 8.17 Are the areas under these infinite curves finite?

Infinite limits of integration

$$A(a) = \lim_{b\to\infty} A(b) = \lim_{b\to\infty} 2 - 2e^{-b/2} = 2$$

$$A(b) = \int_0^b e^{-x/2}\,dx = \ldots = 2 - 2e^{-b/2}$$

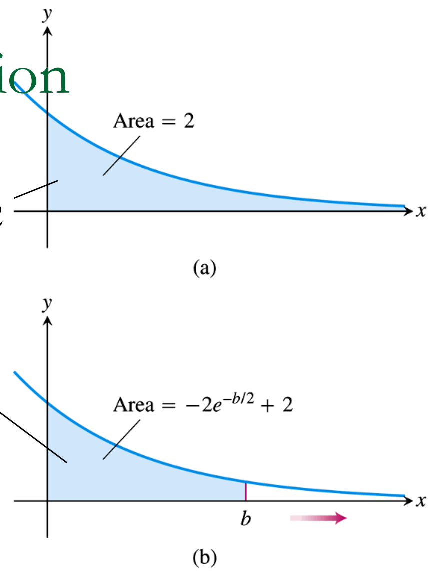


FIGURE 8.18 (a) The area in the first quadrant under the curve $y = e^{-x/2}$ is (b) an improper integral of the first type.

DEFINITION Type I Improper Integrals

Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

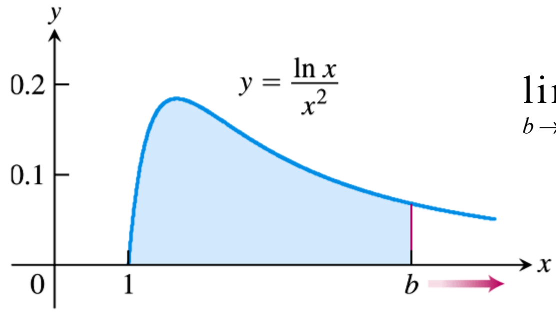
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Example 1 Evaluating an improper integral on [1,∞]

- Is the area under the curve $y=(\ln x)/x^2$ from 1 to ∞ finite? If so, what is it?



$$\lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = ?$$

FIGURE 8.19 The area under this curve is an improper integral (Example 1).

Solution

$$\int_1^b \frac{\ln x}{x}\frac{dx}{x} = \int_1^b \frac{\ln x}{x}\,d(\ln x) = \int_{\ln 1}^{\ln b} \frac{u}{e^u}\,du \qquad ;u = \ln x,\ x = e^u$$

$$\int_0^{\ln b} u\,\underbrace{e^{-u}\,du}_{dw} = u\,\underbrace{(-e^{-u})}_{w}\Big|_0^{\ln b} - \int_0^{\ln b} \underbrace{(-e^{-u})}_{w}\,du$$

$$= ue^{-u}\Big|_{\ln b}^{0} + \int_0^{\ln b} e^{-u}\,du = ue^{-u}\Big|_{\ln b}^{0} - e^{-u}\Big|_0^{\ln b}$$

$$= -\ln b\cdot e^{-\ln b} - (e^{-\ln b} - 1) = -\frac{1}{b}\ln b - \frac{1}{b} + 1$$

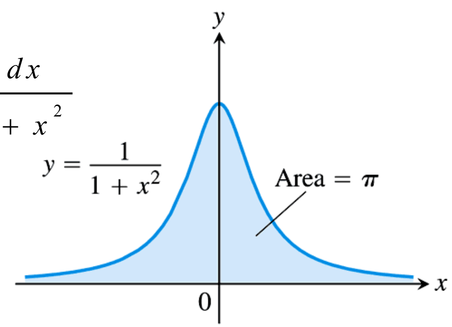
$$\lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{b}\ln b - \frac{1}{b} + 1\right] = 1$$

Example 2 Evaluating an integral on [-∞,∞]

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = ?$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_{-b}^{0} \frac{dx}{1+x^2} + \lim_{b\to\infty}\int_{0}^{b} \frac{dx}{1+x^2}$$

$$= 2\lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2}$$



NOT TO SCALE

FIGURE 8.20 The area under this curve is finite (Example 2).

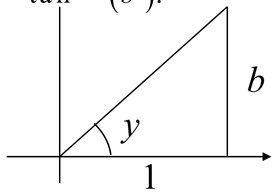
Solution

Using the integral table (Eq. 16)

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int_0^b \frac{dx}{1 + x^2} = \left[\tan^{-1} x\right]_0^b = \tan^{-1}(b) - \tan^{-1} 0 = \tan^{-1}(b).$$

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = 2\lim_{b\to\infty} \tan^{-1} b = 2 \cdot \frac{\pi}{2} = \pi$$



$$y = \tan^{-1} b \Rightarrow b = \tan y$$

$$\lim_{b\to\infty} \tan^{-1} b = \frac{\pi}{2}$$

DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then
$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$
2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$
3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

Example 3 Integrands with vertical asymptotes

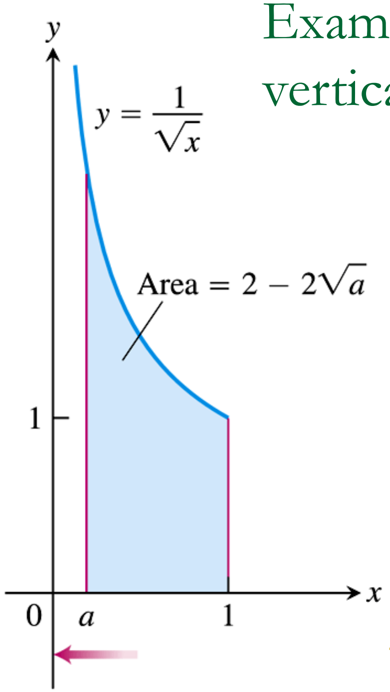
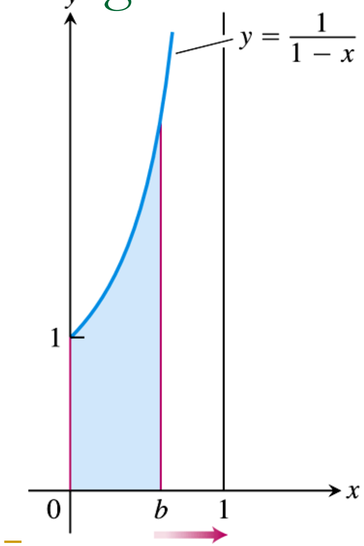


FIGURE 8.21 The area under this curve is

$$\lim_{a\to 0^+} \int_a^1 \left(\frac{1}{\sqrt{x}}\right) dx = 2,$$

an improper integral of the second kind.

Example 4 A divergent improper integral



- Investigate the convergence of
$$\int_0^1 \frac{dx}{1 - x}$$

FIGURE 8.22 The limit does not exist:

$$\int_0^1 \left(\frac{1}{1 - x}\right) dx = \lim_{b\to 1^-} \int_0^b \frac{1}{1 - x}\,dx = \infty$$

The area beneath the curve and above the x-axis for $[0, 1)$ is not a real number (Example 4).

Solution

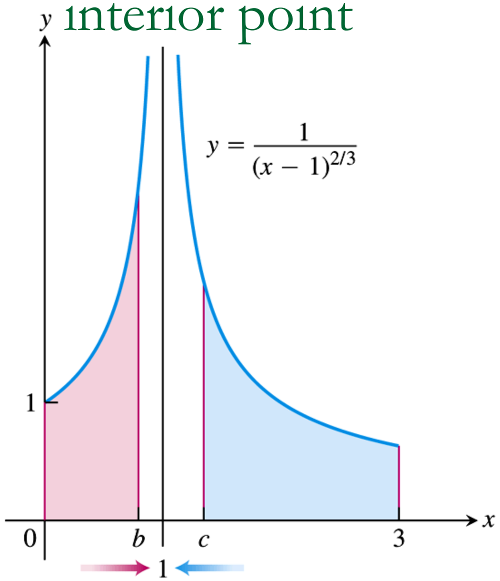
$$\int_0^1 \frac{dx}{1-x} = \lim_{b\to 1^-}\int_0^b \frac{dx}{1-x} = -\lim_{b\to 1^-}\left[\ln|x-1|\right]_0^b$$

$$= -\lim_{b\to 1^-}\left[\ln|b-1| - \ln|0-1|\right]$$

$$= -\lim_{b\to 1^-}\left[\ln|b-1| - \ln|0-1|\right] = \lim_{b\to 1^-}\left[\ln|b-1|^{-1}\right]$$

$$= \lim_{\varepsilon\to 0}\left[\ln\frac{1}{\varepsilon}\right] = \infty$$

Example 5 Vertical asymptote at an interior point



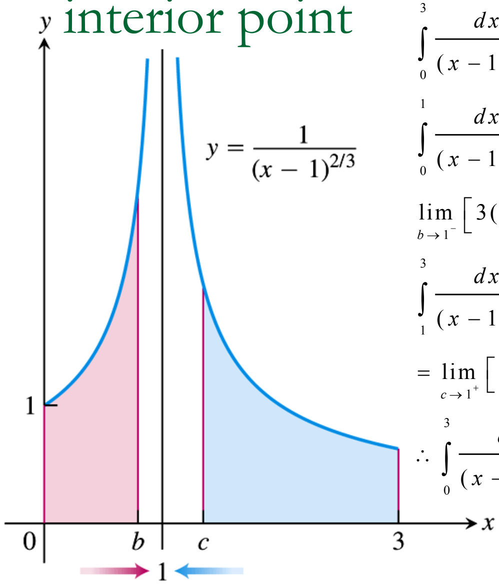
$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = ?$$

FIGURE 8.23 Example 5 shows the convergence of

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = 3 + 3\sqrt[3]{2},$$

so the area under the curve exists (so it is a real number).

Example 5 Vertical asymptote at an interior point



$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-}\left[3(x-1)^{1/3}\right]_0^b =$$

$$\lim_{b\to 1^-}\left[3(b-1)^{1/3} - 3(-1)^{1/3}\right] = \lim_{b\to 1^-}\left[0+3\right] = 3;$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1+}\int_c^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+}\left[3(x-1)^{1/3}\right]_c^3$$

$$= \lim_{c\to 1^+}\left[3(3-1)^{1/3} - 3(c-1)^{1/3}\right] = 3\cdot 2^{2/3}$$

$$\therefore \int_0^3 \frac{dx}{(x-1)^{2/3}} = 3(1+2^{2/3})$$

Example 7 Finding the volume of an infinite solid

- The cross section of the solid in Figure 8.24 perpendicular to the x-axis are circular disks with diameters reaching from the x-axis to the curve $y = e^x$, $-\infty < x < \ln 2$. Find the volume of the horn.

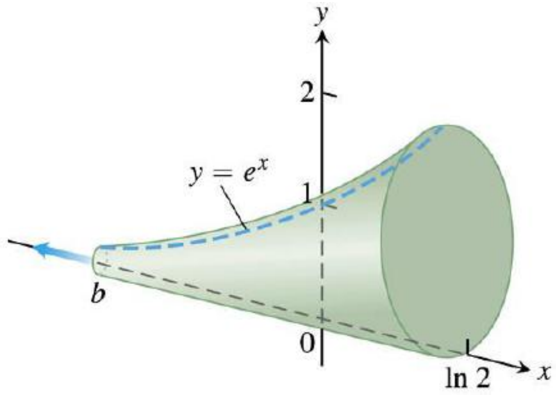


FIGURE 8.24 The calculation in Example 7 shows that this infinite horn has a finite volume.

Example 7 Finding the volume of an infinite solid

volume of a slice of disk of thickness dx, diameter y

$$dV = \pi (y/2)^2 dx$$

$$V = \int_0^V dV = \frac{1}{4}\lim_{b\to -\infty}\int_b^{\ln 2} \pi y(x)^2 dx$$

$$= \frac{1}{4}\lim_{b\to -\infty}\int_b^{\ln 2} \pi e^{2x} dx$$

$$= \frac{1}{8}\lim_{b\to -\infty}\left[\pi e^{2x}\right]_b^{\ln 2}$$

$$= \frac{1}{8}\lim_{b\to -\infty}\left[4\pi - \pi e^{2b}\right]$$

$$= \frac{1}{8}\pi \lim_{b\to -\infty}(4 - e^{2b}) = \frac{\pi}{2}$$

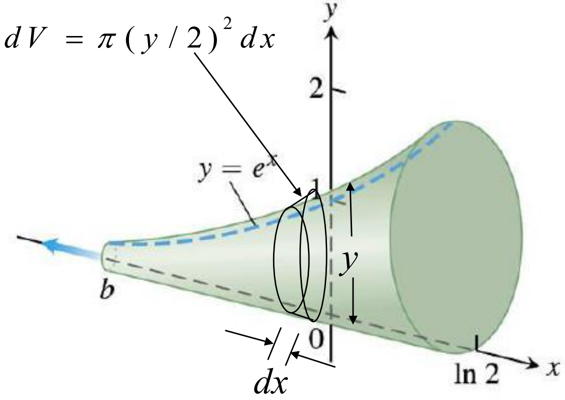


FIGURE 8.24 The calculation in Example 7 shows that this infinite horn has a finite volume.

Chapter 11

Infinite Sequences and Series

11.1

Sequences

What is a sequence

- A sequence is a list of numbers

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

in a given order.

- Each *a* is a ***term*** of the sequence.
- Example of a sequence:
- 2,4,6,8,10,12,…,2*n*,…
- *n* is called the ***index*** of a_n

DEFINITION Infinite Sequence

An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

- In the previous example, a general term a_n of index *n* in the sequence is described by the formula

$$a_n = 2n.$$

- We denote the sequence in the previous example by $\{a_n\} = \{2, 4, 6, 8, \ldots\}$
- In a sequence the order is important:
- 2,4,6,8,… and …,8,6,4,2 are not the same

Other example of sequences

$$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \cdots, \sqrt{n}, \cdots\}, a_n = \sqrt{n};$$

$$\{b_n\} = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots, (-1)^{n+1}\frac{1}{n}, \cdots\}; b_n = (-1)^{n+1}\frac{1}{n};$$

$$\{c_n\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots, \frac{n-1}{n}, \cdots\}; c_n = \frac{n-1}{n};$$

$$\{d_n\} = \{1, -1, 1, -1, 1, \cdots, (-1)^{n+1}, \cdots\}; d_n = (-1)^{n+1};$$

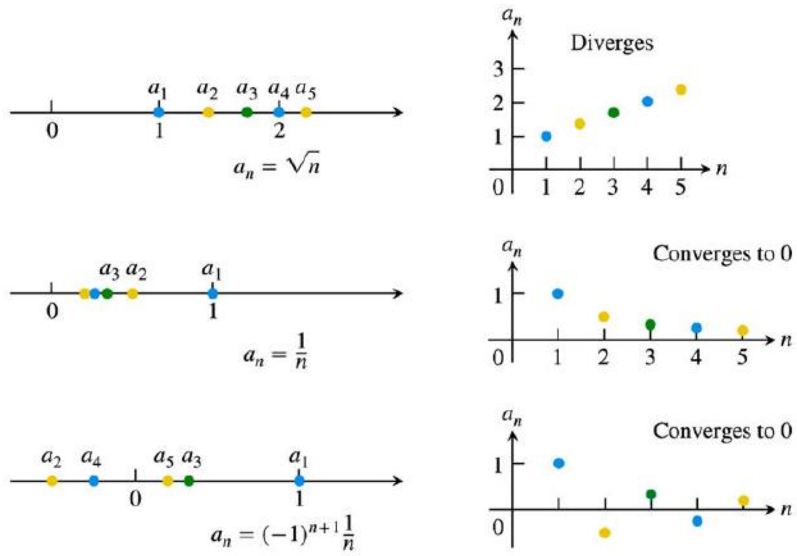


FIGURE 11.1 Sequences can be represented as points on the real line or as points in the plane where the horizontal axis n is the index number of the term and the vertical axis a_n is its value.

DEFINITIONS **Converges, Diverges, Limit**

The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Figure 11.2).

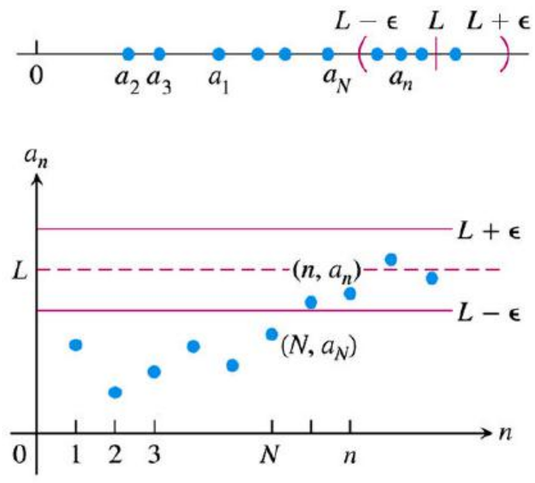


FIGURE 11.2 $a_n \to L$ if $y = L$ is a horizontal asymptote of the sequence of points $\{(n, a_n)\}$. In this figure, all the a_n's after a_N lie within ϵ of L.

DEFINITION Diverges to Infinity

The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad a_n \to \infty.$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \qquad \text{or} \qquad a_n \to -\infty.$$

THEOREM 1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

EXAMPLE 3 Applying Theorem 1

By combining Theorem 1 with the limits of Example 1, we have:

(a) $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = -1 \cdot \lim_{n\to\infty} \frac{1}{n} = -1 \cdot 0 = 0$ — Constant Multiple Rule and Example 1a

(b) $\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n} = 1 - 0 = 1$ — Difference Rule and Example 1a

(c) $\lim_{n\to\infty} \frac{5}{n^2} = 5 \cdot \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 5 \cdot 0 \cdot 0 = 0$ — Product Rule

(d) $\lim_{n\to\infty} \frac{4 - 7n^6}{n^6 + 3} = \lim_{n\to\infty} \frac{(4/n^6) - 7}{1 + (3/n^6)} = \frac{0 - 7}{1 + 0} = -7.$ — Sum and Quotient Rules

THEOREM 2 The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

EXAMPLE 4 Applying the Sandwich Theorem

Since $1/n \to 0$, we know that

(a) $\dfrac{\cos n}{n} \to 0$ because $-\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n}$;

(b) $\dfrac{1}{2^n} \to 0$ because $0 \le \dfrac{1}{2^n} \le \dfrac{1}{n}$;

(c) $(-1)^n \dfrac{1}{n} \to 0$ because $-\dfrac{1}{n} \le (-1)^n \dfrac{1}{n} \le \dfrac{1}{n}$.

THEOREM 3 The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

- Example 6: Applying theorem 3 to show that the sequence $\{2^{1/n}\}$ converges to 0.

- Taking $a_n = 1/n$, $\Rightarrow \lim_{n\to\infty} a_n = 0 \equiv L$
- Define $f(x)=2^x$. Note that $f(x)$ is continuous on $x=L$, and is defined for all $x= a_n = 1/n$
- According to Theorem 3,
- $\lim_{n\to\infty} f(a_n) = f(L)$
- LHS: $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(1/n) = \lim_{n\to\infty} 2^{1/n}$
- RHS = $f(L) = 2^L = 2^0 = 1$
- Equating LHS = RHS, we have $\lim_{n\to\infty} 2^{1/n} = 1$
- $\Rightarrow$ the sequence $\{2^{1/n}\}$ converges to 1

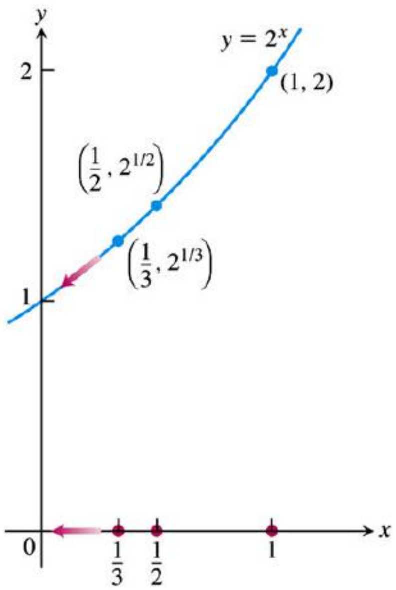


FIGURE 11.3 As $n \to \infty$, $1/n \to 0$ and $2^{1/n} \to 2^0$ (Example 6).

THEOREM 4

Suppose that $f(x)$ is a function defined for all $x \ge n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \ge n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L.$$

- Example 7: Applying l'Hopital rule
- Show that $\lim_{n \to \infty} \frac{\ln n}{n} = 0$
- Solution: The function $f(x) = \frac{\ln x}{x}$ is defined for $x \geq 1$ and agrees with the sequence $\{a_n = (\ln n)/n\}$ for $n \geq 1$.
- Applying l'Hopital rule on $f(x)$:

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \frac{1}{x} = 0$$

- By virtue of Theorem 4,

$$\lim_{x \to \infty} \frac{\ln x}{x} = 0 \Rightarrow \lim_{n \to \infty} a_n = 0$$

Example 9 Applying l'Hopital rule to determine convergence

Does the sequence whose nth term is $a_n = \left(\frac{n+1}{n-1}\right)^n$ converge?

If so, find $\lim_{n \to \infty} a_n$.

Solution: Use l'Hopital rule

Let $f(x) = \left(\frac{x+1}{x-1}\right)^x$ so that $f(n) = a_n$ for $n \geq 1$.

$$\to \ln f(x) = x \ln \left(\frac{x+1}{x-1}\right)$$

$$\lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} x \ln \left(\frac{x+1}{x-1}\right) = \lim_{x \to \infty} \frac{\ln \left(\frac{x+1}{x-1}\right)}{1/x}$$

$$= \lim_{x \to \infty} \frac{\left(\frac{-2}{x^2-1}\right)}{-1/x^2} = \lim_{x \to \infty} \frac{2x^2}{x^2-1} = 2$$

By virtue of Theorem 4, $\lim_{x \to \infty} \ln f(x) = 2 \Rightarrow$

$$\lim_{x \to \infty} f(x) = \exp(2) \Rightarrow \lim_{n \to \infty} a_n = \exp(2)$$

THEOREM 5

The following six sequences converge to the limits listed below:

1. $\lim_{n \to \infty} \frac{\ln n}{n} = 0$
2. $\lim_{n \to \infty} \sqrt[n]{n} = 1$
3. $\lim_{n \to \infty} x^{1/n} = 1 \qquad (x > 0)$
4. $\lim_{n \to \infty} x^n = 0 \qquad (|x| < 1)$
5. $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$
6. $\lim_{n \to \infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

All of the results in Theorem 5 can be proven using Theorem 4. See if you can show some of them yourself.

Example 10

- (a) $(\ln n^2)/n = 2 (\ln n) / n \rightarrow 2 \cdot 0 = 0$
- (b) $\sqrt[n]{n^2} = n^{2/n} = (n^{1/n})^2 \rightarrow (1)^2$
- (c) $\sqrt[n]{3n} = \sqrt[n]{3} \cdot \sqrt[n]{n} = 3^{1/n} \cdot n^{1/n} \rightarrow 1 \cdot 1 = 1$
- (d) $\left(-\frac{1}{2}\right)^n \rightarrow 0$
- (e) $\left(\frac{n-2}{n}\right)^n = \left(1 + \frac{(-2)}{n}\right)^n \rightarrow e^{-2}$
- (f) $\frac{100^n}{n!} \rightarrow 0$

DEFINITION Nondecreasing Sequence
A sequence $\{a_n\}$ with the property that $a_n \le a_{n+1}$ for all n is called a **nondecreasing sequence**.

- Example 12 Nondecreasing sequence
- (a) 1,2,3,4,…,n,…
- (b) ½, 2/3, ¾, 4/5 , …,$n/(n+1)$,…

  (nondecreasing because $a_{n+1}-a_n \ge 0$)
- (c) {3} = {3,3,3,…}

- Two kinds of nondecreasing sequences: bounded and non-bounded.

DEFINITIONS Bounded, Upper Bound, Least Upper Bound
A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \le M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

- Example 13 Applying the definition for boundedness
- (a) 1,2,3,…,n,…has no upper bound
- (b) ½, 2/3, ¾, 4/5 , …,$n/(n+1)$,…is bounded from above by $M = 1$.
- Since no number less than 1 is an upper bound for the sequence, so 1 is the least upper bound.

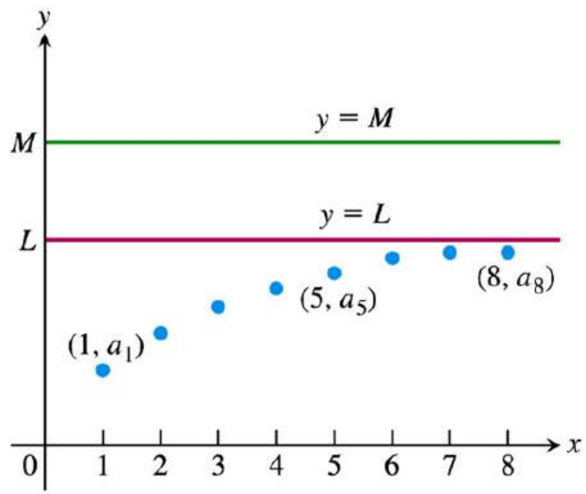


FIGURE 11.4 If the terms of a nondecreasing sequence have an upper bound M, they have a limit $L \le M$.

THEOREM 6 The Nondecreasing Sequence Theorem

A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

- If a non-decreasing sequence converges it is bounded from above.
- If a non-decreasing sequence is bounded from above it converges.
- In Example 13 (b) {½, 2/3, ¾, 4/5 , …,*n*/(*n*+1),…} is bounded by the least upper bound *M* = 1. Hence according to Theorem 6, the sequence converges, and the limit of convergence is the least upper bound 1.

11.2

Infinite Series

DEFINITIONS Infinite Series, *n*th Term, Partial Sum, Converges, Sum

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$\vdots$$
$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$
$$\vdots$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Example of a partial sum formed by a sequence $\{a_n = 1/2^{n-1}\}$

Partial sum		Suggestive expression for partial sum	Value
First:	$s_1 = 1$	$2 - 1$	1
Second:	$s_2 = 1 + \frac{1}{2}$	$2 - \frac{1}{2}$	$\frac{3}{2}$
Third:	$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$2 - \frac{1}{4}$	$\frac{7}{4}$
⋮	⋮	⋮	⋮
*n*th:	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$

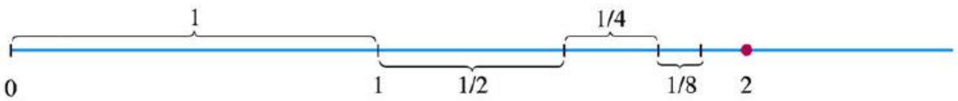


FIGURE 11.5 As the lengths 1, $^1/_2$, $^1/_4$, $^1/_8$, . . . are added one by one, the sum approaches 2.

Short hand notation for infinite series

$$\sum_{n}^{\infty} a_n, \sum_{k=1}^{\infty} a_k \text{ or } \sum a_n$$

- The infinite series is either converge or diverge

Geometric series

- Geometric series are the series of the form

  $a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$
- a and $r = a_{n+1}/a_n$ are fixed numbers and $a \neq 0$. r is called the ratio.
- Three cases can be classified: $r < 1$, $r > 1$, $r = 1$.

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Proof of $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ for $|r| < 1$

Assume $r \neq 1$.

$$s_n = \sum_{k=1}^{k=n} ar^{k-1} \qquad = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$rs_n = r\left(a + ar + ar^2 + \ldots + ar^{n-1}\right) = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$$

$$s_n - rs_n = a - ar^n = a\left(1 - r^n\right)$$

$$s_n = a\left(1 - r^n\right)/\left(1 - r\right)$$

$$\text{If } |r|<1: \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a\left(1-r^n\right)}{1-r} = \frac{a}{1-r} \text{ (By theorem 5.4, } \lim_{n\to\infty} r^n = 1 \text{ for } |r|<1\text{)}$$

For cases $|r| \geq 1$

$$\text{If } |r| > 1: \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \infty \text{ (Because } |r^n| \to \infty \text{ if } |r| > 1$$

$$\text{If } r = 1: s_n = a + ar^1 + ar^2 + \ldots + ar^{n-1} = na$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} na = a \lim_{n\to\infty} n = \infty$$

EXAMPLE 1 Index Starts with $n = 1$

The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1-(1/3)} = \frac{1}{6}.$$

Example 2 Index starts with n=0

- The series $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \frac{5}{4^0} - \frac{5}{4^1} + \frac{5}{4^2} - \frac{5}{4^3} + \ldots$

  is a geometric series with a=5, r=-(1/4).
- It converges to $s_\infty = a/(1-r) = 5/(1+1/4) = 4$

- Note: Be reminded that no matter how complicated the expression of a geometric series is, the series is simply completely specified by r and a. In other words, if you know r and a of a geometric series, you know almost everything about the series.

Example 4

$$5.232323\cdots = 5.\overline{23} = 5.\dot{2}\dot{3}$$

Express the above decimal as a ratio of two integers.

$$5.\dot{2}\dot{3} = 5 + [\cdots]$$

$$[\cdots] = 0.23 + 0.0023 + 0.000023 + \cdots$$

$$= 0.23(\cdots) = \frac{23}{100}(\cdots)$$

$$(\cdots) = 1 + 0.01 + 0.0001 + \cdots = \frac{a}{1-r} = \frac{1}{1-0.01} = \frac{1}{1-\frac{1}{100}} = \frac{1}{\frac{99}{100}} = \frac{100}{99}$$

$$5.\dot{2}\dot{3} = \frac{23}{100}\frac{100}{99} = \frac{23}{99}$$

Example 5 Telescopic series

- Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$
- **Solution**

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)}$$

$$s_k = \sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \frac{1}{n} - \frac{1}{(n+1)} =$$

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots \left(\frac{1}{k-1} - \frac{1}{k}\right) + \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= 1 - \frac{1}{k+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k \to \infty} s_k = 1$$

Divergent series

- Example 6

$$\sum n^2 = 1 + 2 + 4 + 16 + \ldots n^2 + \ldots$$

diverges because the partial sums s_n grows beyond every number L

$$\sum \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \ldots + \frac{n+1}{n} + \ldots$$

diverges because each term is greater than 1,

$$\Rightarrow \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \ldots + \frac{n+1}{n} + \ldots > \sum_{n=1}^{\infty} 1 \to \infty$$

Note

- In general, when we deal with a series, there are two questions we would like to answer:
- (1) the existence of the limit of the series $s_\infty = \sum_{k=1}^{\infty} a_k$
- (2) In the case where the limit of the series exists, what is the value of this limit?

- The tests that will be discussed in the following only provide the answer to question (1) but not necessarily question (2).

Theorem 7 (not very useful to test the convergence of a series)

THEOREM 7

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

- Let S be the convergent limit of the series, i.e. $\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n = S$
- When n is large, s_n and s_{n-1} are close to S
- This means $a_n = s_n - s_{n-1} \to a_n = S - S = 0$ as $n \to \infty$

The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Comment: useful to spot almost instantly if a series is divergent.

Example 7 Applying the *n*th-term test

$$(a)\sum_{n=1}^{\infty} n^2 \text{ diverges because } \lim_{n\to\infty} n^2 = \infty, \text{ i.e. } \lim_{n\to\infty} a_n \text{ fail to exist.}$$

$$(b)\sum_{n=1}^{\infty} \frac{n+1}{n} \text{ diverges because } \lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0.$$

$$(c)\sum_{n=1}^{\infty} (-1)^{n+1} \text{ diverges because } \lim_{n\to\infty} (-1)^{n+1} \text{ fail to exist.}$$

$$(d)\sum_{n=1}^{\infty} \frac{-n}{2n+5} \text{ diverges because } \lim_{n\to\infty} \frac{-n}{2n+5} = \frac{-1}{2} \neq 0 \text{ (l'Hopital rule)}$$

A question

- Will the series converge if $a_n \rightarrow 0$ as $n \rightarrow \infty$?

Example 8 $a_n \rightarrow 0$ but the series diverges

$$1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{2 \text{ terms}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{4 \text{ terms}} + \ldots + \underbrace{\frac{1}{2^n} + \frac{1}{2^n} + \frac{1}{2^n} + \ldots + \frac{1}{2^n}}_{2^n \text{ terms}} + \ldots$$

- The terms are grouped into clusters that add up to 1, so the partial sum increases without bound → the series diverges
- Yet $a_n = 2^{-n} \rightarrow 0$

THEOREM 8

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum k a_n = k \sum a_n = kA$ (Any number k).

- Corollary:
- Every nonzero constant multiple of a divergent series diverges
- If Σa_n converges and Σb_n diverges, then $\Sigma(a_n+b_n)$ and $\Sigma(a_n- b_n)$ both diverges.

- Question:
- If Σa_n and Σb_n both diverges, must $\Sigma(a_n \pm b_n)$ diverge?

EXAMPLE 9 Find the sums of the following series.

(a)
$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)$$
$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} \qquad \text{Difference Rule}$$
$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/6)} \qquad \text{Geometric series with } a = 1 \text{ and } r = 1/2, 1/6$$
$$= 2 - \frac{6}{5}$$
$$= \frac{4}{5}$$

(b)
$$\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n} \qquad \text{Constant Multiple Rule}$$
$$= 4\left(\frac{1}{1-(1/2)}\right) \qquad \text{Geometric series with } a = 1, r = 1/2$$
$$= 8$$

11.3

The Integral Test

Nondecreasing partial sums

- Suppose $\{a_n\}$ is a sequence with $a_n > 0$ for all n
- Then, the partial sum $s_{n+1} = s_n + a_n \geq s_n$
- $\Rightarrow$ The partial sum form a nondecreasing sequence

$$\{s_n = \sum_{k=1}^{n} a_k\} = \{s_1, s_2, s_2, \ldots, s_n, \ldots\}$$

- Theorem 6, the Nondecreasing Sequence Theorem tells us that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the partial sums are bounded from above.

Corollary of Theorem 6

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

Comment: To test whether a non-decreasing sequence converges, check whether its partial sum in bounded from above. If it is, the sequence converges.

This is particular useful for sequence with

$$a_n \to 0 \text{ as } n \to \infty$$

for which neither the n-term test nor theorem 7 can be used to conclude the divergence / convergence.

Example 1 The harmonic series

- The series $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{>\frac{2}{4}=\frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{>\frac{4}{8}=\frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{16}\right)}_{>\frac{8}{16}=\frac{1}{2}} + \ldots$ diverges.
- Consider the sequence of partial sum $\{s_1, s_2, s_4, s_{16}, \cdots, s_{2^k}, \cdots\}$

$$s_1 = 1$$
$$s_2 = s_1 + 1/2 > 1 \cdot (1/2)$$
$$s_4 = s_2 + (1/3 + 1/4) > 2 \cdot (1/2)$$
$$s_8 = s_4 + (1/5 + 1/6 + 1/7 + 1/8) > 3 \cdot (1/2)$$
$$\ldots$$
$$s_{2^k} > k \cdot (1/2)$$

- The partial sum of the first 2^k term in the series, $s_n > k/2$, where k=0,1,2,3…
- This means the partial sum, s_n, is not bounded from above.
- Hence, by the virtue of Corollary 6, the harmonic series diverges

THEOREM 9 The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

EXAMPLE 3 The p-Series

Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x. Since

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \frac{1}{1-p}\lim_{b\to\infty}\left(\frac{1}{b^{p-1}} - 1\right)$$

$$= \frac{1}{1-p}(0-1) = \frac{1}{p-1},$$

$b^{p-1} \to \infty$ as $b \to \infty$ because $p - 1 > 0$.

the series converges by the Integral Test.

If $p < 1$, then $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{b\to\infty}(b^{1-p} - 1) = \infty.$$

The series diverges by the Integral Test.

If $p = 1$, we have the (divergent) harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

The p-series with $p = 1$ is the **harmonic series**

We have convergence for $p > 1$ but divergence for every other value of p

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

Example 4 A convergent series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2+1} \text{ is convergent by the integral test:}$$

Let $f(x) = \frac{1}{x^2+1}$, so that $f(n) = a_n = \frac{1}{n^2+1}$. $f(x)$ is continuos, positive, decreasing for all $x \geq 1$.

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x^2+1}dx = \ldots = \lim_{b\to\infty} \tan^{-1} x \Big]_1^b = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by the integral test.

Caution

- The integral test only tells us whether a given series converges or otherwise
- The test DOES NOT tell us what the convergent limit of the series is (in the case where the series converges), as the series and the integral need not have the same value in the convergent case.

11.4

Comparison Tests

THEOREM 10 The Comparison Test

Let $\sum a_n$ be a series with no negative terms.

(a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N.

(b) $\sum a_n$ diverges if there is a divergent series of nonnegative terms $\sum d_n$ with $a_n \geq d_n$ for all $n > N$, for some integer N.

EXAMPLE 1 Applying the Comparison Test

(a) The series

$$\sum_{n=1}^{\infty} \frac{5}{5n-1}$$

diverges because its nth term

$$\frac{5}{5n-1} = \frac{1}{n - \frac{1}{5}} > \frac{1}{n}$$

is greater than the nth term of the divergent harmonic series.

EXAMPLE 1 Applying the Comparison Test

(b) The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

converges because its terms are all positive and less than or equal to the corresponding terms of

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots.$$

The geometric series on the left converges and we have

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1-(1/2)} = 3.$$

Caution

- The comparison test only tell us whether a given series converges or otherwise
- The test DOES NOT tell us what the convergent limit of the series is (in the case where the series converges), as the two series need not have the same value in the convergent case

THEOREM 11 Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

EXAMPLE 2 Using the Limit Comparison Test

Which of the following series converge, and which diverge?

(a) $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

(b) $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

Solution

(a) Let $a_n = (2n+1)/(n^2+2n+1)$. For large n, we expect a_n to behave like $2n/n^2 = 2/n$ since the leading terms dominate for large n, so we let $b_n = 1/n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2,$$

$\sum a_n$ diverges by Part 1 of the Limit Comparison Test. We could just as well have taken $b_n = 2/n$, but $1/n$ is simpler.

Example 2 continued

(b) Let $a_n = 1/(2^n - 1)$. For large n, we expect a_n to behave like $1/2^n$, so we let $b_n = 1/2^n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - (1/2^n)} = 1,$$

$\sum a_n$ converges by Part 1 of the Limit Comparison Test.

Caution

- The limit comparison test only tell us whether a given series converges or otherwise
- The test DOES NOT tell us what the convergent limit of the series is (in the case where the series converges)

11.5

The Ratio and Root Tests

THEOREM 12 The Ratio Test

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho .$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

EXAMPLE 1 Applying the Ratio Test

Investigate the convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

Solution

(a)

For the series $\sum_{n=0}^{\infty} (2^n + 5)/3^n$,

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5}$$

$$= \frac{1}{3} \cdot \left(\frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}}\right) \to \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

The series converges because $\rho = 2/3$ is less than 1.

Solution

(b)

If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2n + 2)!}{(n + 1)!(n + 1)!}$

$$\frac{a_{n+1}}{a_n} = \frac{n!n!(2n + 2)(2n + 1)(2n)!}{(n + 1)!(n + 1)!(2n)!}$$

$$= \frac{(2n + 2)(2n + 1)}{(n + 1)(n + 1)} = \frac{4n + 2}{n + 1} \to 4.$$

The series diverges because $\rho = 4$ is greater than 1.

Caution

- The ratio test only tell us whether a given series converges or otherwise
- The test DOES NOT tell us what the convergent limit of the series is (in the case where the series converges)

THEOREM 13 The Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

EXAMPLE 3 Applying the Root Test

Which of the following series converges, and which diverges?

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ **(b)** $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ **(c)** $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Solution

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{(\sqrt[n]{n})^2}{2} \to \frac{1}{2} < 1.$

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges because $\sqrt[n]{\frac{2^n}{n^2}} = \frac{2}{(\sqrt[n]{n})^2} \to \frac{2}{1} > 1.$

11.6

Alternating Series, Absolute and Conditional Convergence

Alternating series

- A series in which the terms are alternately positive and negative

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\cdots+\frac{(-1)^{n+1}}{n}+\cdots$$

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+\frac{(-1)^{n}4}{2^{n}}+\cdots$$

$$1-2+3-4+5-6+\cdots(-1)^{n+1}n+\cdots$$

THEOREM 14 The Alternating Series Test (Leibniz's Theorem)

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

The alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges because it satisfies the three requirements of Leibniz's theorem.

THEOREM 15 The Alternating Series Estimation Theorem

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 14, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the numerical value of the first unused term. Furthermore, the remainder, $L - s_n$, has the same sign as the first unused term.

EXAMPLE 2 We try Theorem 15 on a series whose sum we know:

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} = 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}\ \Big|+\frac{1}{256}-\cdots.$$

The theorem says that if we truncate the series after the eighth term, we throw away a total that is positive and less than $1/256$. The sum of the first eight terms is 0.6640625. The sum of the series is

$$\frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}.$$

The difference, $(2/3) - 0.6640625 = 0.0026041666\ldots$, is positive and less than $(1/256) = 0.00390625$.

DEFINITION Absolutely Convergent
A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the correspon-ding series of absolute values, $\sum|a_n|$, converges.

Example:

The geometric series

$$\sum_{n=1}^{\infty} 1\left(-\frac{1}{2}\right)^{n-1} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$$ converges absolutely since

the corresponding absolute series

$$\sum_{n=1}^{\infty} \left|1\left(-\frac{1}{2}\right)^{n-1}\right| = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$ converges

DEFINITION Conditionally Convergent
A series that converges but does not converge absolutely **converges conditionally**.

Example:

The alternative harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$ converges (by virture of Leibniz Theorem)

But the correspoinding absolute series

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$ diverges (a harmonic series)

Hence, by definition, the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

converges conditionally.

THEOREM 16 The Absolute Convergence Test
If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

In other words, if a series converges absolutely, it converges.

In the previous example, we shown that the geometric series $\sum_{n=1}^{\infty} 1\left(-\frac{1}{2}\right)^{n-1}$

converges absolutely. Hence, by virtue of the absolute convergent test, the series

$\sum_{n=1}^{\infty} 1\left(-\frac{1}{2}\right)^{n-1}$ converges.

Caution

- All series that are absolutely convergent converges.
- But the converse is not true, namely, not all convergent series are absolutely convergent.
- Think of series that is conditionally convergent. These are convergent series that are not absolutely convergent.

EXAMPLE 3 Applying the Absolute Convergence Test

(a)

For $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$, the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots.$$

The original series converges because it converges absolutely.

EXAMPLE 3 Applying the Absolute Convergence Test

(b)

For $\sum_{n=1}^{\infty}\frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$,

the corresponding series of absolute values is

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \cdots,$$

which converges by comparison with $\sum_{n=1}^{\infty}(1/n^2)$ because $|\sin n| \le 1$ for every n.

The original series converges absolutely; therefore it converges.

THEOREM 17 The Rearrangement Theorem for Absolutely Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

1. **The nth-Term Test:** Unless $a_n \to 0$, the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. **p-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge? If yes, so does $\sum a_n$, since absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

11.7

Power Series

DEFINITIONS **Power Series, Center, Coefficients**

A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \qquad (1)$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \qquad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

EXAMPLE 1 A Geometric Series

Taking all the coefficients to be 1 in Equation (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This is the geometric series with first term 1 and ratio x. It converges to $1/(1-x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1. \qquad (3)$$

Mathematica simulation

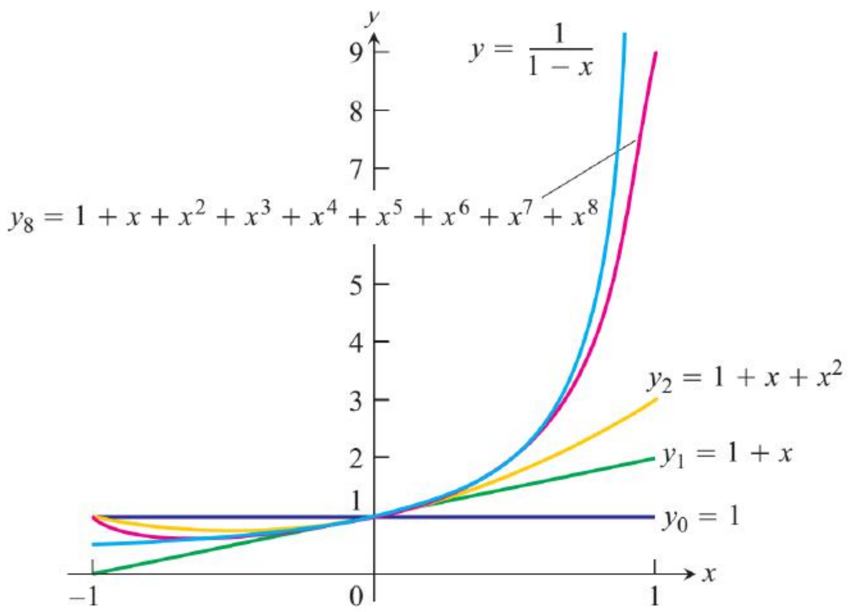


FIGURE 11.10 The graphs of $f(x) = 1/(1-x)$ and four of its polynomial approximations (Example 1).

EXAMPLE 3 Testing for Convergence Using the Ratio Test

For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(b) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

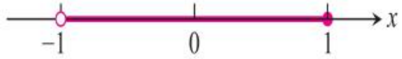
(c) $\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(d) $\sum_{n=0}^{\infty}n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

Solution Apply the Ratio Test to the series $\sum|u_n|$, where u_n is the nth term of the series in question.

(a) $\left|\frac{u_{n+1}}{u_n}\right| = \frac{n}{n+1}|x| \to |x|.$

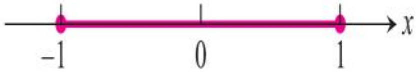
The series converges absolutely for $|x| < 1$. It diverges if $|x| > 1$ because the nth term does not converge to zero. At $x = 1$, we get the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges. At $x = -1$ we get $-1 - 1/2 - 1/3 - 1/4 - \cdots$, the negative of the harmonic series; it diverges. Series (a) converges for $-1 < x \le 1$ and diverges elsewhere.



(b)

$\left|\frac{u_{n+1}}{u_n}\right| = \frac{2n-1}{2n+1}x^2 \to x^2.$

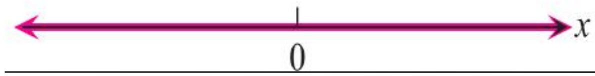
The series converges absolutely for $x^2 < 1$. It diverges for $x^2 > 1$ because the nth term does not converge to zero. At $x = 1$ the series becomes $1 - 1/3 + 1/5 - 1/7 + \cdots$, which converges by the Alternating Series Theorem. It also converges at $x = -1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x = -1$ is the negative of the value at $x = 1$. Series (b) converges for $-1 \le x \le 1$ and diverges elsewhere.



(c)

$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 \text{ for every } x.$

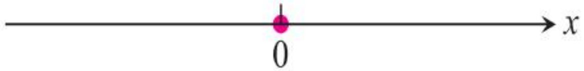
The series converges absolutely for all x.



(d)

$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \to \infty \text{ unless } x = 0.$

The series diverges for all values of x except $x = 0$.



The radius of convergence of a power series

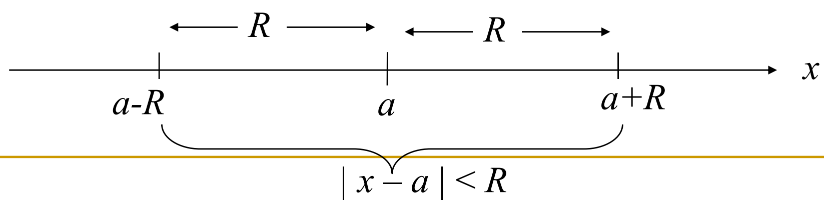
THEOREM 18 The Convergence Theorem for Power Series

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges for $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.

COROLLARY TO THEOREM 18

The convergence of the series $\sum c_n (x - a)^n$ is described by one of the following three possibilities:

1. There is a positive number R such that the series diverges for x with $|x - a| > R$ but converges absolutely for x with $|x - a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).



- *R* is called the radius of convergence of the power series
- The interval of radius *R* centered at *x* = *a* is called the interval of convergence
- The interval of convergence may be open, closed, or half-open: [*a-R, a+R*], (*a-R, a+R*), [*a-R, a+R*) or (*a-R, a+R*]
- A power series converges for all *x* that lies within the interval of convergence.

How to Test a Power Series for Convergence

1. *Use the Ratio Test (or nth-Root Test) to find the interval where the series converges absolutely*. Ordinarily, this is an open interval

$$|x - a| < R \quad \text{or} \quad a - R < x < a + R.$$

2. *If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint*, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. *If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x - a| > R$* (it does not even converge conditionally), because the *n*th term does not approach zero for those values of *x*.

See example 3 (previous slides, where we determined their interval of convergence)

THEOREM 19 The Term-by-Term Differentiation Theorem
If $\sum c_n(x-a)^n$ converges for $a-R<x<a+R$ for some $R>0$, it defines a function f:

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n, \qquad a-R<x<a+R.$$

Such a function f has derivatives of all orders inside the interval of convergence. We can obtain the derivatives by differentiating the original series term by term:

$$f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$f''(x)=\sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at every interior point of the interval of convergence of the original series.

EXAMPLE 4 Applying Term-by-Term Differentiation

Find series for $f'(x)$ and $f''(x)$ if

$$f(x)=\frac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots+x^n+\cdots$$

$$=\sum_{n=0}^{\infty} x^n, \qquad -1<x<1$$

Solution

$$f'(x)=\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+\cdots+nx^{n-1}+\cdots$$

$$=\sum_{n=1}^{\infty} nx^{n-1}, \qquad -1<x<1$$

$$f''(x)=\frac{2}{(1-x)^3}=2+6x+12x^2+\cdots+n(n-1)x^{n-2}+\cdots$$

$$=\sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1<x<1$$

Caution

- Power series is term-by-term differentiable
- However, in general, not all series is term-by-term differentiable, e.g. the trigonometric series $\sum_{n=1}^{\infty}\frac{\sin(n!x)}{n^2}$ is not (it's not a power series)

A power series can be integrated term by term throughout its interval of convergence

THEOREM 20 The Term-by-Term Integration Theorem
Suppose that

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$$

converges for $a-R<x<a+R$ $(R>0)$. Then

$$\sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

converges for $a-R<x<a+R$ and

$$\int f(x)\,dx=\sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}+C$$

for $a-R<x<a+R$.

EXAMPLE 5 A Series for $\tan^{-1} x$, $-1 \le x \le 1$

Identify the function

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad -1 \le x \le 1.$$

Solution We differentiate the original series term by term and get

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots, \qquad -1 < x < 1.$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

We can now integrate $f'(x) = 1/(1 + x^2)$ to get

$$\int f'(x)\,dx = \int \frac{dx}{1+x^2} = \tan^{-1} x + C.$$

The series for $f(x)$ is zero when $x = 0$, so $C = 0$. Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x, \qquad -1 < x < 1. \qquad (7)$$

In Section 11.10, we will see that the series also converges to $\tan^{-1} x$ at $x = \pm 1$.

EXAMPLE 6 A Series for $\ln(1 + x)$, $-1 < x \le 1$

The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x \qquad \text{Theorem 20}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x < 1.$$

It can also be shown that the series converges at $x = 1$ to the number ln 2, but that was not guaranteed by the theorem.

THEOREM 21 The Series Multiplication Theorem for Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

EXAMPLE 7 Multiply the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}, \qquad \text{for } |x| < 1,$$

by itself to get a power series for $1/(1 - x)^2$, for $|x| < 1$.

Solution Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + \cdots + x^n + \cdots = 1/(1-x)$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = 1 + x + x^2 + \cdots + x^n + \cdots = 1/(1-x)$$

and

$$c_n = \underbrace{a_0 b_n + a_1 b_{n-1} + \cdots + a_k b_{n-k} + \cdots + a_n b_0}_{n+1 \text{ terms}}$$

$$= \underbrace{1 + 1 + \cdots + 1}_{n+1 \text{ ones}} = n + 1.$$

Then, by the Series Multiplication Theorem,

$$A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1) x^n$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots$$

is the series for $1/(1 - x)^2$. The series all converge absolutely for $|x| < 1$.

Notice that Example 4 gives the same answer because

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

11.8

Taylor and Maclaurin Series

Series Representation

- In the previous topic we see that an infinite series represents a function. The converse is also true, namely:

  A function that is infinitely differentiable $f(x)$ can be expressed as a power series $\sum_{n=1}^{\infty} b_n (x-a)^n$

- We say: The function $f(x)$ generates the power series $\sum_{n=1}^{\infty} b_n (x-a)^n$
- The power series generated by the infinitely differentiable function is called Taylor series.
- The Taylor series provide useful polynomial approximations of the generating functions

Finding the Taylor series representation

- In short, given an infinitely differentiable function $f(x)$, we would like to find out what is the Taylor series representation of $f(x)$, i.e. what is the coefficients of b_n in $\sum_{n=1}^{\infty} b_n (x-a)^n$
- In addition, we would also need to work out the interval of x in which the Taylor series representation of $f(x)$ converges.
- In generating the Taylor series representation of a generating function, we need to specify the point $x=a$ at which the Taylor series is to be generated.

DEFINITIONS Taylor Series, Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

Note: Maclaurin series is effectively a special case of Taylor series with $a = 0$.

Example 1 Finding a Taylor series

- Find the Taylor series generated by $f(x)=1/x$ at $a= 2$. Where, if anywhere, does the series converge to $1/x$?
- $f(x) = x^{-1}$; $f'(x) = -x^{-2}$; $f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}$
- The Taylor series is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(2)}{k!}(x-2)^k = \sum_{k=0}^{\infty}\frac{(-1)^k k!\, x^{-(k+1)}}{k!}\Bigg|_{x=2}(x-2)^k =$$

$$(-1)^0 2^{-1}(x-2)^0 + (-1)^1 2^{-2}(x-2)^1 + (-1)^2 2^{-3}(x-2)^2 + \ldots (-1)^k 2^{-(k+1)}(x-2)^k + \ldots =$$

$$1/2 - (x-2)/4 + (x-2)^2/8 + \ldots (-1)^k (x-2)^k/2^{(k+1)} + \ldots$$

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(2)}{k!}(x-2)^k = 1/2 - (x-2)/4 + (x-2)^2/8 + \ldots (-1)^k (x-2)^k/2^{(k+1)} + \ldots$$

This is a geometric series with $r = -(x-2)/2$,

Hence, the Taylor series converges for $|r| = |(x-2)/2| < 1$,

or equivalently, $0 < x < 4$.

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(2)}{k!}(x-2)^k = \frac{a}{1-r} = \frac{1/2}{1-(-(x-2)/2)} = \frac{1}{x}$$

$\Rightarrow$ the Taylor series $1/2 - (x-2)/4 + (x-2)^2/8 + \ldots (-1)^k (x-2)^k/2^{(k+1)} + \ldots$

converges to $\frac{1}{x}$ for $0 < x < 4$.

**Mathematica simulation*

Taylor polynomials

- Given an infinitely differentiable function f, we can approximate $f(x)$ at values of x near a by the Taylor polynomial of f, i.e. $f(x)$ can be approximated by $f(x) \approx P_n(x)$, where

$$P_n(x) = \sum_{k=0}^{k=n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- $P_n(x)$ = Taylor polynomial of degree n of f generated at $x=a$.
- $P_n(x)$ is simply the first n terms in the Taylor series of f.
- The remainder, $|R_n(x)| = |f(x) - P_n(x)|$ becomes smaller if higher order approximation is used
- In other words, the higher the order n, the better is the approximation of $f(x)$ by $P_n(x)$
- In addition, the Taylor polynomial gives a close fit to f near the point $x = a$, but the error in the approximation can be large at points that are far away.

DEFINITION Taylor Polynomial of Order n

Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Example 2 Finding Taylor polynomial for e^x at $x = 0$

$$f(x) = e^x \rightarrow f^{(n)}(x) = e^x$$

$$P_n(x) = \sum_{k=0}^{k=n} \frac{f^{(k)}(x)}{k!}\Bigg|_{x=0} x^k = \frac{e^0}{0!}x^0 + \frac{e^0}{1!}x^1 + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \ldots \frac{e^0}{n!}x^n$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!}$$ This is the Taylor polynomial of order n for e^x

If the limit $n \rightarrow \infty$ is taken, $P_n(x) \rightarrow$ Taylor series.

The Taylor series for e^x is $1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$,

In this special case, the Taylor series for e^x converges to e^x for all x.

(To be proven later)

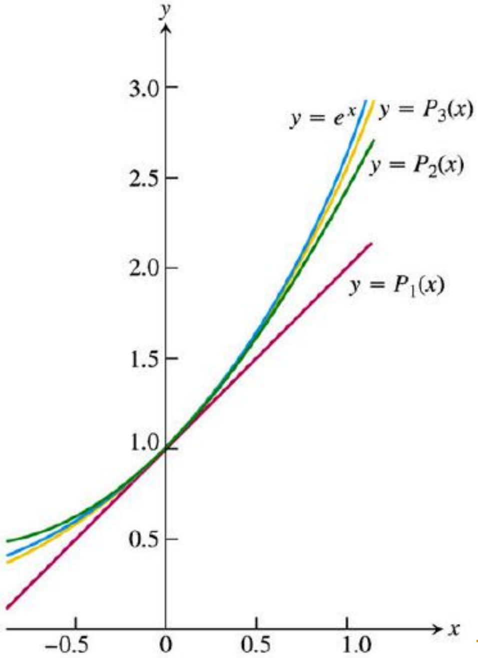


FIGURE 11.12 The graph of $f(x) = e^x$ and its Taylor polynomials
$P_1(x) = 1 + x$
$P_2(x) = 1 + x + (x^2/2!)$
$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!)$.
Notice the very close agreement near the center $x = 0$ (Example 2).

*Mathematica simulation

EXAMPLE 3 Finding Taylor Polynomials for cos x

Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

Solution The cosine and its derivatives are

$$f(x) = \cos x, \qquad f'(x) = -\sin x,$$
$$f''(x) = -\cos x, \qquad f^{(3)}(x) = \sin x,$$
$$\vdots \qquad\qquad \vdots$$
$$f^{(2n)}(x) = (-1)^n \cos x, \qquad f^{(2n+1)}(x) = (-1)^{n+1} \sin x.$$

At $x = 0$, the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0.$$

The Taylor series generated by f at 0 is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$
$$= 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

This is also the Maclaurin series for cos x. In Section 11.9, we will see that the series converges to cos x at every x.

Because $f^{(2n+1)}(0) = 0$, the Taylor polynomials of orders $2n$ and $2n + 1$ are identical:

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Figure 11.13 shows how well these polynomials approximate $f(x) = \cos x$ near $x = 0$. Only the right-hand portions of the graphs are given because the graphs are symmetric about the y-axis.

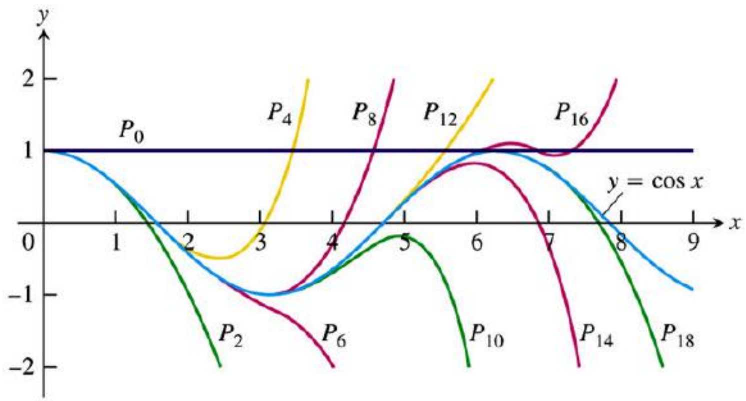


FIGURE 11.13 The polynomials

$$P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$$

converge to cos x as $n \rightarrow \infty$. We can deduce the behavior of cos x arbitrarily far away solely from knowing the values of the cosine and its derivatives at $x = 0$ (Example 3).

*Mathematica simulation

11.9

Convergence of Taylor Series;
Error Estimates

- When does a Taylor series converge to its generating function?
- ANS:
  The Taylor series converge to its generating function if the |remainder| =
  $|R_n(x)| = |f(x)-P_n(x)| \rightarrow 0$ as $n \rightarrow \infty$

Taylor's Formula

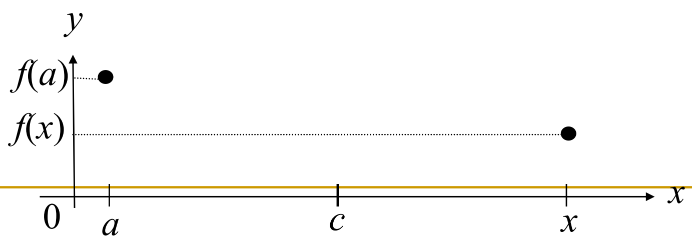
If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \quad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \quad (2)$$

$R_n(x)$ is called the remainder of order n



$f(x) = P_n(x) + R_n(x)$ for each x in I.

If $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$, $P_n(x)$ converges to $f(x)$, then we can write

$$f(x) = \lim_{n \to \infty} P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Example 1 The Taylor series for e^x revisited

- Show that the Taylor series generated by $f(x)=e^x$ at $x=0$ converges to $f(x)$ for every value of x.
- Note: This can be proven by showing that $|R_n| \to 0$ when $n \to \infty$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!} + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \text{ for some } c \text{ between 0 and } x$$

$$|R_n(x)| = \left| \frac{e^c}{(n+1)!} x^{n+1} \right|.$$

If $x > 0, 0 < c < x$

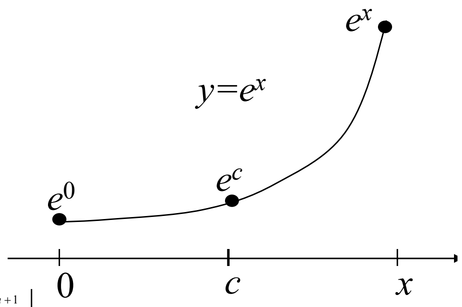
$$\Rightarrow 1 = e^0 < e^c < e^x \to \left| \frac{x^{n+1}}{(n+1)!} \right| < \left| \frac{e^c}{(n+1)!} x^{n+1} \right| < \left| \frac{e^x x^{n+1}}{(n+1)!} \right|$$

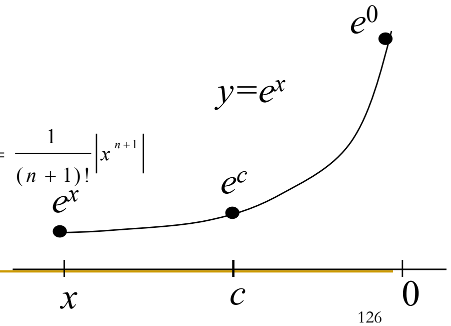
$$\to |R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \text{ for } x > 0.$$

If $x < 0, x < c < 0$

$$\Rightarrow e^x < e^c < e^0 \to \left| \frac{e^c}{(n+1)!} x^{n+1} \right| < \left| \frac{e^0 x^{n+1}}{(n+1)!} \right| = \left| \frac{x^{n+1}}{(n+1)!} \right| = \frac{1}{(n+1)!} |x^{n+1}|$$

$$\to |R_n(x)| < \frac{|x^{n+1}|}{(n+1)!} \text{ for } x < 0$$





Combining the result of both $x > 0$ and $x < 0$,

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \text{ when } x > 0,$$

$$|R_n(x)| < \frac{|x^{n+1}|}{(n+1)!} \text{ when } x < 0$$

Hence, irrespective of the sign of x, $\lim_{n\to\infty} |R_n(x)| = 0$ and the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ converge to } e^x \text{ for every } x.$$

THEOREM 5
The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

THEOREM 1
Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

THEOREM 23 The Remainder Estimation Theorem
If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x - a|^{n+1}}{(n+1)!}.$$

If this condition holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

EXAMPLE 3 The Taylor Series for $\cos x$ at $x = 0$ Revisited

Show that the Taylor series for $\cos x$ at $x = 0$ converges to $\cos x$ for every value of x.

Solution We add the remainder term to the Taylor polynomial for $\cos x$ (Section 11.8, Example 3) to obtain Taylor's formula for $\cos x$ with $n = 2k$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + R_{2k}(x).$$

Because the derivatives of the cosine have absolute value less than or equal to 1, the Remainder Estimation Theorem with $M = 1$ gives

$$|R_{2k}(x)| \le 1 \cdot \frac{|x|^{2k+1}}{(2k+1)!}.$$

For every value of x, $R_{2k} \to 0$ as $k \to \infty$. Therefore, the series converges to $\cos x$ for every value of x. Thus,

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \tag{5}$$

EXAMPLE 6 Calculate e with an error of less than 10^{-6}.

Solution We can use the result of Example 1 with $x = 1$ to write

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1),$$

with

$$R_n(1) = e^c \frac{1}{(n+1)!} \quad \text{for some } c \text{ between 0 and 1.}$$

For the purposes of this example, we assume that we know that $e < 3$. Hence, we are certain that

$$\frac{1}{(n+1)!} < R_n(1) < \frac{3}{(n+1)!}$$

because $1 < e^c < 3$ for $0 < c < 1$.

By experiment we find that $1/9! > 10^{-6}$, while $3/10! < 10^{-6}$. Thus we should take $(n+1)$ to be at least 10, or n to be at least 9. With an error of less than 10^{-6},

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{9!} \approx 2.718282.$$ ■

EXAMPLE 7 For what values of x can we replace $\sin x$ by $x - (x^3/3!)$ with an error of magnitude no greater than 3×10^{-4}?

Solution Here we can take advantage of the fact that the Taylor series for $\sin x$ is an alternating series for every nonzero value of x. According to the Alternating Series Estimation Theorem (Section 11.6), the error in truncating

$$\sin x = x - \frac{x^3}{3!} \,\Big|\, + \frac{x^5}{5!} - \cdots$$

after $(x^3/3!)$ is no greater than

$$\left|\frac{x^5}{5!}\right| = \frac{|x|^5}{120}.$$

Therefore the error will be less than or equal to 3×10^{-4} if

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \quad \text{or} \quad |x| < \sqrt[5]{360 \times 10^{-4}} \approx 0.514. \quad \text{Rounded down, to be safe}$$

The Alternating Series Estimation Theorem tells us something that the Remainder Estimation Theorem does not: namely, that the estimate $x - (x^3/3!)$ for $\sin x$ is an underestimate when x is positive because then $x^5/120$ is positive.

Figure 11.15 shows the graph of $\sin x$, along with the graphs of a number of its approximating Taylor polynomials. The graph of $P_3(x) = x - (x^3/3!)$ is almost indistinguishable from the sine curve when $-1 \le x \le 1$.

11.10

Applications of Power Series

The binomial series for powers and roots

- Consider the Taylor series generated by $f(x) = (1+x)^m$, where m is a constant:

$$f(x) = (1+x)^m$$

$$f'(x) = m(1+x)^{m-1},\ f''(x) = m(m-1)(1+x)^{m-2},$$

$$f'''(x) = m(m-1)(m-2)(1+x)^{m-3},$$

$$\vdots$$

$$f^{(k)}(x) = m(m-1)(m-2)\ldots(m-k+1)(1+x)^{m-k};$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\infty} \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!}x^k$$

$$= 1 + mx + m(m-1)x^2 + m(m-1)(m-2)x^3 + \ldots + \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!}x^k + \ldots$$

The binomial series for powers and roots

$$f(x) = (1+x)^m$$

$$= 1 + mx + m(m-1)x^2 + m(m-1)(m-2)x^3 + \ldots + \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!}x^k + \ldots$$

- This series is called the binomial series, converges absolutely for $|x| < 1$. (The convergence can be determined by using Ratio test, $\left|\frac{u_{k+1}}{u_k}\right| = \left|\frac{m-k}{k+1}x\right| \to |x|$

In short, the binomial series is the Taylor series for $f(x) = (1+x)^m$, where m a constant

The Binomial Series

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k \geq 3.$$

EXAMPLE 2 Using the Binomial Series

We know from Section 3.8, Example 1, that $\sqrt{1+x} \approx 1 + (x/2)$ for $|x|$ small. With $m = 1/2$, the binomial series gives quadratic and higher-order approximations as well, along with error estimates that come from the Alternating Series Estimation Theorem:

$$(1+x)^{1/2} = 1 + \frac{x}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots.$$

Substitution for x gives still other approximations. For example,

$$\sqrt{1-x^2} \approx 1 - \frac{x^2}{2} - \frac{x^4}{8} \qquad \text{for } |x^2| \text{ small}$$

$$\sqrt{1-\frac{1}{x}} \approx 1 - \frac{1}{2x} - \frac{1}{8x^2} \qquad \text{for } \left|\frac{1}{x}\right| \text{ small, that is, } |x| \text{ large.}$$

Taylor series representation of ln x at $x = 1$

- $f(x)=\ln x$; $f'(x) = x^{-1}$;
- $f''(x) = (-1)(1)x^{-2}$; $f'''(x) = (-1)^2(2)(1)x^{-3} \ldots$
- $f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$;

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(x)}{n!}\bigg|_{x=1}(x-1)^n = \frac{f^{(0)}(x)}{0!}\bigg|_{x=1}(x-1)^0 + \sum_{n=1}^{\infty}\frac{f^{(n)}(x)}{n!}\bigg|_{x=1}(x-1)^n$$

$$= \frac{\ln 1}{0!} + \sum_{n=1}^{\infty}\frac{(-1)^{(n-1)}(n-1)!x^{-n}}{n!}\bigg|_{x=1}(x-1)^n = 0 + \sum_{n=1}^{\infty}\frac{(-1)^{(n-1)}(1)^{-n}}{n}(x-1)^n$$

$$= \frac{(-1)^0}{1}(x-1)^1 + \frac{(-1)^1}{2}(x-1)^2 + \frac{(-1)^2}{3}(x-1)^3 + \ldots$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots + (-1)^n\frac{1}{n}(x-1)^n + \ldots$$

**Mathematica simulation*

EXAMPLE 7 Limits Using Power Series

Evaluate

$$\lim_{x\to 1}\frac{\ln x}{x-1}.$$

Solution We represent ln x as a Taylor series in powers of $x - 1$. This can be accomplished by calculating the Taylor series generated by ln x at $x = 1$ directly or by replacing x by $x - 1$ in the series for ln $(1 + x)$ in Section 11.7, Example 6. Either way, we obtain

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \cdots,$$

from which we find that

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\left(1 - \frac{1}{2}(x-1) + \cdots\right) = 1.$$

EXAMPLE 8 Limits Using Power Series

Evaluate

$$\lim_{x\to 0}\frac{\sin x - \tan x}{x^3}.$$

Solution The Taylor series for sin x and tan x, to terms in x^5, are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$$

Hence,

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \cdots = x^3\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

and

$$\lim_{x\to 0}\frac{\sin x - \tan x}{x^3} = \lim_{x\to 0}\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

$$= -\frac{1}{2}.$$

TABLE 11.1 Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty}x^n, \quad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty}(-1)^n x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}, \quad -1 < x \le 1$$

$$\ln\frac{1+x}{1-x} = 2\tanh^{-1}x = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right) = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}, \quad |x| < 1$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \le 1$$

Binomial Series

$$(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \cdots + \frac{m(m-1)(m-2)\cdots(m-k+1)x^k}{k!} + \cdots$$

$$= 1 + \sum_{k=1}^{\infty}\binom{m}{k}x^k, \quad |x| < 1,$$

where

$$\binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2!}, \quad \binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!} \quad \text{for } k \ge 3.$$

Note: To write the binomial series compactly, it is customary to define $\binom{m}{0}$ to be 1 and to take $x^0 = 1$ (even in the usually excluded case where $x = 0$), yielding $(1+x)^m = \sum_{k=0}^{\infty}\binom{m}{k}x^k$. If m is a *positive integer*, the series terminates at x^m and the result converges for all x.

11.11

Fourier Series

'Weakness' of power series approximation

- In the previous lesson, we have learnt to approximate a given function using power series approximation, which give good fit if the approximated power series representation is evaluated near the point it is generated
- For point far away from the point the power series being generated, the approximation becomes poor
- In addition, the series approximation works only within the interval of convergence. Outside the interval of convergence, the series representation fails to represent the generating function
- Furthermore, power series approximation can not represent satisfactorily a function that has a jump discontinuity.
- Fourier series, our next topic, provide an alternative to overcome such shortage

Suppose we wish to approximate a function f on the interval $[0, 2\pi]$ by a sum of sine and cosine functions,

$$f_n(x) = a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots + (a_n \cos nx + b_n \sin nx)$$

or, in sigma notation,

$$f_n(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx). \qquad (1)$$

We would like to choose values for the constants $a_0, a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots, b_n$ that make $f_n(x)$ a "best possible" approximation to $f(x)$. The notion of "best possible" is defined as follows:

1. $f_n(x)$ and $f(x)$ give the same value when integrated from 0 to 2π.
2. $f_n(x) \cos kx$ and $f(x) \cos kx$ give the same value when integrated from 0 to 2π $(k = 1, \ldots, n)$.
3. $f_n(x) \sin kx$ and $f(x) \sin kx$ give the same value when integrated from 0 to 2π $(k = 1, \ldots, n)$.

We chose f_n so that the integrals on the left remain the same when f_n is replaced by f, so we can use these equations to find $a_0, a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots, b_n$ from f:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx \qquad (2)$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx\, dx, \qquad k = 1, \ldots, n \qquad (3)$$

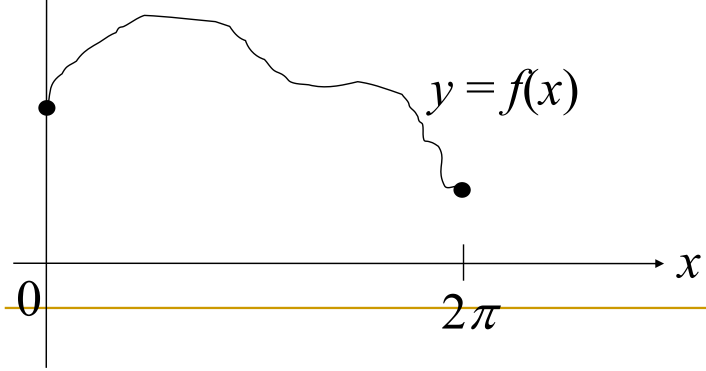
$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx\, dx, \qquad k = 1, \ldots, n \qquad (4)$$

A function $f(x)$ defined on $[0, 2\pi]$ can be represented by a Fourier series

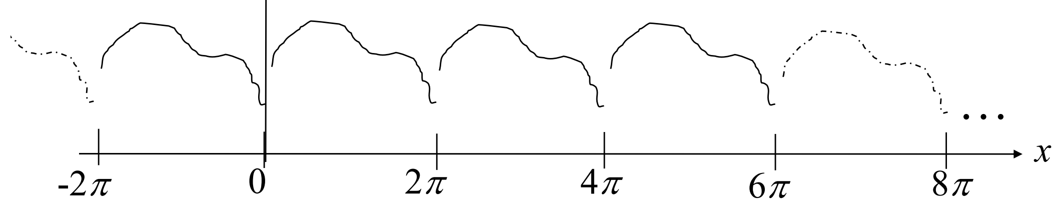
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} f_k(x) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k \cos kx + b_k \sin kx$$

$$= a_0 + \lim_{n\to\infty} \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx, \longleftarrow$$ Fourier series representation of $f(x)$

$$0 \le x \le 2\pi.$$



$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k \cos kx + b_k \sin kx, \infty < x < \infty$$



If $-\infty < x < \infty$, the Fourier series $\lim_{n\to\infty} \sum_{k=0}^{n} f_k(x) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k \cos kx + b_k \sin kx$

acutally represents a periodic function $f(x)$ of a period of $L = 2\pi$,

Orthogonality of sinusoidal functions

m, k nonzero integer.

If $m = k$,

$$\int_0^{2\pi} \cos mx \cos kx dx = \int_0^{2\pi} \cos mx \cos mx dx = \frac{1}{2}\int_0^{2\pi} 1 + \cos(2mx) dx = \frac{1}{2}\left[x + \frac{\sin 2mx}{2m}\right]_0^{2\pi} = \pi.$$

$$\int_0^{2\pi} \sin mx \sin kx dx = \int_0^{2\pi} \sin^2 mx dx = \pi$$

If $m \ne k$,

$\int_0^{2\pi} \cos mx \cos kx dx = 0, \int_0^{2\pi} \sin mx \sin kx dx = 0$.(can be proven using, say, integration by parts or formula for the product of two sinusoidal functions).

In addtion, $\int_0^{2\pi} \sin mx dx = \int_0^{2\pi} \cos mx dx = 0$.

Also, $\int_0^{2\pi} \sin mx \cos kx dx = 0$ for all m, k. We say sin and cos functions are orthogonal to each other.

Derivation of a_0

$$f_n(x) = a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$

Integrate both sides with respect to x from $x = 0$ to $x = 2\pi$

$$\int_0^{2\pi} f_n(x) dx = \int_0^{2\pi} a_0 dx + \sum_{k=1}^{n} \int_0^{2\pi} a_k \cos kx dx + b_k \sin kx dx =$$

$$\int_0^{2\pi} a_0 dx + \sum_{k=1}^{n} a_k \int_0^{2\pi} \cos kx dx + \sum_{k=1}^{n} b_k \int_0^{2\pi} \sin kx dx$$

$$= 2\pi a_0 + 0 + 0 = 2\pi a_0$$

$$\Rightarrow 2\pi a_0 = \int_0^{2\pi} f_n(x) dx.$$

For large enough n, f_n gives a good representation of f, hence we can replace f_n by f:

$$\Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x) dx$$

Derivation of a_k, $k \geq 1$

$$f_n(x) = a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$

Multiply both sides by $\cos mx$ (m nonzero integer), and integrate with respect to x from $x = 0$ to $x = 2\pi$. By doing so, the integral $\int_0^{2\pi} \cos mx \sin kx dx$ get 'killed off' due to the orthogality property of the sinusoidal functions.

In addtion, $\int_0^{2\pi} \cos mx \cos kx dx$ will also gets 'killed off' except for the case $m = k$.

$$\int_0^{2\pi} f(x) \cos mx dx =$$

$$\int_0^{2\pi} a_0 \cos mx dx + \sum_{k=1}^{n} a_k \int_0^{2\pi} \cos kx \cos mx dx + \sum_{k=1}^{n} b_k \int_0^{2\pi} \sin kx \cos mx dx$$

$$= 0 + a_m \int_0^{2\pi} \cos mx \cos mx dx + 0 = \pi a_m$$

$$\Rightarrow a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx \ dx.$$

Derivation of b_k, $k \geq 1$

b_k is simularly derived by multiplying both sides by $\sin mx$ (m nonzero integer), and integrate with respect to x from $x = 0$ to $x = 2\pi$.

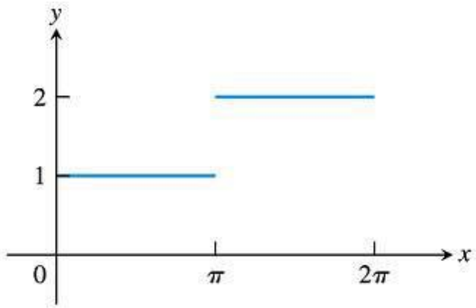
$$\int_0^{2\pi} f(x) \sin mx dx =$$

$$\int_0^{2\pi} a_0 \sin mx dx + \sum_{k=1}^{n} a_k \int_0^{2\pi} \cos kx \sin mx dx + \sum_{k=1}^{n} b_k \int_0^{2\pi} \sin kx \sin mx dx$$

$$= 0 + 0 + b_m \int_0^{2\pi} \sin mx \sin mx dx = \pi b_m$$

$$\Rightarrow b_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin mx \ dx.$$

- Fourier series can represent some functions that cannot be represented by Taylor series, e.g. step function such as



(a)

FIGURE 11.16 (a) The step function

$$f(x) = \begin{cases} 1, & 0 \leq x \leq \pi \\ 2, & \pi < x \leq 2\pi \end{cases}$$

EXAMPLE 1 Finding a Fourier Series Expansion

Fourier series can be used to represent some functions that cannot be represented by Taylor series; for example, the step function f shown in Figure 11.16a.

$$f(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \pi \\ 2, & \text{if } \pi < x \leq 2\pi. \end{cases}$$

The coefficients of the Fourier series of f are computed using Equations (2), (3), and (4).

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{2\pi} \left(\int_0^{\pi} 1\, dx + \int_{\pi}^{2\pi} 2\, dx \right) = \frac{3}{2}$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx\, dx$$

$$= \frac{1}{\pi} \left(\int_0^{\pi} \cos kx\, dx + \int_{\pi}^{2\pi} 2 \cos kx\, dx \right)$$

$$= \frac{1}{\pi} \left(\left[\frac{\sin kx}{k} \right]_0^{\pi} + \left[\frac{2 \sin kx}{k} \right]_{\pi}^{2\pi} \right) = 0, \quad k \geq 1$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx\, dx$$

$$= \frac{1}{\pi} \left(\int_0^{\pi} \sin kx\, dx + \int_{\pi}^{2\pi} 2 \sin kx\, dx \right)$$

$$= \frac{1}{\pi} \left(\left[-\frac{\cos kx}{k} \right]_0^{\pi} + \left[-\frac{2 \cos kx}{k} \right]_{\pi}^{2\pi} \right)$$

$$= \frac{\cos k\pi - 1}{k\pi} = \frac{(-1)^k - 1}{k\pi}.$$

So

$$a_0 = \frac{3}{2}, \quad a_1 = a_2 = \cdots = 0,$$

and

$$b_1 = -\frac{2}{\pi}, \quad b_2 = 0, \quad b_3 = -\frac{2}{3\pi}, \quad b_4 = 0, \quad b_5 = -\frac{2}{5\pi}, \quad b_6 = 0, \ldots$$

The Fourier series is

$$\frac{3}{2} - \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

Notice that at $x = \pi$, where the function $f(x)$ jumps from 1 to 2, all the sine terms vanish, leaving $3/2$ as the value of the series. This is not the value of f at π, since $f(\pi) = 1$. The Fourier series also sums to $3/2$ at $x = 0$ and $x = 2\pi$. In fact, all terms in the Fourier series are periodic, of period 2π, and the value of the series at $x + 2\pi$ is the same as its value at x. The series we obtained represents the periodic function graphed in Figure 11.16b, with domain the entire real line and a pattern that repeats over every interval of width 2π. The function jumps discontinuously at $x = n\pi, n = 0, \pm 1, \pm 2, \ldots$ and at these points has value $3/2$, the average value of the one-sided limits from each side. The convergence of the Fourier series of f is indicated in Figure 11.17.

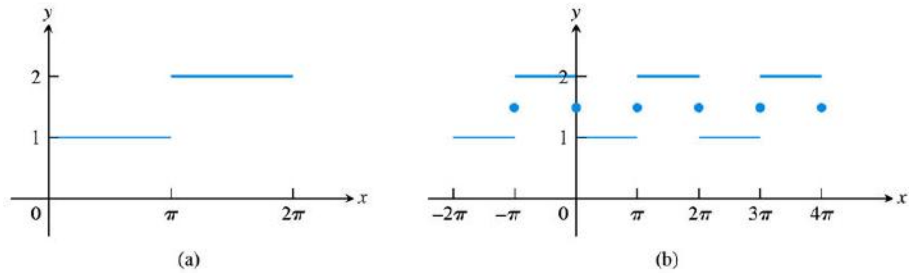


FIGURE 11.16 (a) The step function

$$f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 2, & \pi < x \le 2\pi \end{cases}$$

(b) The graph of the Fourier series for f is periodic and has the value $3/2$ at each point of discontinuity (Example 1).

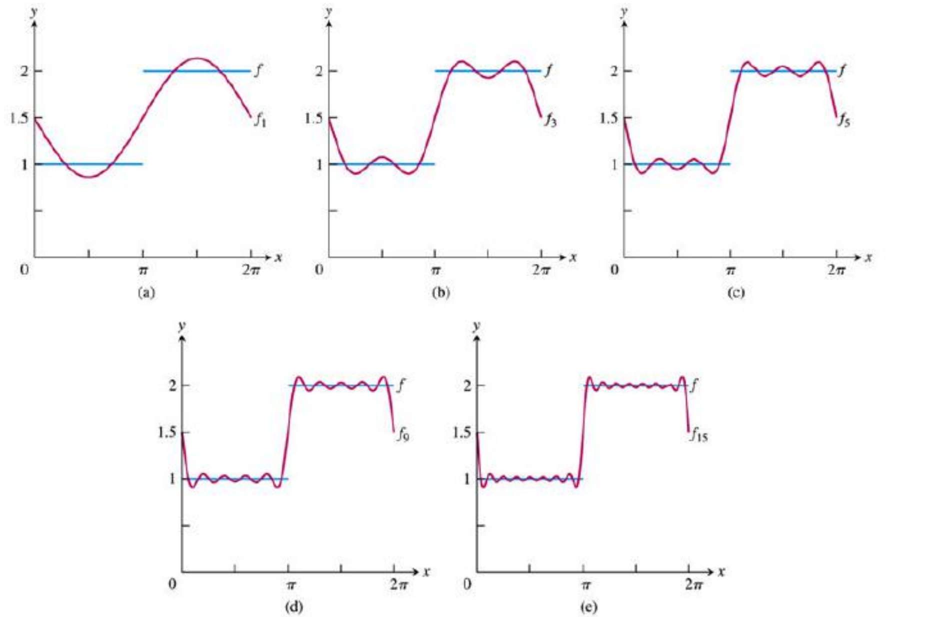


FIGURE 11.17 The Fourier approximation functions f_1, f_3, f_5, f_9, and f_{15} of the function $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 2, & \pi < x \le 2\pi \end{cases}$ in Example 1.

THEOREM 24 Let $f(x)$ be a function such that f and f' are piecewise continuous on the interval $[0, 2\pi]$. Then f is equal to its Fourier series at all points where f is continuous. At a point c where f has a discontinuity, the Fourier series converges to

$$\frac{f(c^+) + f(c^-)}{2}$$

where $f(c^+)$ and $f(c^-)$ are the right- and left-hand limits of f at c.

Fourier series representation of a function defined on the general interval [a,b]

- For a function defined on the interval [0,2π], the Fourier series representation of $f(x)$ is defined as $f(x) = a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$
- How about a function defined on an general interval of [a,b] where the period is $L=b-a$ instead of 2π? Can we still use $a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$ to represent $f(x)$ on [a,b]?

Fourier series representation of a function defined on the general interval [a,b]

- For a function defined on the interval of [a,b] the Fourier series representation on [a,b] is actually

$$a_0 + \sum_{k=1}^{n} a_k \cos\frac{2\pi kx}{L} + b_k \sin\frac{2\pi kx}{L}x$$

$$a_0 = \frac{1}{L}\int_a^b f(x)dx$$

$$a_m = \frac{2}{L}\int_a^b f(x)\cos\frac{2\pi mx}{L}dx$$

$$b_m = \frac{2}{L}\int_a^b f(x)\sin\frac{2\pi mx}{L}dx, m \text{ positive integer}$$

- $L=b-a$

Derivation of a_0

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cos\frac{2\pi kx}{L} + b_k \sin\frac{2\pi kx}{L}x$$

$$\int_a^b f(x)dx = \int_a^b a_0 dx + \sum_{k=1}^{n}\int_a^b a_k \cos\frac{2\pi kx}{L}dx + b_k \sin\frac{2\pi kx}{L}dx =$$

$$\int_a^b a_0 dx + \sum_{k=1}^{n} a_k \int_a^b \cos\frac{2\pi kx}{L}dx + \sum_{k=1}^{n} b_k \int_a^b \sin\frac{2\pi kx}{L}dx$$

$$= a_0(b-a)$$

$$\Rightarrow a_0 = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{L}\int_a^b f(x)dx$$

Derivation of a_k

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cos\frac{2\pi kx}{L} + b_k \sin\frac{2\pi kx}{L}x$$

$$\int_a^b f(x)\cos\frac{2\pi mx}{L}dx$$

$$= \int_a^b a_0 \cos\frac{2\pi mx}{L}dx + \sum_{k=1}^{n}\int_a^b \left(a_k \cos\frac{2\pi kx}{L}\cos\frac{2\pi mx}{L}dx + b_k \sin\frac{2\pi kx}{L}\cos\frac{2\pi mx}{L}\right)dx$$

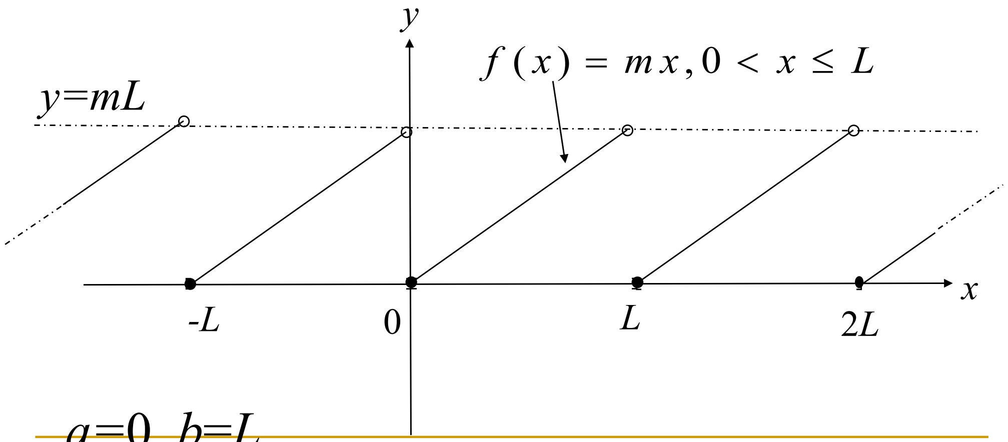
$$= 0 + a_m \int_a^b \cos^2\frac{2\pi mx}{L}dx + 0 = a_m \frac{L}{2}$$

$$\Rightarrow a_m = \frac{2}{L}\int_a^b f(x)\cos\frac{2\pi mx}{L}dx$$

Similarly,

$$b_m = \frac{2}{L}\int_a^b f(x)\sin\frac{2\pi mx}{L}dx$$

Example:



$a=0,\ b=L$

$$a_0 = \frac{1}{L}\int_a^b f(x)dx = \frac{1}{L}\int_a^b mx\,dx = \frac{m}{2L}\left(b^2 - a^2\right) = \frac{mL}{2}$$

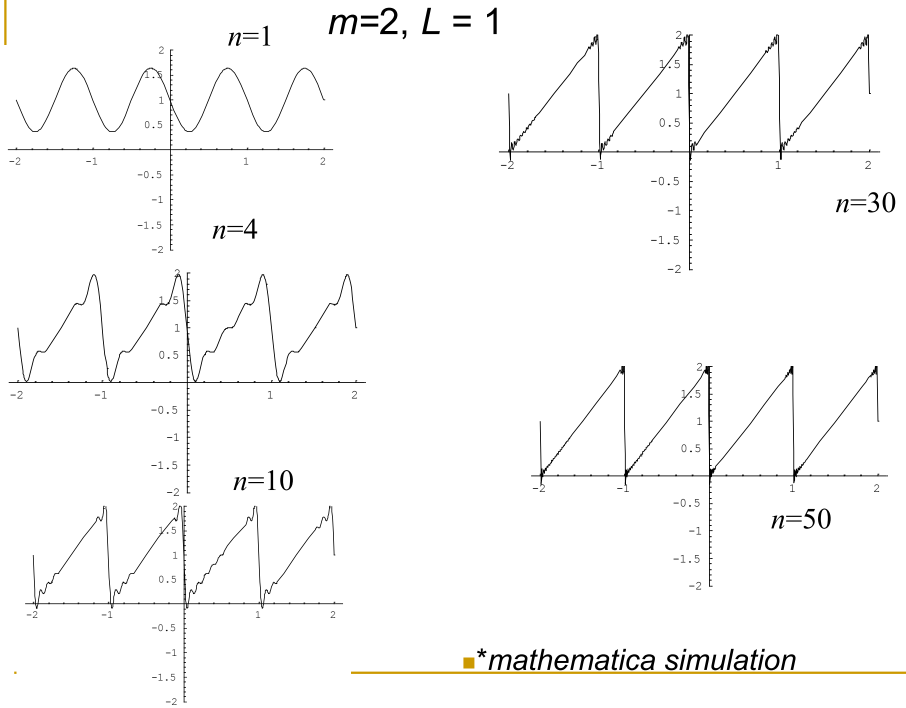
$$a_k = \frac{2}{L}\int_a^b mx\cos\frac{2\pi kx}{L}dx = \frac{2m}{L}\int_a^b x\cos\frac{2\pi kx}{L}dx = \frac{2m}{L}\frac{L^2\left(\cos 2k\pi - 1\right)}{4k^2\pi^2} = 0;$$

$$b_k = \frac{2}{L}\int_a^b f(x)\sin\frac{2\pi kx}{L}dx = \frac{2m}{L}\int_0^L x\sin\frac{2\pi kx}{L}dx$$

$$= \frac{2m}{L}\cdot L^2\left(\frac{-2k\pi\cos(2k\pi) + \sin 2k\pi}{4k^2\pi^2}\right) = \frac{-mL}{k\pi};$$

$$f(x) = mx = \frac{mL}{2} - \frac{mL}{\pi}\sum_{k=1}^{n}\frac{\sin 2\pi kx}{k}$$

$$= mL\left(\frac{1}{2} - \frac{\sin 2\pi x}{\pi} - \frac{\sin 4\pi x}{2\pi} - \frac{\sin 6\pi x}{3\pi} - \ldots - \frac{\sin 2n\pi x}{n\pi} + \ldots\right)$$



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CALCULUS ASSIGNMENT

QUESTIONS FOR ZCA 110

Tutorial 1 (Chapter 1)
Thomas' Calculus 11th edition

Exercise 1.3

Functions and Their Graphs

find the domain and range of each function.

1. $f(x) = 1 + x^2$

3. $F(t) = \dfrac{1}{\sqrt{t}}$

5. $g(z) = \sqrt{4 - z^2}$

Finding Formulas for Functions

13. Express the edge length of a cube as a function of the cube's diagonal length d. Then express the surface area and volume of the cube as a function of the diagonal length.

Functions and Graphs

Find the domain and graph the functions

16. $f(x) = 1 - 2x - x^2$

18. $g(x) = \sqrt{-x}$

22. Graph the following equations and explain why they are not graphs of functions of *x*.

a. $|x| + |y| = 1$ **b.** $|x + y| = 1$

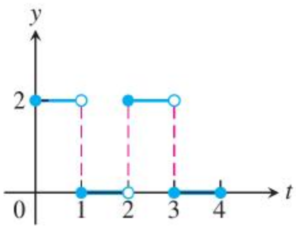
Piecewise-Defined Functions

Graph the function

24. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

27. Find a formula for each function graphed.

b.



Exercise 1.4

Recognizing Functions

In Exercises 3, identify each function as a constant function, linear function, power function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function. Remember that some functions can fall into more than one category.

3. a. $y = \dfrac{3 + 2x}{x - 1}$ **b.** $y = x^{5/2} - 2x + 1$

c. $y = \tan \pi x$ **d.** $y = \log_7 x$

Increasing and Decreasing Functions

Graph the functions. What symmetries, if any, do the graphs have? Specify the intervals over which the function is increasing and the intervals where it is decreasing.

16. $y = (-x)^{3/2}$

18. $y = -x^{2/3}$

Even and Odd Functions

Say whether the function is even, odd, or neither. Give reasons for your answer.

19. $f(x) = 3$

21. $f(x) = x^2 + 1$

23. $g(x) = x^3 + x$

EXERCISES 1.5
Sums, Differences, Products, and Quotients

Find the domains and ranges of f, g, f + g , and $f \cdot g$.

2. $f(x) = \sqrt{x+1}, \quad g(x) = \sqrt{x-1}$

Composites of Functions

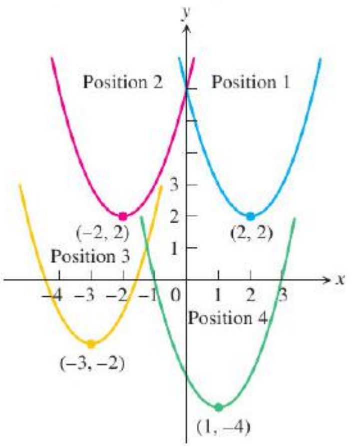
6. If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find
a. $f(g(½))$
b. $g(f(1/2))$

12. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$\frac{1}{x-1}$	$\lvert x \rvert$	?
b.	?	$\frac{x-1}{x}$	$\frac{x}{x+1}$
c.	?	$\sqrt{x}$	$\lvert x \rvert$
d.	$\sqrt{x}$	?	$\lvert x \rvert$

Shifting Graphs

17. Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.
a. $y = (x-1)^2 - 4$ **b.** $y = (x-2)^2 + 2$
c. $y = (x+2)^2 + 2$ **d.** $y = (x+3)^2 - 2$



Graph the functions

39. $y = \sqrt[3]{x-1} - 1$

48. $y = \frac{1}{(x+1)^2}$

Vertical and Horizontal Scaling

Exercises below tell by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation
for the stretched or compressed graph.

51. $y = x^2 - 1$, stretched vertically by a factor of 3

55. $y = \sqrt{x+1}$, compressed horizontally by a factor of 4

EXERCISES 1.6

Radians, Degrees, and Circular Arcs

4. If you roll a 1-m-diameter wheel forward 30 cm over level ground, through what angle will the wheel turn? Answer in radians (to the nearest tenth) and degrees (to the nearest degree).

Evaluating Trigonometric Functions

5. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					

Find the other two if x lies in the specified interval.

7. $\sin x = \frac{3}{5}, \quad x \in \left[\frac{\pi}{2}, \pi\right]$

Graphing Trigonometric Functions

Graph the functions the *ts*-plane (*t*-axis horizontal, *s*-axis vertical). What is the period of each function? What symmetries do the graphs have?

23. $s = \cot 2t$

25. $s = \sec\left(\frac{\pi t}{2}\right)$

Additional Trigonometric Identities

Use the addition formulas to derive the identity.

31. $\cos\left(x - \frac{\pi}{2}\right) = \sin x$

Using the Addition Formulas

Express the given quantity in terms of sin *x* and cos *x*.

39. $\cos(\pi + x)$

Using the Double-Angle Formulas

Find the function values

47. $\cos^2 \frac{\pi}{8}$

49. $\sin^2 \frac{\pi}{12}$

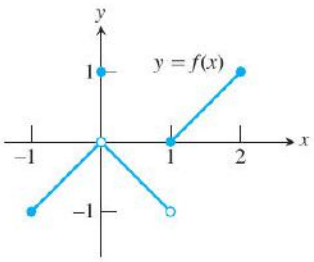
Tutorial 2 (Chapter 2)
Thomas' Calculus 11th edition

Exercise 2.1

Limits from Graphs

3. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

   a. $\lim_{x\to 0} f(x)$ exists.
   b. $\lim_{x\to 0} f(x) = 0$.
   c. $\lim_{x\to 0} f(x) = 1$.
   d. $\lim_{x\to 1} f(x) = 1$.
   e. $\lim_{x\to 1} f(x) = 0$.
   f. $\lim_{x\to x_0} f(x)$ exists at every point x_0 in $(-1, 1)$.



Existence of Limits

9. If $\lim_{x\to 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of f at $x = 1$? Explain.

10. If $f(1) = 5$, must $\lim_{x\to 1} f(x)$ exist? If it does, then must $\lim_{x\to 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x\to 1} f(x)$? Explain.

Limits by Substitution

Find the limits by substitution.

27. $\lim_{x\to \pi/2} x \sin x$

Average Rates of Change

Find the average rate of change of the function over the given interval or intervals.

33. $R(\theta) = \sqrt{4\theta + 1}; \quad [0, 2]$

Exercise 2.2

Limit Calculations

Find the limits.

17. $\lim_{h\to 0} \frac{\sqrt{3h + 1} - 1}{h}$

19. $\lim_{x\to 5} \frac{x - 5}{x^2 - 25}$

27. $\lim_{u\to 1} \frac{u^4 - 1}{u^3 - 1}$

Using Limit Rules

39. Suppose $\lim_{x\to c} f(x) = 5$ and $\lim_{x\to c} g(x) = -2$. Find

   a. $\lim_{x\to c} f(x)g(x)$

Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

occur frequently in calculus. Evaluate this limit for the given value of x and function f.

47. $f(x) = \sqrt{x}, \quad x = 7$

Using the Sandwich Theorem

49. If $\sqrt{5 - 2x^2} \le f(x) \le \sqrt{5 - x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.

EXERCISES 2.3

Centering Intervals About a Point

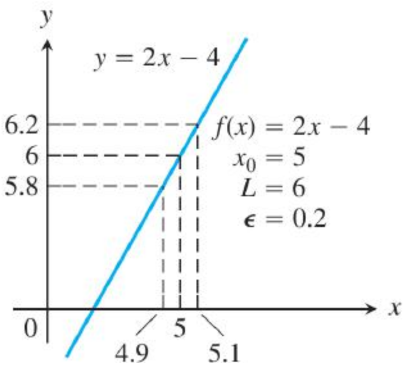
In Exercises 1–6, sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Rightarrow a < x < b$.

4. $a = -7/2, \quad b = -1/2, \quad x_0 = -3/2$

Finding Deltas Graphically

use the graphs to find a $\delta > 0$ such that for all x

7.



Finding Deltas Algebraically

Each of Exercises 15–30 gives a function $f(x)$ and numbers L, x_0 and $\epsilon > 0$. In each case, find an open interval about x_0 on which the inequality $|f(x) - L| < \epsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \epsilon$ holds.

15. $f(x) = x + 1, \quad L = 5, \quad x_0 = 4,$

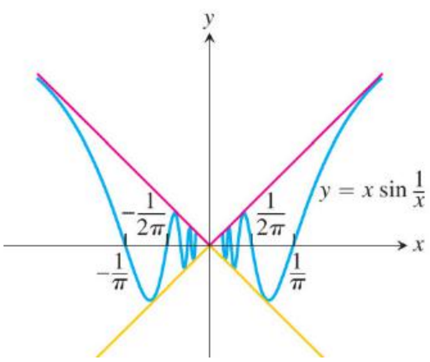
More on Formal Limits

31. $f(x) = 3 - 2x, \quad x_0 = 3, \quad \epsilon = 0.02$

Prove the limit statements

43. $\lim_{x \to 1} \frac{1}{x} = 1$

49. $\lim_{x \to 0} x \sin \frac{1}{x} = 0$



EXERCISES 2.4

Finding Limits Graphically

7. a. Graph $f(x) = \begin{cases} x^3, & x \neq 1 \\ 0, & x = 1. \end{cases}$

b. Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.

c. Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, why not?

Finding One-Sided Limits Algebraically

Find the limit

15. $\lim_{h \to 0^+} \frac{\sqrt{h^2 + 4h + 5} - \sqrt{5}}{h}$

Using $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Find the limits

33. $\lim_{\theta \to 0} \frac{\sin \theta}{\sin 2\theta}$

Calculating Limits as $x \to \pm\infty$

In Exercises 37–42, find the limit of each function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

41. $h(x) = \frac{-5 + (7/x)}{3 - (1/x^2)}$

Limits of Rational Functions

In Exercises 47–56, find the limit of each rational function **(a)** $x \to \infty$ and **(b)** as $x \to -\infty$.

49. $f(x) = \frac{x + 1}{x^2 + 3}$

EXERCISES 2.5

Infinite Limits

Find the limits.

13. $\lim_{x \to (\pi/2)^-} \tan x$

Additional Calculations

Find the limits.

25. $\lim \left(\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}} \right)$ as

a. $x \to 0^+$

c. $x \to 1^+$

Graphing Rational Functions

Graph the rational functions in Exercises 27–38. Include the graphs and equations of the asymptotes and dominant terms.

36. $y = \frac{x^2 - 1}{2x + 4}$

Inventing Functions

Find a function that satisfies the given conditions and sketch its graph. (The answers here are not unique. Any function that satisfies the conditions is acceptable. Feel free to use formulas defined in pieces if that will help.)

43. $\lim_{x \to \pm\infty} f(x) = 0$, $\lim_{x \to 2^-} f(x) = \infty$, and $\lim_{x \to 2^+} f(x) = \infty$

Graphing Terms

The function is given as the sum or difference of two terms. First graph the terms (with the same set of axes). Then, using these graphs as guides, sketch in the graph of the function.

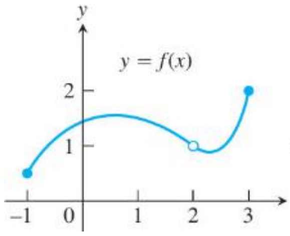
59. $y = \tan x + \frac{1}{x^2}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

EXERCISES 2.6

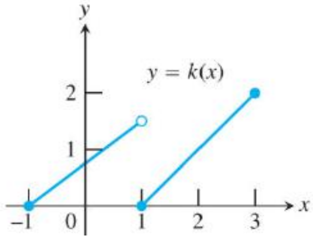
Continuity from Graphs

In the exercises below, say whether the function graphed is continuous on [-1, 3] . If not, where does it fail to be continuous and why?

1.



4.



Applying the Continuity Test

At which points do the functions fail to be continuous? At which points, if any, are the discontinuities removable? Not removable? Give reasons for your

17. $y = |x - 1| + \sin x$

20. $y = \frac{x + 2}{\cos x}$

Composite Functions

Find the limits. Are the functions continuous at the point being approached?

29. $\lim_{x \to \pi} \sin(x - \sin x)$

EXERCISES 2.7

Slopes and Tangent Lines

Find an equation for the tangent to the curve at the given point. Then sketch the curve and tangent together.

5. $y = 4 - x^2, \quad (-1, 3)$

Find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

18. $f(x) = \sqrt{x + 1}, \quad (8, 3)$

22. $y = \dfrac{x - 1}{x + 1}, \quad x = 0$

Tangent Lines with Specified Slopes

26. Find an equation of the straight line having slope 1/4 that is tangent to the curve $y = \sqrt{x}$.

Rates of Change

30. Ball's changing volume What is the rate of change of the volume of a ball ($V = (4/3)\pi r^3$) with respect to the radius when the radius is $r = 2$?

Tutorial 3 (Chapter 3)
Thomas' Calculus 11th edition

EXERCISES 3.1

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions. Then find the values of the derivatives as specified.

6. $r(s) = \sqrt{2s + 1}\,; \quad r'(0), r'(1), r'(1/2)$

Find the indicated derivatives.

10. $\dfrac{dv}{dt}$ if $v = t - \dfrac{1}{t}$

12. $\dfrac{dz}{dw}$ if $z = \dfrac{1}{\sqrt{3w - 2}}$

Slopes and Tangent Lines

Differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

18. $w = g(z) = 1 + \sqrt{4 - z}, \quad (z, w) = (3, 2)$

Find the values of the derivative.

21. $\left.\dfrac{dr}{d\theta}\right|_{\theta=0}$ if $r = \dfrac{2}{\sqrt{4 - \theta}}$

Differentiability and Continuity on an Interval

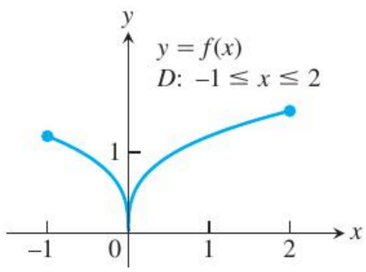
The figure below shows the graph of a function over a closed interval D. At what domain points does the function appear to be

a. differentiable?

b. continuous but not differentiable?

c. neither continuous nor differentiable?

43.



EXERCISES 3.2

Derivative Calculations

Find the derivatives of the functions

24. $u = \dfrac{5x + 1}{2\sqrt{x}}$

28. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

Find the first and second derivatives.

38. $p = \dfrac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$

Using Numerical Values

39. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

a. $\dfrac{d}{dx}(uv)$ **b.** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ **c.** $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d.** $\dfrac{d}{dx}(7v - 2u)$

Slopes and Tangents

41. a. Normal to a curve Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point (2, 1).

b. Smallest slope What is the smallest slope on the curve? At what point on the curve does the curve have this slope?

c. Tangents having specified slope Find equations for the tangents to the curve at the points where the slope of the curve is 8.

EXERCISES 3.3

Motion Along a Coordinate Line

7. Particle motion At time t, the position of a body moving along the s-axis is $s = t^3 - 6t^2 + 9t$ m.

a. Find the body's acceleration each time the velocity is zero.

b. Find the body's speed each time the acceleration is zero.

c. Find the total distance traveled by the body from $t = 0$ to $t = 2$.

Free-Fall Applications

10. Lunar projectile motion A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ meters in t sec.

a. Find the rock's velocity and acceleration at time t. (The acceleration in this case is the acceleration of gravity on the moon.)

b. How long does it take the rock to reach its highest point?

c. How high does the rock go?

d. How long does it take the rock to reach half its maximum height?

e. How long is the rock aloft?

Conclusions About Motion from Graphs

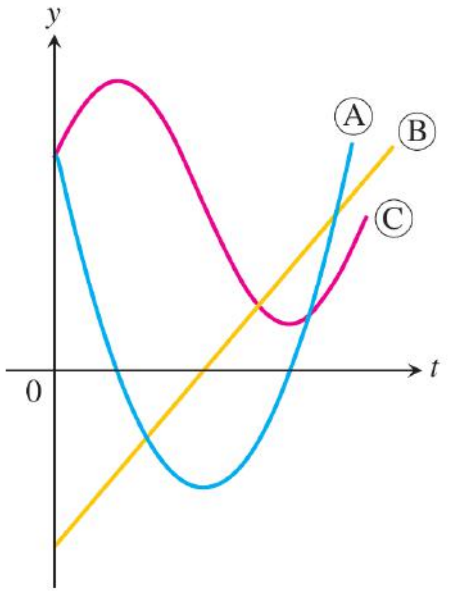


FIGURE 3.21 The graphs for Exercise 21.

35. Find all points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent line is parallel to the line $y = 2x$. Sketch the curve and tangent(s) together, labeling each with its equation.

27. Draining a tank It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth y of fluid in the tank t hours after the valve is opened is given by the formula

$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ m}.$$

a. Find the rate dy/dt (m/h) at which the tank is draining at time t.

b. When is the fluid level in the tank falling fastest? Slowest? What are the values of dy/dt at these times?

EXERCISES 3.4

Derivatives

find dy/dx

5. $y = (\sec x + \tan x)(\sec x - \tan x)$

11. $y = x^2 \sin x + 2x \cos x - 2 \sin x$

25. Find y'' if

a. $y = \csc x$.

Tangent Lines

Graph the curves over the given intervals, together with their tangents at the given values of x. Label each curve and tangent with its equation.

29. $y = \sec x, \quad -\pi/2 < x < \pi/2$

$x = -\pi/3, \pi/4$

Trigonometric Limits

Find the limits

39. $\lim_{x \to 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right)$

40. $\lim_{x \to -\pi/6} \sqrt{1 + \cos(\pi \csc x)}$

EXERCISES 3.5

Derivative Calculations

In Exercises 9–18, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

14. $y = \left(\frac{x}{5} + \frac{1}{5x}\right)^5$

Find the derivatives of the functions

26. $y = \frac{1}{x}\sin^{-5} x - \frac{x}{3}\cos^3 x$

Second Derivatives

Find *y*"

49. $y = \left(1 + \frac{1}{x}\right)^3$

51. $y = \frac{1}{9}\cot(3x - 1)$

Finding Numerical Values of Derivatives

61. Find ds/dt when $\theta = 3\pi/2$ if $s = \cos\theta$ and $d\theta/dt = 5$.

Tangents to Parametrized Curves

94. $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\pi/4$

EXERCISES 3.6

Derivatives of Rational Powers

Find dy/dx

13. $y = \sin\left[(2t + 5)^{-2/3}\right]$

Differentiating Implicitly

Use implicit differentiation to find dy/dx

19. $x^2y + xy^2 = 6$

Second Derivatives

In Exercises 37-42, use implicit differentiation to find dy/dx and then d^2y/dx^2.

41. $2\sqrt{y} = x - y$

Slopes, Tangents, and Normals

Verify that the given point is on the curve and find the lines that are (a) tangent and (b) normal to the curve at the given point.

45. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$

Implicitly Defined Parametrizations

63. $x^2 - 2tx + 2t^2 = 4, \quad 2y^3 - 3t^2 = 4, \quad t = 2$

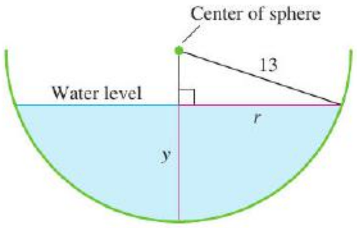
EXERCISES 3.7

12. Changing dimensions in a rectangular box Suppose that the edge lengths x, y, and z of a closed rectangular box are changing at the following rates:

$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$

20. A growing raindrop Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's

19. A draining hemispherical reservoir Water is flowing at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y meters deep.



Linearizing Trigonometric Functions

Find the linearization of f at $x = a$.

11. $f(x) = \sin x$ at (a) $x = 0$, (b) $x = \pi$

The Approximation $(1 + x)^k \approx 1 + kx$

17. Faster than a calculator Use the approximation $(1 + x)^k \approx 1 + kx$ to estimate the following.

a. $(1.0002)^{50}$ **b.** $\sqrt[3]{1.009}$

Derivatives in Differential Form

Find dy.

28. $y = \sec(x^2 - 1)$

Approximation Error

The function f(x) changes value when x changes from x_0 to $x_0 + dx$. Find

31. $f(x) = x^2 + 2x$, $x_0 = 1$, $dx = 0.1$

EXERCISES 3.8

Finding Linearizations

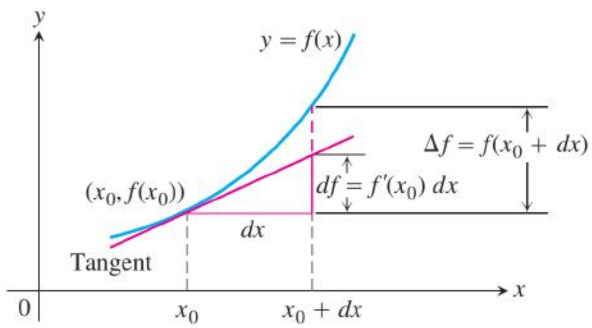
find the linearization $L(x)$ of $f(x)$ at $x = a$.

4. $f(x) = \sqrt[3]{x}$, $a = -8$

Linearization for Approximation

You want linearizations that will replace the functions in the following over intervals that include the given points x_0. To make your subsequent work as simple as possible, you want to center each linearization not at x_0 but at a nearby integer $x = a$ at which the given function and its derivative are easy to evaluate. What linearization do you use in each case?

5. $f(x) = x^2 + 2x$, $x_0 = 0.1$



Tutorial 4 (Chapter 4)
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EXERCISES 4.1

Absolute Extrema on Finite Closed Intervals

Find the absolute maximum and minimum values of the function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

24. $g(x) = -\sqrt{5 - x^2}, \quad -\sqrt{5} \le x \le 0$

Finding Extreme Values

Find the function's absolute maximum and minimum values and say where they are assumed.

44. $y = \dfrac{x + 1}{x^2 + 2x + 2}$

Local Extrema and Critical Points

Find the derivative at each critical point and determine the local extreme values.

48. $y = x^2\sqrt{3 - x}$

54. Let $f(x) = |x^3 - 9x|$.

a. Does $f'(0)$ exist?

b. Does $f'(3)$ exist?

c. Does $f'(-3)$ exist?

d. Determine all extrema of f.

Optimization Applications

Area of an athletic field
62. An athletic field is to be built in the shape of a rectangle *x* units long capped by semicircular regions of radius *r* at the two ends. The field is to be bounded by a 400-m racetrack.
a. Express the area of the rectangular portion of the field as a function of *x* alone or *r* alone (your choice).
b. What values of x and r give the rectangular portion the largest possible area?

EXERCISES 4.2

Finding c in the Mean Value Theorem

Find the value or values of c that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals.

2. $f(x) = x^{2/3}, \quad [0, 1]$

Checking and Using Hypotheses

10. For what values of a, m and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval [0, 2]?

Roots (Zeros)

Show that the function has exactly one zero in the given interval.

15. $f(x) = x^4 + 3x + 1, \quad [-2, -1]$

Finding Functions from Derivatives

25. Suppose that $f'(x) = 2x$ for all x. Find $f(2)$ if

a. $f(0) = 0$ **b.** $f(1) = 0$ **c.** $f(-2) = 3$.

Finding Position from Acceleration

Exercise 43 give the acceleration $a = d^2s/dt^2$, initial velocity and initial position of a body moving on a coordinate line. Find the body's position at time t.

43. $a = -4\sin 2t, \quad v(0) = 2, \quad s(0) = -3$

EXERCISES 4.3

Analyzing f Given f'

Answer the following questions about the functions whose derivatives are given below:
a. What are the critical points of f?
b. On what intervals is f increasing or decreasing?
c. At what points, if any, does f assume local maximum and minimum values?

7. $f'(x) = x^{-1/3}(x + 2)$
8. $f'(x) = x^{-1/2}(x - 3)$

Extremes of Given Functions

a. Find the intervals on which the function is increasing and decreasing.
b. Then identify the function's local extreme values, if any, saying where they are taken on.
c. Which, if any, of the extreme values are absolute?

24. $f(x) = \dfrac{x^3}{3x^2 + 1}$

Extreme Values on Half-Open Intervals

a. Identify the function's local extreme values in the given domain, and say where they are assumed.
b. Which of the extreme values, if any, are absolute?

32. $g(x) = -x^2 - 6x - 9, \quad -4 \le x < \infty$

36. $k(x) = x^3 + 3x^2 + 3x + 1, \quad -\infty < x \le 0$

Theory and Examples

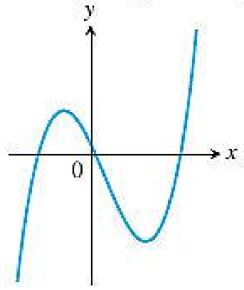
47. As x moves from left to right through the point $c = 2$, is the graph of $f(x) = x^3 - 3x + 2$ rising, or is it falling? Give reasons for your answer.

EXERCISES 4.4

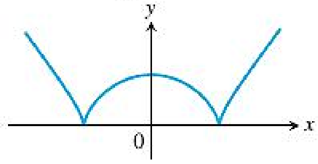
Analyzing Graphed Functions

Identify the inflection points and local maxima and minima of the functions graphed below. Identify the intervals on which the functions are concave up and concave down.

1. $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$



3. $y = \frac{3}{4}(x^2 - 1)^{2/3}$



Graph Equations

Use the steps of the graphing procedure to graph the equations below. Include the coordinates of any local extreme points and inflection points.

40. $y = \sqrt{|x - 4|}$

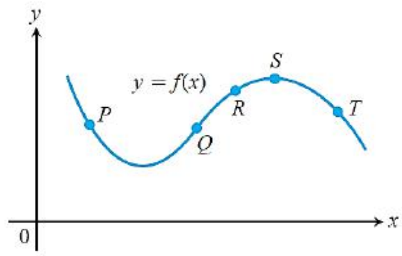
Sketching the General Shape Knowing y'

Each of Exercises below gives the first derivative of a continuous function $= f(x)$. Find y'' and sketch the general shape of the graph of f.

41. $y' = 2 + x - x^2$

Theory and Examples

67. The accompanying figure shows a portion if the graph of a twice-differentiable function $y = f(x)$. At each of the five labelled points, classify y' and y'' as positive, negative, or zero.



75. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2)$$

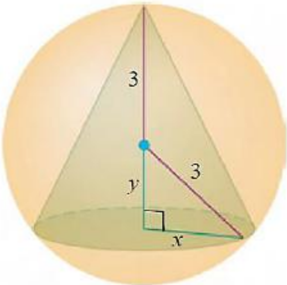
At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection? (Hint: Draw the sign pattern for y'.)

EXERCISES 4.5

Applications in Geometry

6. You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a,0)$ to $(0,\ b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.

12. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



18. A rectangle is to be inscribed under the arch of the curve $y = 4\cos(0.5x)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area?

22. A window is in the form if a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.

EXERCISES 4.6

Finding Limits

In Exercises 1 and 5, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

1. $\lim_{x\to 2} \frac{x-2}{x^2-4}$

Applying l'Hôpital's Rule

Use l'Hôpital's Rule to find the limits in Exercises 22 and 25.

22. $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right)$ **25.** $\lim_{x\to\pm\infty} \frac{3x-5}{2x^2-x+2}$

Theory and Applications

32. ∞/∞ **Form**
Give an example of two differentiable functions f and g with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ that satisfy the following.

a. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 3$ **b.** $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

c. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$

EXERCISES 4.8

Finding Antiderivatives

In Exercises 8 and 14, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

8. **a.** $\frac{4}{3}\sqrt[3]{x}$ **b.** $\frac{1}{3\sqrt[3]{x}}$ **c.** $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$

14. **a.** $\csc^2 x$ **b.** $-\frac{3}{2}\csc^2 \frac{3x}{2}$ **c.** $1 - 8\csc^2 2x$

Finding Indefinite Integrals

In Exercise 31 and 46, find the most general antiderivative or indefinite integral. Check your answer by differentiation.

31. $\int 2x(1 - x^{-3})\,dx$

46. $\int (2\cos 2x - 3\sin 3x)\,dx$

Checking Antiderivative Formulas

Verify the formulas in Exercises 60 by differentiation.

60. $\int \frac{1}{(x+1)^2}\,dx = \frac{x}{x+1} + C$

Theory and Examples

101. Suppose that

$$f(x) = \frac{d}{dx}\left(1 - \sqrt{x}\right) \quad \text{and} \quad g(x) = \frac{d}{dx}(x+2)$$

Find:

a. $\int f(x)\,dx$

b. $\int g(x)\,dx$

c. $\int [-f(x)]\,dx$

d. $\int [-g(x)]\,dx$

e. $\int [f(x) + g(x)]\,dx$

f. $\int [f(x) - g(x)]\,dx$

Tutorial 5 (Chapter 5 and 6)
Thomas' Calculus 11th edition

EXERCISES 5.1

Area

In Exercise 1 use finite approximations to estimate the area under the graph of the function using
a. a lower sum with two rectangles of equal width.
b. a lower sim with four rectangles of equal width.
c. an upper sum with two rectangles of equal width.
d. an upper sum with four rectangles of equal width.

1. $f(x) = x^2$ between $x = 0$ and $x = 1$

Area of a Circle

21. Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of n:

a. 4 (square) **b.** 8 (octagon) **c.** 16
d. Compare the areas in parts (a), (b) and (c) with the area of the circle.

EXERCISES 5.2

Sigma Notation

Write the sums in Exercises 1 without sigma notation. Then evaluate them.

1. $\sum_{k=1}^{2} \frac{6k}{k+1}$

Values of Finite Sums

17. Suppose that $\sum_{k=1}^{n} a_k = -5$ and $\sum_{k=1}^{n} b_k = 6$. Find the values of

c. $\sum_{k=1}^{n} (a_k + b_k)$

Evaluate the sums in Exercise 24.

24. $\sum_{k=1}^{6} (k^2 - 5)$

Limits of Upper Sums

For the functions in Exercise 36, find a formula for the upper sum obtained by dividing the interval $[a,b]$ into n equal subintervals. Then take a limit of this sum as $n \to \infty$ to calculate the area under the curve over $[a,b]$.

36. $f(x) = 2x$ over the interval $[0, 3]$

EXERCISES 5.3

Expressing Limits as Integrals

Express the limits in Exercise 1 as definite integrals.

1. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} c_k^2 \, \Delta x_k$, where P is a partition of $[0, 2]$

Using Properties and Known Values to Find Other Integrals

12. Suppose that $\int_{-3}^{0} g(t)\, dt = \sqrt{2}$. Find

a. $\int_{0}^{-3} g(t)\, dt$ **b.** $\int_{-3}^{0} g(u)\, du$

c. $\int_{-3}^{0} [-g(x)]\, dx$ **d.** $\int_{-3}^{0} \frac{g(r)}{\sqrt{2}}\, dr$

Using Area to Evaluate Definite Integrals

In Exercise 15, graph the integrands and use areas to evaluate the integrals.

15. $\int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$

Evaluations

Use the results of Equations (1) and (3) to evaluate the integrals in Exercise 38.

38. $\int_{0}^{3b} x^2\, dx$

Average Value

In Exercise 55, graph the function and find its average value over the given interval.

55. $f(x) = x^2 - 1 \quad \text{on} \quad [0, \sqrt{3}]$

EXERCISES 5.4

Evaluating Integrals

Evaluate the integrals in Exercises 23 and 25.

23. $\displaystyle\int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2}\, ds$

25. $\displaystyle\int_{-4}^{4} |x|\, dx$

Derivatives of Integrals

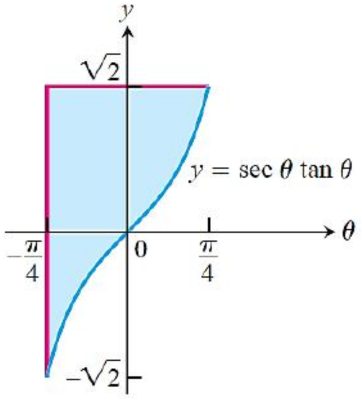
Find dy/dx in Exercise 36.

36. $\displaystyle y = \int_{\tan x}^{0} \frac{dt}{1 + t^2}$

Area

Find the areas of the shaded regions in Exercise 45.

45.



Theory and Examples

62. Find

$$\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{t^4 + 1}\, dt$$

EXERCISES 5.5

Evaluating Integrals

Evaluate the indefinite integrals in Exercise 4 and 11 by using the given substitutions to reduce the integrals to standard form.

4. $\displaystyle\int \left(1 - \cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\, dt, \quad u = 1 - \cos\frac{t}{2}$

11. $\displaystyle\int \csc^2 2\theta \cot 2\theta\, d\theta$

a. Using $u = \cot 2\theta$ **b.** Using $u = \csc 2\theta$

Evaluate the integrals in Exercises 36 and 48.

36. $\displaystyle\int \frac{6\cos t}{(2 + \sin t)^3}\, dt$

48. $\displaystyle\int 3x^5 \sqrt{x^3 + 1}\, dx$

Simplifying Integrals Step by Step

Evaluate the integrals in Exercise 51.

51. $\displaystyle\int \frac{(2r - 1)\cos\sqrt{3(2r - 1)^2 + 6}}{\sqrt{3(2r - 1)^2 + 6}}\, dr$

EXERCISES 5.6

Evaluating Definite Integrals

Use the substitution formula in Theorem 6 to evaluate the integrals in Exercises 7 and 14.

7. a. $\displaystyle\int_{-1}^{1} \frac{5r}{(4 + r^2)^2}\, dr$ **b.** $\displaystyle\int_0^{1} \frac{5r}{(4 + r^2)^2}\, dr$

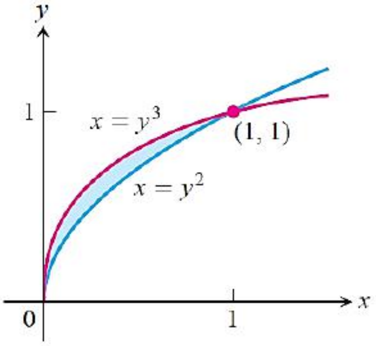
14. a. $\displaystyle\int_{-\pi/2}^{0} \frac{\sin w}{(3 + 2\cos w)^2}\, dw$

b. $\displaystyle\int_0^{\pi/2} \frac{\sin w}{(3 + 2\cos w)^2}\, dw$

Area

Find the total areas of the shaded regions in Exercise 32.

32.



73. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

__**EXERCISES 6.3**__

Length of Parametrized Curves

Find the lengths of the curves in Exercise 1.

1. $x = 1 - t, \quad y = 2 + 3t, \quad -2/3 \leq t \leq 1$

Finding Lengths of Curves

Find the lengths of the curves in Exercises 7 and 16. If you have a grapher, you may want to graph these curves to see what they look like.

7. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$

16. $y = \int_{-2}^{x} \sqrt{3t^4 - 1}\, dt, \quad -2 \leq x \leq -1$

Theory and Applications

27. a. Find a curve through the point (1, 1) whose length integral is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx$$

b. How many such curves are there? Give reasons for your answer.

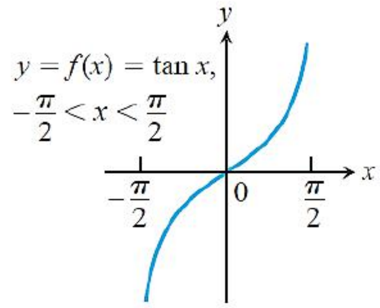
Tutorial 6 (Chapter 7)
Thomas' Calculus 11th edition

EXERCISES 7.1

Graphing Inverse Functions

Exercise 10 shows the graph of a function $y = f(x)$. Copy the graph and draw in the line $y = x$. Then use symmetry with respect to the line $y = x$ to add the graph of f^{-1} to your sketch. (It is not necessary to find a formula for f^{-1}.) Identify the domain and range of f^{-1}.

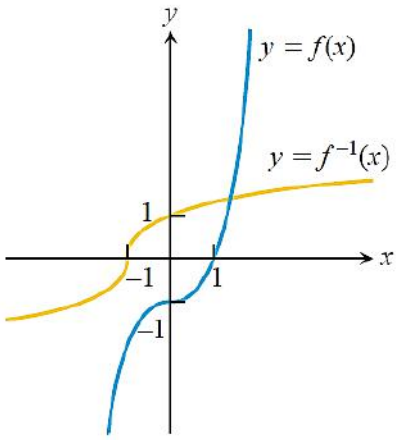
10.



Formulas for Inverse Functions

Exercise 15 gives a formula for a function $y = f(x)$ and shows the graphs of f and f^{-1}. Find a formula for f^{-1} in each case.

15. $f(x) = x^3 - 1$



21. $f(x) = x^3 + 1$

Derivatives of Inverse Functions

In Exercises 25 and 30:

a. Find $f^{-1}(x)$.
b. Graph f and f^{-1} together.
c. Evaluate df/dx at $x = a$ and $\frac{df^{-1}}{dx}$ at $x = f(a)$ to show that at these points $\frac{df^{-1}}{dx} = 1/(\frac{df}{dx})$.

25. $f(x) = 2x + 3, \quad a = -1$

30.
a. Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.
b. Graph h and k over an x-interval large enough to show the graphs intersecting at (2, 2) and (-2, -2). Be sure the picture shows the required symmetry about the line $y = x$.
c. Find the slopes of the tangents to the graphs at h and k at (2, 2) and (-2, -2).
d. What lines are tangent to the curves at the origin?

EXERCISES 7.2

Using the Properties of Logarithms

1. Express the following logarithms in terms of ln 2 and ln 3.

a. ln 0.75 **b.** ln (4/9) **c.** ln (1/2)
d. ln $\sqrt[3]{9}$ **e.** ln $3\sqrt{2}$ **f.** ln $\sqrt{13.5}$

Derivatives of Logarithms

In Exercise 22, find the derivative of y with respect to x, t, or θ, as appropriate.

22. $y = \dfrac{x \ln x}{1 + \ln x}$

Integration

Evaluate the integrals in Exercise 39.

39. $\displaystyle\int \frac{2y\,dy}{y^2 - 25}$

Logarithmic Differentiation

In Exercise 64, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

64. $y = \dfrac{\theta \sin \theta}{\sqrt{\sec \theta}}$

Theory and Applications

69. Locate and identify the absolute extreme values of

a. ln (cos x) on $[-\frac{\pi}{4}, \frac{\pi}{3}]$,

b. cos (ln x) on $[\frac{1}{2}, 2]$.

EXERCISES 7.3

Algebraic Calculations with the Exponential and Logarithm

Find simpler expressions for the quantities in Exercise 2.

2. **a.** $e^{\ln(x^2+y^2)}$ **b.** $e^{-\ln 0.3}$ **c.** $e^{\ln \pi x - \ln 2}$

Solving Equations with Logarithmic or Exponential Terms

In Exercise 10, solve for y in terms of t or x, as appropriate.

10. $\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x)$

In Exercise 16, solve for t.

16. $e^{(x^2)} e^{(2x+1)} = e^t$

Derivatives

In Exercises 23 and 36, find the derivative of y with respect to x, t, or θ, as appropriate.

23. $y = (x^2 - 2x + 2)e^x$

36. $y = \displaystyle\int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt$

Integrals

Evaluate the integrals in Exercises 49 and 56.

49. $\displaystyle\int \frac{e^{\sqrt{r}}}{\sqrt{r}} \, dr$

56. $\displaystyle\int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \csc^2 \theta \, d\theta$

Theory and Applications

67. Find the absolute maximum and minimum values of $f(x) = e^x - 2x$ on [0, 1].

EXERCISES 7.4

Algebraic Calculations With a^x and $\log_a x$

Simplify the expressions in Exercise 4.

4. **a.** $25^{\log_5(3x^2)}$ **b.** $\log_e(e^x)$ **c.** $\log_4(2^{e^x \sin x})$

Derivatives

In Exercises 18 and 29, find the derivative of y with respect to the given independent variable.

18. $y = (\ln \theta)^\pi$

29. $y = \log_3\left(\left(\dfrac{x + 1}{x - 1}\right)^{\ln 3}\right)$

Logarithmic Differentiation

In Exercises 41 and 46, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

41. $y = (\sqrt{t})^t$

46. $y = (\ln x)^{\ln x}$

Integration

Evaluate the integrals in Exercise 65.

65. $\displaystyle\int_0^2 \frac{\log_2(x + 2)}{x + 2} \, dx$

Evaluate the integrals in Exercise 72.

72. $\int_1^{e^x} \frac{1}{t}\,dt$

Theory and Applications

75. Find the area of the region between the curve $y = 2x/(1+x^2)$ and the interval $-2 \le x \le 2$ of the x-axis.

<u>**EXERCISES 7.5**</u>

6. Voltage in a discharging capacitor
Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage V across its terminals and that, if t is measured in seconds,

$$\frac{dV}{dt} = -\frac{1}{40}V.$$

Solve this equation for V, using V_0 to denote the value of V when $t = 0$. How long will it take the voltage to drop to 10% of its original value?

8. Growth of bacteria
A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours there are 10,000 bacteria. At the end of 5 hours there are 40,000. How many bacteria were present initially?

<u>**EXERCISES 7.7**</u>

Common Values of Inverse Trigonometric Functions

Use reference triangles to find the angles in Exercise 6.

6. a. $\cos^{-1}\left(\frac{-1}{2}\right)$ **b.** $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

c. $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

Trigonometric Function Values

13. Given that $\alpha = sin^{-1}(5/13)$, find cos α, tan α, sec α, csc α, and cot α.

Evaluating Trigonometric and Inverse Trigonometric Terms

Find the values in Exercise 26.

26. $\sec\left(\cot^{-1}\sqrt{3} + \csc^{-1}(-1)\right)$

Finding Derivatives

In Exercise 51, find the derivative of y with respect to the appropriate variable.

51. $y = \sin^{-1}\sqrt{2}\,t$

Evaluating Integrals

Evaluating the integrals in Exercise 72.

72. $\int \frac{dx}{\sqrt{1-4x^2}}$

Evaluate the integrals in Exercise 107.

107. $\int \frac{(\sin^{-1}x)^2\,dx}{\sqrt{1-x^2}}$

Integration Formulas

Verify the integration formulas in Exercise 117.

117. $\int \frac{\tan^{-1}x}{x^2}\,dx = \ln x - \frac{1}{2}\ln(1+x^2) - \frac{\tan^{-1}x}{x} + C$

<u>**EXERCISES 7.8**</u>

Hyperbolic Function Values and Identities

Each of Exercise 1 gives a value of sinh x or cosh x. Use the definitions and the identity $\cosh^2 x - \sinh^2 x = 1$ to find the values of the remaining five hyperbolic functions.

1. $\sinh x = -\frac{3}{4}$

Derivatives

In Exercise 16, find the derivative of y with respect to the appropriate variable.

16. $y = t^2 \tanh \frac{1}{t}$

Indefinite Integrals

Evaluate the integrals in Exercise 43.

43. $\int 6 \cosh \left(\frac{x}{2} - \ln 3 \right) dx$

Definite Integrals

Evaluate the integrals in Exercise 60.

60. $\int_0^{\ln 10} 4 \sinh^2 \left(\frac{x}{2} \right) dx$

Evaluating Inverse Hyperbolic Functions and Related Integrals

When hyperbolic function keys are not available on a calculator, it is still possible to evaluate the inverse hyperbolic functions by expressing them as logarithms, as shown here.

$\sinh^{-1} x = \ln \left(x + \sqrt{x^2 + 1} \right),$	$-\infty < x < \infty$
$\cosh^{-1} x = \ln \left(x + \sqrt{x^2 + 1} \right),$	$x \geq 1$
$\tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x},$	$\lvert x \rvert < 1$
$\operatorname{sech}^{-1} x = \ln \left(\frac{1 + \sqrt{1 - x^2}}{x} \right),$	$0 < x \leq 1$
$\operatorname{csch}^{-1} x = \ln \left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{\lvert x \rvert} \right),$	$x \neq 0$
$\coth^{-1} x = \frac{1}{2} \ln \frac{x + 1}{x - 1},$	$\lvert x \rvert > 1$

Use the formulas in the box here to express the numbers in Exercise 66 in terms of natural logarithms.

66. $\operatorname{csch}^{-1} (-1/\sqrt{3})$

Applications and Theory

83. Arc length

Find the length of the segment of the curve $y = (1/2) \cosh 2x$ from $x = 0$ to $x = \ln \sqrt{5}$.

Tutorial 7 (Chapter 8)
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EXERCISES 8.1

Basic Substitutions

Evaluate each integral in Exercise 36 by using a substitution to reduce it to standard form.

36. $\int \frac{\ln x\,dx}{x + 4x\ln^2 x}$

Completing the Square

Evaluate each integral in Exercise 41 by completing the square and using a substitution to reduce it to standard form.

41. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

Improper Fractions

Evaluate each integral in Exercise 50 by reducing the improper fraction and using a substitution (if necessary) to reduce it to standard form.

50. $\int_{-1}^{3} \frac{4x^2-7}{2x+3}\,dx$

Separating Fractions

Evaluate each integral in Exercise 56 by separating the fraction and using a substitution (if necessary) to reduce it to standard form.

56. $\int_{0}^{1/2} \frac{2-8x}{1+4x^2}\,dx$

Multiplying by a Form of 1

Evaluate each integral in Exercise 59 by multiplying by a form of 1 and using a substitution (if necessary) to reduce it to standard form.

59. $\int \frac{1}{\sec\theta + \tan\theta}\,d\theta$

Eliminating Square Roots

Evaluate each integral in Exercise 68 by eliminating the square root.

68. $\int_{\pi/2}^{\pi} \sqrt{1-\sin^2\theta}\,d\theta$

Assorted Integrations

Evaluate each integral in Exercise 82 by using any technique you think is appropriate.

82. $\int \frac{dx}{x\sqrt{3+x^2}}$

Trigonometric Powers

83.
a. Evaluate $\int cos^3\,\theta\,d\theta$. (Hint: $cos^2\,\theta = 1 - sin^2\,\theta$.)
b. Evaluate $\int cos^5\,\theta\,d\theta$.
c. Without actually evaluating the integral, explain how you would evaluate $\int cos^9\,\theta\,d\theta$.

EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercise 1, 19 and 24.

1. $\int x\sin\frac{x}{2}\,dx$

19. $\int_{2/\sqrt{3}}^{2} t\sec^{-1} t\,dt$

24. $\int e^{-2x}\sin 2x\,dx$

Substitution and Integration by Parts

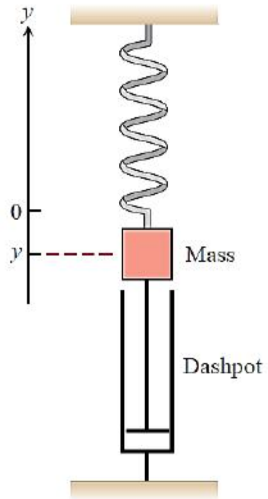
Evaluate the integrals in Exercise 30 by using a substitution prior to integration by parts.

30. $\int z(\ln z)^2\,dz$

37. Average value
A retarding force, symbolized by the dashpot in the figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t}\cos t, \quad t \geq 0.$$

Find the average value of y over the interval $0 \leq t \leq 2\pi$.



Reduction Formulas

In Exercise 41, use integration by parts to establish the reduction formula.

41. $\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx, \quad a \neq 0$

EXERCISES 8.3

Expanding Quotients into Partial Fractions

Expand the quotients in Exercise 6 by partial fractions.

6. $\frac{z}{z^3 - z^2 - 6z}$

Nonrepeated Linear Factors

In Exercise 12, express the integrands as a sum of partial fractions and evaluate the integrals.

12. $\int \frac{2x + 1}{x^2 - 7x + 12}\,dx$

Repeated Linear Factors

In Exercise 20, express the integrands as a sum of partial fractions and evaluate the integrals.

20. $\int \frac{x^2\,dx}{(x - 1)(x^2 + 2x + 1)}$

Irreducible Quadratic Factors

In Exercise 26, express the integrands as a sum of partial fractions and evaluate the integrals.

26. $\int \frac{s^4 + 81}{s(s^2 + 9)^2}\,ds$

Improper Fractions

In Exercise 31, perform long division on the intergrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

31. $\int \frac{9x^3 - 3x + 1}{x^3 - x^2}\,dx$

Evaluating Integrals

Evaluating the integrals in Exercise 38.

38. $\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

EXERCISES 8.4

Products of Powers of Sines and Cosines

Evaluate the integrals in Exercise 6 and 14.

6. $\int_0^{\pi/2} 7\cos^7 t\,dt$

14. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta\,d\theta$

Integrals with Square Roots

Evaluate the integrals in Exercise 22.

22. $\int_{-\pi}^{\pi} (1 - \cos^2 t)^{3/2}\, dt$

Powers of Tan x and Sec x

Evaluate the integrals in Exercise 26.

26. $\int_{0}^{\pi/12} 3 \sec^4 3x\, dx$

Products of Sines and Cosines

Evaluate the integrals in Exercise 38.

38. $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x\, dx$

__**EXERCISES 8.5**__

Basic Trigonometric Substitutions

Evaluate the integrals in Exercise 1, 14 and 28.

1. $\int \frac{dy}{\sqrt{9 + y^2}}$

14. $\int \frac{2\, dx}{x^3 \sqrt{x^2 - 1}}, \quad x > 1$

28. $\int \frac{(1 - r^2)^{5/2}}{r^8}\, dr$

In Exercise 32, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

32. $\int_{1}^{e} \frac{dy}{y\sqrt{1 + (\ln y)^2}}$

Applications

41. Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9 - x^2}/3$.

__**EXERCISES 8.6**__

Using Integral Tables

Use the table of integrals to evaluate the integrals in Exercise 8 and 20.

8. $\int \frac{dx}{x^2 \sqrt{4x - 9}}$

20. $\int \frac{d\theta}{4 + 5 \sin 2\theta}$

Substitution and Integral Tables

In Exercise 45, use a substitution to change the integral into one you can find in the table. Then evaluate the integral.

45. $\int \cot t \sqrt{1 - \sin^2 t}\, dt, \quad 0 < t < \pi/2$

Using Reduction Formulas

Use reduction formulas to evaluate the integrals in Exercise 60.

60. $\int \csc^2 y \cos^5 y\, dy$

Powers of x Times Exponentials

Evaluate the integrals in Exercise 80 using table Formulas 103-106. These integrals can also be evaluated using integration (Section 8.2).

80. $\int x 2^{\sqrt{2x}}\, dx$

Substitutions with Reduction Formulas

Evaluate the integrals in Exercise 81 by making a substitution (possibly trigonometric) and then applying a reduction formula.

81. $\int e^t \sec^3 (e^t - 1)\, dt$

Hyperbolic Functions

Use the integral tables to evaluate the integrals in Exercise 90.

90. $\int x \sinh 5x \, dx$

<u>EXERCISES 8.8</u>

Evaluating Improper Integrals

Evaluate the integrals in Exercises 1 and 26 without using tables.

1. $\int_0^\infty \frac{dx}{x^2 + 1}$

26. $\int_0^1 (-\ln x) \, dx$

Testing for Convergence

In Exercises 35, 50 and 64, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

35. $\int_0^{\pi/2} \tan \theta \, d\theta$

50. $\int_0^\infty \frac{d\theta}{1 + e^\theta}$

64. $\int_{-\infty}^\infty \frac{dx}{e^x + e^{-x}}$

Theory and Examples

65. Find the values of p for which each integral converges.

b. $\int_2^\infty \frac{dx}{x(\ln x)^p}$

Tutorial 8 (Chapter 11)
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EXERCISES 11.1

Finding Terms of a Sequence

Exercise 2 gives a formula for the nth term a_n of a sequence $\{a_n\}$. Find the values of a_1, a_2, a_3, and a_4.

2. $a_n = \dfrac{1}{n!}$

Finding a Sequence's Formula

In Exercise 16, find a formula for the nth term of the sequence.

16. The sequence $1, -\dfrac{1}{4}, \dfrac{1}{9}, -\dfrac{1}{16}, \dfrac{1}{25}, \ldots$

Reciprocals of squares
of the positive integers,
with alternating signs

Finding Limits

Which of the sequences $\{a_n\}$ in Exercises 25, 49 and 80 converge. and which diverge? Find the limit of each convergent sequence.

25. $a_n = \dfrac{1 - 2n}{1 + 2n}$

49. $a_n = \left(1 + \dfrac{7}{n}\right)^n$

80. $a_n = \dfrac{(\ln n)^5}{\sqrt{n}}$

EXERCISES 11.2

Finding nth Partial Sums

In Exercise 1, find a formula for the nth partial sum of each series and use it to find the series' sum if the series converges.

1. $2 + \dfrac{2}{3} + \dfrac{2}{9} + \dfrac{2}{27} + \cdots + \dfrac{2}{3^{n-1}} + \cdots$

Series with Geometric Terms

In Exercise 7, write out the first few terms of each series to show how the series starts. Then find the sum of the series.

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$

Telescoping Series

Find the sum of each series in Exercise 15.

15. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{(4n - 3)(4n + 1)}$

Convergence or Divergence

Is Exercise 23 converge or diverge? Give reasons for your answer. If a series converges, find its sum.

23. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$

Geometric Series

In geometric series in Exercise 41, write out the first few terms of the series to find a and r, and find the sum of the series. Then express the inequality $|r| < 1$ in terms of x and find the values of x for which the inequality holds and the series converges.

41. $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$

Repeating Decimals

Express each of the numbers in Exercise 51 as the ratio of two integers.

51. $0.\overline{23} = 0.23\ 23\ 23 \ldots$

EXERCISES 11.3

Determining Convergence or Divergence

Which of the series in Exercises 1, 9, 10 and 28 converge, and which diverge? Give reasons for your answers. (When you check an answer, remember that there may be more than one way to determine the series' convergence or divergence.)

1. $\sum_{n=1}^{\infty} \frac{1}{10^n}$ **9.** $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

10. $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$ **28.** $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

EXERCISES 11.4

Determining Convergence and Divergence

Which of the series in 1, 10 and 36 converge, and which diverge? Give reasons for your answers.

1. $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$ **10.** $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$

36. $\sum_{n=1}^{\infty} \frac{1}{1 + 2^2 + 3^2 + \cdots + n^2}$

EXERCISES 11.6

Determining Convergence or Divergence

Is Exercise 1 converge or diverge? Give reasons for your answers.

1. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

Absolute Convergence

Which of the series in Exercises 13 and 30 converge absolutely, which converge, and which diverge? Give reasons for your answers.

13. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ **30.** $\sum_{n=1}^{\infty} (-5)^{-n}$

EXERCISES 11.7

Intervals of Convergence

In Exercise 1, 11 and 22, **(a)** find the series' radius and interval of convergence. For what values of x does the series converge **(b)** absolutely, **(c)** conditionally?

1. $\sum_{n=0}^{\infty} x^n$ **11.** $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

22. $\sum_{n=1}^{\infty} (\ln n) x^n$

In Exercise 36, find the series' interval of convergence and, within this interval, the sum of the series as a function of x.

36. $\sum_{n=0}^{\infty} (\ln x)^n$

EXERCISES 11.8

Finding Taylor Polynomials

In Exercises 1 and 4, find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a.

1. $f(x) = \ln x, \quad a = 1$

4. $f(x) = 1/(x + 2), \quad a = 0$

Finding Taylor Series at $x = 0$ (Maclaurin Series)

Find the Maclaurin series for the functions in Exercise 9.

9. e^{-x}

Finding Taylor Series

In Exercises 24 and 28, find the Taylor series generated by f at $x = a$.

24. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2,$

$a = -1$

28. $f(x) = 2^x, \quad a = 1$

EXERCISES 11.9

Taylor Series by Substitution

Use substitution to find the Taylor series at $x = 0$ of the functions in Exercise 1.

1. e^{-5x}

More Taylor Series

Find Taylor series at $x = 0$ for the functions in Exercise 8.

8. $x^2 \sin x$

EXERCISES 11.10

Binomial Series

Find the first four terms of the binomial series for the functions in Exercises 1 and 9.

1. $(1 + x)^{1/2}$

9. $\left(1 + \frac{1}{x}\right)^{1/2}$

Find the binomial series for the functions in Exercise 11.

11. $(1 + x)^4$

EXERCISES 11.11

Finding Fourier Series

In Exercises 1 and 8, find the Fourier series associated with the given functions. Sketch each function.

1. $f(x) = 1 \quad 0 \leq x \leq 2\pi$

8. $f(x) = \begin{cases} 2, & 0 \leq x \leq \pi \\ -x, & \pi < x \leq 2\pi \end{cases}$