

Chapter 2

Realistic projectile motion

2.1 Frictionless motion with Newton's law

$$\frac{dv}{dt} = \frac{F}{m}$$

Newton's law

$$\frac{dE}{dt} = P$$

E total energy of the moving object, P the power supplied into the system.

For a flat course the energy is all kinetic, so $E = \frac{1}{2}mv^2$

$$\int_{v_0}^v v' dv' = \int_0^t \frac{P}{m} dt'$$

$$v = \sqrt{v_0^2 + 2Pt/m}$$

Euler method

$$\frac{dv}{dt} \approx \frac{v_{i+1} - v_i}{\Delta t} + O(\Delta t^2)$$

$$v_{i+1} = v_i + \frac{P}{mv_i} \Delta t$$

$$t_i \equiv i \Delta t$$

Given the velocity at time step i (i.e., v_i), we can calculate an *approximate* value of the velocity at the next step, v_{i+1} .

Pseudocode

- Initialisation: Set values for P , mass m , and time step Δt , and total number of time steps, N , initial velocity v_0 .
- Do the actual calculation
 - $v_{i+1} = v_i + (P/mv_i)\Delta t$
 - $t_{i+1} = t_i + \Delta t$
 - Repeat for N time steps.
- Output the result
sample code 2.1.1, 2.1.2

Adding friction to the Equation of Motion

- Frictionless motion is unrealistic as it predicts velocity shoots to infinity with time.
- Add in drag force, innocently modeled as

$$\vec{F}_{\text{drag}} \approx -B_1 \vec{v} - B_2 \vec{v}^2 .$$

The mass of air moved in time

dt is $m_{\text{air}} \sim \rho A v dt$, where ρ is the density of air and A the frontal area of the object.

kinetic energy is $E_{\text{air}} \sim m_{\text{air}} v^2 / 2$.

$$F_{\text{drag}} v dt = E_{\text{air}}$$

$$F_{\text{drag}} \approx \frac{C \rho A v^2}{2} \quad C \text{ is known as the drag coefficient} \quad B_2 \equiv C \rho A / 2$$

$C \sim 1$, depend on aerodynamics, measured experimentally

- The effect of F_{drag} : $P \rightarrow P - F_{\text{drag}} v$

$$\frac{dE}{dt} = P \longrightarrow \frac{dE}{dt} = P - F_{\text{drag}} v$$

$$v_{i+1} = v_i + \frac{P}{mv_i} \Delta t \longrightarrow v_{i+1} = v_i + \frac{(P - F_{\text{drag}} v_i)}{mv_i} \Delta t$$

$$v_{i+1} = v_i + \frac{\left(P - \frac{C\rho A v_i^2}{2} v_i \right)}{mv_i} \Delta t = v_i + \frac{P}{mv_i} \Delta t - \frac{C\rho A v_i^2}{2m} \Delta t$$

sample code [2.1.3](#)

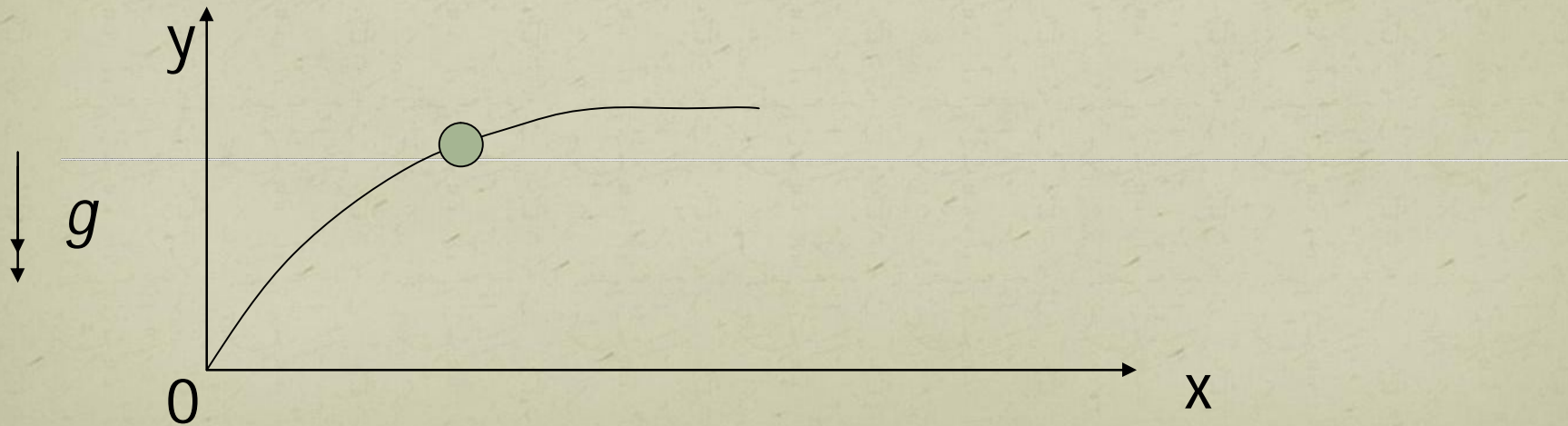
2.2 Projectile motion: The trajectory of a cannon shell

$$\frac{d^2x}{dt^2} = 0$$

$$\frac{d^2y}{dt^2} = -g$$

Two second order differential equations.

Wish to know the position (x, y) and velocity (v_x, v_y) of the projectile at time t , given initial conditions.

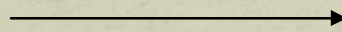


2.2 Projectile motion: The trajectory of a cannon shell

Four first order differential equations.

$$\begin{aligned}\frac{dx}{dt} &= v_x \\ \frac{dv_x}{dt} &= 0 \\ \frac{dy}{dt} &= v_y \\ \frac{dv_y}{dt} &= -g\end{aligned}$$

Euler's
method



Eq. 2.15

$$\begin{aligned}x_{i+1} &= x_i + v_{x,i} \Delta t \\ v_{x,i+1} &= v_{x,i} \\ y_{i+1} &= y_i + v_{y,i} \Delta t \\ v_{y,i+1} &= v_{y,i} - g \Delta t .\end{aligned}$$

Eq. 2.16

2.2 Projectile motion: The trajectory of a cannon shell

Drag force comes in via

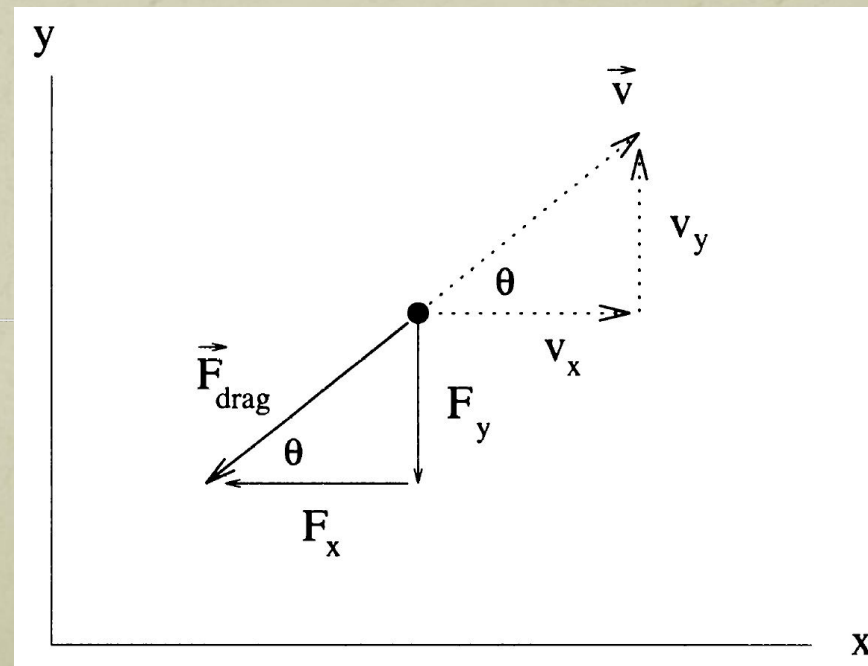
$$F_{\text{drag}} = -B_2 v^2$$

$$F_{\text{drag},x} = F_{\text{drag}} \cos \theta = F_{\text{drag}} v_x / v$$

similar for $F_{\text{drag},y}$.

$$F_{\text{drag},x} = -B_2 v v_x$$

$$F_{\text{drag},y} = -B_2 v v_y$$



Trajectory of a cannon shell with drag force

Adding this force to the equations of motion leads to

$$v_i = (v_{x,i}^2 + v_{y,i}^2)^{1/2}$$

$$x_{i+1} = x_i + v_{x,i} \Delta t$$

$$v_{x,i+1} = v_{x,i} - \frac{B_2 v_i v_{x,i}}{m} \Delta t$$

$$y_{i+1} = y_i + v_{y,i} \Delta t$$

$$v_{y,i+1} = v_{y,i} - g \Delta t - \frac{B_2 v_i v_{y,i}}{m} \Delta t .$$

sample code [2.1.4](#), [2.1.5](#)

Air density correction

- Drag force on a projectile depends on air's density, which in turn depends on the altitude.
- Two types of models for air's density dependence on altitude:
- Isothermal approximation - simple, assume constant temperature throughout, corresponds to zero heat conduction in the air.

$$\rho = \rho_0 \exp(-y/y_0) \quad y_0 = kT / mg \approx 1.0 \times 10^4 \text{ m}$$

y is the altitude,

ρ_0 is density at sea level ($y = 0$).

- Adiabatic approximation - more realistic, assume poor but non-zero thermal conductivity of air.

$$\rho = \rho_0 \left(1 - \frac{ay}{T_0} \right)^\alpha$$

$\alpha \approx 2.5$ for air; $a \approx 6.5 \times 10^{-3} \text{ K/m}$
 T_0 sea level temperature (in K)

Correction to the drag force

- The drag force w/o correction $F_{\text{drag}} = -B_2 v^2$ corresponds to the drag force at sea-level
- It has to be replaced by

$$F_{\text{drag}} = -B_2 v^2 \longrightarrow F_{\text{drag}}^* = F_{\text{drag}} \cdot \left(\frac{\rho}{\rho_0} \right)$$

$$F_{\text{drag},x}^* = F_{\text{drag}}^* \cos \theta = F_{\text{drag}}^* \frac{v_x}{v} = F_{\text{drag}} \cdot \left(\frac{\rho}{\rho_0} \right) \cdot \frac{v_x}{v} = -B_2 v v_x \cdot \left(\frac{\rho}{\rho_0} \right)$$

$$F_{\text{drag},y}^* = -B_2 v v_y \cdot \left(\frac{\rho}{\rho_0} \right)$$

Isothermal approximation:

$$\frac{\rho}{\rho_0} = \exp\left(-\frac{y}{y_0}\right)$$

$$F_{\text{drag},x}^* = -B_2 v v_x \cdot \left(\frac{\rho}{\rho_0}\right) = -B_2 v v_x \exp\left(-\frac{y}{y_0}\right); F_{\text{drag},y}^* = -B_2 v v_y \exp\left(-\frac{y}{y_0}\right)$$

$$y_0 = kT / mg \approx 1.0 \times 10^4 \text{ m}$$

Adiabatic approximation:

$$\frac{\rho}{\rho_0} = \left(1 - \frac{ay}{T_0}\right)^\alpha,$$

$$F_{\text{drag},x}^* = -B_2 v v_x \cdot \left(\frac{\rho}{\rho_0}\right) = -B_2 v v_x \left(1 - \frac{ay}{T_0}\right)^\alpha; F_{\text{drag},y}^* = -B_2 v v_y \left(1 - \frac{ay}{T_0}\right)^\alpha$$

$\alpha \approx 2.5$ for air; $a \approx 6.5 \times 10^{-3} \text{ K/m}$

T_0 sea level temperature (in K)

Trajectory of a cannon shell with drag force,
corrected for altitude dependence of air density

Adding this force to the equations of motion leads to

$$v_i = (v_{x,i}^2 + v_{y,i}^2)^{1/2}$$

$$x_{i+1} = x_i + v_{x,i} \Delta t$$

$$v_{x,i+1} = v_{x,i} - \frac{B_2 v_i v_{x,i}}{m} \Delta t \cdot \left(\frac{\rho}{\rho_0} \right)$$

$$y_{i+1} = y_i + v_{y,i} \Delta t$$

$$v_{y,i+1} = v_{y,i} - g \Delta t - \frac{B_2 v_i v_{y,i}}{m} \Delta t \cdot \left(\frac{\rho}{\rho_0} \right)$$

Curves with thermal and adiabatic correction

- Modify the existing code to produce the curves as in Figure 2.5, page 30, Giodano 2nd edition.

sample code [2.1.6, 2.17](#)
