Chapter 4

The Solar System

4.1 Kepler's Laws

Newton's law applied
to Earth-Sun system: $F_G = \frac{GM_S M_E}{r^2}$

Equation of motion (EoM)

$$F_{G,x} = -\frac{G\,M_S\,M_E}{r^2}\cos\theta = -\frac{G\,M_S\,M_E\,x}{r^3}$$

for *x* component

$$\frac{dx}{dt} = v_x \qquad \frac{dv_x}{dt} = -\frac{G\,M_S\,x}{r^3}$$

Similarly for *y*-component.

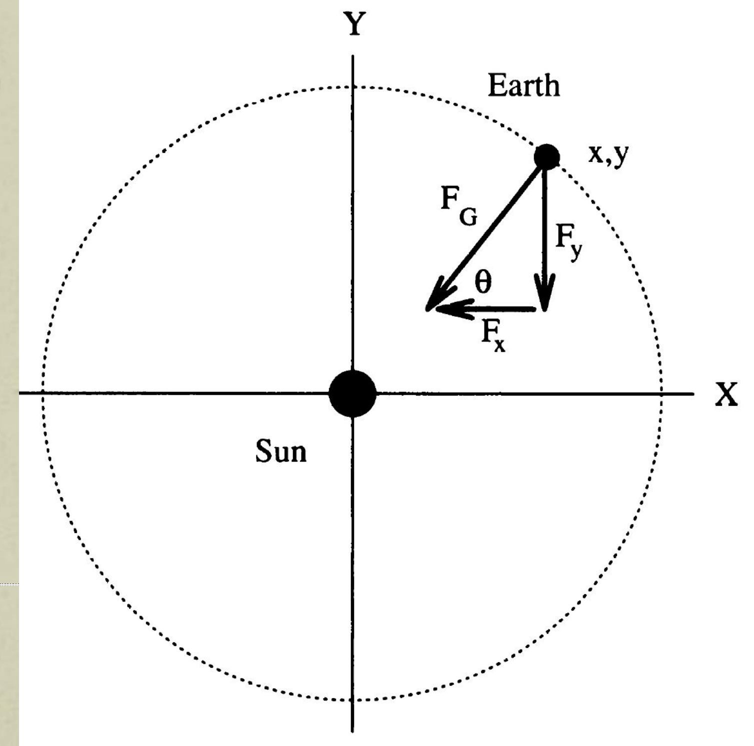


Figure 4.1: Coordinate system for describing the motion of Earth in orbit around the Sun. The Sun is at the origin and Earth is located at coordinates (x, y).

Note: this is a 2-D motion

Constants for the solar system

$R = 1.5 \times 10^{11}$ m (Earth-Sun distance, = 1 A.U.)

$M_\odot = 2.0 \times 10^{30}$ kg (Sun's mass.)

$M_E = 6.0 \times 10^{24}$ kg (Earth's mass.)

For the sake of numerical convenient, GM_S assumes the following value in AU unit

$$\frac{M_E v^2}{r} = F_G = \frac{G M_S M_E}{r^2} \longrightarrow G M_S = v^2 r = 4\pi^2 \text{ AU}^3/\text{yr}^2$$ Eq. (4.6)

Descritisation of the E.o.M (Euler-Cromer method)

Sun-Planet system is a oscillatory (periodic system)

$$\frac{dx}{dt} = v_x \qquad \frac{dv_x}{dt} = -\frac{G M_S x}{r^3} \qquad \frac{dy}{dt} = v_y \qquad \frac{dv_y}{dt} = -\frac{G M_S y}{r^3}$$

$$v_{x,i+1} = v_{x,i} - \frac{4\pi^2 x_i}{r_i^3}\Delta t$$

$$x_{i+1} = x_i + \boxed{v_{x,i+1}}\,\Delta t$$

$$v_{y,i+1} = v_{y,i} - \frac{4\pi^2 y_i}{r_i^3}\Delta t$$

$$y_{i+1} = y_i + \boxed{v_{y,i+1}}\,\Delta t\,,$$

Eq. (4.7)

Descritisation of the E.o.M (Euler-Cromer method)

Pseudocode:

- r[i]=Sqrt[x[i]^2+y[i]^2]
- vx[i+1]=vx[i]-4*(Pi^2)*x[i]*Deltat/r[i]^3,
- vy[i+1]=vy[i]-4*(Pi^2)*y[i]*Deltat/r[i]^3
- x[i+1]=x[i]+Deltat*v[i+1], y[i+1]=y[i]+Deltat*y[i+1]

Initial condition matters

Earth's orbit around the sun: $a = 1$ A.U. (assumed circular orbit)

Period of the Earth around the Sun, $T = 1$ yr

const = $T^2/a^3 = 1$, according to Newton's law.

Initial position: $(x_0, y_0) = (a, 0)$.

Initial velocity at $(a,0)$: $v_{0y} = 2\pi(a/T) = 2\pi/(a \cdot \text{const}) = 2\pi/a$, $v_{0x} = 0$.

- For Mars, e.g., $a = 1.52$ A.U., $T = (a^3/\text{const})^{1/2} = a^{3/2}$ yr (assumed circular orbit).

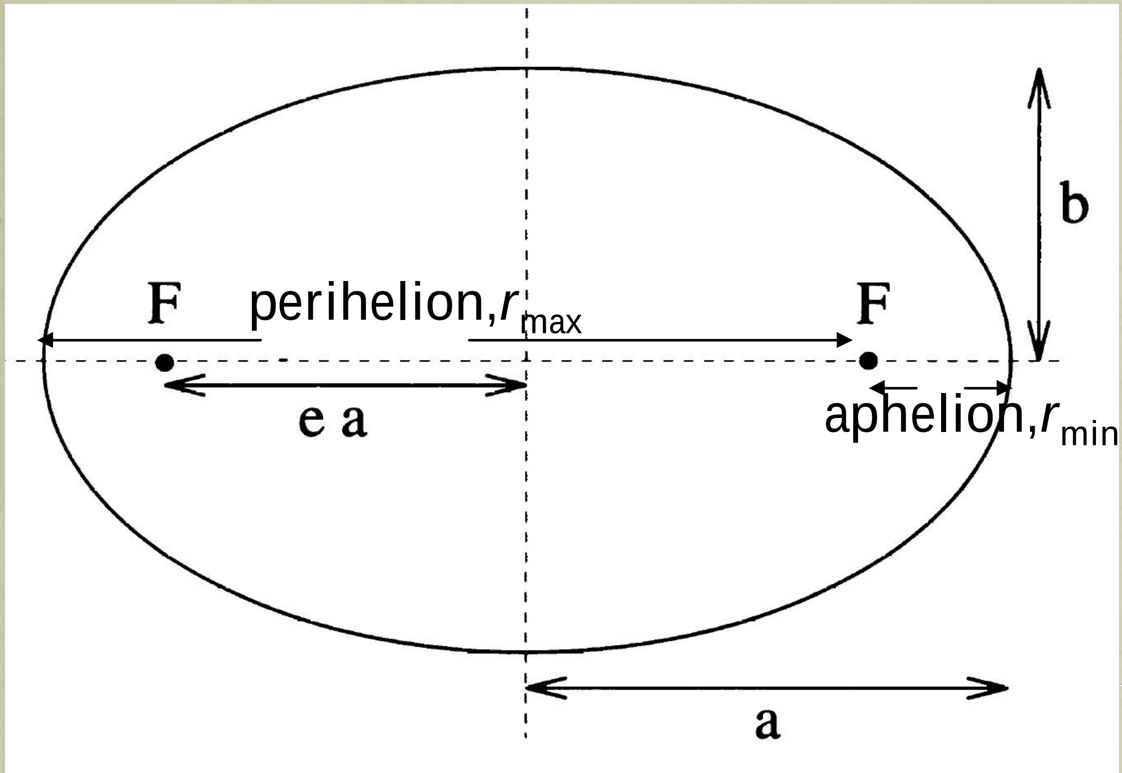
sample code: 4.1.1, 4.1.2

Initial condition deviated from the "perfect" one

If the initial condition at the initial position (a,0) deviates from $v_{0y} = 2\pi/a$, $v_{0x} = 0$, the orbits will become non-circular (elliptical, of non zero eccentricity).

This corresponds to a more generic situation, hence most planets around the Sun are not circular.

Elliptical orbit



sample code: 4.1.3

$$b = a\sqrt{1 - e^2}, \quad r_{\max} = a(1+e); r_{\min} = a(1-e);$$

Figure 4.3: Hypothetical elliptical orbit. A sun lies at one of the foci of the ellipse; both foci are labeled F. The semimajor and semiminor axes are a and b, and the eccentricity is e. This drawing is not to scale.

Initial condition deviated from the "perfect" one

The planet system evolves according to Newton's law (which is in turn simulated via the code).

You can use the simulation to "measure" the value of T^2/a^3 of a planet with an elliptical orbit.

According to Keppler's law, for any planet,

$T^2/a^3 = 1$ yr^2/A.U^3.

“Measuring” period and semimajor

- Now you have a code that gives you the locus of all values {$x(t)$,$y(t)$} from $t = 0$ till $t=t_{final}$.
- How do ask Mathematica to find the period and semimajor?
- Table[{x(t),y(t)},{i,0,tfinal}] is a “list”.

Hint: You may have to learn to abstract information from the list.

Abstract info from List

```
Print[
 Table[txyMrs[[i]], {i, 1, 20}]
]; (*end print*)
```

Sample code: *4.1.4.*

```
{{1.52, 0.}, {1.52197, 0.00509634}, {1.52387, 0.0101924}, {1.52571, 0.0152881},
 {1.52747, 0.0203831}, {1.52917, 0.0254771}, {1.5308, 0.0305701}, {1.53237, 0.0356617},
 {1.53386, 0.0407517}, {1.53529, 0.04584}, {1.53665, 0.0509262}, {1.53795, 0.0560103},
 {1.53918, 0.0610919}, {1.54034, 0.0661709}, {1.54144, 0.071247}, {1.54247, 0.0763201},
 {1.54343, 0.0813899}, {1.54433, 0.0864562}, {1.54516, 0.0915188}, {1.54593, 0.0965775}}
```

```
Print[txyMrs[[1]], " ", txyMrs[[1, 1]], " ", txyMrs[[1, 2]]];
Print[txyMrs[[5]], " ", txyMrs[[5, 1]], " ", txyMrs[[5, 2]]];
```

```
{1.52, 0.} 1.52 0.
{1.52747, 0.0203831} 1.52747 0.0203831
```

"Brute force" way to measure semimajor

- Search from the list txyMrs two points that are separated furthest.
- deltar[i,j]=Sqrt[(txyMrs[[i,1]] - txyMrs[[j,1]])^2 + (txyMrs[[i,2]] - txyMrs[[j,2]])^2] gives distance between point [x(i),y(i)] and [x(j),y(j)].
- Search for all values of (i,j) such that deltar[i,j] is maximum. Can be done by writing a double-loop with index i and j. The largest deltar[i,j] is to 2 × semimajor.
- This is a brute force way of doing the work … (please think of a more intelligent way)

Sample code: *4.1.5, 4.1.6*

How to “measure” period?

- Search from the list txyMrs for two points that overlaps for the first time.
- The index $i=i_T$ when this happens tell us the period,
- $T = i_T \cdot \Delta t$. Use the code to “measure” T yourself.
- Once perihelion is known, the aphelion can be calculated using geometry (semimajor and seminor are orthogonal to each other).
- Eccentricity e and the location of both loci F of the ellipse can also be calculated.

Verifying Keppler's third law

- Now you are ready to write a code to attack assignment 4, where you are asked to "measure" the semimajor, semiminor, eccentricity, foci, perihelion, aphelion and period of a planet.
- Then you can also verify if Keppler's law is obeyed.