particle enters the hole, it leaves the container. Calculate the number of particles in the container as a function of time. Show that this number, which is proportional to the partial pressure of the cream particles varies as $\exp(-t/\tau)$, where τ is the effective time constant for the escape. *Hint:* Reasonable parameter choices are a 50×50 container lattice and a hole 10 units in length along one of the edges.

7.16. Carry out an analysis of the entropy for the nonlinear damped pendulum studied in Chapter 3. Consider the behavior of $\theta(t)$ and divide the possible range for θ into a number of cells (try 100). Simulate the pendulum and calculate a histogram of the number of times the pendulum angle falls into a cell as a function of θ; sample $\theta(t)$ in synchrony with the drive force, as we did in calculating the Poincaré sections. Calculate the entropy using (7.24) as a function of the driving force. You should find that S is small in the periodic regime and large when the pendulum is chaotic. What is S in the period-2 and period-4 regimes?

7.6 CLUSTER GROWTH MODELS

We have spent a good deal of time in this chapter exploring random walks and their connection with diffusion and the approach to equilibrium. Another interesting random process, which turns out to be closely related to random walks, concerns the growth of clusters, such as snowflakes and soot particles. In this section we will examine two different models of cluster growth. The first is known as the Eden model and operates according to the following rules. Consider a two dimensional lattice of points (x, y), where x and y are both integers. These are the allowed locations for the particles that will make up the cluster. We begin by placing a seed particle at the origin ($x = 0, y = 0$); this is our initial cluster. A cluster grows by the addition of particles to its perimeter. Our initial cluster has nearest-neighbor points on the lattice at $(\pm 1, 0)$ and $(0, \pm 1)$. We will refer to such still unoccupied sites that are nearest neighbors of occupied sites as the perimeter sites of the cluster. We next choose one of these perimeter sites at random and place a particle at the chosen location. The cluster now contains two particles and a correspondingly larger perimeter. This process is then repeated; a perimeter site is chosen at random (i.e., all perimeter sites have the same probability of being chosen), and a particle added at that location. We continue this process until a cluster of the desired size is obtained. This is the Eden model of cluster growth.

A typical Eden cluster is shown in Figure 7.17. While it is a little rough around the edges, it is basically a circular disc with a few holes. Note that as the cluster grows these holes tend to fill in, since they are treated on the same footing (they are equally likely to be occupied by the next particle) as the exterior perimeter sites.

The Eden model is sometimes referred to as a "cancer" model, because the clusters grow from within by expanding their borders. However, not all clusters in nature grow in this manner. For example, snowflakes and soot particles grow by the addition of new particles that originate from outside the cluster.[23] This process is captured by a different cluster model, which is known as diffusion-limited aggregation, or DLA.

[23] More precisely, the places where new particles are added depend on processes that take place outside the cluster.

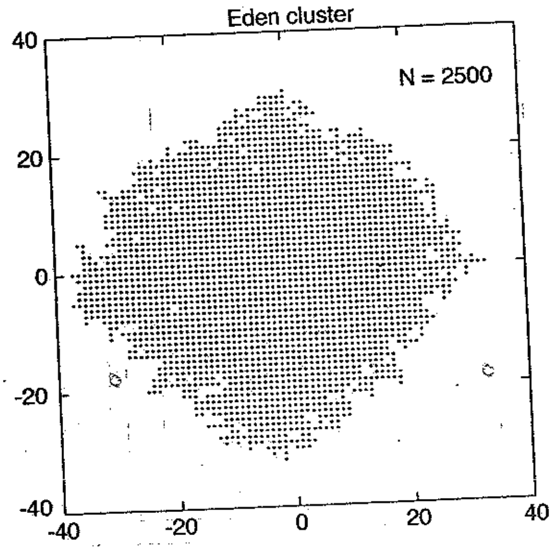

FIGURE 7.17: Eden cluster containing 2500 particles. Note that there are very few "holes" inside the main body of the cluster.

The growth rules for DLA clusters are as follows. We again start with a seed particle at the origin. We then release a particle at a randomly chosen location (x, y) that is some distance away from the seed and let it perform a random walk. If (or when) this walker lands on a perimeter site, it sticks there and becomes part of the cluster.[24] This process is repeated with many walkers until a large cluster is grown. One way to motivate (or justify!) the choice of these growth rules is to consider how a large particle might be built up from smaller particles or molecules in a solution. If the cluster is located well away from any other objects, such as walls or other clusters, small particles will approach it from all directions. In addition, it seems reasonable to assume that small particles will move diffusively as they travel through the solution around the cluster. This process is captured in the DLA growth rules since, as we have already seen, diffusion is equivalent to a random walk. Of course, we can imagine situations in which these rules would not be appropriate. For example, there might be some localized source of the small particles, or perhaps a prevailing current, that would give an overall drift velocity in addition to the random walk. These are perfectly reasonable models and each could be interesting depending in part on possible connections to real systems.

A cluster grown using the DLA rules is shown in Figure 7.18 (we will consider the programming associated with generating such clusters in the exercises). Comparing our DLA cluster with the Eden cluster Figure 7.17, it is obvious that

[24] We may also let the particle stick permanently at a perimeter site with some specified probability that is less than unity (or only after a certain number of visits to that site). Such a model may better describe a situation where the sticking process represents e.g., a chemical reaction, which occurs probabilistically. A problem of this type is explored in the exercises.

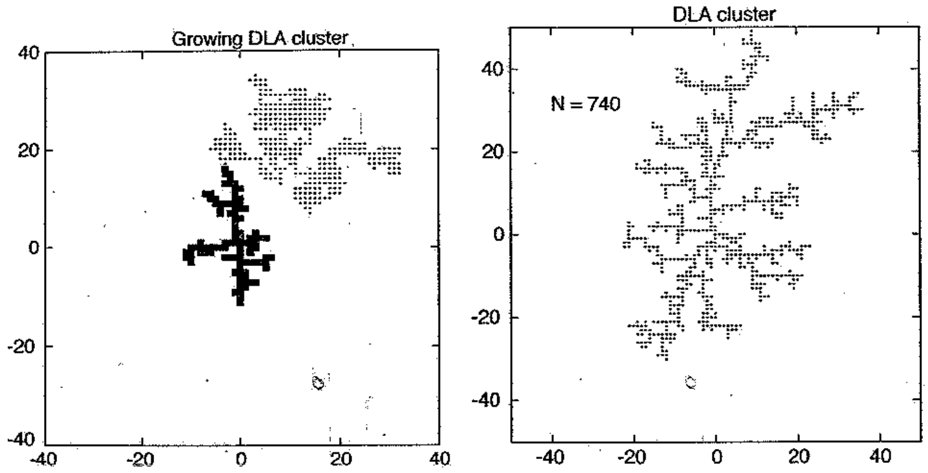

FIGURE 7.18: Left: growing DLA cluster. The filled squares are sites in the cluster and the dots are the lattice sites visited by a particular random walker as it approached. This walker eventually touched the perimeter at the top edge of the cluster and became attached there. Right: DLA cluster containing 740 particles.

they have very different properties. The Eden cluster is, as we have already noted, essentially a solid disk with very few holes and a fairly smooth perimeter. In contrast, the DLA cluster contains many large open spaces and the perimeter is very irregular. These differences are directly connected with the growth rules. For the Eden clusters all perimeter sites, even the interior ones, are equally likely to be filled by the next particle. This tends to fill in any holes or cracks, since those were likely formed long before the outermost parts of the cluster. For DLA it is extremely unlikely that such crevices will be filled in, as the probability that a random walker will manage to navigate past the outermost parts of the cluster on its way deep into a crack is very low. A walker is much more likely to first make contact with the outer edges of the cluster.

This intuitive explanation of the difference between Eden and DLA clusters is useful, but we would like to have a quantitative measure of this difference. This brings us to consider objects that are known as *fractals*, which will be our primary topic for the remainder of this section and the next one as well. Rather than try to give a very general definition of what it means to be a fractal, we will instead introduce a few terms and concepts associated with these objects. A definition will gradually emerge as we proceed.

Let us consider how we might measure the dimensionality of an object. At first this may seem like a silly exercise. Your intuition tells you that straight lines are one-dimensional objects, flat disks are two dimensional, etc. But what about a piece of spaghetti, or a string that is tangled, or a piece of crumpled paper? While your intuition probably would still feel comfortable in these cases, it is instructive to construct an *operational* definition for dimensionality. There are several ways to

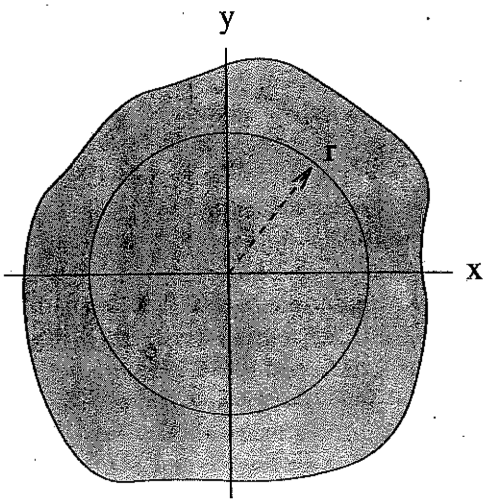

FIGURE 7.19: Method for calculating the effective dimensionality of a cluster, shown as the shaded region. $m(r)$ is the mass contained within a circle of radius r.

approach this problem. In the next section we will discuss simple curves and other objects that are close to being one dimensional. Here we will consider the problem for our Eden and DLA clusters.

Suppose we have a large disk of uniform density that lies in the x–y plane, as illustrated in Figure 7.19. If we consider the mass of the disk that is contained within a circle of radius r, it is easy to see that this is given by

$$m(r) = \sigma \pi r^2 , \tag{7.25}$$

where σ is the mass per unit area and r is small enough that the test circle is entirely contained within the disk. The key point is that the mass scales as r^2, and this 2 is also the dimensionality of the object. If we instead had a straight line or a similar type of curve, the mass would be

$$m(r) = 2 \lambda r , \tag{7.26}$$

where λ is the mass per unit length and r is again small enough that the circle does not go beyond the ends of the line. The mass now scales as r^1, and 1 is again the spatial dimensionality of the object.[25]

These observations form the basis of an operational definition that we can use to calculate the effective dimensionality of a cluster. Our definition is

$$m(r) \sim r^{d_f} , \tag{7.27}$$

[25]Strictly speaking, objects must be of infinite size for the dimensionality obtained in this way to be correct to all scales of r. In that sense, any finite object becomes zero-dimensional at a sufficiently large scale of r.

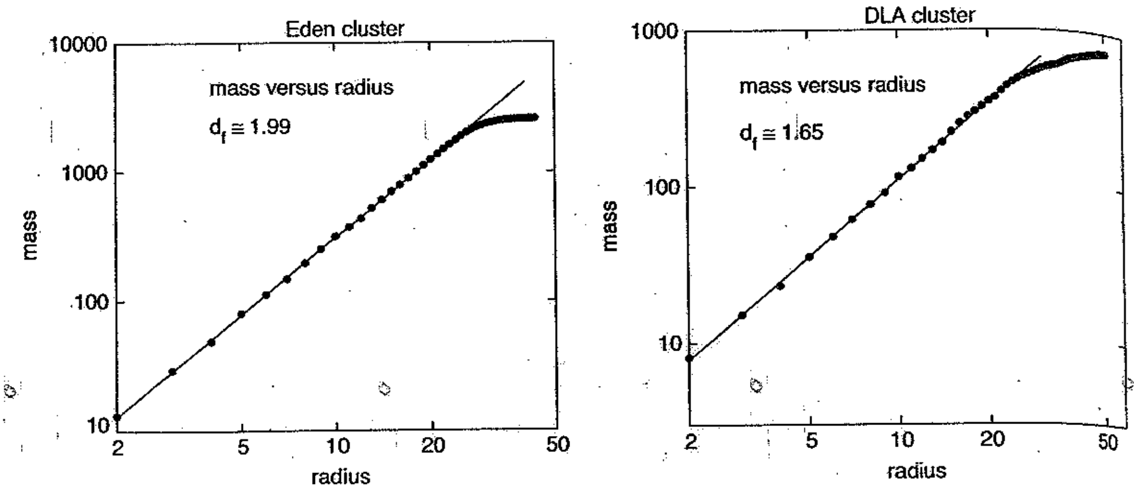

FIGURE 7.20: Left: plot of $\log m$ versus $\log r$ for the Eden cluster in Figure 7.17. The solid line is a least-squares fit whose slope is the fractal dimensionality, which for this cluster was $d_f \approx 1.99$; it was equal to 2 to within the statistical uncertainties. Right: plot of $\log m$ versus $\log r$ for the DLA cluster in Figure 7.18. The solid line is again a least-squares fit whose slope is the fractal dimensionality, which for this cluster was $d_f \approx 1.65$.

where d_f is the effective or *fractal* dimensionality of the object. We have already seen cases that yield $d_f = 1$ (a simple line or curve), and $d_f = 2$ (a solid disk); a solid sphere would be described by $d_f = 3$. It remains for us to devise objects for which d_f is not an integer.

To apply this definition to one of our clusters, we need to calculate the mass inside a circle of radius r, which is centered inside the cluster. For convenience we choose the position of the initial seed particle as the center and call this point the origin.[26] Assuming that all of the particles have the same mass, we can find $m(r)$ by counting the number of particles within a distance r of the origin. Values for the mass as a function of r for our Eden and DLA clusters are shown in Figure 7.20, where we have plotted the results on logarithmic scales. Such plots are useful, since taking the logarithm of both sides of (7.27) yields

$$\log m \sim d_f \log r \,, \tag{7.28}$$

so the slope of a log-log plot is equal to the fractal dimensionality.

For both types of clusters the results for $\log m$ versus $\log r$ are consistent with a straight line and thus with the relation (7.28) for small r. However, the curves flatten out for large r. This is due to the finite size of the clusters. For very large

[26]Strictly speaking, the definition (7.27) should be applied with many test circles, with centers chosen at all possible locations within the cluster. The notion and value of an effective dimensionality should *not* depend on the choice of a specific "center." However, for simplicity in our numerical calculations, we will use the seed particle location as the center. The interested (or skeptical) reader is encouraged to calculate the dimensionality using other choices.

measuring circles (see Figure 7.19) the entire cluster will be inside the circle, and in this case $m(r)$ will be independent of r. For the same reason, when r is only a little less than the maximum "radius" of the cluster, $r_{\max}$, $m(r)$ will be suppressed below its value for the ideal case; that is, for an extremely large (infinite) cluster. In practice, a particular cluster can only be used to estimate $m(r)$ for distances up to about $r_{\max}/2$.

The solid lines in Figure 7.20 are least-squares fits of (7.28) to the results for $m(r)$ out to $r_{\max}/2$. The slopes of these lines are the fractal dimensionalities, and we find $d_f \approx 1.99$ for the Eden cluster and 1.65 for the DLA cluster. To within the statistical errors the Eden cluster has a dimensionality of 2. This is in accord with our intuition; the Eden cluster is essentially just a solid disk. However, the DLA cluster has a fractal dimensionality much less than 2 (and also much greater than 1). Indeed, this is why it is known as a fractal.

In order for a cluster to have an effective dimensionality d_f, which is not an integer, its mass must increase more slowly[27] than r^2. This means that it must contain holes or cracks, as we have observed in the DLA clusters. However, simply containing such open spaces is not enough. A planar object that has a certain, constant fraction of open space would still have $d_f = 2$. In order to have $d_f < 2$, the sizes of these open spaces must *increase* with r. Evidently, DLA clusters have just this property.

EXERCISES

7.17. Write a program to generate DLA clusters and calculate their fractal dimensionality. Here are a few programming suggestions for this problem. We have already seen that random walkers can take a long time to move an appreciable distance and this can make the generation of a DLA cluster very slow if you are not careful. Initially, start new walkers a distance r_{start} away from the origin, by choosing the initial position of a walker at random on a circle of radius r_{start} (but make sure that they are on the lattice).[28] If the walker wanders too far from the cluster, say farther than $1.5 \times r_{\text{start}}$, it may never hit the cluster, so a new walker should be started. As the cluster grows, r_{start} should be increased so that the walkers don't begin too close to the cluster. Try keeping r_{start} at least 5 units larger than the maximum cluster size (that is, the point on the cluster which is farthest from the origin). Also, when the walker is far from the cluster you can let it take steps of length 2 (to speed up the walk), then decrease the step length as it approaches the cluster.

7.18. Grow a DLA cluster using the algorithm described in the previous problem, but instead of letting the walkers start from points on a circle that surrounds the cluster, have all of the walkers begin at a location on the x axis. How does this affect the shape and structure of the cluster?

7.19. Generate a DLA structure using an initial "seed," which is the entire x axis. That is, begin with all of the sites on the x axis occupied and let the walkers begin some distance above this axis. The resulting structure is sometimes used

[27] We assume here that the cluster is grown on a planar lattice and not a three-dimensional one.

[28] Be sure that you pick the initial location of the walker at random from possible locations on the circle. This is most easily done by choosing an angle at random in the range 0–2π and using it to specify the starting point of the walker.

to model the paths followed by electric discharges in a gas (that is, lightening bolts).

***7.20.** Repeat the previous problem, but allow your random walkers to move on a *three-dimensional* lattice. You should find a value of $d_f \sim 2.5$ in this case.

7.21. Generate a DLA cluster as above but let the walker stick to the cluster only with a probability $p < 1$, say, $p = 0.1, 0.3, 0.6$, etc. Describe any trends that you find in the fractal structures as p is varied.

7.22. Generate a DLA cluster using walkers that perform a biased random walk. That is, let your walkers have a higher probability for walking in one particular direction (along the $+x$ direction, for example) than in other directions. This is a biased random walk, as we have considered in an earlier exercise. Study how both d_f and the overall shape of the cluster depend on the magnitude of this drift velocity.

***7.23.** An interesting variation on DLA is to begin with a lattice in which some fraction of sites are occupied with particles, and then let the cluster diffuse and pick up particles as it makes contact with them. Use a square lattice and place particles on sites at random with some probability ($p = 0.1$ is a good choice). Let the cluster perform a random walk and whenever a perimeter site is occupied by a particle, that particle then becomes part of the cluster. Generate clusters in this way and calculate their fractal dimensionality. You should find $d_f \sim 1.7$, which is about the same as a DLA cluster. Interestingly, d_f for this cluster-diffusion model seems to vary with p. Calculate d_f for other values of p and show that d_f becomes larger (it should approach ~ 1.95) for large values of p. This calculation was first performed by Voss (see Voss [1984]).

7.7 FRACTAL DIMENSIONALITIES OF CURVES

While DLA clusters may be nice to look at, we must still ask what it is about fractals that makes them interesting from a *physics* point of view. We will discuss this question in due course, but it is useful to first to consider another problem concerning fractal objects. It is convenient to introduce this problem using a class of regular fractals that are known as Koch curves. In contrast to the fractal clusters grown using the DLA model, Koch curves are generated by *deterministic* rules. While such regular fractals do not have a direct connection with physics, they are useful for learning more about fractals, as we will now see.

Perhaps the simplest way to define a Koch curve is through the examples in Figure 7.21, which shows a family of such curves. The first member of the family is shown at the bottom and is just a straight line of length L; we will refer to this as a Koch curve of order one. The second-order Koch curve (the second curve from the bottom) is derived from the first-order curve by replacing the straight section with four segments of length $L/3$, oriented with respect to the original (first-order section) as shown. The third-order curve is obtained from the second-order curve by replacing *each* of its straight sections by four more segments, with lengths $L/9$. The fourth and higher-order curves are obtained in an analogous manner. This procedure can be used to obtain curves of arbitrary order.

The Koch curves are thus defined *recursively*. A member of the series is generated from the preceding member by replacing each of its straight sections by four new segments, as described above. We can clearly generate other types of