Chapter 2

Realistic projectile motion

2.1 Frictionless motion with Newton's law

$$\frac{dv}{dt} = \frac{F}{m}$$

Newton's law

$$\frac{dE}{dt} = P$$

E total energy of the moving object, *P* the power supplied into the system.

For a flat course the energy is all kinetic, so $E = \frac{1}{2}mv^2$.

$$\int_{v_0}^{v} v'\,dv' = \int_0^t \frac{P}{m}\,dt'$$

$$v = \sqrt{v_0^2 + 2Pt/m}$$

Euler method

$$\frac{dv}{dt} \approx \frac{v_{i+1} - v_i}{\Delta t} + O(\Delta t^2)$$

$$v_{i+1} = v_i + \frac{P}{m v_i} \Delta t$$

$$t_i \equiv i \, \Delta t$$

Given the velocity at time step i (i.e., v_i), we can calculate an *approximate* value of the velocity at the next step, v_{i+1}.

Pseudocode

- Initialisation: Set values for *P*, mass *m*, and time step Δt, and total number of time steps, *N*, initial velocity v_0.

- Do the actual calculation
  - $v_{i+1} = v_i + (P/mv_i)\Delta t$
  - $t_{i+1} = t_i + \Delta t$
  - Repeat for *N* time steps.

- Output the result

Adding friction to the Equation of Motion

- Frictionless motion is unrealistic as it predicts velocity shoots to infinity with time.
- Add in drag force, innocently modeled as

$$\vec{F}_{\text{drag}} \approx -B_1 \vec{v} - B_2 \vec{v}^2 .$$

The mass of air moved in time

dt is $m_{\text{air}} \sim \rho A v dt$, where ρ is the density of air and A the frontal area of the object.

kinetic energy is $E_{\text{air}} \sim m_{\text{air}} v^2/2$.

$$F_{\text{drag}} v dt = E_{\text{air}}$$

$$F_{\text{drag}} \approx \frac{C\rho A v^2}{2}$$

C is known as the drag coefficient $B_2 \equiv C\rho A/2$

C ~ 1, depend on aerodynamics, measured experimentally

- The effect of F_{drag} is to modify $P \rightarrow P - F_{drag}\, v$

$$\frac{dE}{dt} = P \longrightarrow \frac{dE}{dt} = P - F_{drag} v$$

$$v_{i+1} = v_i + \frac{P}{m v_i}\Delta t \longrightarrow v_{i+1} = v_i + \frac{\left(P - F_{drag} v_i\right)}{m v_i}\Delta t$$

$$v_{i+1} = v_i + \frac{\left(P - \frac{C\rho A v_i^2}{2} v_i\right)}{m v_i}\Delta t = v_i + \frac{P}{m v_i}\Delta t - \frac{C\rho A v_i^2}{2m}\Delta t$$

1D free fall with drag force

- Develop a code that shows the variation of velocity with time for a 1D object, which is constantly being pumped in energy at a given rate fix rate, *P*, and an non-zero initial velocity v_0.
- For a object undergoing 1D free fall motion along the vertical direction,

$$P = -dU/dt = -d/dt(mgy) = -mgv_y$$

So that

$$v_{y,i+1} = v_i + \frac{P}{mv_{y,i}}\Delta t - \frac{C\rho A v_{y,i}^2}{2m}\Delta t$$

$$= v_{y,i} - \frac{mgv_{y,i}}{mv_{y,i}}\Delta t - \frac{C\rho A v_{y,i}^2}{2m}\Delta t$$

$$= v_{y,i} - g\Delta t - \frac{C\rho A v_{y,i}^2}{2m}\Delta t$$

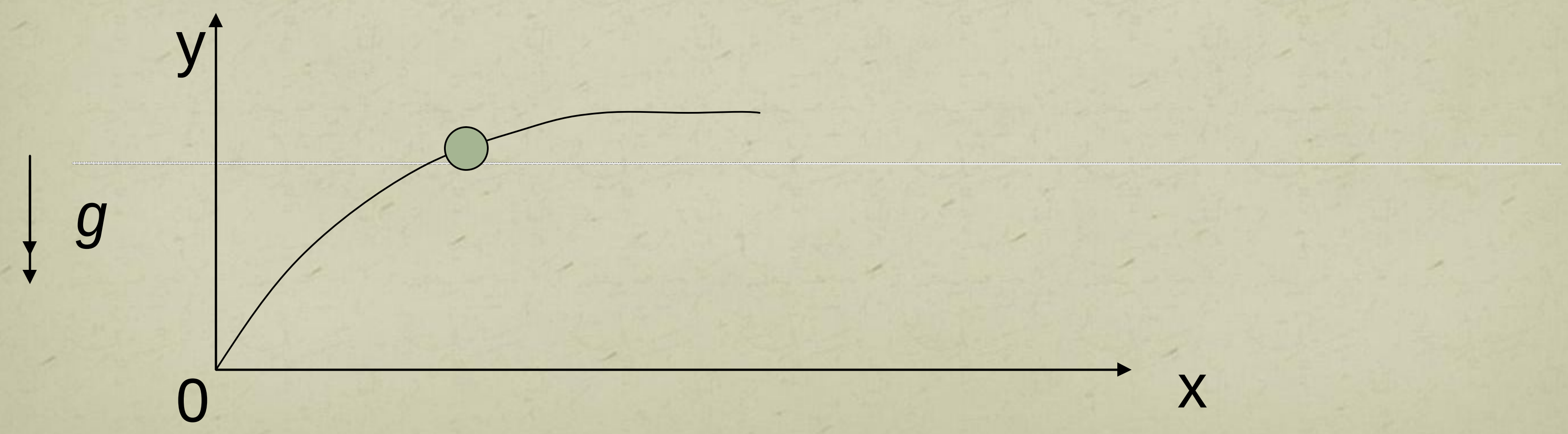
2.2 Projectile motion: The trajectory of a cannon shell

$$\frac{d^2x}{dt^2} = 0$$

$$\frac{d^2y}{dt^2} = -g$$

Two second order differential equations.

Wish to know the position (x,y) and velocity (v_x,v_y) of the projectile at time t, given initial conditions.



2.2 Projectile motion: The trajectory of a cannon shell

Four first order differential equations.

$$\frac{dx}{dt} = v_x$$
$$\frac{dv_x}{dt} = 0$$
$$\frac{dy}{dt} = v_y$$
$$\frac{dv_y}{dt} = -g$$

Eq. 2.15

Euler's method →

$$x_{i+1} = x_i + v_{x,i}\,\Delta t$$
$$v_{x,i+1} = v_{x,i}$$
$$y_{i+1} = y_i + v_{y,i}\,\Delta t$$
$$v_{y,i+1} = v_{y,i} - g\,\Delta t\,.$$

Eq. 2.16

2.2 Projectile motion: The trajectory of a cannon shell

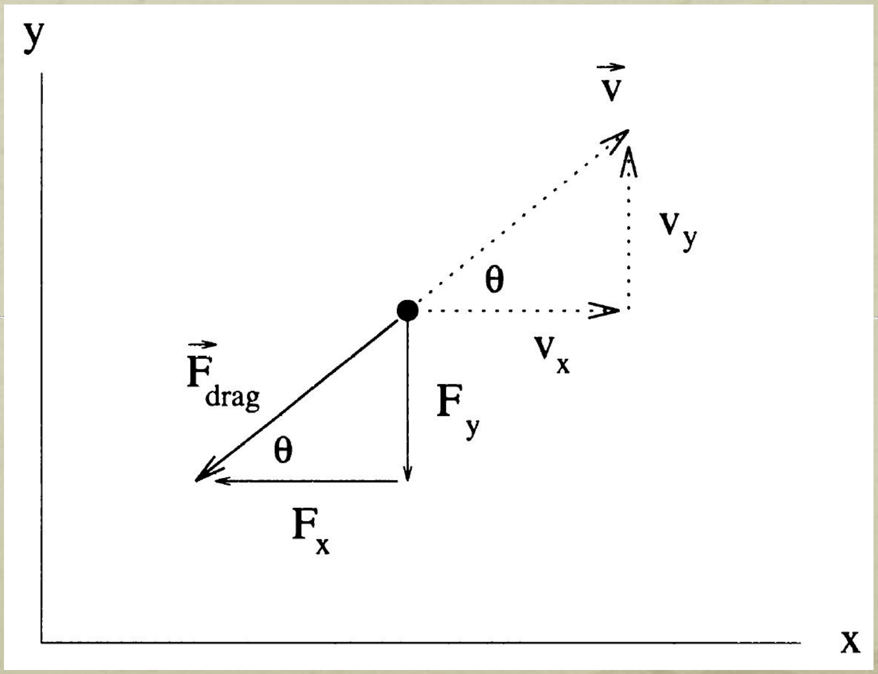
Drag force comes in via

$$F_{drag} = -B_2 v^2$$

$$F_{drag,x} = F_{drag} \cos\theta = F_{drag} v_x / v ,$$

similar for $F_{drag,y}$.

$$F_{drag,x} = -B_2 v v_x$$
$$F_{drag,y} = -B_2 v v_y$$



Trajectory of a cannon shell with drag force

Adding this force to the equations of motion leads to

$$v_i = (v_{x,i}^2 + v_{y,i}^2)^{1/2}$$

$$x_{i+1} = x_i + v_{x,i}\,\Delta t$$

$$v_{x,i+1} = v_{x,i} - \frac{B_2\, v_i v_{x,i}}{m}\,\Delta t$$

$$y_{i+1} = y_i + v_{y,i}\,\Delta t$$

$$v_{y,i+1} = v_{y,i} - g\,\Delta t - \frac{B_2\, v_i v_{y,i}}{m}\,\Delta t\,.$$

Air density correction

- Drag force on a projectile depends on air's density, which in turn depends on the altitude.
- Two types of models for air's density dependence on altitude:
- Isothermal approximation - simple, assume constant temperature throughout, corresponds to zero heat conduction in the air. S

$$\rho = \rho_0 \exp(-y/y_0) \quad y_0 = kT/mg \approx 1.0\times10^4\,\mathrm{m}$$

y is the altitude, m mass of air's molecule ρ_0 is density at sea level ($y = 0$).

•Adiabatic approximation - more realistic, assume poor but non-zero thermal conductivity of air.

$$\rho = \rho_0\left(1 - \frac{ay}{T_0}\right)^{\alpha}$$

$\alpha \approx 2.5$ for air; $a \approx 6.5\times10^{-3}\,\mathrm{K/m}$

T_0 sea level temperature (in K)

Correction to the drag force

- The drag force w/o correction $F_{\text{drag}} = -B_2 v^2$ corresponds to the drag force at sea-level, with $B_2 = -\frac{1}{2} C \rho_0 A$
- For general altitude, it has to be modified:

$$F_{\text{drag}} = -B_2 v^2 = -\frac{1}{2} C \rho_0 A v^2$$

$$F^*_{\text{drag}} = -\frac{1}{2} C \rho A v^2 = -\frac{1}{2} C \rho_0 A v^2 \frac{\rho}{\rho_0} = -\frac{\rho}{\rho_0} B_2 v^2$$

$$F^*_{\text{drag},x} = F^*_{\text{drag}} \cos\theta = F^*_{\text{drag}} \frac{v_x}{v} = F_{\text{drag}} \cdot \left(\frac{\rho}{\rho_0}\right) \cdot \frac{v_x}{v} = -B_2 v v_x \cdot \left(\frac{\rho}{\rho_0}\right)$$

$$F^*_{\text{drag},y} = -B_2 v v_y \cdot \left(\frac{\rho}{\rho_0}\right)$$

Isothermal approximation:

$$\frac{\rho}{\rho_0} = \exp\left(-\frac{y}{y_0}\right)$$

$$F^*_{\text{drag},x} = -B_2 v v_x \cdot \left(\frac{\rho}{\rho_0}\right) = -B_2 v v_x \exp\left(-\frac{y}{y_0}\right); F^*_{\text{drag},y} = -B_2 v v_y \exp\left(-\frac{y}{y_0}\right)$$

$$y_0 = kT / mg \approx 1.0 \times 10^4 \text{ m}$$

Adiabatic approximation:

$$\frac{\rho}{\rho_0} = \left(1 - \frac{ay}{T_0}\right)^{\alpha},$$

$$F^*_{\text{drag},x} = -B_2 v v_x \cdot \left(\frac{\rho}{\rho_0}\right) = -B_2 v v_x \left(1 - \frac{ay}{T_0}\right)^{\alpha}; F^*_{\text{drag},y} = -B_2 v v_y \left(1 - \frac{ay}{T_0}\right)^{\alpha}$$

$$\boldsymbol{a} \approx 6.5 \times 10^{-3} \text{ K/m}$$

$\alpha \approx 2.5$ for air;

T_0 sea level temperature (in K)

Trajectory of a cannon shell with drag force, corrected for altitude dependence of air density

Adding this force to the equations of motion leads to

$$v_i = (v_{x,i}^2 + v_{y,i}^2)^{1/2}$$

$$x_{i+1} = x_i + v_{x,i}\,\Delta t$$

$$v_{x,i+1} = v_{x,i} - \frac{B_2\, v_i v_{x,i}}{m}\,\Delta t \cdot \left(\frac{\rho}{\rho_0}\right)$$

$$y_{i+1} = y_i + v_{y,i}\,\Delta t$$

$$v_{y,i+1} = v_{y,i} - g\,\Delta t - \frac{B_2\, v_i v_{y,i}}{m}\,\Delta t \cdot \left(\frac{\rho}{\rho_0}\right)$$

Curves with thermal and adiabatic correction

- Modify the existing code to produce the curves as in Figure 2.5, page 30, Giordano 2nd edition.