Chapter 3

Simple Harmonic Motion

3.1 Simple Harmonic Motion

Force on the pendulum

$$F_\theta = -m\, g\, \sin\theta$$

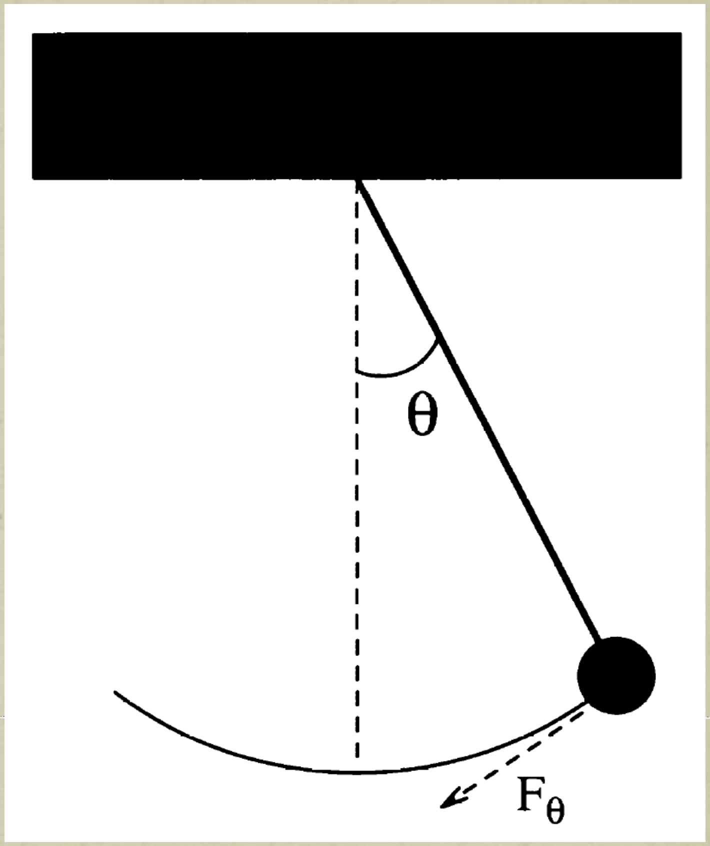
Equation of motion (EoM)

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta \qquad \sin\theta \approx \theta$$

Analytical solution:

$$\theta = \theta_0 \sin(\Omega t + \phi)$$

$\Omega = \sqrt{g/\ell}$, and θ_0 and ϕ are constants determined by initial conditions.



The period of the SHO is given by $T = 2\pi\sqrt{\frac{l}{g}}$

Euler method to solve the EoM

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta$$

↓

$$\frac{d\omega}{dt} = -\frac{g}{\ell}\theta\,,$$

$$\frac{d\theta}{dt} = \omega\,,$$

Run t for a few period (T) to see the oscillatory behavior

Set Δt, θ_0, ω_0, l, g, iterate

$$\omega_{i+1} = \omega_i - \frac{g}{\ell}\theta_i\,\Delta t\,,$$

$$\theta_{i+1} = \theta_i + \omega_i\,\Delta t\,.$$

$t_{i+1} = t_i + \Delta t$

sample code 3.1.1

Total energy of SHO

The total energy of the SHO in can be calculated as

$$E_{i+1} = K_{i+1} + U_{i+1} = \frac{1}{2} m \left(l\omega_{i+1} \right)^2 + mgl \left(1 - \cos\theta_{i+1} \right)$$

$$\approx \frac{1}{2} m l^2 \omega_{i+1}{}^2 + mgl \left[1 - \left(1 - \frac{\theta_{i+1}^2}{2} \right) \right]$$

$$= \frac{1}{2} m l^2 \omega_{i+1}{}^2 + \frac{1}{2} mgl\theta_{i+1}^2$$

- You may use the Mathematica command Series[Cos[x],{x,0,2}] to obtain the Taylor series expansion of cosθ at $\theta = 0$. This is amount to making the assumption that $\theta \rightarrow 0$.

Euler method fails in this case

- The oscillation amplitude increases with time → Euler method is not suitable for oscillatory motion (but may be good for projectile motion which is non-periodic)
- The peculiar effect of the numerical solution can also be investigated from the energy of the oscillator when Euler method is applied:
- $E \approx (1/2) m\, l^2 [\omega^2 + (g/l)\theta^2]$ (with $\theta \rightarrow 0$)
- Descretise the equation by applying Euler method,
- $E_{i+1} = E_i + (1/2) mgl^2 [\omega_i^2 + (g/l)\theta_i^2](\Delta t)^2$
- → E increases with i → this is an undesirable result.

Euler vs. Euler-Cromer method

- In the Euler method:
- ω_{i+1} is calculated based on previous values ω_i and θ_i
- So is θ_{i+1} is calculated based on previous values ω_i and θ_i

$$\omega_{i+1} = \omega_i - \frac{g}{\ell}\theta_i \Delta t ,$$

$$\theta_{i+1} = \theta_i + \omega_i \Delta t .$$

- In the Euler-Cromer method:
- ω_{i+1} is calculated based on previous values ω_i and θ_i
- but θ_{i+1} is calculated based on present values ω_{i+1} and previous value θ_i

$$\omega_{i+1} = \omega_i - \frac{g}{\ell}\theta_i \Delta t ,$$

$$\theta_{i+1} = \theta_i + \omega_{i+1} \Delta t$$

sample code 3.2.2

Euler vs Euler-Cromer

Euler method: forward differencing

$$\omega_i \approx \frac{\theta_{i+1} - \theta_i}{\Delta t} \Rightarrow \theta_{i+1} \approx \theta + \omega_i \Delta t_i$$

Euler-Cromer method: backward differencing

$$\omega_{i+1} \approx \frac{\theta_{i+1} - \theta_i}{\Delta t} \Rightarrow \theta_{i+1} \approx \theta + \omega_{i+1} \Delta t_i$$

Why Euler-Cromer works better?

- Because it conserver energy over each complete period of motion.

3.2 Forced SHM

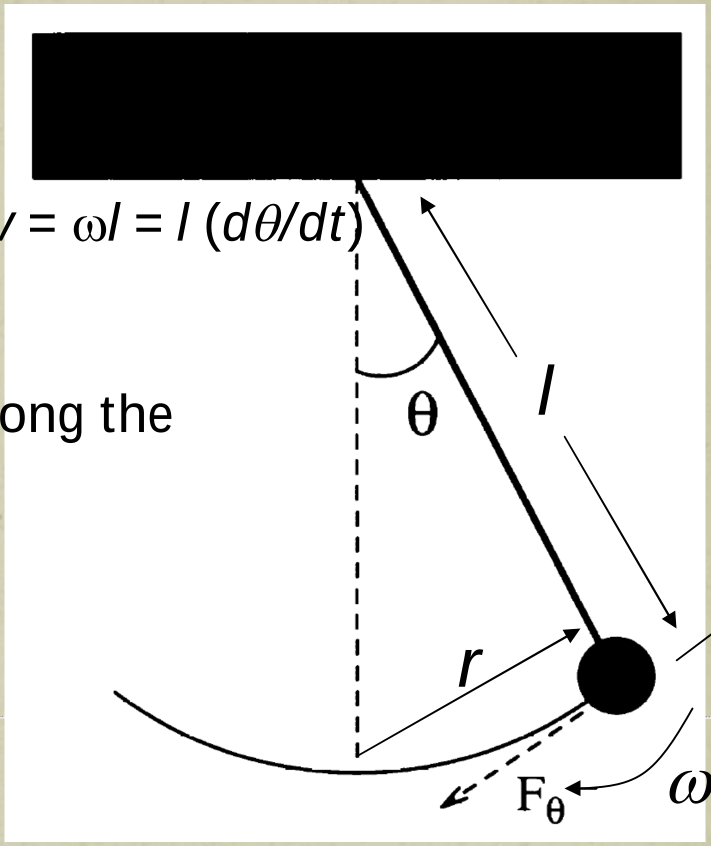
Drag force on a moving object, $f_d = -kv$

For a pendulum, instantaneous velocity $v = \omega l = l\,(d\theta/dt)$

Hence, $f_d = -kl\,(d\theta/dt)$.

The net force on the forced pendulum along the tangential direction

$$F = -mg\sin\theta - kl\,(d\theta/dt).$$



$$F = -mg\sin\theta - kl\frac{d\theta}{dt} \approx -mg\theta - kl\frac{d\theta}{dt};$$

$$m\frac{d^2r}{d^2t} \approx m\frac{d^2}{dt^2}(l\theta) = ml\frac{d^2\theta}{dt^2};$$

$$F = m\frac{d^2r}{dt^2} \rightarrow \boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - \frac{k}{m}\frac{d\theta}{dt} \equiv -\frac{g}{l}\theta - q\frac{d\theta}{dt}; q = \frac{k}{m}}$$

E.o.M

Analytical solution

Underdamped regime (small damping). Still oscillate, but amplitude decay slowly over many period before dying totally.

$$\theta(t) = \theta_0 e^{-qt/2} \sin\left(\varphi + t\sqrt{\Omega^2 - q^2/4}\right)$$

$\Omega = \sqrt{g/l}$ the natural frequency of the system

Overdamped regime (very large damping), decay slowly over several period before dying totally. θ is dominated by exponential term.

$$\theta(t) = \theta_0 e^{-\left(qt/2 \pm \sqrt{q^2/4 - \Omega^2}\right)t}$$

Critically damped regime, intermediate between under- and overdamping case.

$$\theta(t) = (\theta_0 + Ct)e^{-qt/2}$$

Cromer-Euler descretisation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} - \text{Eq. (1)}$$

Let $\omega = \frac{d\theta}{dt}$. In term of ω, Eq. (1) becomes

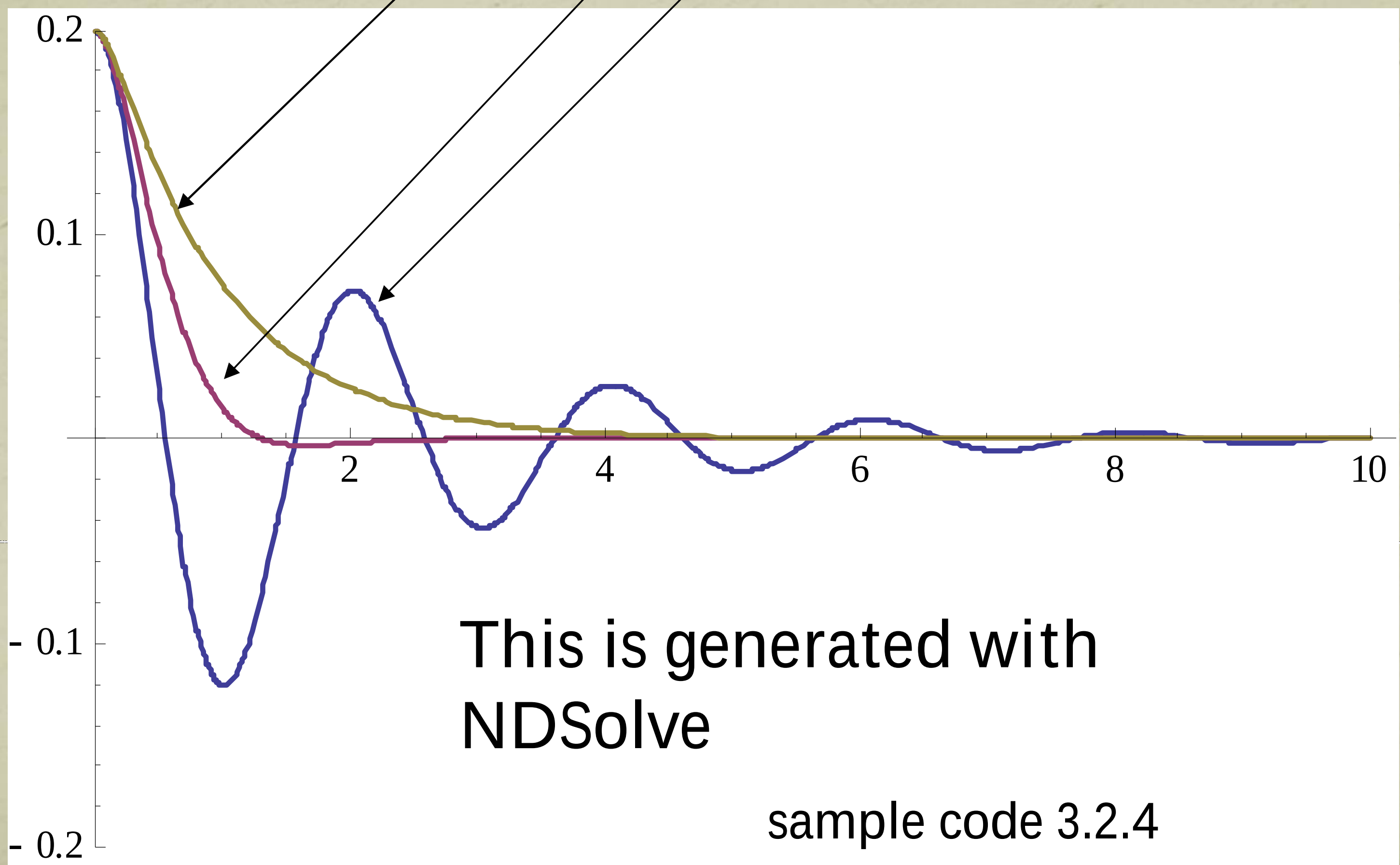
$$\frac{d\omega}{dt} = -\frac{g}{l}\theta - q\omega$$

$$\rightarrow \frac{\omega_{i+1} - \omega_i}{\Delta t} = -\frac{g}{l}\theta_i - q\omega_i$$

$$\rightarrow \omega_{i+1} = \omega_i - \left(\frac{g}{l}\theta_i + q\omega_i\right)\Delta t$$

From $\omega = \frac{d\theta}{dt} \rightarrow \theta_{i+1} = \theta_i + \omega_{i+1}\Delta t$ (Euler-Cromer) sample code 3.2.3

Numerical solution of the damped oscillator for q = 10,5,1, *l*=1.0 m



Adding driving force

$F_\theta = -m g \sin\theta$ - $kl\,(d\theta/dt) + F_D \sin(\Omega_D t)$

Ω_D frequency of the applied force

$$F = -mg\sin\theta - kl\frac{d\theta}{dt} \approx -mg\theta - kl\frac{d\theta}{dt} + F_D\sin\left(\Omega_D t\right);$$

$$m\frac{d^2r}{dt^2} \approx m\frac{d^2}{dt^2}\left(l\theta\right) = ml\frac{d^2\theta}{dt^2};$$

$$F = m\frac{d^2r}{dt^2} \rightarrow \frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - \frac{k}{m}\frac{d\theta}{dt} + \frac{F_D}{m}\sin\left(\Omega_D t\right) \equiv -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D}{m}\sin\left(\Omega_D t\right)$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D}{m}\sin\left(\Omega_D t\right)$$

Analytical solution

$$\theta(t) = \theta_0 \sin(\Omega_D t + \phi)$$

$$\theta_0 = \frac{F_D / m}{\sqrt{\left(\Omega^2 - \Omega_D^2\right)^2 + \left(q\Omega_D\right)^2}}$$

Resonance happens when $\Omega_D = \Omega$

Euler-Kromer descretisation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D}{m}\sin\left(\Omega_D t\right)$$

$$\frac{d\omega}{dt} = -\frac{g}{l}\theta - q\omega + \frac{F_D}{m}\sin\left(\Omega_D t\right) \rightarrow \frac{\omega_{i+1} - \omega_i}{\Delta t} = -\frac{g}{l}\theta_i - q\omega_i + \frac{F_D}{m}\sin\left(\Omega_D t_i\right)$$

$$\rightarrow \omega_{i+1} = \omega_i - \left(\frac{g}{l}\theta_i + q\omega_i - \frac{F_D}{m}\sin\left(\Omega_D i\Delta t\right)\right)\Delta t$$

$$\text{From } \omega = \frac{d\theta}{dt} \rightarrow \theta_{i+1} = \theta_i + \omega_{i+1}\Delta t \qquad \text{(Euler-Cromer)}$$

sample code 3.2.5

Second order Runge-Kutta method

- Consider a generic second order differential equation.

$$\frac{d^2 y(t)}{dt^2} = f(y)$$

- It can be numerically solved using second order Runge-Kutta method.
- First, split the second order DE into two first order parts:

$$v(t) = \frac{dy(t)}{dt} \qquad \frac{dv(t)}{dt} = f(y)$$

Algorithm

- Set initial conditions:
- Calculate $y' = y_i + \frac{1}{2} v_i \Delta t$
- Calculate $v' = v_i + \frac{1}{2} f\left(y'\right) \Delta t$
- Calculate $y_{i+1} = y_i + v' \Delta t$
- Calculate $v_{i+1} = v_i + f\left(y'\right) \Delta t$

RK2 for SHO

For the special case of SHO, the RK2 algorithm is translated into the following

$$\frac{d^2y(t)}{dt^2} = f(y) \qquad \equiv \frac{d^2\theta(t)}{dt^2} = -\left(\frac{g}{l}\right)\theta(t)$$

$$v(t) = \frac{dy(t)}{dt}, \qquad \equiv \omega(t) = \frac{d\theta(t)}{dt}$$

$$\frac{dv(t)}{dt} = f(y) \qquad \equiv \frac{d\omega(t)}{dt} = -\left(\frac{g}{l}\right)\theta(t)$$

Set initial conditions: $y_0 \equiv t(0), v_0 \equiv v(0)$ $\qquad \theta_0 \equiv \theta(0), \omega_0 \equiv \omega(0)$

Calculate $y' = y_i + \frac{1}{2}v_i\Delta t$ $\qquad \theta' = \theta_i + \frac{1}{2}\omega_i\Delta t$

Calculate $v' = v_i + \frac{1}{2}f(y')\Delta t$ $\qquad \omega' = \omega_i + \frac{1}{2}\left(-\frac{g}{l}\theta'\right)\Delta t$

Calculate $y_{i+1} = y_i + v'\Delta t$ $\qquad \theta_{i+1} = \theta_i + \omega'\Delta t$

Calculate $v_{i+1} = v_i + f(y')\Delta t$ $\qquad \omega_{i+1} = \omega_i + \left(-\frac{g}{l}\theta'\right)\Delta t$

Use RK2 to calculate the total energy of SHO

- We have seen previously that energy as calculated using Euler method is unstable.
- Now try to calculate *E* again, using the values of ω_i and θ_i as calculated using RK2.

$$E_{i+1} = \frac{1}{2} m l^2 {\omega_{i+1}}^2 + \frac{1}{2} m g l \theta_{i+1}^2$$

- You will have to do this in the HA3.