Chapter 4

The Solar System

4.1 Kepler's Laws

Newton's law applied to Earth-Sun system: $F_G = \dfrac{GM_S M_E}{r^2}$

Equation of motion (EoM)

$$F_{G,x} = -\frac{G\,M_S\,M_E}{r^2}\cos\theta = -\frac{G\,M_S\,M_E\,x}{r^3}$$

for *x* component

$$\frac{dx}{dt} = v_x \qquad \frac{dv_x}{dt} = -\frac{G\,M_S\,x}{r^3}$$

Similarly for *y*-component.

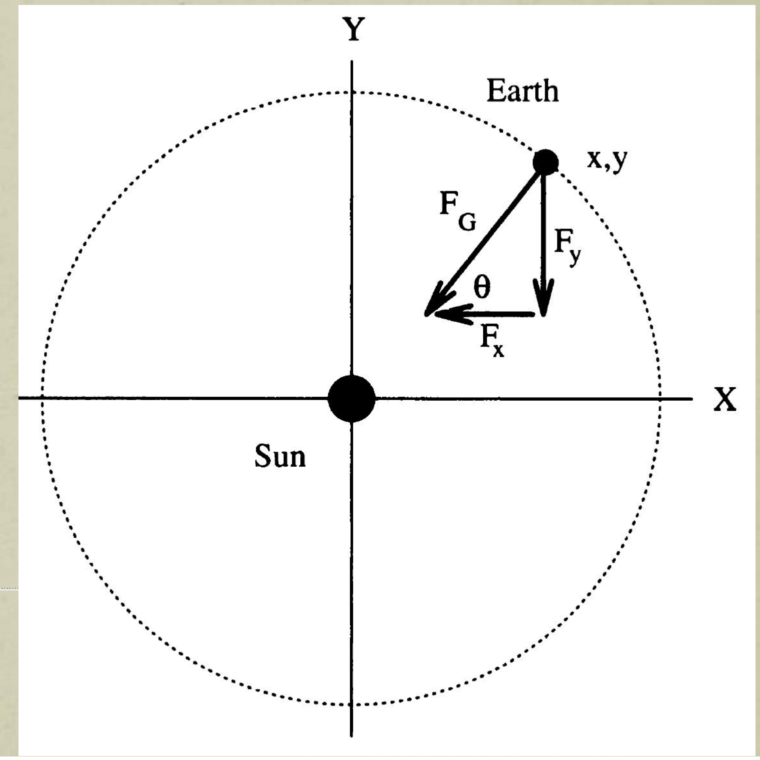


Figure 4.1: Coordinate system for describing the motion of Earth in orbit around the Sun. The Sun is at the origin and Earth is located at coordinates (x, y).

Note: this is a 2-D motion

Constants for the solar system

$R = 1.5 \times 10^{11}$ m (Earth-Sun distance, = 1 A.U.)

$M_S = 2.0 \times 10^{30}$ kg (Sun's mass.)

$M_E = 6.0 \times 10^{24}$ kg (Earth's mass.)

For the sake of numerical convenient, GM_S assumes the following value in AU unit

$$G M_S = 4\pi^2 \text{ AU}^3/\text{yr}^2$$

Eq. (4.6)

GM_S in AU unit

- For the Sun-Earth system,

$$\frac{GM_S M_E}{R^2} = M_E \omega^2 R \Rightarrow GM_S = \omega^2 R^3$$

- $\omega = 2\pi/T$. In AU unit, T=1year, hence $\omega = 2\pi$/yr.
- R = 1 AU in AU unit for Sun-Earth system.

$$GM_S = \omega^2 R^3 = \left(4\pi^2 / \text{yr}^2\right)\text{AU}^3 = 4\pi^2 \text{AU}^3 / \text{yr}^2$$

Descritisation of the E.o.M (Euler-Cromer method)

Sun-Planet system is a oscillatory (periodic system)

$$\frac{dx}{dt} = v_x \qquad \frac{dv_x}{dt} = -\frac{G M_S x}{r^3} \qquad \frac{dy}{dt} = v_y \qquad \frac{dv_y}{dt} = -\frac{G M_S y}{r^3}$$

$$\begin{aligned}
v_{x,i+1} &= v_{x,i} - \frac{4\pi^2 x_i}{r_i^3}\Delta t \\
x_{i+1} &= x_i + \boxed{v_{x,i+1}}\,\Delta t \\
v_{y,i+1} &= v_{y,i} - \frac{4\pi^2 y_i}{r_i^3}\Delta t \\
y_{i+1} &= y_i + \boxed{v_{y,i+1}}\,\Delta t\,,
\end{aligned}$$

Eq. (4.7)

A simple model of planetary system

First, model the solar system with an approximation: all the orbits of the planet are circular, with constant radius. This approximation will be relaxed to a more general form later.

We will use a to represent the Sun-planet distance in this simple model. In this case, $R = a$, a constant. (Note that when this approximation is relaxed, i.e., in a more general model, R will be a variable).

The ratio T^2/R^3 is a constant.

According to Newton's law, the ratio T^2/R^3 is a constant. for all planets in the Solar system. (Can you show this?)

In AU unit, the constant ratio $T^2/R^3 = 1\ \mathrm{yr}^2/\mathrm{AU}^3$.

Show this. Use the fact that, in AU unit,

$$GM_S = 4\pi^2 \mathrm{AU}^3 / \mathrm{yr}^2$$

Initial condition matters

When simulate planetary motion in this simple model,

Initial position: $(x_0,y_0) = (a,0)$.

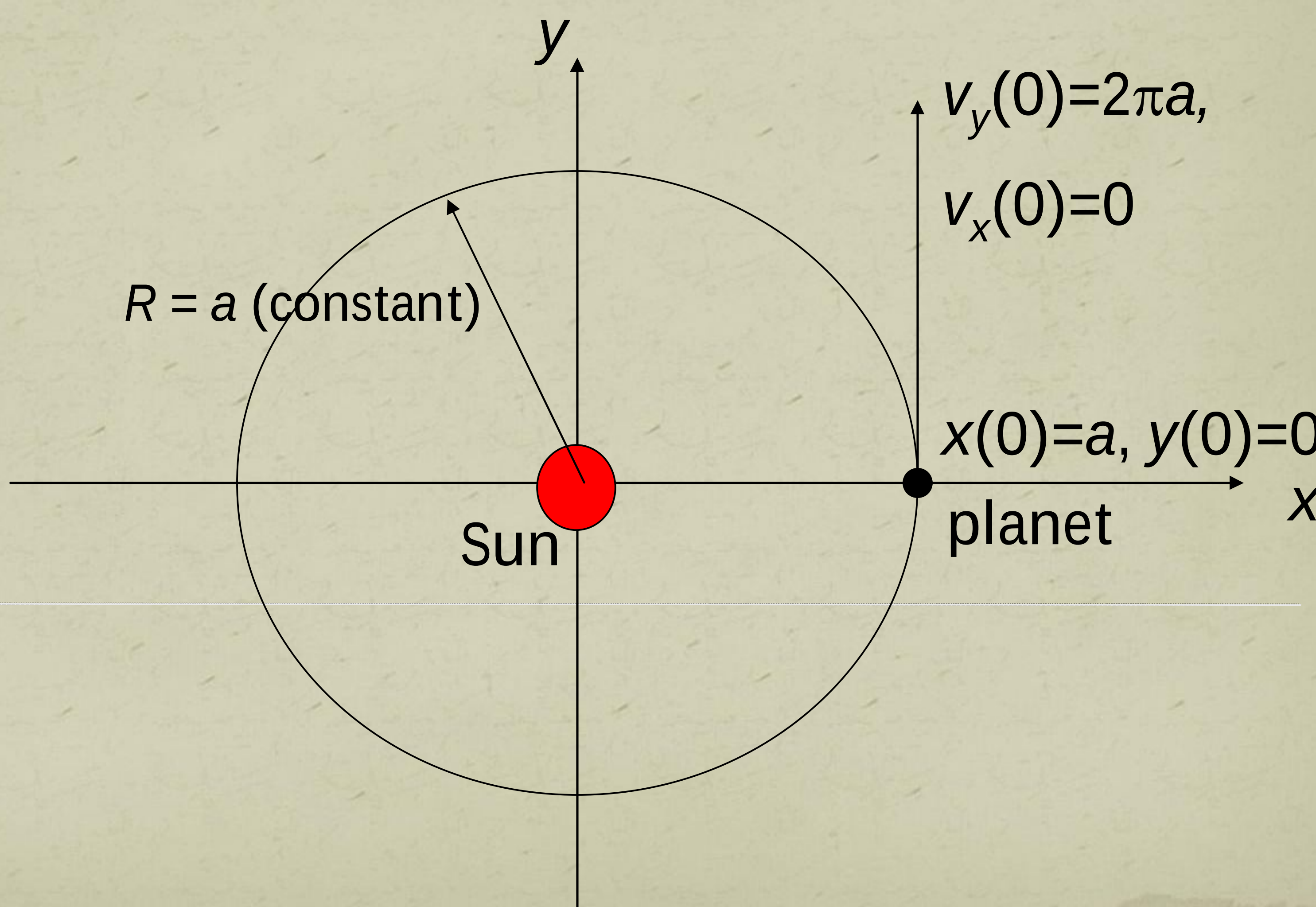
Initial velocity at $(a,0)$: $v_{0y} = 2\pi/a$, $v_{0x} = 0$. (do you know why we set $v_{0y} = 2\pi/a$?)

For Earth, $a_E = 1.52$ A.U, $T_E = 1$ yr.

For Mars, $a_{Mars} = 1.52$ A.U.. What is the period of Mars in unit of yr?

ANS: $(1.52)^{3/2}$ yr.

Initial conditions for vector position and velocity



Exercise

- Write a code to calculate the orbits of Earth and Mars:
- (1) In your code, superimpose both orbits in a same frame (static).
- (2) Display a video of the evolution of the locus of both planets.

- Assume both begin with the following initial conditions:
- For Earth:

$(x_0,y_0)_E = (a_E,0) = (1\text{ AU},0);$

$(v_{0y},v_{0x})_E = (2\pi/a_E,\ 0) = (2\pi/1\text{AU},\ 0).$

- For Mars:

$(x_0,y_0)_{Mars} = (a_{Mars},0) = (1.52\text{ AU},0);$

$(v_{0y},v_{0x})_{Mars} = (2\pi/a_{Mars},\ 0) = (2\pi/1.52\text{AU},\ 0).$

Deviation from the "perfect" initial conditions.

- If the initial condition at the initial position (a_0,0) deviates from $v_{0y} = 2\pi/a_0$, $v_{0x} = 0$, the orbits will become non-circular (elliptical, of non zero eccentricity). Here a_0 is a generic parameter representing the average distance of a planet from the Sun. For example, $a_0 = 1$ AU for Earth, =1.52 AU for Mars.

- You could modify your code in previous exercise to see this effect yourself. To do so, we introduce two parameters, β and γ, to be defined in the following slid.

- A non zero value of γ and $\beta \neq 1$ correspond to a more generic situation. Most planets' orbits around the Sun are not circular.

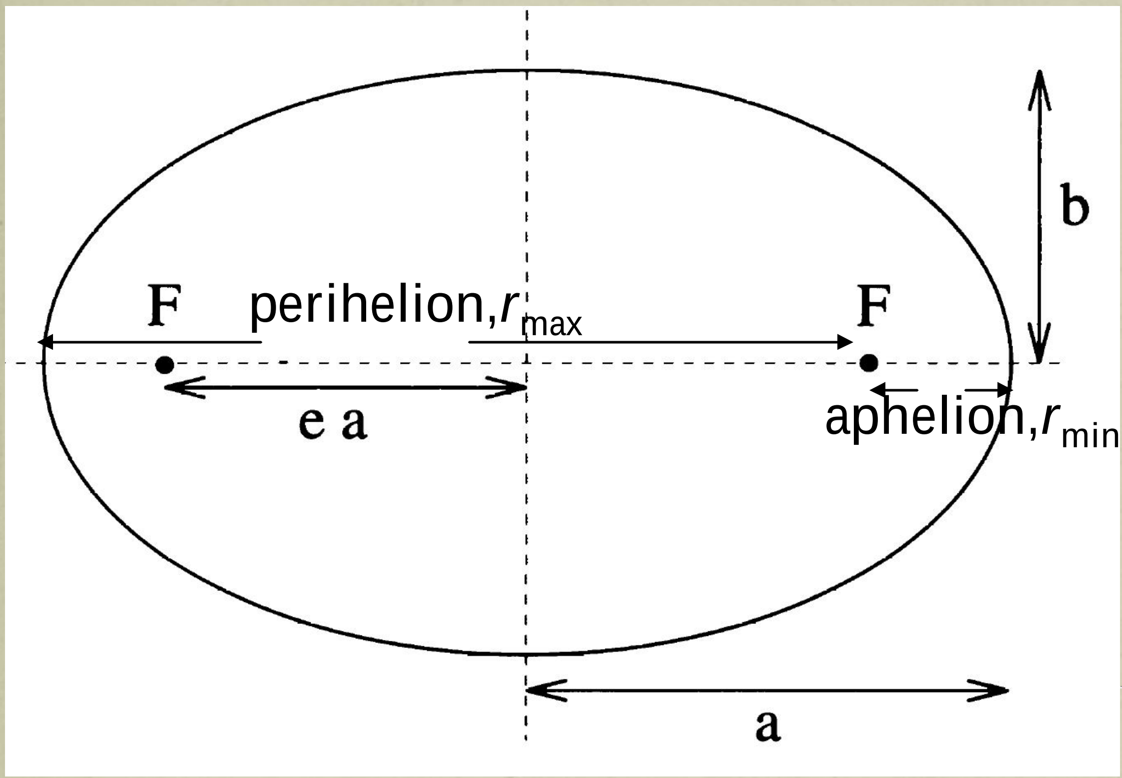
How to parametrise the initial velocities

- First, we define the initial speed v'_0 as follows:
- $v'_0 = \beta v_0$ where $v_0 = 2\pi a_0 / T' = 2\pi / a_0^{1/2}$
- where we have used $T' = a^{3/2}$ is obtained via the Keppler's law in AU unit, $T'^2 / a_0^3 = 1$.
- The components of the initial velocities are such that
- $v_x(o) = v'_0 \cos \gamma$; $v_x(o) = v'_0 \sin \gamma$, so that $v_x(o)^2 + v_y(o)^2 = (v'_0)^2 = \beta^2 v_0^2$
- In the limit $\beta = 1$, $\gamma = 0$, we get back the 'perfect' circular case.
- The initial KE of the planet is therefore
- $K(0) = (1/2) m \beta^2 v_0^2$

Escape velocities

- A planet if kicked started with too large a KE will fly off from the bounding of the Sun's gravity and never return.
- Mathematically, at $r = a_0$, this means $K(0) > |U|$
- The initial speed when this happens is called the escape velocity, which can be estimated as follows
- $\frac{1}{2}mv'^2_0 > GMm/a_0 \Rightarrow \beta > \sqrt{2} = 1.414$.
- That is, if the parameter approaches 1.414, the planet will fly off and never return. If we track the period of a planet, we will find that as $\beta \rightarrow 1.414$, $T \rightarrow$ infinity.

Elliptical orbit



sample code: 4.1.3

$$b = a\sqrt{1 - e^2}, \quad r_{max} = a(1+e); r_{min} = a(1-e);$$

Figure 4.3: Hypothetical elliptical orbit. A sun lies at one of the foci of the ellipse; both foci are labeled F. The semimajor and semiminor axes are a and b, and the eccentricity is e. This drawing is not to scale.

"Measuring" the constant ratio T^2/a^3

In the following, *a* denotes semimajor.

Our simulation uses Newton's law to evolve the planetary system. You can regard the computer simulation as an simulated version of real life "experiment".

For example, you can try to "measure" the value of T^2/a^3 of a planet with an elliptical orbit in the simulation.

According to Keppler's law, the ratio T^2/a^3 for all planets circulating the same Sun must be equal to 1 (in AU unit).

When you ``measure'' the ratio of T^2/a^3 of your simulation, you should get a value of 1.00.

"Measuring" period and semimajor

To ``measure'' the ratio of T^2/a^3 for a planet, you need first set a planet into a elliptical orbit. Then "measure" (i) the period T, (ii) the semimajor a.

- Your code should have calculated the locus (i.e., all the points $\{x(\mathrm{t}), y(\mathrm{t})\}$) from $t = 0$ till $t = t_{\mathrm{final}}$.
- How to "measure"? Write codes to instruct Mathematica to find the period and semimajor for you.
- You should repeat the ``measurement'' of the ratio for a few different planets, e.g, Earth, Mars, Jupiter, etc.

Measuring semimajor

How to “measure” period?

Verifying Keppler’s third law

- Now you are ready to write a code to attack assignment 4, where you are asked to “measure” the semimajor, semiminor, eccentricity, foci, perihelion, aphelion and period of a planet.
- Then you can also verify if Keppler’s law is obeyed.