Chapter 2

Realistic projectile motion

2.1 Frictionless motion with Newton's law

$$\frac{dv}{dt} = \frac{F}{m}$$

Newton's law

$$\frac{dE}{dt} = P$$

E total energy of the moving object, *P* the power supplied to the system.

For a flat course the energy is all kinetic, so $E = \frac{1}{2}mv^2$.

$$\int_{v_0}^{v} v'\,dv' = \int_0^t \frac{P}{m}\,dt'$$

$$v = \sqrt{v_0^2 + 2Pt/m}$$

Assuming *P* a constant.

Euler method

$$\frac{dv}{dt} \approx \frac{v_{i+1} - v_i}{\Delta t} + O(\Delta t^2)$$

$$v = \sqrt{v_0^2 + 2Pt/m}$$

gives

$$v_{i+1} = v_i + \frac{P}{mv_i}\Delta t$$

$$t_i \equiv i\,\Delta t$$

Given the velocity at time step i (i.e., v_i), we can calculate an *approximate* value of the velocity at the next step, v_{i+1}.

Pseudocode

- Initialisation: Set values for *P*, mass *m*, and time step Δt, and total number of time steps, *N*, initial velocity v_0.

- Do the actual calculation
  - $v_{i+1} = v_i + (P/mv_i)\Delta t$
  - $t_{i+1} = t_i + \Delta t$
  - Repeat for *N* time steps.

- Output the result

Adding friction to the Equation of Motion

- Frictionless motion is unrealistic as it predicts velocity shoots to infinity with time.
- Add in drag force, innocently modeled as

$$F_{\text{drag}} \approx -B_1 v - B_2 v^2 .$$

The mass of air moved in time dt is $m_{\text{air}} \sim \rho A v dt$, where ρ is the density of air and A the frontal area of the object.

kinetic energy is $E_{\text{air}} \sim m_{\text{air}} v^2/2$.

$$F_{\text{drag}} v dt = E_{\text{air}}$$

$$F_{\text{drag}} \approx -\frac{C \rho A v^2}{2}$$

C is known as the drag coefficient $B_2 \equiv C\rho A/2$

C ~ 1, depend on aerodynamics, measured experimentally

- The effect of F_{drag} is to replace P by $P - F_{drag}\, v$

$$\frac{dE}{dt} = P \quad \longrightarrow \quad \frac{dE}{dt} = P - F_{drag} v$$

$$v_{i+1} = v_i + \frac{P}{m v_i}\Delta t \quad \longrightarrow \quad v_{i+1} = v_i + \frac{\left(P - F_{drag} v_i\right)}{m v_i}\Delta t$$

$$v_{i+1} = v_i + \frac{\left(P - \frac{C\rho A v_i^2}{2} v_i\right)}{m v_i}\Delta t = v_i + \frac{P}{m v_i}\Delta t - \frac{C\rho A v_i^2}{2m}\Delta t$$

1D free fall with drag force

- Develop a code that shows the variation of velocity with time for a 1D object, which is constantly being pumped in energy at a given rate fix rate, *P*, and an non-zero initial velocity v_0.
- For a object undergoing 1D free fall motion along the vertical direction,

$$P = dK/dt = -dU/dt = -d/dt(mgy) = -mgv_y$$

So that

$$v_{y,i+1} = v_i + \frac{P}{mv_{y,i}}\Delta t - \frac{C\rho A v_{y,i}^2}{2m}\Delta t$$

$$= v_{y,i} - \frac{mgv_{y,i}}{mv_{y,i}}\Delta t - \frac{C\rho A v_{y,i}^2}{2m}\Delta t$$

$$= v_{y,i} - g\Delta t - \frac{C\rho A v_{y,i}^2}{2m}\Delta t$$

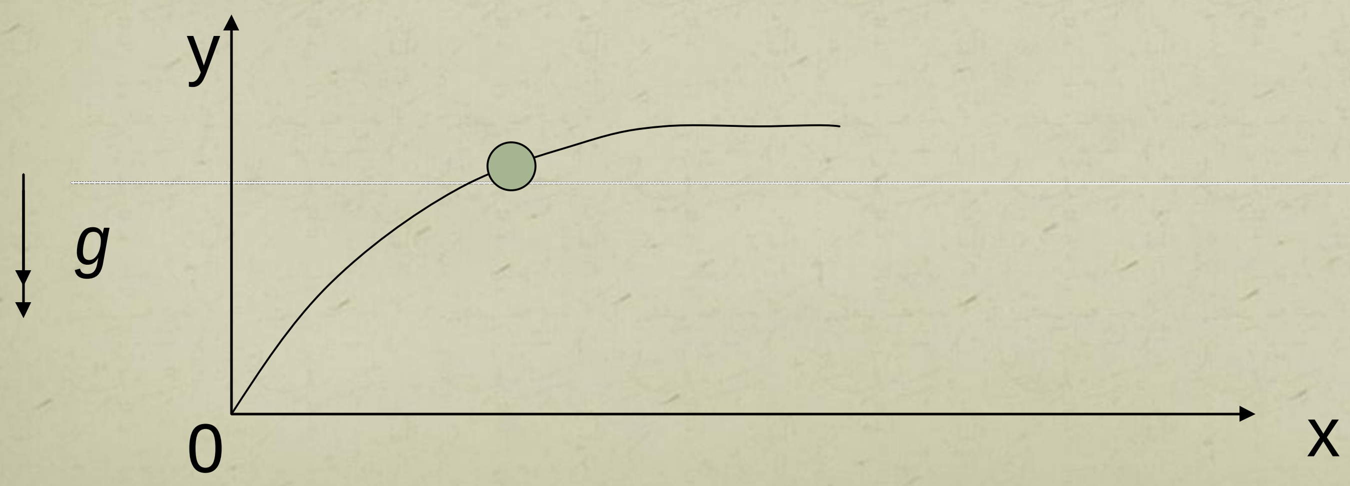
2.2 Projectile motion: The trajectory of a cannon shell

$$\frac{d^2x}{dt^2} = 0$$

$$\frac{d^2y}{dt^2} = -g$$

Two second order differential equations.

Wish to know the position (x,y) and velocity (v_x,v_y) of the projectile at time t, given initial conditions.



2.2 Projectile motion: The trajectory of a cannon shell (cont.)

Four first order differential equations.

$$\frac{dx}{dt} = v_x$$

$$\frac{dv_x}{dt} = 0$$

$$\frac{dy}{dt} = v_y$$

$$\frac{dv_y}{dt} = -g$$

Eq. 2.15

Euler's method →

$$x_{i+1} = x_i + v_{x,i}\,\Delta t$$

$$v_{x,i+1} = v_{x,i}$$

$$y_{i+1} = y_i + v_{y,i}\,\Delta t$$

$$v_{y,i+1} = v_{y,i} - g\,\Delta t\,.$$

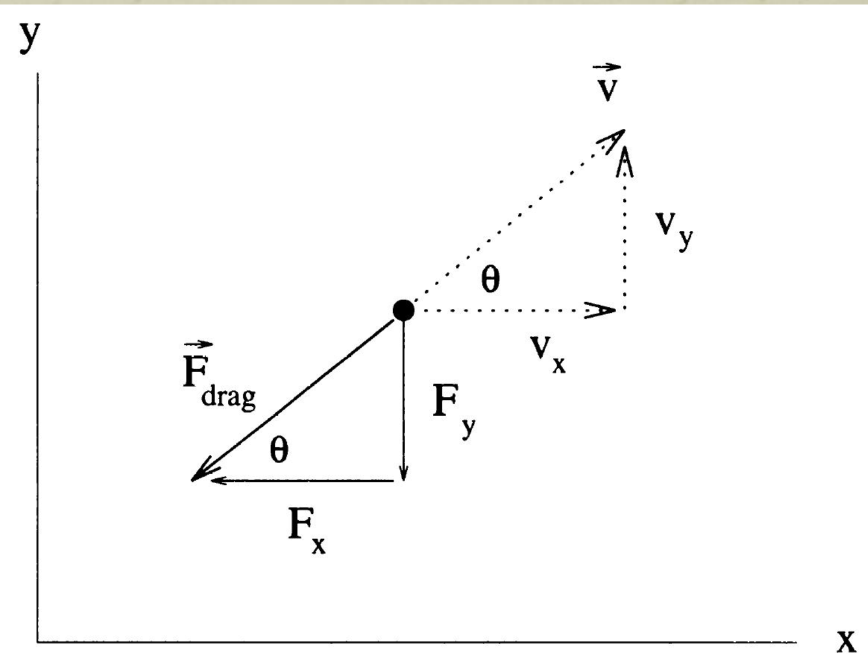
Eq. 2.16

Inclusion of drag force

$$F_{\text{drag}} = -B_2 v^2$$

$$F_{\text{drag},x} = F_{\text{drag}} \cos\theta = F_{\text{drag}} v_x / v,$$

similar for $F_{\text{drag},y}$.



$$F_{\text{drag},x} = -B_2 v v_x$$

$$F_{\text{drag},y} = -B_2 v v_y$$

Trajectory of a cannon shell with drag force

Adding this force to the equations of motion leads to

$v_i = (v_{x,i}^2 + v_{y,i}^2)^{1/2}$

$$x_{i+1} = x_i + v_{x,i}\,\Delta t$$

$$v_{x,i+1} = v_{x,i} - \frac{B_2\, v_i v_{x,i}}{m}\,\Delta t$$

$$y_{i+1} = y_i + v_{y,i}\,\Delta t$$

$$v_{y,i+1} = v_{y,i} - g\,\Delta t - \frac{B_2\, v_i v_{y,i}}{m}\,\Delta t\,.$$

Air density correction

- Drag force on a projectile depends on air's density, which in turn depends on the altitude.
- Two types of models for air's density dependence on altitude:
- Isothermal approximation - simple, assume constant temperature throughout, corresponds to zero heat conduction in the air.

$$\rho = \rho_0 \exp(-y/y_0) \qquad y_0 = kT/mg \approx 1.0\times 10^4\,\text{m}$$

y is the altitude, m mass of air's molecule ρ_0 is density at sea level ($y = 0$).

•Adiabatic approximation - more realistic, assume poor but non-zero thermal conductivity of air.

$$\rho = \rho_0 \left(1 - \frac{\boldsymbol{a}y}{\boldsymbol{T}_0}\right)^{\alpha}$$

$\alpha \approx 2.5$ for air; $\boldsymbol{a} \approx 6.5\times 10^{-3}\,\text{K/m}$

T_0 sea level temperature (in K)

Correction to the drag force

- The drag force w/o correction $F_{\text{drag}} = -B_2 v^2$ corresponds to the drag force at sea-level, with $B_2 = -\frac{1}{2} C \rho_0 A$
- For general altitude, it has to be modified:

$$F_{\text{drag}} = -B_2 v^2 = -\frac{1}{2} C \rho_0 A v^2$$

$$F^*_{\text{drag}} = -\frac{1}{2} C \rho A v^2 = -\frac{1}{2} C \rho_0 A v^2 \frac{\rho}{\rho_0} = -\frac{\rho}{\rho_0} B_2 v^2$$

$$F^*_{\text{drag},x} = F^*_{\text{drag}} \cos\theta = F^*_{\text{drag}} \frac{v_x}{v} = F_{\text{drag}} \cdot \left(\frac{\rho}{\rho_0}\right) \cdot \frac{v_x}{v} = -B_2 v v_x \cdot \left(\frac{\rho}{\rho_0}\right)$$

$$F^*_{\text{drag},y} = -B_2 v v_y \cdot \left(\frac{\rho}{\rho_0}\right)$$

Isothermal approximation:

$$\frac{\rho}{\rho_0} = \exp\left(-\frac{y}{y_0}\right)$$

$$F^*_{\text{drag},x} = -B_2 v v_x \cdot \left(\frac{\rho}{\rho_0}\right) = -B_2 v v_x \exp\left(-\frac{y}{y_0}\right); F^*_{\text{drag},y} = -B_2 v v_y \exp\left(-\frac{y}{y_0}\right)$$

$$y_0 = kT / mg \approx 1.0 \times 10^4 \text{ m}$$

Adiabatic approximation:

$$\frac{\rho}{\rho_0} = \left(1 - \frac{ay}{T_0}\right)^{\alpha},$$

$$F^*_{\text{drag},x} = -B_2 v v_x \cdot \left(\frac{\rho}{\rho_0}\right) = -B_2 v v_x \left(1 - \frac{ay}{T_0}\right)^{\alpha}; F^*_{\text{drag},y} = -B_2 v v_y \left(1 - \frac{ay}{T_0}\right)^{\alpha}$$

$$\boldsymbol{a} \approx 6.5 \times 10^{-3} \text{ K/m}$$

$\alpha \approx 2.5$ for air;

T_0 sea level temperature (in K)

Trajectory of a cannon shell with drag force, corrected for altitude dependence of air density

Adding this force to the equations of motion leads to

$$v_i = (v_{x,i}{}^2 + v_{y,i}{}^2)^{1/2}$$

$$x_{i+1} = x_i + v_{x,i}\,\Delta t$$

$$v_{x,i+1} = v_{x,i} - \frac{B_2\, v_i v_{x,i}}{m}\,\Delta t \cdot \left(\frac{\rho}{\rho_0}\right)$$

$$y_{i+1} = y_i + v_{y,i}\,\Delta t$$

$$v_{y,i+1} = v_{y,i} - g\,\Delta t - \frac{B_2\, v_i v_{y,i}}{m}\,\Delta t \cdot \left(\frac{\rho}{\rho_0}\right)$$

Curves with thermal and adiabatic correction

- Modify the existing code to produce the curves as in Figure 2.5, page 30, Giordano 2nd edition.

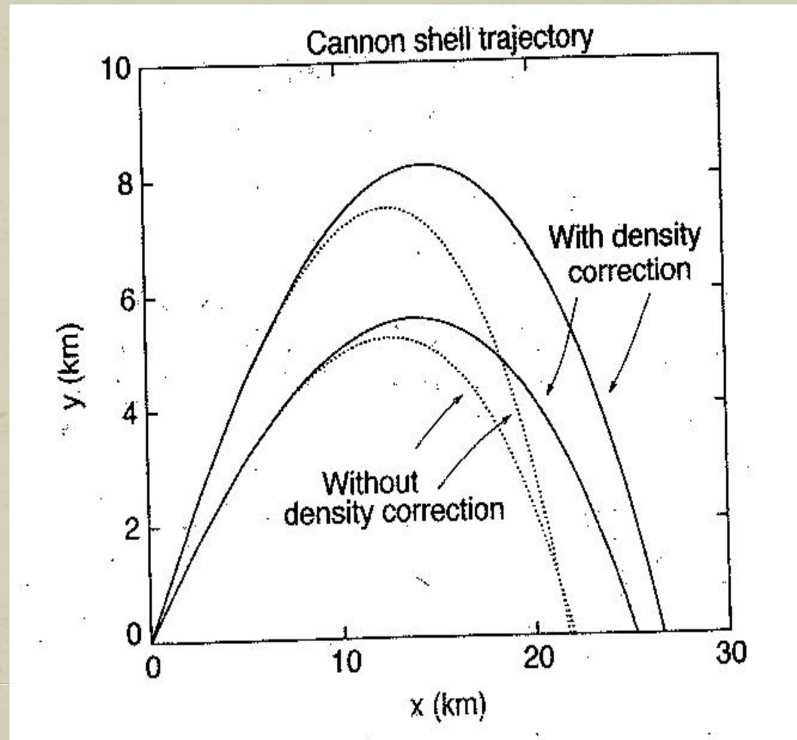


FIGURE 2.5: Trajectory of a cannon shell with (solid curves) and without (dotted curves) the effect of the lower air density at high altitudes taken into account. In all cases the air resistance was included with $B_2/m = 4 \times 10^{-5}\ \mathrm{m}^{-1}$, and the initial speed was 700 $\mathrm{m/s}$. The lower two curves were for a firing angle of $35°$, while for the top curves it was $45°$. Note that the x and y scales are different.