Lecture 6 Waves

the wave equation.

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

We treat both x and t as discrete variables with $x = i\Delta x$ and $t = n\Delta t$.

$$y(i, n) \equiv y(x = i\Delta x, t = n\Delta t)$$

After discretising, wave equation takes the form

$$\frac{y(i, n+1) + y(i, n-1) - 2\,y(i, n)}{(\Delta t)^2} \approx c^2 \left[\frac{y(i+1, n) + y(i-1, n) - 2\,y(i, n)}{(\Delta x)^2}\right]. \quad (6.2)$$

Rearranging (6.2) we can express $y(i, n+1)$ in terms of y at previous time steps,

$$y(i, n+1) = 2\,[1 - r^2]\,y(i, n) - y(i, n-1) + r^2\,[y(i+1, n) + y(i-1, n)]\ , \quad (6.3)$$

$r \equiv c\Delta t/\Delta x.$

we know the string configuration at time steps n and $n-1$, we can calculate the configuration at step $n+1$.

Fixed ends boundary conditions :

`y(i,n)` to be zero at the ends of the string (at `i=0` and `i=nmax`),

Instability: a simulation with $r > 1$ yields waves

Input: c, Δx, Δt, x_{ini}, x_{last}; initial profiles y(i,0), y(i,-1). Boundary conditions: y(x_{ini}, n)=0, y(x_{last}, n)=0.

Constraint for stability: $r \equiv c\Delta t/\Delta x$.<1.

initial string a (single) gaussian profile,

$$y_0(x) = \exp[-k(x - x_0)^2]$$

use $x_0 = 0.3$ m and $k = 1000$ m^{-2}, from $x = 0$ to 1 m.

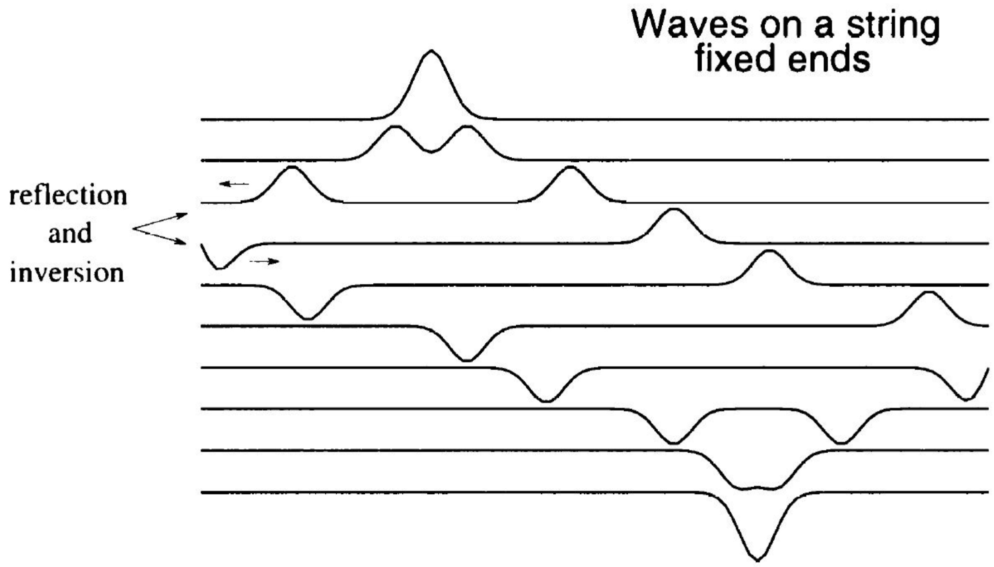

Figure 6.1: Waves propagating on a string with fixed ends. The string had a length of 1 m and the simulation used the values $c = 300$ m/s, $\Delta x = 0.01$ m, and $\Delta t = \Delta x/c$. The initial string profile is given at the top, and successive traces (moving from top to bottom) show the string at progressively later times. An example of reflection and inversion of a wavepacket when it reaches the left end of the string is indicated.

Sample code 6.1.1

Initialisation
Define Initial Profile
Time stepping (evolution) ← Boundary conditions
Output (visualization)

String travelling with different velocities at different sections on the string

Consider a string that is composed of two segments on which the waves have different velocities. This would arise, for example, if we tie together two strings of different thicknesses (i.e., different values for the mass per unit length). If we were dealing with electromagnetic (light) waves, this would correspond to light traveling from a material with one index of refraction into a material with a different index. We can model this problem with (6.3) and the program developed above by letting c be dependent on position. To be specific, we consider a string 1 m long described by a wave velocity c_1 for $0 \leq x \leq 0.5$, and c_2 elsewhere. This means that r in (6.3) has different values on the two halves of the string.

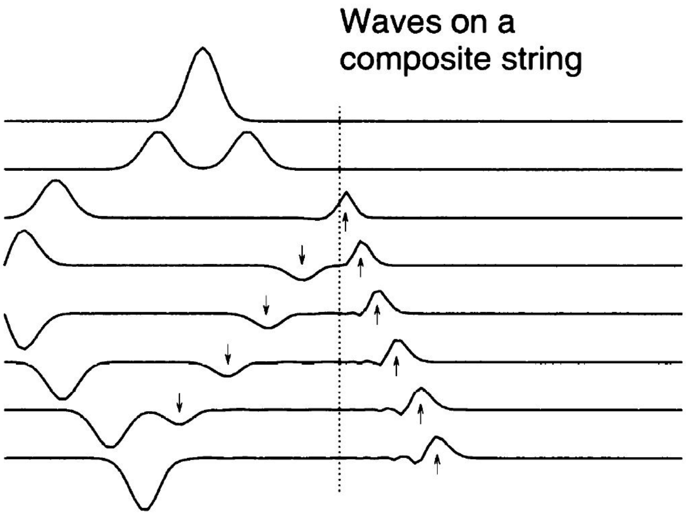

Figure 6.2: Waves propagating on a composite string with fixed ends. The string was 1 m long, with $c_1 = 300$ m/s on the left half of the string and $c_2 = 150$ m/s on the right half (the dotted line shows where the change in c occurs). We used $\Delta x = 0.01$ m and $\Delta t = \Delta x / c_1$. The arrows indicate pieces of the wavepacket that are reflected by, and transmitted through, the point where the propagation velocity changes.

Sample code 6.1.2