Chapter 7
Solving first order differential equation numerically

Example of first order differential equation commonly encountered in physics

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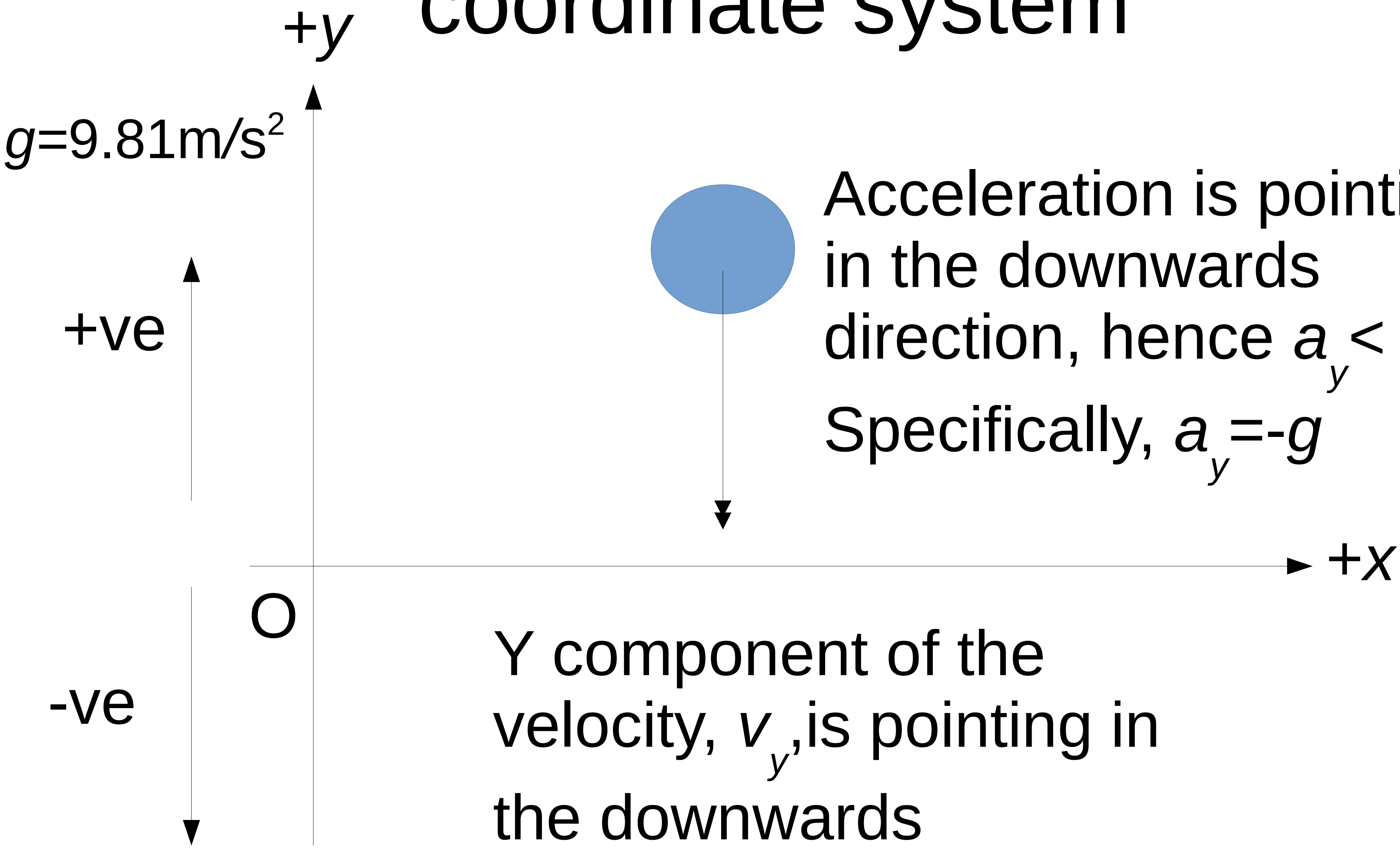
- $$\frac{dv_y}{dt} = -g \,;\, \frac{dy}{dt} = v_y$$
- $$m\frac{dv}{dt} = -mg - \eta v$$
- $$\frac{dN}{dx} = -\lambda N \,;\, \frac{dN}{dt} = -\frac{N}{\tau}$$
- $$m\frac{dv}{dx} = -k\,x$$
-
- Do you recognize these equations?

Definition of directions in our coordinate system



Acceleration is pointing in the downwards direction, hence $a_y < 0$.

Specifically, $a_y = -g$

Y component of the velocity, v_y, is pointing in the downwards direction, hence $v_y < 0$

General form of first order differential equations

$$\frac{df(t)}{dt} = G(t)$$

$dN(t)/dt = -N(t)\ /\tau$

$G(t) \equiv -N(t)/\tau$

$f(t) \equiv N(t)$

$dv(x)/dx = -(k/m)x$

$G(t) \equiv -(k/m)x$

$f(t) \equiv v(x)$

$dv(t)/dt = -g - \eta v(t)/m$

$G(t) \equiv -g - \eta v(t)/m$

$f(t) \equiv v(t)$

Analytical solution of $\frac{dv_y}{dt} = -g$

- ZCA 101 mechanics, kinematic equation for a free fall object

$$\frac{dv_y}{dt} = -g$$

What is the solution, i.e., $v_y = v_y(t)$?

$$\frac{dv_y}{dt} = -g$$

$$\Rightarrow \int \frac{dv_y}{dt} dt = -\int g \, dt$$

$$\int dv_y = v_y = -\int g \, dt = -g t + c$$

$$\Rightarrow v_y = v_y(t) = -g t + c$$

Analytical solution of $\frac{dv_y}{dt} = -g$

- To completely solve this first order differential equation, i.e., to determine v_y as a function of t, and the arbitrary constant c, a boundary value or initial value of v_y at a given time t is necessary. Usually (but not necessarily) $v_y(0)$, i.e., the value of v_y at t=0 has to be assumed.

$$\frac{dv_y}{dt} = -g$$

$$\int_{v_y(0)}^{v(t)} dv = -\int_0^t g\,dt = -g\,t$$

$$\Rightarrow v_y = v_y(t) = -g\,t + v_y(0)$$

Analytical solution of $\frac{dy}{dt} = v_y$

- Assume $v_y = v_y(t)$ a known function of t.
- To completely solve the equation so that we can know what is the function $y(t)$, we need to know the value of $y(0)$.

$$\int_{y(0)}^{y(t)} dy = \int_0^t v_y \, dt$$

$$\Rightarrow y(t) - y(0) = \int_0^t v_y \, dt$$

Analytical solution of $\frac{dy}{dt} = v_y$

- In free fall without drag force, $v_y(t) = v_y(0) - gt$.

  The complete solution takes the form

$$y(t) - y(0) = \int_0^t v_y \, dt = \int_0^t \left(v_y(0) - g t\right) dt$$

$$y(t) = y(0) + v_y(0) t - \frac{1}{2} g t^2$$

Boundary condition

- In general, to completely solve a first order differential equation for a function with single variable, a boundary condition value must be provided.
- Generalising such argument, two boundary condition values must be supplied in order to completely solve a second order differential equation.
- *n* boundary condition values must be supplied in order to completely solve a *n*-th order differential equation.
- Hence, supply boundary condition values are necessary when numerically solving a differential equation.

Analytical solution of a free fall object in a viscous medium

- $$m\frac{dv}{dt}=-mg-\eta v$$
- Boundary condition: *v*=0 at *t* = 0.
- $$\int_{v(0)}^{v(t)}\frac{dv}{dt}dt=\int_{v(0)}^{v(t)}dv=\int_0^t\left(-g-\frac{\eta}{m}v\right)dt$$

$$\Rightarrow\int_{v(0)=0}^{v(t)}\frac{dv}{\left(-g-\frac{\eta}{m}v\right)}=\int_0^t dt$$

$$\Rightarrow v(t)=-\left(\frac{mg}{\eta}\right)\left[1-\exp\left(\frac{-\eta t}{m}\right)\right]$$

Number of beta particles penetrating a medium (recall your first year lab experiments)

$$\frac{dN}{dx} = -\lambda N$$

$$\int_{N_0}^{N} \frac{dN}{N} = -\int_{0}^{x} \lambda \, dx$$

$$N(x) = N_0 \exp[-\lambda x]$$

Number of radioactive particle remained after time *t* (recall your first year lab experiments. τ: half-life)

$$\frac{dN}{dt} = -\frac{N}{\tau}$$

$$\int_{N_0}^{N} \frac{dN}{N} = -\int_{0}^{t} \frac{1}{\tau}\, dt$$

$$N(t) = N_0 \exp[-\frac{t}{\tau}]$$

Relation of speed vs. displacement in SHM

$$E = K + P = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

$$\Rightarrow \frac{dE}{dx} = \frac{d}{dx} \left(\frac{1}{2} m v^2 + \frac{1}{2} k x^2 \right)$$

$$\Rightarrow m \frac{dv}{dx} = -k\, x$$

The solution is

$$v(x) = v_0 + \frac{k\, x_0^2}{2} m - \frac{k}{2m} x^2$$

$$bondary\ condition: v = v_0\ at\ x = x_0$$

DSolve and NDSolve

Now, we would learn how to solve these first order differential equations

Dsolve (symbolically)

and

NDSolve (numerically)

See Dsolve_example.nb,NDSolve_example.nb

Now, how would you write your own algorithm to solve first order differential equations numerically?

$$\frac{df(t)}{dt} = G(t)$$

Euler’s method for discreteising a first order differential equation into a difference equation

Discretising the differential equation

In Euler method, where the differentiation of a function at time *t* is approximated as

$$\frac{df(t)}{dt} \simeq \frac{f(t+\Delta t)-f(t)}{\Delta t} = \frac{f(t_{i+1})-f(t_i)}{\Delta t}, \qquad t_i = i\Delta t, \quad t_{i+1}-t_i = \Delta t,$$

$$\frac{df(t_i)}{dt} \simeq \frac{f(t_{i+1})-f(t_i)}{\Delta t}; t_i = i\Delta t \Rightarrow f(t_{i+1}) \simeq f(t_i) + \frac{df(t_i)}{dt}\Delta t = f(t_i) + G(t_i)\Delta t$$

Essentially, Euler's method says

Differential equation $$\frac{df(t)}{dt} = G(t)$$

Discretise

$$f(t_{i+1}) \simeq f(t_i) + G(t_i)\Delta t$$

Difference equation

Boundary condition: the numerical value at $f(t_0)$ has to be supplied.

Discretising the differential equation

$dN(t)/dt = -N(t)/\tau$ ⟷ c.f $\frac{df(t)}{dt} = G(t)$

- $dN(t)/dt = -N(t)/\tau$ is discretised into a difference equation,

$$N(t+\Delta t) \approx N(t) - \frac{N(t)}{\tau}\Delta t$$

$$\Rightarrow N(t_{i+1}) \approx N(t_i) - \frac{N(t_i)}{\tau}\Delta t$$

c.f $f(t_{i+1}) \simeq f(t_i) + G(t_i)\Delta t$

$N(t_i) \rightarrow f(t_i)$, $\frac{N(t_i)}{\tau} \rightarrow G(t_i)$

which is suitable for numerical manipulation using computer.

There are many different way to discretise a differential equation.

Euler's method is just among the easiest of them.

The code's structure

Initialisation:

Assign (i) $N(t=0)$, τ, (ii) Number of steps, Nstep, (iii) Time when to stop, say tfinal= 10τ.

- The global error is of the order O ~ Δt
- Δt = tfinal/Nstep
- In principle, the finer the time interval Δt is, the numerical solution becomes more accurate.
- $t_i = t_0 + i\Delta t$; i=0,1,2,...,tfinal
- Calculate $N(t_1) = N(t_0) - \Delta t\, N(t_0)/\tau$.
- Then calculate $N(t_2) = N(t_1) - \Delta t\, N(t_1)/\tau$
- $N(t_3) = N(t_2) - \Delta t\, N(t_2)/\tau$, ...
- Stop when t = tfinal.
- Plot the output: $N(t)$ as function of t.

See code Euler_nucleidecay.nb

Sample Euler codes

- Euler_dydt.nb
- Euler_dvdt.nb
- Euler_freefall_dragforce.nb
- Euler_SHM_v_vs_x.nb

Exercise

- A freely falling object through a fluid medium can alternatively be modeled such that the drag force is proportional to the square of its speed. The first order differential equation for such an object is given by
- $$\frac{dv}{dt} = -g + k v^2$$
- Let *k*=0.01, and the boundary condition is *v*(0) = -20 m/s.
- Use DSolve to obtain the analytical solution for *v*(*t*).
- Develop a code that implements Euler's method to numerically solve the equation for tfinal=20.0s.
- Overlap your numerical solution on top of the analytically obtained plot. Both should agree to each other.

Exercise

- Solve the vertical position of the falling object numerically as a function of time *t* in the previous exercise using Euler's method. You must make use of the analytical expression of *v*(*t*) as obtained from your solution for that exercise. Assume the initial condition *y*(0)=0.

$$\frac{dy(t)}{dt}=v(t)$$

- What is the value of *y*(tfinal)?