Chapter 9

Solving Second order differential equations numerically, 2

Online lecture materials

- •The online lecture notes by Dr. Tai-Ran Hsu of San José State University,
- http://www.engr.sjsu.edu/trhsu/Chapt [er%204%20Second%20order%20DEs.p](http://www.engr.sjsu.edu/trhsu/Chapter 4 Second order DEs.pdf) df
- provides a very clear explanation of the solutions and applications of some typical second order differential equations.

2nd Order Homogeneous DEs

$$
\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0
$$

with **TWO** given conditions **The solutions**

Case 1:
$$
a^2 - 4b > 0
$$
:
\n
$$
u(x) = e^{-\frac{ax}{2}} \left(c_1 e^{\sqrt{a^2 - 4b} x/2} + c_2 e^{-\sqrt{a^2 - 4b} x/2} \right)
$$

Case 2: $a^2 - 4b < 0$:

$$
u(x) = e^{-\frac{ax}{2}} \left[A \sin\left(\frac{1}{2}\sqrt{4b-a^2}\right) x + B \cos\left(\frac{1}{2}\sqrt{4b-a^2}\right) x \right]
$$

Case 3: $a^2 - 4b = 0$: - A special case

$$
u(x) = c_1 e^{-\frac{ax}{2}} + c_2 x e^{-\frac{ax}{2}} = (c_1 + c_2 x) e^{-\frac{ax}{2}}
$$
(4.12)

where c_1 , c_2 , A and B are arbitrary constants to be determined by given conditions

Example 4.1 Solve the following differential equation

$$
\frac{d^2u(x)}{dx^2} + 5\frac{du(x)}{dx} + 6u(x) = 0
$$

$$
u(x) = e^{-5x/2} \left(c_1 e^{x/2} + c_2 e^{-x/2} \right) = c_1 e^{-2x} + c_2 e^{-3x}
$$

where c_1 and c_2 are arbitrary constants to be determined by given conditions

Example 4.2

$$
\frac{d^{2}u(x)}{dx^{2}} + 6\frac{du(x)}{dx} + 9u(x) = 0
$$

$$
u(0) = 2
$$

$$
\left. \frac{du(x)}{dx} \right|_{x=0} = 0
$$

$$
u(x) = 2(1+3x)e^{-3x}
$$

DSolve

•**DSolve** of Mathematica can provide analytical solution to a generic second order differential equation. See Math built in 2ODE.nb.

Typical second order, non-homogeneous ordinary differential equations

$$
\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)
$$
\n(4.25)

\nNon-homogeneous term

Solution of Equation (4.25) consists **TWO** components:

Solution u(x) =
$$
\begin{bmatrix} \text{Complementary} \\ \text{solution } u_{h}(x) \end{bmatrix} + \begin{bmatrix} \text{Particular} \\ \text{solution } u_{p}(x) \end{bmatrix}
$$

u(x) = u_h(x) + u_p(x)

Typical second order, non-homogeneous ordinary differential equations

$$
\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)
$$
\nNon-homogeneous term

$$
u(x) = u_h(x) + u_p(x)
$$

$$
\frac{d^2 u_h(x)}{dx^2} + a \frac{du_h(x)}{dx} + bu_h(x) = 0
$$

There is **NO** fixed rule for deriving $u_p(x)$

Example 4.6

$$
\frac{d^2y(x)}{dx^2} - \frac{dy(x)}{dx} - 2y(x) = \sin 2x
$$

 $y(x) = y_h(x) + y_p(x)$

$$
\frac{d^2 y_h(x)}{dx^2} - \frac{dy_h(x)}{dx} - 2y_h(x) = 0
$$

$$
y_h(x) = c_1 e^{-x} + c_2 e^{2x}
$$

Guess: $y_p(x) = A \sin 2x + B \cos 2x$ After some algebra $y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 e^{2x} + \left(-\frac{3}{20} Sin 2x + \frac{1}{20} Cos 2x\right)$

Example 4.8

$$
\frac{d^2u(x)}{dx^2} + 4u(x) = 2\sin 2x
$$

$$
u(x) = u_h(x) + u_p(x) = c_1 \cos 2x + c_2 \sin 2x - \frac{x}{2} \cos 2x
$$

 \mathbf{r} \sim

Simple Harmonic pendulum as a special case of second order DE

Force on the pendulum $F_{\theta} = -m g \sin \theta$

for small oscillation, $\sin \theta \approx \theta$.

Equation of motion (EoM)

 dv_{θ}

 dt

 $-mgsin\theta$ =m

 $F_{\theta}=ma_{\theta}$

 \boldsymbol{d}

 $\frac{dr}{ }$

 dt

 dt

 $=$ m

The period of the SHO is given by *g l* $T = 2\pi$ $d^2\theta$ dt^2 \approx $$ $g\theta$ \boldsymbol{l} *r n*(*x*)

 $\approx m$

 d^2

 $\frac{u}{dt^2}$ (10

l

 θ

Simple Harmonic pendulum as a special case of second order DE (cont.)

$$
\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)
$$
\n
$$
x \equiv t
$$
\n
$$
u(x) \equiv \theta(t)
$$
\n
$$
a \equiv 0
$$
\n
$$
b \equiv \frac{g}{l}
$$
\n
$$
n(x) \equiv 0
$$
\n
$$
\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l}
$$

Simple Harmonic pendulum as a special case of second order DE (cont.)

$$
\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l}
$$

Analytical solution:

 $\theta = \theta_0 \sin(\Omega t + \phi)$

 $\Omega = \sqrt{g/l}$ natural frequency of the pendulum; θ_0 and ϕ are constant determined by boundary conditions

Simple Harmonic pendulum with drag force as a special case of second order DE

Drag force on a moving object, $f_d = -kv$

For a pendulum, instantaneous velocity $v = \omega l = l (d\theta/dt)$ Hence, $f_d = -kl \, (d\theta/dt)$.

The net force on the forced pendulum along the tangential direction

$$
F_{\theta} = -m g \sin \theta - k l (d\theta/dt).
$$

$$
F_{\theta} = -mg \sin \theta - kl \frac{d\theta}{dt} \approx -mg \theta - kl \frac{d\theta}{dt};
$$

Simple Harmonic pendulum with dr
\nforce as a special case of second order
\n
$$
Drag force on a moving object, f_d = -kv
$$
\nFor a pendulum, instantaneous velocity $v = \omega l = l (d\theta/dt)$
\nHence, $f_d = - kl (d\theta/dt)$.
\nThe net force on the forced pendulum along the
\ntangential direction
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$$
F_{\theta} = -m g \sin \theta - kl (d\theta/dt).
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$$
F_{\theta} = -m g \sin \theta - kl (d\theta/dt).
$$
\n
$$
F_{\theta} = -m g \sin \theta - kl \frac{d\theta}{dt} \approx -m g \theta - kl \frac{d\theta}{dt};
$$
\n
$$
m \frac{d^2 r}{d^2 t} \approx m \frac{d^2 \theta}{dt^2} (l\theta) = m l \frac{d^2 \theta}{dt^2};
$$
\n
$$
F_{\theta} = m \frac{d^2 r}{d^2 t} \rightarrow \frac{d^2 \theta}{dt^2} = -\frac{g}{l} \theta - \frac{k}{m} \frac{d\theta}{dt} = -\frac{g}{l} \theta - q \frac{d\theta}{dt}; q = \frac{k}{m}
$$

Simple Harmonic pendulum with drag force as a special case of second order DE (cont.)

Analytical solution

 $(t) = \theta_0 e^{-4i\pi^2}$ 2) mall damping)

wly over many p
 $\frac{1}{2} \sin \left(\varphi + t \sqrt{\Omega^2 - \frac{q^2}{4}} \right)$

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slowly over m
 $e^{-qt/2} \sin \left(\varphi + t \sqrt{\Omega^2} \right)$

the natural frequence

very large dangers 4 **DN**

: (small damping). Still oscillate,

lowly over many period before
 $e^{-qt/2} \sin \left(\varphi + t \sqrt{\Omega^2 - \frac{q^2}{4}} \right)$

the natural frequency of the system

very large damping), decay slowly

ore dying totally. θ is dominate **1**
 small damping). Stil
 qu/2 sin $\left[\varphi + t\sqrt{\Omega^2 - \frac{q^2}{4}}\right]$
 e natural frequency of the system **ution**
 t t t t g t t t y slowly over mapply over the <i>t e^{t} *e* $e^{-qt/2}$ sin $\left(\varphi + t\sqrt{\Omega^2 - t}\right)$
 $\qquad = \sqrt{\frac{g}{l}}$ the natural frequency in the extra distribution of the form of the set of the set $\Omega = \sqrt{\frac{g}{h}}$ the natural frequency of the system **Dution**
egime (small damping). Still oscillate,
ecay slowly over many period before
 $\theta(t) = \theta_0 e^{-qt/2} \sin \left(\varphi + t \sqrt{\Omega^2 - \frac{q^2}{4}} \right)$
 $\Omega = \sqrt{\frac{g}{t}}$ the natural frequency of the system
ime (very large damping), decay slowl Il damping). Still oscillate,
over many period before
 $\left[\begin{matrix} \varphi+t\sqrt{\Omega^2-\frac{q^2}{4}}\end{matrix}\right]$
al frequency of the system
rge damping), decay slowly tion

me (small damping). Still oscillate,

y slowly over many period before

= $\theta_0 e^{-q t/2} \sin \left(\rho + t \sqrt{\Omega^2 - \frac{q^2}{4}} \right)$
 $\sqrt{\frac{g}{l}}$ the natural frequency of the system

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over many period before
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al frequency of the system
rge damping), decay slowly
ing totally. θ is dominated **Dution**

egime (small damping). Still oscillate,

ecay slowly over many period before
 $\theta(t) = \theta_0 e^{-qt/2} \sin \left(\theta + t \sqrt{\Omega^2 - \frac{q^2}{4}} \right)$
 $\Omega = \sqrt{\frac{g}{l}}$ the natural frequency of the system

ime (very large damping), decay sl Underdamped regime (small damping). Still oscillate, but amplitude decay slowly over many period before dying totally. α
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yver man
 $\int_{\varphi + t}^{\varphi + t} \sqrt{\Omega^2 - \frac{q}{4}}$

1 frequency of
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ng totally
 $\frac{qt}{2} \pm t \sqrt{\frac{q^2}{4} - \Omega^2}$ ution

ime (small damping). Still oscillate,

ay slowly over many period before
 $\frac{1}{\sqrt{e}} e^{-\theta} e^{-\theta/2} \sin \left(\theta + i \sqrt{\Omega^2 - \frac{q^2}{4}} \right)$
 $\frac{\sqrt{\frac{8}{t}}}{\sqrt{\frac{8}{t}}}$ the natural frequency of the system

e (very large damping), d all damping). Still oscillate,

over many period before
 $\left[\varphi + i\sqrt{\Omega^2 - \frac{q^2}{4}}\right]$

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or a all damping). Still oscillate,

y over many period before
 $\lim_{\ln\left(\varrho + t\sqrt{\Omega^2 - \frac{q^2}{4}}\right)}$
 $\lim_{\ln\left(\frac{1}{2} \ln\left(\frac{\Omega}{\Omega}\right)^2 + \frac{q^2}{4}\right)}$

arge damping), decay slowly

ying totally. θ is dominated
 $-\left(\frac{q t}{2} \pm t\sqrt{\frac{q$ (small damping). Still oscillate,

owly over many period before
 $e^{-qt/2} \sin \left(\varphi + t \sqrt{\Omega^2 - \frac{q^2}{4}} \right)$

he natural frequency of the system

ery large damping), decay slowly

ore dying totally. θ is dominated
 $= \theta_0 e^{-$

2 by exponential term. $\left[\frac{qt}{2}t\sqrt{q^2-\Omega^2}\right]$ Overdamped regime (very large damping), decay slowly over several period before dying totally. θ is dominated $\frac{d}{dx}$
 $\frac{1}{4}$
 $\frac{1}{4}$
 $\frac{1}{4}$
 $\frac{1}{2}$
 $\frac{1}{2}$ $g = \theta_0 e^{-\theta} \sin \left(\frac{\theta + t \sqrt{\Omega} - \frac{\pi}{4}}{4} \right)$
 $= \sqrt{\frac{g}{t}}$ the natural frequency of the system
 the (very large damping), decay slowly
 before dying totally. θ **is dominated**
 n.
 $\theta(t) = \theta_0 e^{-\frac{\left(\frac{gt}{2} + t \sqrt{\frac{g^2$ the natural frequency of the system

erry large damping), decay slowly

pre dying totally. θ is dominated
 $= \theta_0 e^{-\left(\frac{qt}{2} \pm t \sqrt{\frac{q^2}{4} - \Omega^2}\right)}$

me, intermediate between

ing case.
 $= (\theta_0 + c_t) e^{-\frac{qt}{2}}$

$$
(t) = \theta_0 e^{-\left(2 \frac{1}{2} t\right)}
$$

Critically damped regime, intermediate between under- and overdamping case.

$$
\theta(t) = (\theta_0 + Ct) e^{-\frac{qt}{2}}
$$

See 20DE Pendulum.nb where DSolve solves the three cases of a damped pendulum analytically.

Adding driving force to the damped oscillator: forced oscillator ding driving force

cillator: forced os
 $-m g \sin \theta - kl (d\theta/dt) + F_D \sin(\theta$
 $\sin \theta - kl \frac{d\theta}{dt} + F_D \sin (\Omega_D t)$
 $\frac{2r}{2t} \approx m \frac{d^2}{dt^2} (l\theta) = m l \frac{d^2 \theta}{dt^2};$
 $\frac{2r}{2t} = m l \frac{d^2 \theta}{dt^2} \approx -mg \theta - kl \frac{d\theta}{dt}$
 $\frac{g}{dt} \theta - q \frac{d\theta}{dt} + \frac{F_D \sin (\Omega$ ding driving force

illator: forced osc
 $-m g \sin \theta - kl (d\theta/dt) + F_D \sin(\Omega$
 $\sin \theta - kl \frac{d\theta}{dt} + F_D \sin (\Omega_D t)$
 $\frac{d^2 \theta}{dt^2} = m \frac{d^2 \theta}{dt^2} (l\theta) = m l \frac{d^2 \theta}{dt^2};$
 $\frac{d^2 r}{dt^2} = m l \frac{d^2 \theta}{dt^2} \approx -mg \theta - kl \frac{d\theta}{dt}$
 $\frac{d^2 \theta}{dt^2} = -mg \theta - kl \frac$ ding drivil

cillator: fo
 $-m g \sin \theta - kl \frac{d \theta}{dt} +$
 $\frac{2r}{2t} \approx m \frac{d^2}{dt^2} (l \theta)$
 $\frac{2r}{2t} = m l \frac{d^2 \theta}{dt^2} \approx - \frac{g}{dt} \theta - q \frac{d \theta}{dt} + \frac{F_D}{dt}$ ding drivir

cillator: for
 $-m g \sin \theta$ - kl (dt
 $\sin \theta$ - kl $\frac{d \theta}{dt}$ + B
 $\frac{r}{2t} \approx m \frac{d^2}{dt^2} (l\theta) =$
 $\frac{r^2}{2t} = m l \frac{d^2 \theta}{dt^2} \approx -1$
 $\frac{g}{dt} \theta - q \frac{d \theta}{dt} + \frac{F_D}{dt}$ ding driving force to the damped
illator: forced oscillator
 $-m g \sin \theta - kl \frac{d\theta}{dt} + F_D \sin(\Omega_D t) \frac{\Omega_D f$ requency of
 $\sin \theta - kl \frac{d\theta}{dt} + F_D \sin(\Omega_D t) \approx -mg \theta - kl \frac{d\theta}{dt} + F_D \sin(\Omega_D t);$
 $\frac{r}{t} \approx m \frac{d^2}{dt^2} (l\theta) = m l \frac{d^2\theta}{dt^2};$
 $\frac{r}{t} =$ the dator
 Ω_D frequence

the applied j
 $g \theta - k l \frac{d \theta}{dt}$
 $\sin (\Omega_D t)$ **ightary**
 s in (2 D)
 $\frac{d^2\theta}{dt^2}$
 $= m l \frac{d^2\theta}{dt^2}$
 $= m l \frac{d^2\theta}{dt^2}$
 $= m l \frac{d^2\theta}{dt^2}$ **g force to the damped

ced oscillator**
 $\frac{d}{dt} + F_D \sin(\Omega_D t)$ $\frac{\Omega_D frequency \ of}{the applied force}$
 $\frac{d}{dt} \sin(\Omega_D t) \approx -mg\theta - kl \frac{d\theta}{dt} + F_D \sin(\Omega_D t);$
 $m l \frac{d^2\theta}{dt^2};$
 $n g \theta - kl \frac{d\theta}{dt} + F_D \sin(\Omega_D t)$
 $\frac{\sin(\Omega_D t)}{dt}; q = \frac{k}{t}$ **ator**
 Ω_D *frequency of*
 n g θ *- kl* $\frac{d\theta}{dt}$ + *F*_{*l*}
 p sin ($\Omega_p t$) *D* **COPPERENT DATA**
 D **COPPERENT A**
 D $\frac{d\theta}{dt}$ + $F_D \sin(\Omega_D t) \approx$
 $F_D \sin(\Omega_D t) \approx$
 $F_D \sin(\Omega_D t) \sin(\Omega_D t)$
 D $\frac{d\theta}{dt}$ + $m \ln \theta = k l \frac{d\theta}{dt} + m \frac{d\theta}{dt}$
 D $\frac{d\theta}{dt}$ = $m \ln \theta = k l \frac{d\theta}{dt}$
 D $m \ln \theta = k l \frac{d\theta}{dt}$ *dding driving force to the damped*
 oscillator: forced oscillator
 $F_{\theta} = -m g \sin \theta - kl (d\theta/dt) + F_D \sin(\Omega_D t) - \frac{\Omega_D freeuency of}{the applied force}$
 $F_{\theta} = -m g \sin \theta - kl \frac{d \theta}{dt} + F_D \sin (\Omega_D t) \approx -m g \theta - kl \frac{d \theta}{dt} + F_D \sin (\Omega_D t);$
 $F_{\theta} = m \frac{d^2 r}{d^2 t} \approx m \frac{d^2 \$ **Adding driving force to the data of the dividend in the matrice of the dividend of** $F_{\theta} = -mg \sin \theta - kl (d\theta/dt) + F_D \sin(\Omega_D t)$ $\frac{\Omega_D f$ **requeres** $F_{\theta} = -mg \sin \theta - kl \frac{d\theta}{dt} + F_D \sin (\Omega_D t) \approx -mg \theta - kl \frac{d}{dt}$ **
 F_{\theta} = m \frac{d^2 r}{d^2 t} \approx m \frac{d^2}{dt^** iving force to the damped

forced oscillator
 $k l (d\theta/dt) + F_D \sin(\Omega_D t)$ $\Omega_D frequency of$
 $\frac{\theta}{t} + F_D \sin(\Omega_D t) \approx -mg \theta - kl \frac{d\theta}{dt} + F_D \sin(\Omega_D t);$
 $l\theta$) = $m l \frac{d^2 \theta}{dt^2};$
 $\Rightarrow -mg \theta - kl \frac{d\theta}{dt} + F_D \sin(\Omega_D t)$
 $\frac{F_D \sin(\Omega_D t)}{m l}; q = \frac{k}{m}$ ator: forced oscillator

g sin θ - kl (d0/dt) + F_D sin($\Omega_D t$) $\Omega_D f$ requency of
 θ - kl $\frac{d\theta}{dt}$ + F_D sin($\Omega_D t$) ∞ - $m g \theta$ - kl $\frac{d\theta}{dt}$ + F_D sin($\Omega_D t$);
 $m \frac{d^2}{dt^2}$ ($i\theta$) = $m l \frac{d^2\theta}{dt^2$ Fiving force to the damped

: forced oscillator
 $-kl \frac{d\theta}{dt} + F_D \sin(\Omega_D t)$ $\frac{\Omega_D frequency \theta f}{d t}$
 $\frac{d\theta}{dt} + F_D \sin(\Omega_D t) \approx -mg\theta - kl \frac{d\theta}{dt} + F_D \sin(\Omega_D t)$
 $\cdot (l\theta) = ml \frac{d^2\theta}{dt^2}$;
 $\frac{\theta}{2} \approx -mg\theta - kl \frac{d\theta}{dt} + F_D \sin(\Omega_D t)$
 $+ \frac{F_D \sin(\Omega_D t)}{ml$ Adding driving force to the damped

oscillator: forced oscillator
 $F_0 = -m g \sin \theta - kl (d\theta/dt) + F_D \sin(\Omega_D t) \frac{\Omega_D f \text{ requires } \theta_f}{\theta h \text{ applied force}}$
 $= -m g \sin \theta - kl \frac{d\theta}{dt} + F_D \sin (\Omega_D t) \approx -m g \theta - kl \frac{d\theta}{dt} + F_D \sin (\Omega_D t);$
 $= m \frac{d^2 r}{d^2 t} \approx m \frac{d^2}{dt^2} (l\theta)$ Adding driving force to the damped

oscillator: forced oscillator
 $F_e = -m g \sin \theta - kl (d\theta/dt) + F_D \sin(\Omega_D t)$ $\frac{\Omega_D frequency of}{the applied force}$
 $= -m g \sin \theta - kl \frac{d\theta}{dt} + F_D \sin (\Omega_D t) \approx -m g \theta - kl \frac{d\theta}{dt} + F_D \sin (\Omega_D t);$
 $= m \frac{d^2 r}{d^2 t} \approx m \frac{d^2}{dt^2} (l\theta) = m l \frac{d^2 \$ Adding driving force to the damped

oscillator: forced oscillator
 $F_e = -m g \sin \theta - kl (d\theta/dt) + F_D \sin(\Omega_D t)$ Ω_Df requency of
 $F_e = -m g \sin \theta - kl (d\theta/dt) + F_D \sin(\Omega_D t) \approx -m g \theta - kl \frac{d \theta}{dt} + F_D \sin (\Omega_D t)$
 $= m \frac{d^2 r}{d^2 t} \approx m \frac{d^2}{dt^2} (l\theta) = m l \frac{$ Adding driving force to the damped

oscillator: forced oscillator
 $F_e = -m g \sin \theta - kl (d\theta/dt) + F_D \sin(\Omega_D t)$ Ω_Df requency of
 $F_e = -m g \sin \theta - kl (d\theta/dt) + F_D \sin(\Omega_D t) \approx -m g \theta - kl \frac{d \theta}{dt} + F_D \sin (\Omega_D t)$
 $= m \frac{d^2 r}{d^2 t} \approx m \frac{d^2}{dt^2} (l\theta) = m l \frac{$ Adding driving force to the damped

Dscillator: forced oscillator
 $\hat{b}_b = -m g \sin \theta - kl (d\theta/dt) + F_D \sin(\Omega_D t)$ $\frac{\Omega_D frequency \text{ of}}{\text{the applied force}}$
 $-m g \sin \theta - kl \frac{d\theta}{dt} + F_D \sin (\Omega_D t) \approx -m g \theta - kl \frac{d\theta}{dt} + F_D \sin (\Omega_D t);$
 $m \frac{d^2 r}{d^2 t} \approx m \frac{d^2}{dt^2} (l\theta)$

 $F_{\theta} = -m g \sin \theta - kl (d\theta/dt) + F_D \sin(\Omega_D t)$ Ω_D *frequency of the applied force*

 θ_{θ} = -mg sin θ - kl - + F_{D} sin $(\Omega_{D} t) \approx$ -mg θ - kl - + F_{D} sin $(\Omega_{D} t)$;

$$
F_{\theta} = m \frac{d^2 r}{d^2 t} \approx m \frac{d^2}{dt^2} (l\theta) = m l \frac{d^2 \theta}{dt^2};
$$

$$
F_{\theta} = m \frac{d^2 r}{d^2 t} = m l \frac{d^2 \theta}{dt^2} \approx -mg \theta - kl \frac{d \theta}{dt} + F_D \sin(\Omega_D t)
$$

$$
\frac{d^2\theta}{dt^2} \approx -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{ml}; q = \frac{k}{m}
$$

Analytical solution

Analytical solution
\n
$$
\theta(t) = \theta_0 \sin(\Omega_D t + \phi)
$$
\n
$$
\theta_0 = \frac{F_D / (m l)}{\sqrt{(\Omega^2 - \Omega_D^2)^2 + (q \Omega_D)^2}}
$$
\nonance happens when $\Omega_D = \Omega = \sqrt{g/l}$

Resonance happens when $\Omega_D = \Omega = \sqrt{g/l}$ $\frac{1}{\sqrt{2}}$
 $\Omega_p = \Omega = \sqrt{g/l}$

Forced oscillator: An example of non homogeneous 2nd order DE

ced oscillator: An example of homogeneous 2nd order DE

\n
$$
\frac{d^{2}u(x)}{dx^{2}} + a\frac{du(x)}{dx} + bu(x) = n(x)
$$
\n
$$
x \equiv t
$$
\n
$$
u(x) \equiv \theta(t)
$$
\n
$$
a \equiv q
$$
\n
$$
b \equiv \frac{g}{l}
$$
\n
$$
n(x) \equiv \frac{F_{D}\sin(\Omega_{D}t)}{ml}
$$
\n
$$
\frac{d^{2}\theta}{dt^{2}} = -\frac{g}{l} \theta - q\frac{d\theta}{dt} + \frac{F_{D}\sin(\Omega_{D}t)}{ml}
$$

Exercise: Forced oscillator

$$
\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{ml}
$$

Solution: Forced oscillate
 $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{m l}$

e to solve the forced oscillator. Plot of

analytical solutions of $\theta(t)$ for t fron
 $\frac{2\pi}{\Omega}$, $\Omega = \sqrt{g/l}$, for $\Omega_D = 0.01\Omega$, 0.5. $\begin{aligned} \mathbf{H}^{2}\theta &= -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_{D}\sin(\Omega_{D}t)}{m\theta} \\ \text{Let } & \theta & \text{ is a constant.} \\ \text{and, the force of the force of the direction, and the force is given by the equation: \\ & \mathbf{H}^{2}\pi/\Omega, \ \Omega & = \sqrt{g/l}, \ \text{for} \ \Omega_{D} = 0.01. \\ & \text{Assume the boundary condition} \end{aligned}$ **Se:** Forced oscillator
 $\frac{\theta}{2} = -\frac{g}{l} \theta - q \frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{m l}$

so solve the forced oscillator. Plot on the same

solve the forced oscillator. Plot on the same

salytical solutions of $\theta(t)$ for t from o to 10 **2:** Forced oscillator
= $-\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{m l}$
solve the forced oscillator. Plot on the same
ytical solutions of $\theta(t)$ for t from o to 10 T,
 θ , $\Omega = \sqrt{g/l}$, for $\Omega_D = 0.01\Omega$, 0.5 Ω , 0.99 Ω , Use DSolve to solve the forced oscillator. Plot on the same graph the analytical solutions of $\theta(t)$ for *t* from o to 10 *T*, where *T*= $2\pi/\Omega$, $\Omega = \sqrt{g/l}$, for $\Omega_D = 0.01\Omega$, 0.5 Ω , 0.99 Ω , 1.5 Ω , 4 Ω . Assume the boundary conditions $\theta(t=0)=0$; $d\theta/dt(t=0)=0$; $m=l=F_D=1$; $q=0$.

See forced_Pendulum.nb.

Second order Runge-Kutta (RK2) method

Consider a generic second order differential equation. *fa* (RK2)
G (*u*)
G (*u*)
sing secon
dit the seco

 2 2 *d u x d x*

der Runge
 *d*²*u* (*x*
 *d*²
 *d*² **paramidier and providing**
 *v**x**x**y**(x)* **=** $\frac{1}{2}$ *d* 2 *d*
 d 3 *d x*
 d x (RK2) m
 d d d d d v (*x*)
 d v (*x*) = *d x* ntial
order
d order DI
G (*u*) *differential*
 second order
 $\frac{y(x)}{dx} = G(u)$ It can be numerically solved using second order Runge-Kutta method. First, split the second order DE into two first order parts:

$$
v(x) = \frac{du(x)}{dx} \qquad \qquad \frac{dv(x)}{dx} = G(u)
$$

Algorithm

Set boundary conditions: $u(x=x_0)=u_0$, $u'(x=x_0)=v(x=x_0)=v_0$.

Translating the SK2 algorithm into the case of simple pendulum K2 algorithm
 $G(u) = -\frac{g}{l} \theta(t)$

l

translating the SK2 algorithm in
\n**mple pendulum**
\n
$$
\frac{d^{2}u(x)}{dx^{2}} = G(u)
$$
\n
$$
G(u) = -\frac{g}{l}\theta(t)
$$
\n
$$
F(x) = \frac{du(x)}{dx}
$$
\n
$$
\frac{dv(x)}{dx} = G(u)
$$
\n
$$
G(u) = -\frac{g}{l}\theta(t)
$$
\n
$$
F(x) = \frac{du(x)}{dx}
$$
\nboundary conditions:

\nSet by
\n
$$
x_{0} = u_{0} + \frac{1}{2}v_{i}\Delta x
$$

\n
$$
\tilde{v} = v_{i} + \frac{1}{2}G(\tilde{u})\Delta x
$$
\n
$$
\tilde{\omega} = \tilde{u}
$$

Set boundary conditions: $u(x=x_0) = u_0, u'(x=x_0) = v(x=x_0) = v_0$

$$
v(x) = \frac{du(x)}{dx}
$$

\n
$$
\frac{dv(x)}{dx} = G(u)
$$

\nboundary conditions:
\n
$$
=x_o
$$

$$
=u_o, u'(x=x_o) = v(x=x)
$$

\n
$$
\tilde{u} = u_i + \frac{1}{2}v_i \Delta x
$$

\n
$$
\tilde{v} = v_i + \frac{1}{2}G(\tilde{u})\Delta x
$$

\n
$$
u_{i+1} = u_i + v \Delta x
$$

\n
$$
v_{i+1} = v_i + G(\tilde{u}) \Delta x
$$

 $v(x) = \frac{du(x)}{dx}$
 $\frac{dv(x)}{dx} = G(u)$
 ooundary conditions:
 $\tilde{x}_0 = u_i, u^2(x=x_0) = v(x=x_0) = v_0$
 $\tilde{u} = u_i + \frac{1}{2}v_i\Delta x$
 $\tilde{v} = v_i + \frac{1}{2}G(\tilde{u})\Delta x$
 $u_{i+1} = u_i + \tilde{v}\Delta x$
 $\tilde{v}_{i+1} = v_i + G(\tilde{u})\Delta x$
 $\tilde{\omega}_{i+1} = v_i + G(\tilde{u})\$ $d^2\theta(t)$ dt^2 = − $g\theta$ $\mathcal{I}_{\mathcal{I}}$ (t) to the
 $\frac{d^2\theta(t)}{dt^2} =$
(*t*) = $\frac{d\theta(t)}{dt}$ he cas
 $\frac{d\theta}{dt} = -\frac{g\theta}{l}$ $t) = -$ **0 the case**
 $\frac{\theta(t)}{dt^2} = -\frac{g\theta}{l}$
 $\frac{d\theta(t)}{dt}$
 $\frac{\theta(t)}{dt} = -\frac{g\theta(t)}{l}$
 andary condition
 $\theta_o, \theta'(t=t_o) = \omega(t_o)$ $\theta(t)$ $\omega(t) = \frac{\sqrt{t}}{t}$ **10 the case of**
 $\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l}$
 $\frac{d\theta(t)}{dt} = -\frac{g\theta(t)}{l}$
 $\frac{d\omega(t)}{dt} = -\frac{g\theta(t)}{l}$

oundary conditions: to the case of
 $\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l}$
 $\frac{d\theta(t)}{dt} = -\frac{g\theta(t)}{l}$

undary conditions:
 $\frac{g\theta(t)}{dt} = -\frac{g\theta(t)}{l}$ Set boundary conditions: $\theta(t=t_o) = \theta_o, \ \theta'(t=t_o) = \omega(t=t_o) = \omega_o$ 1 2 $\ddot{}$ *dt*² *l*
 $\omega(t) = \frac{d\theta(t)}{dt}$
 $\frac{d\omega(t)}{dt} = -\frac{g\theta(t)}{l}$

boundary conditions:
 $= t_o$) = θ_o , $\theta'(t=t_o) = \omega(t=t_o) = \omega_o$
 $\tilde{\theta} = \theta_i + \frac{1}{2}\omega_i \Delta t$
 $\tilde{\omega} = \omega_i + \frac{1}{2} \left(-\frac{g\theta}{d}\right) \Delta t$ $\frac{d \omega(t)}{dt} = -\frac{g \theta(t)}{l}$
 boundary conditions:
 $= t_o$) = θ_o , $\theta'(t=t_o) = \omega(t=t_o) = \omega_o$
 $\tilde{\theta} = \theta_i + \frac{1}{2}\omega_i \Delta t$
 $\tilde{\omega} = \omega_i + \frac{1}{2}\left(-\frac{g \theta}{l}\right) \Delta t$
 $\theta_{i+1} = \theta_i + \tilde{\omega} \Delta t$
 $\theta_{i+1} = \omega_i + \left(-\frac{g \theta}{l}\right) \Delta t$ $\begin{array}{cc} i & 2 & l \end{array}$ $g(\theta)$ | *t l* θ) $rac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l}$
 $\omega(t) = \frac{d\theta(t)}{dt}$
 $rac{d\omega(t)}{dt} = -\frac{g\theta(t)}{l}$

boundary conditions:
 $= t_o$) = θ_o , $\theta'(t=t_o) = \omega(t=t_o) = \omega_o$
 $\tilde{\theta} = \theta_i + \frac{1}{2}\omega_i \Delta t$
 $\tilde{\omega} = \omega_i + \frac{1}{2}\left(-\frac{g\tilde{\theta}}{l}\right)\Delta t$
 $\theta_{i+1} = \theta_i + \tilde{\$ = $-\frac{g\theta}{l}$
 $\frac{g\theta(t)}{dt}$
 $-\frac{g\theta(t)}{l}$

r conditions:
 $(t=t_o) = \omega(t=t_o) = \omega_o$
 $\frac{\omega_i \Delta t}{l}$
 $\frac{g\theta}{l} \Delta t$ $\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l}$
 $\omega(t) = \frac{d\theta(t)}{dt}$
 $\omega(t) = -\frac{g\theta(t)}{l}$

oundary conditions:
 $\omega(t) = \theta_0, \theta'(t=t_0) = \omega(t=t_0) = \omega_0$
 $= \theta_i + \frac{1}{2}\omega_i \Delta t$
 $= \omega_i + \frac{1}{2}\left(-\frac{g\theta}{l}\right)\Delta t$
 $= \theta_i + \omega \Delta t$ = $-\frac{\partial}{\partial t}$
 $\frac{g \theta(t)}{dt}$
 $-\frac{g \theta(t)}{t}$
 $\frac{g}{dt}$
 $\frac{f}{dt}$
 $\frac{f}{dt}$
 $\frac{g \theta}{dt}$
 $\frac{g \theta}{t}$
 $\frac{g \theta}{t}$
 $\frac{g \theta}{t}$
 $\frac{g \theta}{t}$ **boundary**
 $i t_o$) = θ_o , θ
 $\tilde{\theta} = \theta_i + \frac{1}{2} \theta_i$
 $\tilde{\theta} = \omega_i + \frac{1}{2} \theta_i + \theta_i$
 $\theta_{i+1} = \theta_i + \theta_i$ $g(\theta)$ | *t l* θ) $\omega(t) = \frac{d\omega(t)}{dt}$
 $\frac{d\omega(t)}{dt} = -\frac{g\theta(t)}{t}$

t boundary conditions:
 $(t=t_0) = \theta_0$, $\theta'(t=t_0) = \omega(t=t_0) = \omega_0$
 $\tilde{\theta} = \theta_i + \frac{1}{2}\omega_i \Delta t$
 $\tilde{\omega} = \omega_i + \frac{1}{2}\left(-\frac{g\tilde{\theta}}{t}\right) \Delta t$
 $\theta_{i+1} = \theta_i + \tilde{\omega} \Delta t$
 $\omega_{i+1} = \omega_i + \left(\frac{\theta(t)}{dt}$
 $-\frac{g\theta(t)}{l}$

y conditions:
 $\theta'(t=t_o) = \omega(t=t_o) = \omega_o$
 $\omega_i \Delta t$
 $-\left(-\frac{g\theta}{l}\right) \Delta t$
 $\omega \Delta t$
 $\left(-\frac{g\theta}{l}\right) \Delta t$ (t) = $\frac{dV}{dt}$
 $\frac{\omega(t)}{dt} = -\frac{g\theta(t)}{t}$

undary conditions:

= θ_o , $\theta'(t=t_o) = \omega(t=t_o) = \omega_o$
 $\theta_i + \frac{1}{2}\omega_i \Delta t$
 $\omega_i + \frac{1}{2}\left(-\frac{g\theta}{t}\right) \Delta t$

= $\theta_i + \omega \Delta t$

= $\omega_i + \left(-\frac{g\theta}{t}\right) \Delta t$ dt
 $-\frac{g \theta(t)}{l}$

y conditions:
 $y'(t=t_0) = \omega(t=t_0) = \omega_0$
 $\omega_i \Delta t$
 $-\left(-\frac{g \theta}{l}\right) \Delta t$
 $\omega \Delta t$
 $\left(-\frac{g \theta}{l}\right) \Delta t$ algorithm into the case of
= $-\frac{g}{l} \theta(t)$ $\frac{d^2 \theta(t)}{dt^2} = -\frac{g\theta}{l}$

Exercise: Develop a code to implement SK2 for the case of the simple pendulum. Boundary conditions: $\omega(0) = \sqrt{\frac{g}{l}}$; $\theta(0)$ le to implement !
ble p<u>e</u>ndulum.
 $(0) = \sqrt{\frac{g}{l}}$; $\theta(0) = 0$ de to implement SK2

uple pendulum.
 $\omega(0) = \sqrt{\frac{g}{l}}$: $\theta(0) = 0$

See [pendulum_RK2.nb](http://www2.fizik.usm.my/tlyoon/teaching/ZCE111/1415SEM2/notes/pendulum_RK2.nb)

Translating the SK2 algorithm into the case of damped pendulum 2 **he cas**
 $\frac{\partial(t)}{\partial t} = -t$
 $\frac{d\theta(t)}{dt}$ ase of
 $-\frac{g\theta}{l}$ –
 $q\omega(t)$
ditions:
 $e^{j\omega(t-t_0)}$

i

i i u u v x () 2 *d u x G u d x d u x v x d x d v x G u d x* () () *g G u t q t l* ^q ^w

Set boundary conditions:
\n
$$
u(x=x_0)=u_0, u'(x=x_0)=v(x=x_0)=v_0
$$

\n $\tilde{u} = u_1 + \frac{1}{2}v_1\Delta x$

$$
\tilde{v} = v_i + \frac{1}{2} G (\tilde{u}) \Delta x
$$

$$
u_{i+1} = u_i + \tilde{v} \Delta x
$$

$$
v = v_i + G (\tilde{u}) \Delta x
$$

$$
\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l} - q\frac{d\theta}{dt}
$$

into the case of

\n
$$
\frac{d^{2}\theta(t)}{dt^{2}} = -\frac{g\theta}{l} - q\frac{d\theta}{dt}
$$
\n
$$
\omega(t) = \frac{d\theta(t)}{dt}
$$
\n
$$
\frac{d\omega(t)}{dt} = -\frac{g\theta}{l} - q\omega(t)
$$
\nboundary conditions:

\n
$$
= t_{o} = \theta_{o}, \ \theta'(t = t_{o}) = \omega(t = t_{o}) = \omega_{o}
$$

 $v(x) = \frac{du(x)}{dx}$
 $\frac{dv(x)}{dx} = G(u)$
 $\frac{dv(x)}{dx} = G(u)$
 $\frac{d\theta(t)}{dt} = -\frac{g\theta}{t} - q\omega(t)$
 boundary conditions:
 $x_0 = u_0, u'(x=x_0) = v(x=x_0) = v_0$
 $\frac{d\theta(t)}{dt} = -\frac{g\theta}{t} - q\omega(t)$
 $\frac{d\theta(t)}{dt} = -\frac{g\theta}{t} - q\omega(t)$
 boundary conditions Set boundary conditions: $\theta(t=t_o) = \theta_o, \ \theta'(t=t_o) = \omega(t=t_o) = \omega_o$ $\tilde{\theta} = \theta_i + \frac{1}{2} \omega_i \Delta t$ 2 $($ $\frac{d}{dt^2} - \frac{d\theta(t)}{dt} - q \frac{d\theta(t)}{dt}$
 $\frac{d\theta(t)}{dt} = -\frac{g\theta}{l} - q\omega(t)$

boundary conditions:
 t_o) = θ_o , $\theta'(t=t_o) = \omega(t=t_o) = \omega_o$
 $\theta = \theta_i + \frac{1}{2}\omega_i\Delta t$
 $\theta_i - \frac{g}{2l}\theta(t)\Delta t$ *i i* 1 (*i i j j i t*₀)= θ_0 , θ' (*t*=*t*₀)= ω ₀ θ ₀ θ _{*i*} θ = θ_i + $\frac{1}{2}\omega_i\Delta t$
 $-\theta_i \hat{\theta}$ $\Delta t \Rightarrow \hat{\omega} = \frac{\left(\omega_i - \frac{g}{2l}\hat{\theta}(t)\Delta t\right)}{\left(1 + \frac{1}{2}q\Delta t\right)}$
 $\theta_{i+1} = \theta_i + \hat{\omega}\Delta t$
 $rac{d\theta}{dt}$
= ω_o
 $(\theta \Delta t)$ $1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ \sim $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ m into
 $\omega(t)$
 $\omega(t)$
 $\frac{d\omega(t)}{dt}$

Set bour
 $\theta(t) = \frac{1}{\omega}$
 $\theta(t) - q\omega$
 $\theta_{i+1} = \omega_i$ 2 $\begin{pmatrix} 1 & 1 \end{pmatrix}$ $1+ -q \Delta t$ 2) *i g* \sim) $\frac{d\theta}{dt}$
 $\frac{d\theta}{dt}$
 $\frac{d\theta}{dt}$ $g \left[\begin{array}{cc} 2l & -l \end{array} \right]$ \sim $\left[\begin{array}{cc} l & 2l \end{array} \right]$ **n** into the
 $\frac{d^2\theta(t)}{dt^2}$
 $\omega(t) = \frac{\frac{d\theta(t)}{dt}}{\frac{d\theta(t)}{dt}} = -\frac{g\theta}{l}$

et boundary co
 $(t=t_o) = \theta_o, \theta'(t-\theta_o)$
 $\theta = \theta_i + \frac{1}{2}\omega_i$
 $t = \theta_i + \omega \Delta$
 $\theta_{i+1} = \theta_i + \omega \Delta$
 $\theta_{i+1} = \omega_i + \left(-\frac{g\theta}{l}\right)$ *l* $q \frac{d\theta}{dt}$
 $(\vec{\theta}(t) \Delta t)$
 $q \Delta t$ Se of
 $-\frac{g\theta}{l} - q\frac{d\theta}{dt}$
 $=\omega(t=t_o) = \omega_o$
 $\omega_i - \frac{g}{2l}\tilde{\theta}(t)\Delta t$
 $\left(1 + \frac{1}{2}q\Delta t\right)$ **12 algorithm into the case of**
 n
 $\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l} - q\frac{d\theta}{dt}$
 $\frac{\omega(t)}{dt^2} = -\frac{g\theta}{l} - q\frac{d\theta}{dt}$
 $\frac{\omega(t)}{t} = -\frac{\omega(t)}{l}$

Set boundary conditions:
 $\omega(t) = \omega_0 + \frac{1}{2} \left(-\frac{\omega(t)}{l} - \frac{\omega(t)}{l} \right)$

Set boun 2 algorithm into the case of
 $rac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l} - q\frac{d\theta}{dt}$
 $\omega(t) = \frac{d\theta(t)}{dt}$
 $\frac{\omega(t)}{dt} = -\frac{g\theta}{l} - q\omega(t)$

Set boundary conditions:
 $\theta(t=t_0) = \theta_0$, $\theta'(t=t_0) = \omega(t=t_0) = \omega_0$
 $\hat{\theta} = \theta_1 + \frac{1}{2}\omega_1\Delta t$
 $= \omega_1$ dt
 i et bound
 $\theta(t=t_o) = \theta_o$
 $\hat{\theta} = \theta_i$
 $(t) - q\hat{\omega} \Delta t$
 $\theta_{i+1} = \theta_i$
 $\theta_{i+1} = \omega_i + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $g \theta$ \sim $\frac{1}{2}$ *q* ω (*t*)

ditions:
 ω)= $\omega(t=t_o)$ =
 t
 $\omega_i - \frac{g}{2l} \tilde{\theta}$ (
 $\left(1 + \frac{1}{2} q l\right)$
 $q \tilde{\omega}$) Δt *l* θ ~ 1 $rac{d\theta(t)}{dt^2} = -\frac{d\theta(t)}{dt}$
 $\phi(t) = \frac{d\theta(t)}{dt}$
 $\left[\frac{d\phi(t)}{dt} = -\frac{g\theta}{t} - q\omega(t)\right]$

Set boundary conditions:
 $\theta(t=t_0) = \theta_0, \theta'(t=t_0) = \omega_0(t=t_0) = \omega_0$
 $\hat{\theta} = \theta_i + \frac{1}{2}\omega_i\Delta t$
 $\hat{\theta}(t) - q\hat{\omega}\right)\Delta t \Rightarrow \hat{\omega} = \frac{\left(\omega_i - \frac{g}{2t}\hat$ $\frac{\partial \theta(t)}{\partial t^2} = -\frac{\partial \theta}{l} - q \frac{d\theta}{dt}$
 $= -\frac{\frac{d\theta(t)}{dt}}{l}$
 $= -\frac{\frac{g\theta}{l}}{l} - q\omega(t)$

dary conditions:
 $\frac{1}{\rho}, \theta'(t=t_o) = \omega(t=t_o) = \omega_o$
 $\frac{1}{t} - \omega_i \Delta t$
 $\Delta t \approx \omega \omega \frac{d\theta(t)}{dt}$
 $\frac{1}{t} + \omega \Delta t$
 $\left(-\frac{g\theta}{l} - q\omega\right) \Delta t$ $\frac{d\theta(t)}{dt^2} = -\frac{1}{l} - q\frac{d\theta(t)}{dt}$
 $\frac{d\theta(t)}{dt} = -\frac{g\theta}{l} - q\omega(t)$

boundary conditions:
 t_o) = θ_o , $\theta'(t=t_o) = \omega(t=t_o) = \omega_o$
 $\theta = \theta_i + \frac{1}{2}\omega_i\Delta t$
 $\theta = \theta_i + \frac{1}{2}\omega_i\Delta t$
 $\left(1 + \frac{1}{2}q\Delta t\right)$
 $\theta_{i+1} = \theta_i + \omega_i\Delta t$

Exercise:

Develop a code to implement SK2 for the case of a pendulum experiencing a drag force, with damping coefficient $q = 0.1^*$ (4 Ω), $\Omega = \sqrt{g/l}$, $l = 1.0$ m. Boundary conditions: $\theta(0) = 0.2$; $\omega(t = 0) = 0$; *d* implement SK2 for

encing a drag force,

* (4Ω), Ω= $\sqrt{g/l}$, l =

ons: $\theta(0) = 0.2$; ω(
 $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt}$ *d* implement SK2 for
 α the time a drag force
 α^* (42), $\Omega = \sqrt{g/l}$,
 α ons: $\theta(0) = 0.2$; α
 $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt}$
 α
 α , α , α implement SK2 for the case of a

ncing a drag force, with damping

(4 Ω), $\Omega = \sqrt{g/l}$, $l = 1.0$ m.

ns: $\theta(0) = 0.2$; $\omega(t = 0) = 0$;
 $\frac{\theta}{2} = -\frac{g}{l} \theta - q \frac{d\theta}{dt}$

$$
\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\,\frac{d\theta}{dt}
$$

See [pendulum_RK2.nb](http://www2.fizik.usm.my/tlyoon/teaching/ZCE111/1415SEM2/notes/pendulum_RK2.nb)

Exercise:

Develop a code to implement SK2 for the case of a forced pendulum experiencing no drag force but a driving force $F_D \sin(\Omega_D t)$, $\Omega = \sqrt{g/l}$, $l = 1.0$ m, $m=1$ kg; F_D = 1N; Ω_D =0.99 Ω ; Boundary conditions: $\theta(0) = 0.0$; $\omega(t = 0) = 0$; **E**:

s code to implement SK2 for the case of a

ndulum experiencing no drag force but a

rce $F_D \sin(\Omega_D t)$, $\Omega = \sqrt{g/l}$, $l = 1.0$ m,
 $D = 1$ N; $\Omega_D = 0.99 \Omega$;

conditions: $\theta(0) = 0.0$; $\omega(t = 0) = 0$;
 $\frac{d^2 \theta}{dt^2} = -\frac{g}{l}$ 2:

code to implement SK2 for the case

dulum experiencing no drag force b

ce $F_D \sin(\Omega_D t)$, $\Omega = \sqrt{g/l}$, $l = 1.0$ m
 $\eta = 1$ N; $\Omega_D = 0.99 \Omega$;

conditions: $\theta(0) = 0.0$; $\omega(t = 0) =$
 $\frac{d^2 \theta}{dt^2} = -\frac{g}{l} \theta - q \frac{d\theta}{dt} + \frac{F$ ode to implement SK2 for the case of a

ulum experiencing no drag force but a
 $F_D \sin(\Omega_D t)$, $\Omega = \sqrt{g/l}$, $l = 1.0$ m,

1N; $\Omega_D = 0.99 \Omega$;

nditions: $\theta(0) = 0.0$; $\omega(t = 0) = 0$;
 $\frac{\theta}{2} = -\frac{g}{l} \theta - q \frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{$ to implement SK2 for the case of a

m experiencing no drag force but a

sin($\Omega_D t$), $\Omega = \sqrt{g/l}$, $l = 1.0$ m,

V; Ω_D =0.99 Ω ;

itions: $\theta(0) = 0.0$; $\omega(t = 0) = 0$;
 $= -\frac{g}{l} \theta - q \frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{m l}$

$$
\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{ml}
$$

Exercise: Stability of the total energy a SHO in RK2.

$$
\omega = \frac{d\theta}{dt}
$$
, angular velocity. *M*=1kg; *l*=1m.

$$
E_{i+1} = K_{i+1} + U_{i+1} = \frac{1}{2} m \left(l\omega_{i+1} \right)^2 + mgl \left(1 - \cos \theta_{i+1} \right)
$$

$$
\approx \frac{1}{2} ml^2 \omega_{i+1}^2 + mgl \left[1 - \left(1 - \frac{\theta_{i+1}^2}{2} \right) \right]
$$

$$
= \frac{1}{2} ml^2 \omega_{i+1}^2 + \frac{1}{2} mgl \theta_{i+1}^2
$$

User your RK2 code to track the total energy for *t* running from *t*=0 till t=25*T*; *T*= $\sqrt{g/l}$. Boundary conditions: E_{i} should remain constant throughout all t_{i} .

ulated as

v for *t* running

litions: $\omega(0) = \sqrt{\frac{g}{l}}$; $\theta(0) = 0$

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unning
 $\omega(0) = \sqrt{\frac{g}{l}}$; $\theta(0) = 0$

Velocity verlet algorithm for solving the Newton second law

Newton's second law

•Given a position-dependent force acting on a particle, **F**(**r**), Newton second law determines the acceleration of a particle, **a**.

d 2 **r**/d*t* ²**= F**(**r**)/*m*

- •This is a second order differential equation.
- •Given **F**(**r**), we wish to know what are the subsequent evolution of **r**(*t*), **v**(*t*) beginning from the boundary values of **r**(0), **v**(0).
- •Previously we solve the second order DE using RK2 to obtain **r**(*t*), **v**(*t*).

Verlet algorithms

- •The equation d²r/dt²=**F**(r)/m can be integrated to obtain **r**(*t*), **v**(*t*), via a numerical scheme: Verlet algoritm
- •Three types: ordinary Verlet, velocity Verlet, leap frog verlet.
- •In the following, we shall denote the RHS as $\mathsf{F}(\mathsf{r})/m \to \mathsf{a}$ So that d 2 **r**/d*t* ²=**F**(**r**)/*m* **↓** d 2 **r**/d*t* ²= **a**

• See http://en.wikipedia.org/wiki/Verlet_integration

Velocity Verlet algorithm

1. Calculate:

 $\vec{x}(t+\Delta t) = \vec{x}(t) + \vec{v}(t)\Delta t + \frac{1}{2}\vec{a}(t)\Delta t^2$

- 2. Derive $\vec{a}(t + \Delta t)$ from the interaction potential using $\vec{x}(t + \Delta t)$
- 3. Calculate:

 $\vec{v}(t+\Delta t) = \vec{v}(t) + \frac{1}{2} (\vec{a}(t) + \vec{a}(t+\Delta t)) \Delta t$

Note, however, that this algorithm assumes that acceleration $\vec{a}(t + \Delta t)$ only depends on position $\vec{x}(t + \Delta t)$, and does not depend on velocity $\vec{v}(t+\Delta t)$.

Global error in velocity Verlet algorithm

•The global (cumulative) error in *x* over a constant interval of time is given by

 $\Delta x \sim O(\Delta t^2)$

•Because the velocity is determined in a noncumulative way from the positions in the Verlet integrator, the global error in velocity is also

 $\Delta v \sim O(\Delta t^2)$

Exercise: SHO

- •Solve for *x*(*t*), *v*(*t*), for a simple harmonic oscillator with constant *k*, mass *m,* initial displacement x_0 , and velocity $v_0 = 0$, using velocity Verlet algorithm.
- •For SHO, $F = -kx \Rightarrow a = -(k/m)x$

$$
d^2x/dt^2 = -(k/m)x
$$

See [verlet_algorithm_samples.nb](http://www2.fizik.usm.my/tlyoon/teaching/ZCE111/1415SEM2/notes/verlet_algorithm_samples.nb)

Exercise: 2D free-fall projectile

- •Solve for *x*(*t*), *y*(*t*), *v*(*t*) of a 2D free-fall projectile with initial speed v_0 =0 and launching angle θ_0 using velocity Verlet algorithm.
- For 2D projectile motion, $\boldsymbol{F} = m g \hat{y} + 0 \hat{x}$.

$$
\mathbf{a} = \mathbf{F}/m
$$

\n
$$
a_x \hat{x} + a_y \hat{y} = 0 \hat{x} - g \hat{y}
$$

\n
$$
d^2x/dt^2 \hat{x} + d^2y/dt^2 \hat{x} = -g \hat{y}
$$

\n
$$
\Rightarrow d^2x/dt^2 = 0, \quad d^2y/dt^2 = -g
$$

See <u>verlet</u> algorithm samples.nb

Exercise: Scattering of a projectile charge via Coulomb force

Notation for scattering of a projectile charge via Coulomb force

$$
\begin{aligned}\n\bullet \mathbf{r}_Q &= (x_Q, y_Q); \mathbf{r}_q = (x, y) \\
\bullet \mathbf{F} &= k \frac{qQ}{(x - x_Q)^2 + (y - y_Q)^2} \hat{r}; \quad \hat{r} = \frac{\mathbf{r}_q - \mathbf{r}_Q}{|\mathbf{r}_q - \mathbf{r}_Q|} \\
\bullet \Rightarrow\n\end{aligned}
$$

$$
\bullet a_y = \frac{1}{m} \mathbf{F} \cdot \hat{\mathbf{y}} = \frac{k}{m} \frac{qQ}{(x - x_Q)^2 + (y - y_Q)^2} \hat{r} \cdot \hat{\mathbf{y}} = \frac{d^2 y}{dt^2},
$$

$$
\bullet a_x = \frac{1}{m} \mathbf{F} \cdot \hat{\mathbf{x}} = \frac{k}{m} \frac{qQ}{(x - x_Q)^2 + (y - y_Q)^2} \hat{r} \cdot \hat{\mathbf{x}} = \frac{d^2 x}{dt^2}.
$$

The equations required by Verlet algorithm

$$
\boldsymbol{r}_Q = (x_Q, y_Q); \boldsymbol{r}_q = (x, y) \quad \hat{r} = \frac{\boldsymbol{r}_q - \boldsymbol{r}_Q}{|\boldsymbol{r}_q - \boldsymbol{r}_Q|}
$$

$$
\frac{d^2x}{dt^2} = \frac{kqQ}{m} \frac{\hat{r} \cdot \hat{x}}{(x - x_Q)^2 + (y - y_Q)^2}
$$

See [verlet_algorithm_2D_coulomb_scatterings.nb](http://www2.fizik.usm.my/tlyoon/teaching/ZCE111/1415SEM2/notes/verlet_algorithm_2D_coulomb_scatterings.nb)

Störmer-Verlet integration algorithm

$$
\vec{x}_{n+1} = 2\vec{x}_n - \vec{x}_{n-1} + \vec{a}_n(\Delta t)^2
$$

$$
\vec{v}_{n+1} = (\vec{x}_{n+1} - \vec{x}_n)/\Delta t
$$

- Another variant of Verlet algoritm
- Use this for integrating dynamical system with a velocity-dependent acceleration, such as Lorentz force on a moving charge particle.
- The cumulative error in the velocity is larger than that in velocity Verlet algorithm

Exercise: Charge moving in a magnetic field

- •A charge (mass *m* and charge *q*) moving with velocity **v** =(*v^x* , *v^y* , *v^z*) in a magnetic field **B**=(*B^x* , *By* , *B^z*) experiences a velocity-dependent Lorentz force $F=(F_x, F_y, F_z) = q \mathbf{v} \times \mathbf{B}$. Develop a code based on the Störmer-Verlet integration algorithm to simulate the dynamical path of the charge particle moving through the magnetic field. Assume: *q=+*1 unit, mass $m = 1$ unit, initially located at $(0,0,0)$, initial velocity (v_{0x} , v_{0y} , v_{0z}), v_{0x} = v_{0y} =0.1 unit, v_{0z} =0.05 unit, **B**=(0, 0, B_z), B_z = 0.1 unit. You should see a helical trajectory circulating about the *z*-direction.
- verlet algorithm 3D coulomb helix.nb