Chapter 9

Solving Second order differential equations numerically, 2

Online lecture materials

- The online lecture notes by Dr. Tai-Ran Hsu of San José State University,

http://www.engr.sjsu.edu/trhsu/Chapter%204%20Second%20order%20DEs.pdf

provides a very clear explanation of the solutions and applications of some typical second order differential equations.

2^{nd} Order Homogeneous DEs

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0$$

with TWO given conditions

The solutions

Case 1: $a^2 - 4b > 0$:

$$u(x) = e^{-\frac{ax}{2}}\left(c_1 e^{\sqrt{a^2 - 4b}\; x/2} + c_2 e^{-\sqrt{a^2 - 4b}\; x/2}\right)$$

Case 2: $a^2 - 4b < 0$:

$$u(x) = e^{-\frac{ax}{2}}\left[A\,Sin\left(\frac{1}{2}\sqrt{4b - a^2}\right)x + B\,Cos\left(\frac{1}{2}\sqrt{4b - a^2}\right)x\right]$$

Case 3: $a^2 - 4b = 0$: — A special case

$$u(x) = c_1 e^{-\frac{ax}{2}} + c_2 x e^{-\frac{ax}{2}} = (c_1 + c_2 x) e^{-\frac{ax}{2}} \qquad (4.12)$$

where c_1, c_2, A and B are arbitrary constants to be determined by given conditions

Example 4.1 Solve the following differential equation

$$\frac{d^2u(x)}{dx^2} + 5\frac{du(x)}{dx} + 6u(x) = 0$$

$$u(x) = e^{-5x/2}\left(c_1 e^{x/2} + c_2 e^{-x/2}\right) = c_1 e^{-2x} + c_2 e^{-3x}$$

where c_1 and c_2 are arbitrary constants to be determined by given conditions

Example 4.2

$$\frac{d^2u(x)}{dx^2} + 6\frac{du(x)}{dx} + 9u(x) = 0$$

$$u(0) = 2$$

$$\left.\frac{du(x)}{dx}\right|_{x=0} = 0$$

$$u(x) = 2(1+3x)e^{-3x}$$

DSolve

- **DSolve** of Mathematica can provide analytical solution to a generic second order differential equation. See Math_built_in_2ODE.nb.

Typical second order, non-homogeneous ordinary differential equations

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x) \qquad (4.25)$$

Non-homogeneous term

Solution of Equation (4.25) consists **TWO** components:

Solution u(x) = Complementary solution $u_h(x)$ + Particular solution $u_p(x)$

$$u(x) = u_h(x) + u_p(x)$$

Typical second order, non-homogeneous ordinary differential equations

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x) \qquad (4.25)$$

Non-homogeneous term

$u(x) = u_h(x) + u_p(x)$

$$\frac{d^2u_h(x)}{dx^2} + a\frac{du_h(x)}{dx} + bu_h(x) = 0$$

There is **NO** fixed rule for deriving $u_p(x)$

Example 4.6

$$\frac{d^2 y(x)}{dx^2} - \frac{dy(x)}{dx} - 2y(x) = Sin\, 2x$$

$$y(x) = y_h(x) + y_p(x)$$

$$\frac{d^2 y_h(x)}{dx^2} - \frac{dy_h(x)}{dx} - 2y_h(x) = 0$$

$$y_h(x) = c_1\, e^{-x} + c_2\, e^{2x}$$

Guess: $y_p(x) = A\ Sin\ 2x + B\ Cos\ 2x$

After some algebra

$$y(x) = y_h(x) + y_p(x) = c_1\, e^{-x} + c_2\, e^{2x} + \left(-\frac{3}{20} Sin\, 2x + \frac{1}{20} Cos\, 2x \right)$$

Example 4.8

$$\frac{d^2u(x)}{dx^2} + 4u(x) = 2\,Sin\,2x$$

$$u(x) = u_h(x) + u_p(x) = c_1\,Cos\,2x + c_2\,Sin\,2x - \frac{x}{2}Cos\,2x$$

Simple Harmonic pendulum as a special case of second order DE

Force on the pendulum $F_\theta = -m\,g\,\sin\theta$

for small oscillation, $\sin\theta \approx \theta.$

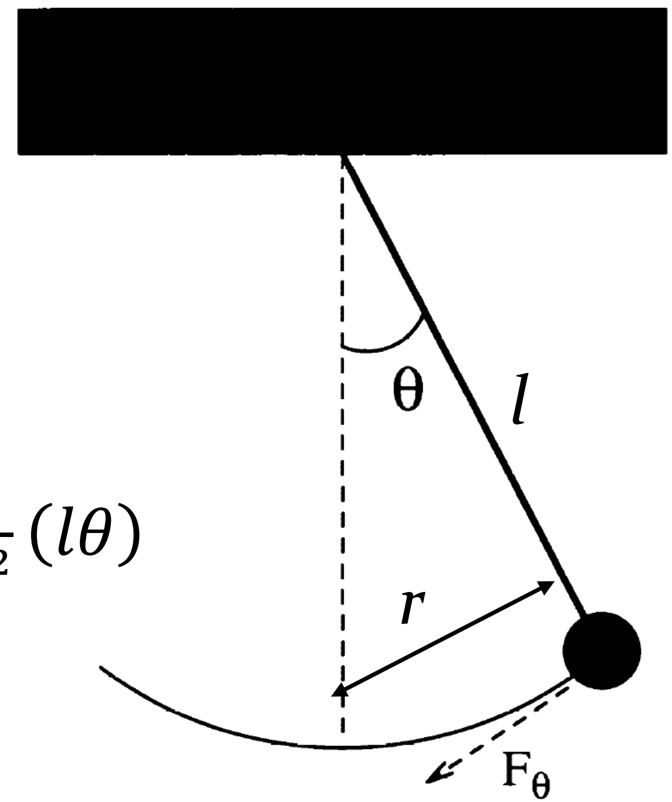
Equation of motion (EoM)

$$F_\theta = ma_\theta$$

$$-mgsin\theta{=}m\frac{dv_\theta}{dt} = m\frac{d}{dt}\left(\frac{dr}{dt}\right) \approx m\frac{d^2}{dt^2}(l\theta)$$

$$\frac{d^2\theta}{dt^2} \approx -\frac{g\theta}{l}$$

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)$$



The period of the SHO is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Simple Harmonic pendulum as a special case of second order DE (cont.)

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)$$

$$x \equiv t$$
$$u(x) \equiv \theta(t)$$
$$a \equiv 0$$
$$b \equiv \frac{g}{l}$$
$$n(x) \equiv 0$$

$$\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l}$$

Simple Harmonic pendulum as a special case of second order DE (cont.)

$$\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l}$$

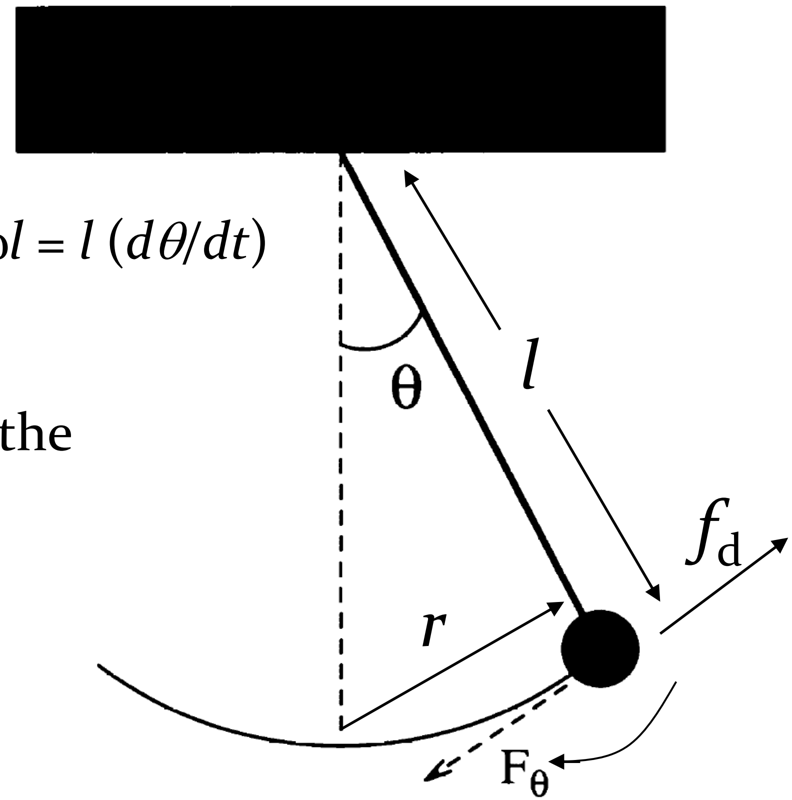
Analytical solution:

$$\theta = \theta_0 \sin(\Omega t + \phi)$$

$\Omega = \sqrt{g/l}$ natural frequency of the pendulum;
θ_0 and ϕ are constant determined by boundary conditions

Simple Harmonic pendulum with drag force as a special case of second order DE

Drag force on a moving object, $f_d = - kv$



For a pendulum, instantaneous velocity $v = \omega l = l\,(d\theta/dt)$

Hence, $f_d = - kl\,(d\theta/dt)$.

The net force on the forced pendulum along the tangential direction

$$F_\theta = -mg\sin\theta - kl\,(d\theta/dt).$$

$$F_\theta = -mg\sin\theta - kl\frac{d\theta}{dt} \approx -mg\theta - kl\frac{d\theta}{dt};$$

$$m\frac{d^2 r}{d^2 t} \approx m\frac{d^2}{dt^2}(l\theta) = ml\frac{d^2\theta}{dt^2};$$

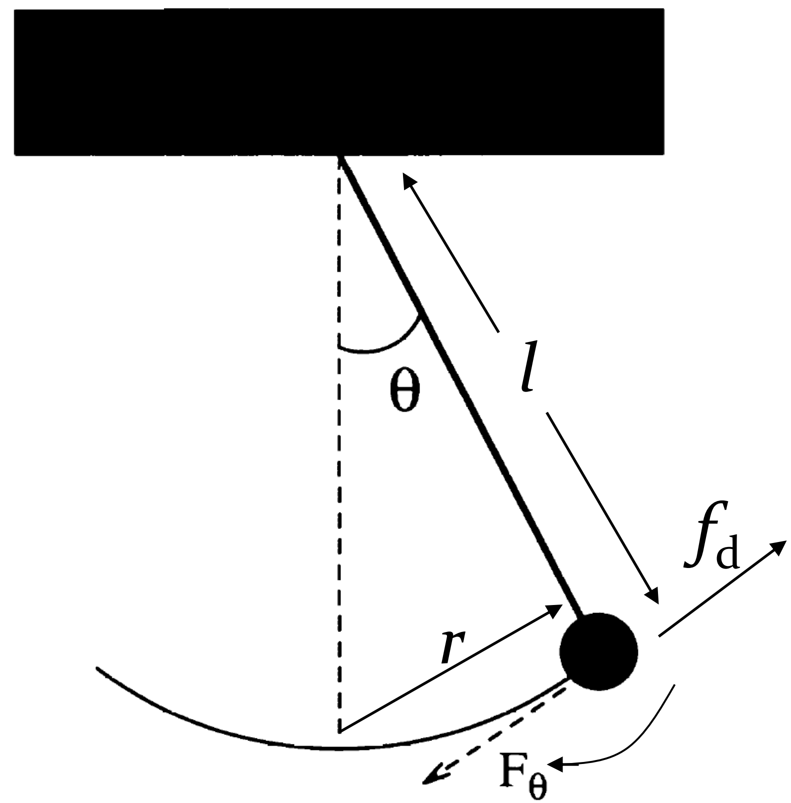
$$F_\theta = m\frac{d^2 r}{d^2 t} \rightarrow \frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - \frac{k}{m}\frac{d\theta}{dt} \equiv -\frac{g}{l}\theta - q\frac{d\theta}{dt}; q = \frac{k}{m}$$

Simple Harmonic pendulum with drag force as a special case of second order DE (cont.)

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)$$

$$x \equiv t$$
$$u(x) \equiv \theta(t)$$
$$a \equiv q$$
$$b \equiv \frac{g}{l}$$
$$n(x) \equiv 0$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt}; q = \frac{k}{m}$$



Analytical solution

Underdamped regime (small damping). Still oscillate, but amplitude decay slowly over many period before dying totally.

$$\theta(t) = \theta_0 e^{-qt/2} \sin\left(\varphi + t\sqrt{\Omega^2 - \frac{q^2}{4}}\right)$$

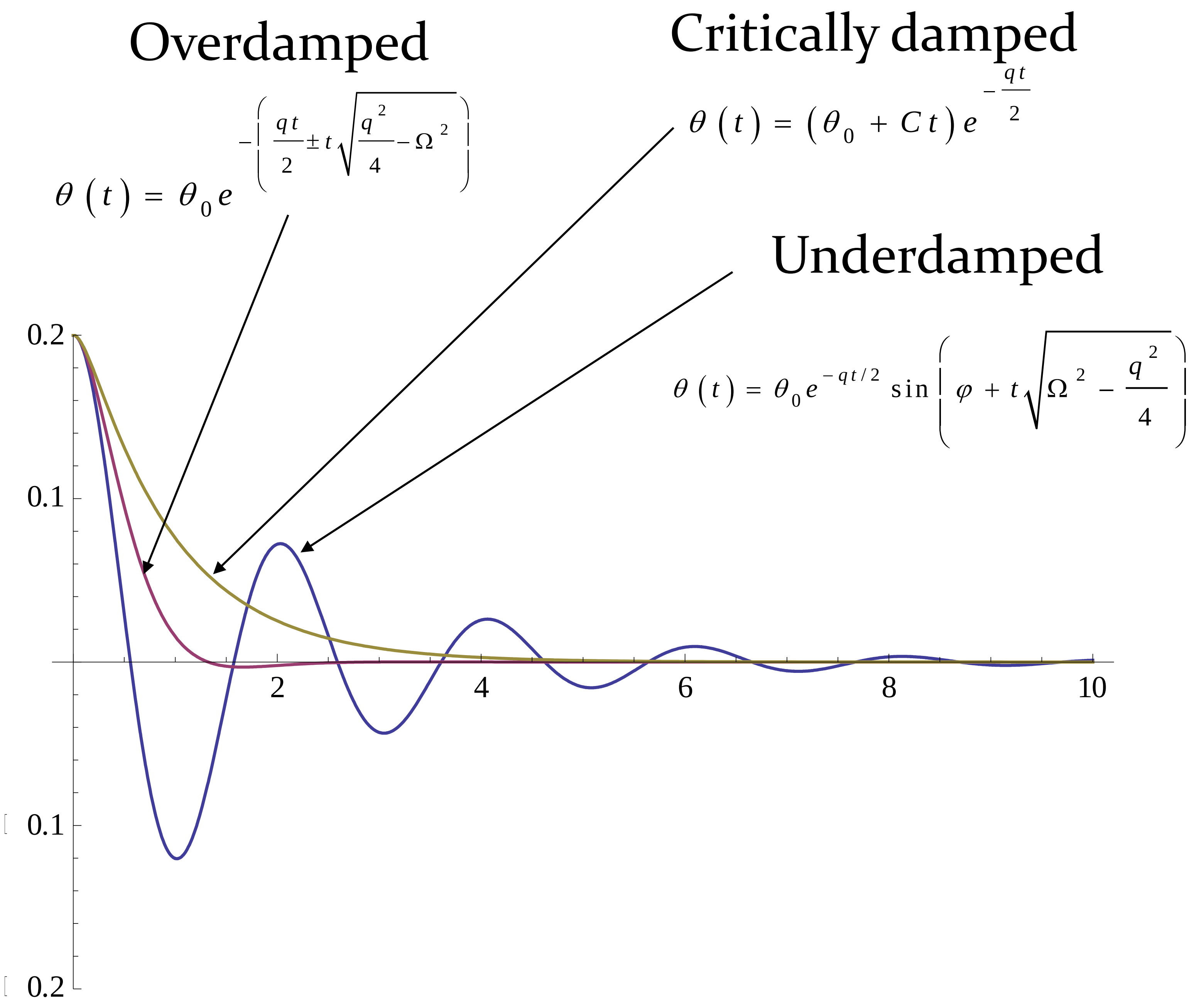
$$\Omega = \sqrt{\frac{g}{l}} \text{ the natural frequency of the system}$$

Overdamped regime (very large damping), decay slowly over several period before dying totally. θ is dominated by exponential term.

$$\theta(t) = \theta_0 e^{-\left(\frac{qt}{2} \pm t\sqrt{\frac{q^2}{4} - \Omega^2}\right)}$$

Critically damped regime, intermediate between under- and overdamping case.

$$\theta(t) = (\theta_0 + Ct) e^{-\frac{qt}{2}}$$


Overdamped
$\theta(t) = \theta_0 e^{-\left(\frac{qt}{2} \pm t\sqrt{\frac{q^2}{4} - \Omega^2}\right)}$
Critically damped
$\theta(t) = (\theta_0 + Ct)e^{-\frac{qt}{2}}$
Underdamped
$\theta(t) = \theta_0 e^{-qt/2} \sin\left(\varphi + t\sqrt{\Omega^2 - \frac{q^2}{4}}\right)$
0.2
0.1
0.1
0.2
2
4
6
8
10

See 2ODE_Pendulum.nb where **DSolve** solves the three cases of a damped pendulum analytically.

Adding driving force to the damped oscillator: forced oscillator

$$F_\theta = -mg\sin\theta - kl\,(d\theta/dt) + F_D\sin(\Omega_D t)$$

Ω_D *frequency of the applied force*

$$F_\theta = -mg\sin\theta - kl\frac{d\theta}{dt} + F_D\sin(\Omega_D t) \approx -mg\theta - kl\frac{d\theta}{dt} + F_D\sin(\Omega_D t);$$

$$F_\theta = m\frac{d^2r}{d^2t} \approx m\frac{d^2}{dt^2}(l\theta) = ml\frac{d^2\theta}{dt^2};$$

$$F_\theta = m\frac{d^2r}{d^2t} = ml\frac{d^2\theta}{dt^2} \approx -mg\theta - kl\frac{d\theta}{dt} + F_D\sin(\Omega_D t)$$

$$\frac{d^2\theta}{dt^2} \approx -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D\sin(\Omega_D t)}{ml};\; q = \frac{k}{m}$$

Analytical solution

$$\theta(t) = \theta_0 \sin\left(\Omega_D t + \phi\right)$$

$$\theta_0 = \frac{F_D / (m l)}{\sqrt{\left(\Omega^2 - \Omega_D^2\right)^2 + \left(q \Omega_D\right)^2}}$$

Resonance happens when $\Omega_D = \Omega = \sqrt{g / l}$

Forced oscillator: An example of non homogeneous 2nd order DE

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)$$

$$x \equiv t$$

$$u(x) \equiv \theta(t)$$

$$a \equiv q$$

$$b \equiv \frac{g}{l}$$

$$n(x) \equiv \frac{F_D \sin(\Omega_D t)}{ml}$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{ml}$$

Exercise: Forced oscillator

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{ml}$$

Use DSolve to solve the forced oscillator. Plot on the same graph the analytical solutions of $\theta(t)$ for t from 0 to 10 T, where $T = 2\pi/\Omega$, $\Omega = \sqrt{g/l}$, for $\Omega_D = 0.01\Omega, 0.5\Omega, 0.99\Omega, 1.5\Omega, 4\Omega$. Assume the boundary conditions $\theta(t=0)=0$; $d\theta/dt(t=0)=0$; $m=l=F_D=1$; $q=0$.

See forced_Pendulum.nb.

Second order Runge-Kutta (RK2) method

Consider a generic second order differential equation.

$$\frac{d^2 u(x)}{dx^2} = G(u)$$

It can be numerically solved using second order Runge-Kutta method. First, split the second order DE into two first order parts:

$$v(x) = \frac{du(x)}{dx} \qquad \frac{dv(x)}{dx} = G(u)$$

Algorithm

Set boundary conditions: $u(x=x_o)=u_o$, $u'(x=x_o)=v(x=x_o)=v_o$.

calculate $$\tilde{u} = u_i + \frac{1}{2} v_i \Delta x$$

calculate $$\tilde{v} = v_i + \frac{1}{2} G(\tilde{u}) \Delta x$$

calculate $$u_{i+1} = u_i + \tilde{v} \Delta x$$

calculate $$v_{i+1} = v_i + G(\tilde{u}) \Delta x$$

Translating the SK2 algorithm into the case of simple pendulum

$$\frac{d^2 u(x)}{dx^2} = G(u)$$

$$G(u) \equiv -\frac{g}{l}\theta(t)$$

$$v(x) = \frac{du(x)}{dx}$$

$$\frac{dv(x)}{dx} = G(u)$$

Set boundary conditions:
$u(x=x_o)=u_o$, $u'(x=x_o)=v(x=x_o)=v_o$

$$\tilde{u} = u_i + \frac{1}{2} v_i \Delta x$$

$$\tilde{v} = v_i + \frac{1}{2} G(\tilde{u}) \Delta x$$

$$u_{i+1} = u_i + \tilde{v} \Delta x$$

$$v_{i+1} = v_i + G(\tilde{u}) \Delta x$$

$$\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l}$$

$$\omega(t) = \frac{d\theta(t)}{dt}$$

$$\frac{d\omega(t)}{dt} = -\frac{g\theta(t)}{l}$$

Set boundary conditions:
$\theta(t=t_o)= \theta_o$, $\theta'(t=t_o)=\omega(t=t_o)=\omega_o$

$$\tilde{\theta} = \theta_i + \frac{1}{2}\omega_i \Delta t$$

$$\tilde{\omega} = \omega_i + \frac{1}{2}\left(-\frac{g\tilde{\theta}}{l}\right)\Delta t$$

$$\theta_{i+1} = \theta_i + \tilde{\omega}\Delta t$$

$$\omega_{i+1} = \omega_i + \left(-\frac{g\tilde{\theta}}{l}\right)\Delta t$$

Exercise: Develop a code to implement SK2 for the case of the simple pendulum.
Boundary conditions: $\omega(0) = \sqrt{\frac{g}{l}}; \theta(0) = 0$

See pendulum_RK2.nb

Translating the SK2 algorithm into the case of damped pendulum

$$\frac{d^2u(x)}{dx^2} = G(u)$$

$$\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l} - q\frac{d\theta}{dt}$$

$$v(x) = \frac{du(x)}{dx}$$

$$\omega(t) = \frac{d\theta(t)}{dt}$$

$$\frac{dv(x)}{dx} = G(u)$$

$$G(u) \equiv -\frac{g}{l}\theta(t) - q\omega(t)$$

$$\frac{d\omega(t)}{dt} = -\frac{g\theta}{l} - q\omega(t)$$

Set boundary conditions:

$u(x=x_o)=u_o$, $u'(x=x_o)=v(x=x_o)=v_o$

Set boundary conditions:

$\theta(t=t_o)= \theta_o$, $\theta'(t=t_o)=\omega(t=t_o)=\omega_o$

$$\tilde{u} = u_i + \frac{1}{2}v_i\Delta x$$

$$\tilde{\theta} = \theta_i + \frac{1}{2}\omega_i\Delta t$$

$$\tilde{v} = v_i + \frac{1}{2}G(\tilde{u})\Delta x$$

$$\tilde{\omega} = \omega_i + \frac{1}{2}\left(-\frac{g}{l}\tilde{\theta}(t) - q\tilde{\omega}\right)\Delta t \Rightarrow \tilde{\omega} = \frac{\left(\omega_i - \frac{g}{2l}\tilde{\theta}(t)\Delta t\right)}{\left(1 + \frac{1}{2}q\Delta t\right)}$$

$$u_{i+1} = u_i + \tilde{v}\Delta x$$

$$\theta_{i+1} = \theta_i + \tilde{\omega}\Delta t$$

$$v_{i+1} = v_i + G(\tilde{u})\Delta x$$

$$\omega_{i+1} = \omega_i + \left(-\frac{g\tilde{\theta}}{l} - q\tilde{\omega}\right)\Delta t$$

Exercise:

Develop a code to implement SK2 for the case of a pendulum experiencing a drag force, with damping coefficient q= 0.1* (4Ω), Ω= $\sqrt{g/l}, l = 1.0$ m.

Boundary conditions: $\theta(0) = 0.2; \omega(t = 0) = 0;$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt}$$

See pendulum_RK2.nb

Exercise:

Develop a code to implement SK2 for the case of a forced pendulum experiencing no drag force but a driving force$F_D \sin(\Omega_D t)$, $\Omega = \sqrt{g/l}$, $l = 1.0$ m, m=1kg; $F_\mathrm{D} = 1\mathrm{N}$; Ω_D=0.99 Ω;
Boundary conditions: $\theta(0) = 0.0; \omega(t = 0) = 0;$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{ml}$$

Exercise: Stability of the total energy a SHO in RK2.

$\omega = \frac{d\theta}{dt}$, angular velocity. *M*=1kg; *l*=1m.

The total energy of the SHO in can be calculated as

$$E_{i+1} = K_{i+1} + U_{i+1} = \frac{1}{2}m\left(l\omega_{i+1}\right)^2 + mgl\left(1 - \cos\theta_{i+1}\right)$$

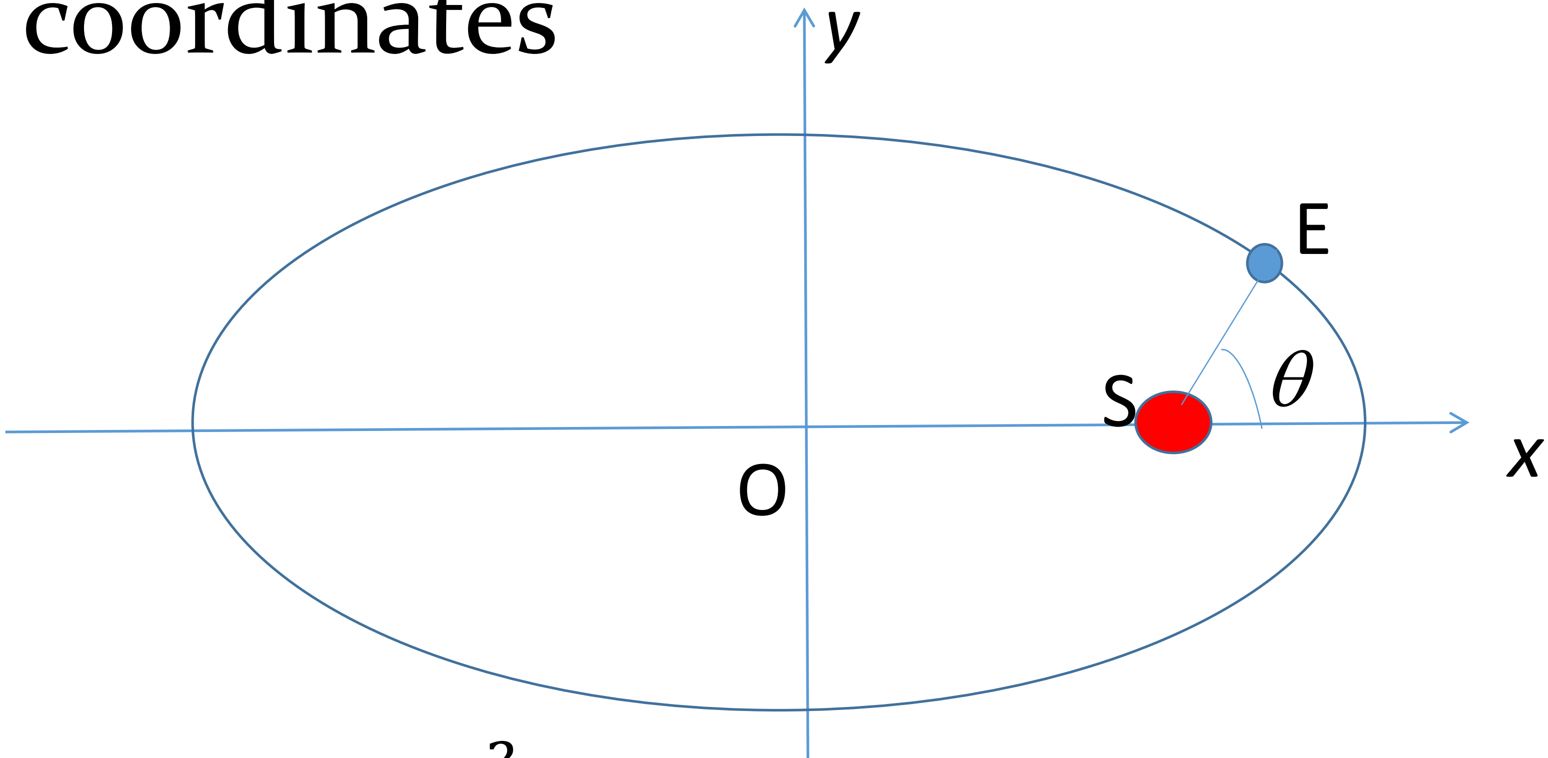
$$\approx \frac{1}{2}ml^2\omega_{i+1}{}^2 + mgl\left[1 - \left(1 - \frac{\theta_{i+1}^2}{2}\right)\right]$$

$$= \frac{1}{2}ml^2\omega_{i+1}{}^2 + \frac{1}{2}mgl\theta_{i+1}^2$$

User your RK2 code to track the total energy for t running from t=0 till t=25T; T=$\sqrt{g/l}$. Boundary conditions: $\omega(0) = \sqrt{\frac{g}{l}}; \theta(0) = 0$

E_i should remain constant throughout all t_i.

Exercise: Develop a RK2 code for planetary motion in polar coordinates



$$m\ddot{r} - mr\left(\frac{L}{mr^2}\right)^2 = -\frac{GMm}{r^2} \qquad mr^2\dot{\theta} = L$$

where $$\ddot{r} = \frac{d^2r}{dt^2};\ \dot{\theta} = \frac{d\theta}{dt}$$

Velocity verlet algorithm for solving the Newton second law

Newton's second law

- Given a position-dependent force acting on a particle, $\mathbf{F}(\mathbf{r})$, Newton second law determines the acceleration of a particle, $\mathbf{a}$.

$$d^2\mathbf{r}/dt^2 = \mathbf{F}(\mathbf{r})/m$$

- This is a second order differential equation.
- Given $\mathbf{F}(\mathbf{r})$, we wish to know what are the subsequent evolution of $\mathbf{r}(t)$, $\mathbf{v}(t)$ beginning from the boundary values of $\mathbf{r}(0)$, $\mathbf{v}(0)$.
- Previously we solve the second order DE using RK2 to obtain $\mathbf{r}(t)$, $\mathbf{v}(t)$.

Verlet algorithms

- The equation $d^2\mathbf{r}/dt^2=\mathbf{F}(\mathbf{r})/m$ can be integrated to obtain $\mathbf{r}(t)$, $\mathbf{v}(t)$, via a numerical scheme: Verlet algoritm
- Three types: ordinary Verlet, velocity Verlet, leap frog verlet.
- In the following, we shall denote the RHS as

$$\mathbf{F}(\mathbf{r})/m \longrightarrow \mathbf{a}$$

So that $d^2\mathbf{r}/dt^2=\mathbf{F}(\mathbf{r})/m$

$$\downarrow$$

$$d^2\mathbf{r}/dt^2= \mathbf{a}$$

- See http://en.wikipedia.org/wiki/Verlet_integration

Velocity Verlet algorithm

1. Calculate:
   $\vec{x}(t+\Delta t) = \vec{x}(t) + \vec{v}(t)\,\Delta t + \frac{1}{2}\,\vec{a}(t)\,\Delta t^2$
2. Derive $\vec{a}(t+\Delta t)$ from the interaction potential using $\vec{x}(t+\Delta t)$
3. Calculate:
   $\vec{v}(t+\Delta t) = \vec{v}(t) + \frac{1}{2}\,(\vec{a}(t) + \vec{a}(t+\Delta t))\,\Delta t$

Note, however, that this algorithm assumes that acceleration $\vec{a}(t+\Delta t)$ only depends on position $\vec{x}(t+\Delta t)$, and does not depend on velocity $\vec{v}(t+\Delta t)$.

Global error in velocity Verlet algorithm

- The global (cumulative) error in x over a constant interval of time is given by

$$\Delta x \sim O(\Delta t^2)$$

- Because the velocity is determined in a non-cumulative way from the positions in the Verlet integrator, the global error in velocity is also

$$\Delta v \sim O(\Delta t^2)$$

Exercise: SHO

- Solve for $x(t)$, $v(t)$, for a simple harmonic oscillator with constant k, mass m, initial displacement x_0, and velocity $v_0 = 0$, using velocity Verlet algorithm.
- For SHO, $F = -kx \Rightarrow a = -(k/m)x$

$$d^2x/dt^2 = -(k/m)x$$

See verlet_algorithm_samples.nb

Exercise: 2D free-fall projectile

- Solve for $x(t)$, $y(t)$, $v(t)$ of a 2D free-fall projectile with initial speed v_0=0 and launching angle θ_0 using velocity Verlet algorithm.
- For 2D projectile motion, $\boldsymbol{F} = -mg\hat{y} + 0\hat{x}.$

$$\mathbf{a} = \boldsymbol{F}/m$$

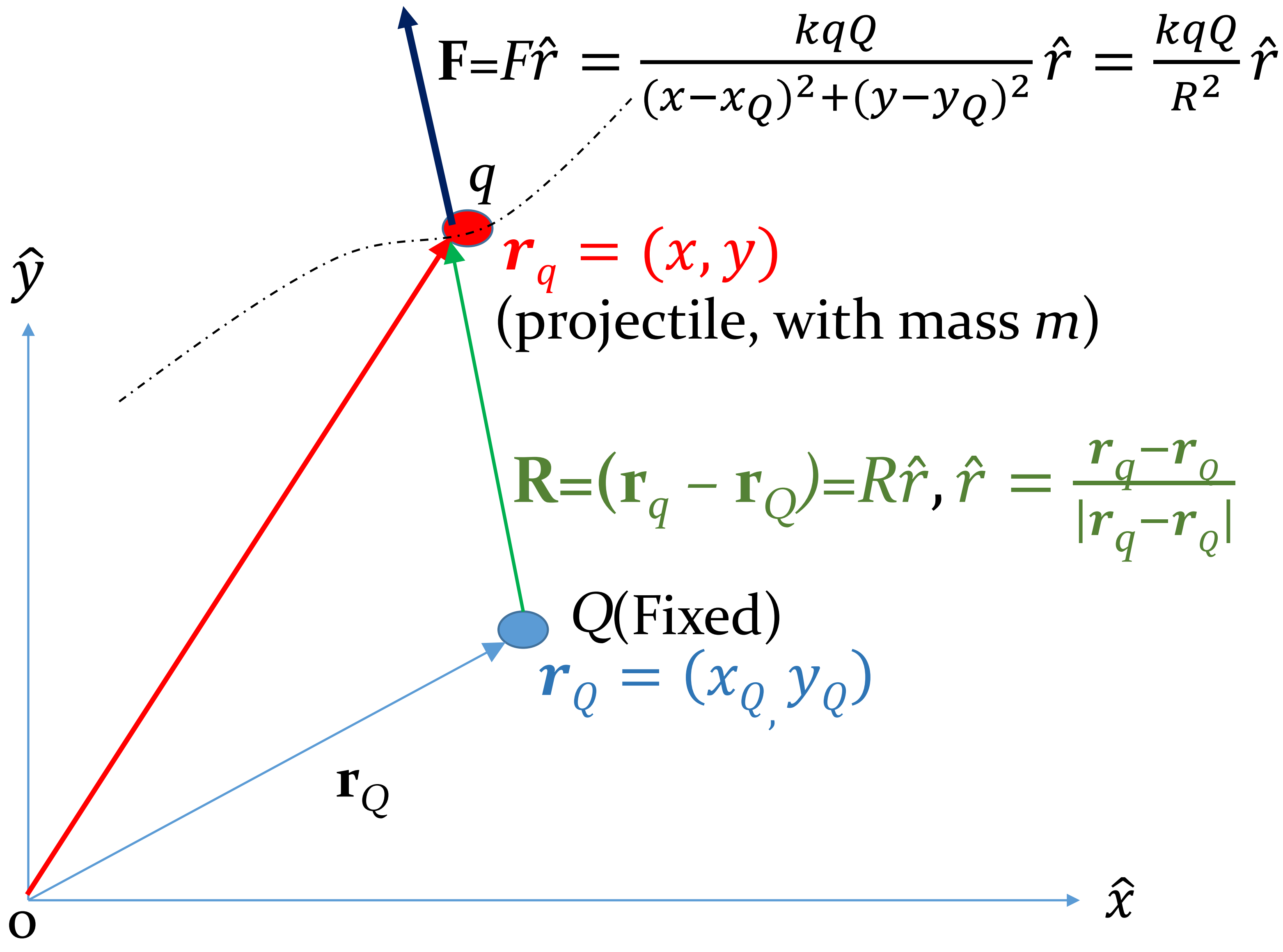
$$a_x\hat{x} + a_y\hat{y} = 0\hat{x} - g\hat{y}$$

$$\mathrm{d}^2x/\mathrm{d}t^2\hat{x} + \mathrm{d}^2y/\mathrm{d}t^2\hat{x} = -g\hat{y}$$

$$\Rightarrow \mathrm{d}^2x/\mathrm{d}t^2 = 0, \qquad \mathrm{d}^2y/\mathrm{d}t^2 = -g$$

See verlet_algorithm_samples.nb

Exercise: Scattering of a projectile charge via Coulomb force



Notation for scattering of a projectile charge via Coulomb force

- $\boldsymbol{r}_Q = (x_{Q,}\, y_Q); \boldsymbol{r}_q = (x, y)$
- $\boldsymbol{F} = k\frac{qQ}{(x - x_Q)^2 + (y - y_Q)^2}\hat{r}; \quad \hat{r} = \frac{\boldsymbol{r}_q - \boldsymbol{r}_Q}{|\boldsymbol{r}_q - \boldsymbol{r}_Q|}$
- $\Rightarrow$
- $a_y = \frac{1}{m}\boldsymbol{F} \cdot \hat{y} = \frac{k}{m}\frac{qQ}{(x - x_Q)^2 + (y - y_Q)^2}\hat{r} \cdot \hat{y} = \frac{d^2y}{dt^2},$
- $a_x = \frac{1}{m}\boldsymbol{F} \cdot \hat{x} = \frac{k}{m}\frac{qQ}{(x - x_Q)^2 + (y - y_Q)^2}\hat{r} \cdot \hat{x} = \frac{d^2x}{dt^2}.$

The equations required by Verlet algorithm

- $\boldsymbol{r}_Q = (x_Q, y_Q); \boldsymbol{r}_q = (x, y)$ $\quad \hat{r} = \dfrac{\boldsymbol{r}_q - \boldsymbol{r}_Q}{|\boldsymbol{r}_q - \boldsymbol{r}_Q|}$

$$\frac{d^2y}{dt^2} = \frac{kqQ}{m}\frac{\hat{r}\cdot\hat{y}}{(x-x_Q)^2+(y-y_Q)^2},$$

$$\frac{d^2x}{dt^2} = \frac{kqQ}{m}\frac{\hat{r}\cdot\hat{x}}{(x-x_Q)^2+(y-y_Q)^2}$$

See verlet_algorithm_2D_coulomb_scatterings.nb

Störmer-Verlet integration algorithm

$$\vec{x}_{n+1} = 2\vec{x}_n - \vec{x}_{n-1} + \vec{a}_n(\Delta t)^2$$

$$\vec{v}_{n+1} = (\vec{x}_{n+1} - \vec{x}_n)/\Delta t$$

- Another variant of Verlet algoritm
- Use this for integrating dynamical system with a velocity-dependent acceleration, such as Lorentz force on a moving charge particle.
- The cumulative error in the velocity is larger than that in velocity Verlet algorithm

Exercise: Charge moving in a magnetic field

- A charge (mass m and charge q) moving with velocity **v** =(v_x, v_y , v_z) in a magnetic field **B**=(B_x, B_y , B_z) experiences a velocity-dependent Lorentz force **F**=(F_x, F_y , F_z) = q **v** × **B**. Develop a code based on the Störmer-Verlet integration algorithm to simulate the dynamical path of the charge particle moving through the magnetic field. Assume: q=+1 unit, mass m = 1 unit, initially located at (0,0,0), initial velocity (v_{0x},v_{0y}, v_{0z}), v_{0x}=v_{0y}=0.1 unit, v_{0z} =0.05 unit, **B**=(0, 0, B_z), B_z = 0.1 unit. You should see a helical trajectory circulating about the z-direction.
- verlet_algorithm_3D_coulomb_helix.nb