Chapter 7

Solving first order differential equation numerically

Example of first order differential equation commonly encountered in physics

$$\frac{dv_y}{dt} = -g; \frac{dy}{dt} = v_y$$

$$m\frac{dv}{dt} = -mg - \eta v$$

$$\frac{dN}{dx} = -\lambda N; \frac{dN}{dt} = -\frac{N}{\tau}$$

$$m\frac{dv}{dx} = -kx$$

- Do you recognize these equations?

General form of first order differential equations

$$\frac{df(t)}{dt} = G(t)$$

$dN(t)/dt = -N(t)\ /\tau$
$G(t) \equiv -N(t)/\tau$
$f(t) \equiv N(t)$

$dv(x)/dx = -(k/m)x$
$t \equiv x$
$d/dt \equiv d/dx$
$G(t) \equiv -(k/m)x$
$f(t) \equiv v(x)$

$dv(t)/dt = -g-\eta v(t)/m$
$G(t) \equiv -g-\eta v(t)/m$
$f(t) \equiv v(t)$

Analytical solution of $\frac{dv_y}{dt} = -g$

- ZCA 101 mechanics, kinematic equation for a free fall object

$$\frac{dv_y}{dt} = -g$$

- What is the solution, i.e., $v_y = v_y(t)$?

$$\frac{dv_y}{dt} = -g$$

$$\Rightarrow \int \frac{dv_y}{dt} dt = -\int g\, dt$$

$$\int dv_y = v_y = -\int g\, dt = -gt + c$$

$$\Rightarrow v_y = v_y(t) = -gt + c$$

Analytical solution of $\frac{dv_y}{dt} = -g$

- To completely solve this first order differential equation, i.e., to determine v_y as a function of t, and the arbitrary constant c, a boundary value or initial value of v_y at a given time t is necessary. Usually (but not necessarily) $v_y(0)$, i.e., the value of v_y at t=0 has to be assumed.

$$\frac{dv_y}{dt} = -g$$

$$\int_{v_y(0)}^{v_y(t)} dv = -\int_0^t g\,dt = -gt$$

$$\Rightarrow v_y = v_y(t) = -gt + v_y(0)$$

Analytical solution of $\frac{dy}{dt} = v_y$

- Assume $v_y = v_y(t)$ a known function of t.
- To completely solve the equation so that we can know what is the function $y(t)$, we need to know the value of $y(0)$.

$$\int_{y(0)}^{y(t)} dy = \int_0^t v_y \, dt$$

$$\Rightarrow y(t) - y(0) = \int_0^t v_y \, dt$$

Analytical solution of $\frac{dy}{dt} = v_y$

- In free fall without drag force, $v_y(t)=v_y(0)-gt$.
- The complete solution takes the form

$$y(t) - y(0) = \int_0^t v_y \, dt = \int_0^t \left(v_y(0) - gt\right) dt$$

$$y(t) = y(0) + v_y(0)t - \frac{1}{2}gt^2$$

Boundary condition

- In general, to completely solve a first order differential equation for a function with single variable, a boundary condition value must be provided.
- Generalising such argument, two boundary condition values must be supplied in order to completely solve a second order differential equation.
- *n* boundary condition values must be supplied in order to completely solve a *n*-th order differential equation.
- Hence, supplying boundary condition values are necessary when numerically solving a differential equation.

Analytical solution of a free fall object in a viscous medium

$$m\frac{dv}{dt} = -mg - \eta v$$

- Boundary condition: v=0 at t = 0.

$$\int_{v(0)}^{v(t)} \frac{dv}{dt} dt = \int_{v(0)}^{v(t)} dv = \int_0^t \left(-g - \frac{\eta}{m} v\right) dt$$

$$\Rightarrow \int_{v(0)=0}^{v(t)} \frac{dv}{\left(-g - \frac{\eta}{m} v\right)} = \int_0^t dt$$

$$\Rightarrow v(t) = -\left(\frac{mg}{\eta}\right)\left[1 - \exp\left(\frac{-\eta t}{m}\right)\right]$$

Number of beta particles penetrating a medium (recall your first year lab experiments)

$$\frac{dN}{dx} = -\lambda N$$

$$\int_{N_0}^{N} \frac{dN}{N} = -\int_{0}^{x} \lambda \, dx$$

$$N(x) = N_0 \exp[-\lambda x]$$

Number of radioactive particle remained after time *t* (recall your first year lab experiments. τ: half-life)

$$\frac{dN}{dt} = -\frac{N}{\tau}$$

$$\int_{N_0}^{N} \frac{dN}{N} = -\int_{0}^{t} \frac{1}{\tau} dt$$

$$N(t) = N_0 \exp\left[-\frac{t}{\tau}\right]$$

Relation of speed vs. displacement in SHM

$$E = K + P = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$\Rightarrow \frac{dE}{dx} = \frac{d}{dx}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = 0$$

$$\Rightarrow m\frac{dv}{dx} = -kx$$

The solution is

$$v(x) = v_0 + \frac{kx_0^2}{2}m - \frac{k}{2m}x^2$$

bondary condition: $v = v_0 \; at \; x = x_0$

DSolve and NDSolve

Now, we would learn how to solve these first order differential equations

DSolve[] (symbolically)

and

NDSolve[] (numerically)

Sample codes:

C7_NSolve_example.nb

C7_NDSolve_example.nb

Now, how would you write your own algorithm to solve first order differential equations numerically?

$$\frac{df(t)}{dt} = G(t)$$

Euler's method for discreteising a first order differential equation into a difference equation

Discretising the differential equation

In Euler method, where the differentiation of a function at time t is approximated as

$$\frac{df(t)}{dt} \simeq \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{f(t_{i+1}) - f(t_i)}{\Delta t}, \qquad t_i = i\Delta t, \quad t_{i+1} - t_i = \Delta t,$$

$$\frac{df(t_i)}{dt} \simeq \frac{f(t_{i+1}) - f(t_i)}{\Delta t}; t_i = i\Delta t \Rightarrow f(t_{i+1}) \simeq f(t_i) + \frac{df(t_i)}{dt}\Delta t = f(t_i) + G(t_i)\Delta t$$

Essentially, Euler's method says

Differential equation

$$\frac{df(t)}{dt} = G(t)$$

Discretise

$$f(t_{i+1}) \simeq f(t_i) + G(t_i)\Delta t$$

Difference equation

Boundary condition:
the numerical value at $f(t_0)$ has to be supplied.

Discretising the differential equation

$$dN(t)/dt = -N(t)\,/\,\tau \quad \longleftrightarrow \quad \frac{df(t)}{dt} = G(t)$$

c.f

- $dN(t)/dt = -N(t)\,/\,\tau$ is discretised into a difference equation,
- $N(t+\Delta t) \approx N(t) - \dfrac{N(t)}{\tau}\Delta t$

c.f

$$\Rightarrow N(t_{i+1}) \approx N(t_i) - \frac{N(t_i)}{\tau}\Delta t \quad \longleftrightarrow \quad f(t_{i+1}) \simeq f(t_i) + G(t_i)\Delta t$$

$N(t_i) \rightarrow f(t_i)$, $\dfrac{N(t_i)}{\tau} \rightarrow G(t_i)$

- which is suitable for numerical manipulation using computer.
- There are many different way to discretise a differential equation.
- Euler's method is just among the easiest of them.

The code's structure $\frac{dN}{dt} = -\frac{N}{\tau}$

Initialisation:

Assign (i) $N\,(t{=}0)$, τ, (ii) Number of steps, Nstep, (iii) Time when to stop, say tfinal= 10 τ.

- The global error is of the order O ~ Δt
- Δt = tfinal/Nstep
- In principle, the finer the time interval Δt is, the numerical solution becomes more accurate.
- $t_i = t_0 + i\Delta t$; i=0,1,2,…,tfinal
- Calculate $N\,(t_1) = N\,(t_0) - \Delta t\, N\,(t_0)/\tau$.
- Then calculate $N\,(t_2) = N\,(t_1) - \Delta t\, N\,(t_1)/\tau$
- $N\,(t_3) = N\,(t_2) - \Delta t\, N\,(t_2)/\tau$, …
- Stop when t = tfinal.
- Plot the output: $N\,(t)$ as function of t.

Sample code: [C7_Euler_nucleidecay.nb]

Exercise

- Develop your own version of Euler method to solve

$$m\frac{dv}{dt} = -mg - \eta v$$

$$m\frac{dv}{dx} = -kx$$

- Note that your code must handle the boundary condition properly.
- Also use **Dsolve[]** to generate the analytical solutions. Overlap the analytical solution on top of the numerical solutions using Euler method to show that you have obtained the correct result.

Sample Euler codes

- C7_Euler_freefall_dragforce.nb

$$m\frac{dv}{dt} = -mg - \eta v$$

- C7_Euler_SHM_v_vs_x.nb

$$m\frac{dv}{dx} = -kx$$