# Chapter 8

## Solving Second order differential equations numerically

# Online lecture materials

- The online lecture notes by Dr. Tai-Ran Hsu of San José State University,

http://www.engr.sjsu.edu/trhsu/Chapter%204%20Second%20order%20DEs.pdf

provides a very clear explanation of the solutions and applications of some typical second order differential equations.

# $2^{nd}$ Order Homogeneous DEs

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0$$

with TWO given conditions

## The solutions

Case 1: $a^2 - 4b > 0$:

$$u(x) = e^{-\frac{ax}{2}}\left(c_1 e^{\sqrt{a^2 - 4b}\; x/2} + c_2 e^{-\sqrt{a^2 - 4b}\; x/2}\right)$$

Case 2: $a^2 - 4b < 0$:

$$u(x) = e^{-\frac{ax}{2}}\left[A\,Sin\left(\frac{1}{2}\sqrt{4b - a^2}\right)x + B\,Cos\left(\frac{1}{2}\sqrt{4b - a^2}\right)x\right]$$

Case 3: $a^2 - 4b = 0$: — A special case

$$u(x) = c_1 e^{-\frac{ax}{2}} + c_2 x e^{-\frac{ax}{2}} = (c_1 + c_2 x) e^{-\frac{ax}{2}} \tag{4.12}$$

where $c_1$, $c_2$, A and B are arbitrary constants to be determined by given conditions

**Example 4.1** Solve the following differential equation

$$\frac{d^2u(x)}{dx^2} + 5\frac{du(x)}{dx} + 6u(x) = 0$$

$$u(x) = e^{-5x/2}\left(c_1 e^{x/2} + c_2 e^{-x/2}\right) = c_1 e^{-2x} + c_2 e^{-3x}$$

where $c_1$ and $c_2$ are arbitrary constants to be determined by given conditions

## Example 4.2

$$\frac{d^2u(x)}{dx^2} + 6\frac{du(x)}{dx} + 9u(x) = 0$$

u(0) = 2

$$\left.\frac{du(x)}{dx}\right|_{x=0} = 0$$

$$u(x) = 2(1+3x)e^{-3x}$$

# Typical second order, non-homogeneous ordinary differential equations

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x) \qquad (4.25)$$

Non-homogeneous term

Solution of Equation (4.25) consists **TWO** components:

Solution u(x) = Complementary solution $u_h(x)$ + Particular solution $u_p(x)$

$$u(x) = u_h(x) + u_p(x)$$

# Typical second order, non-homogeneous ordinary differential equations

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x) \qquad (4.25)$$

Non-homogeneous term

$u(x) = u_h(x) + u_p(x)$

$$\frac{d^2u_h(x)}{dx^2} + a\frac{du_h(x)}{dx} + bu_h(x) = 0$$

There is **NO** fixed rule for deriving $u_p(x)$

# Example 4.6

$$\frac{d^2 y(x)}{dx^2} - \frac{dy(x)}{dx} - 2y(x) = Sin\,2x$$

$$y(x) = y_h(x) + y_p(x)$$

$$\frac{d^2 y_h(x)}{dx^2} - \frac{dy_h(x)}{dx} - 2y_h(x) = 0$$

$$y_h(x) = c_1\, e^{-x} + c_2\, e^{2x}$$

Guess: $y_p(x) = A\ Sin\ 2x + B\ Cos\ 2x$

After some algebra

$$y(x) = y_h(x) + y_p(x) = c_1\, e^{-x} + c_2 e^{2x} + \left( -\frac{3}{20} Sin\,2x + \frac{1}{20} Cos\,2x \right)$$

# Example 4.8

$$\frac{d^2u(x)}{dx^2} + 4u(x) = 2\,Sin\,2x$$

$$u(x) = u_h(x) + u_p(x) = c_1\,Cos\,2x + c_2\,Sin\,2x - \frac{x}{2}Cos\,2x$$

# Simple Harmonic pendulum as a special case of second order DE

Force on the pendulum $F_\theta = -m\, g\, \sin\theta$

for small oscillation, $\sin\theta \approx \theta.$

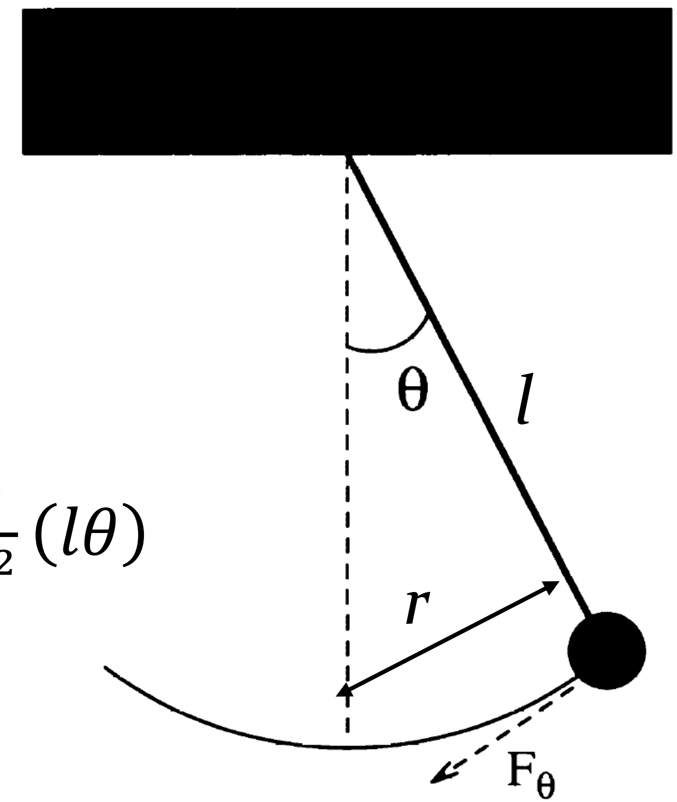
Equation of motion (EoM)

$$F_\theta = m a_\theta$$

$$-mgsin\theta = m\frac{dv_\theta}{dt} = m\frac{d}{dt}\left(\frac{dr}{dt}\right) \approx m\frac{d^2}{dt^2}(l\theta)$$

$$\frac{d^2\theta}{dt^2} \approx -\frac{g\theta}{l}$$

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)$$




The period of the SHO is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

# Simple Harmonic pendulum as a special case of second order DE (cont.)

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)$$

$$x \equiv t$$

$$u(x) \equiv \theta(t)$$

$$a \equiv 0$$

$$b \equiv \frac{g}{l}$$

$$n(x) \equiv 0$$

$$\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l}$$

# Simple Harmonic pendulum as a special case of second order DE (cont.)

$$\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l}$$

Analytical solution:

$$\theta = \theta_0 \sin(\Omega t + \phi)$$

$\Omega = \sqrt{g/l}$ natural frequency of the pendulum;
$\theta_0$ and $\phi$ are constant determined by boundary conditions

# Simple Harmonic pendulum with drag force as a special case of second order DE

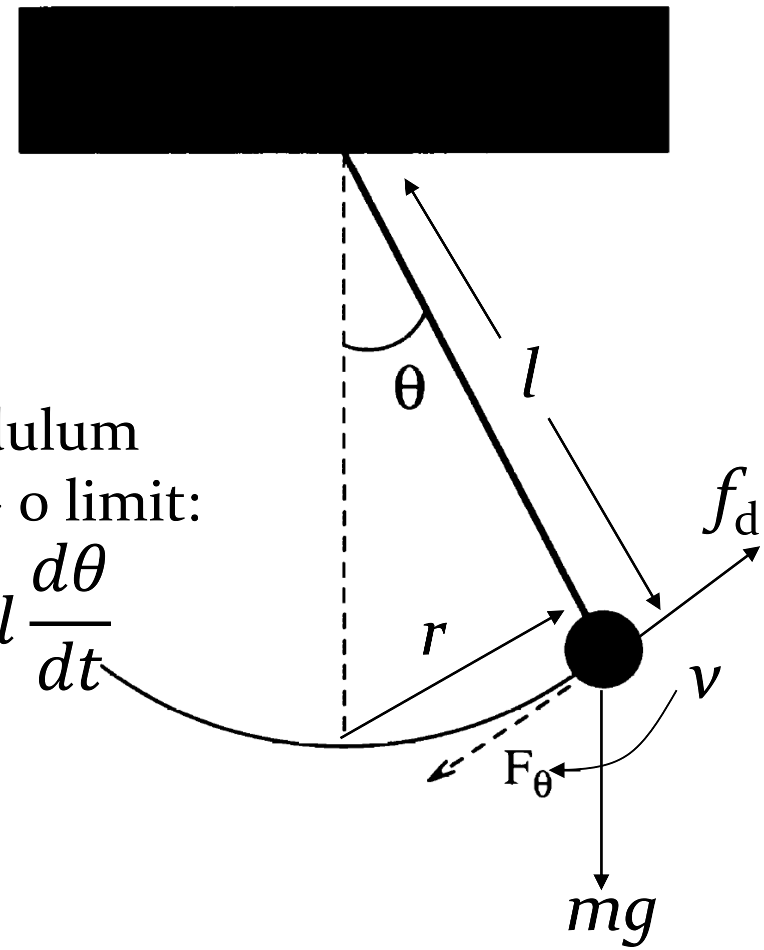
Drag force on a moving object, $f_d = -kv$

For a pendulum, instantaneous velocity

$v = \omega l = l\,(d\theta/dt)$

Hence, $f_d = -kl\,(d\theta/dt)$.

Consider the net force on the forced pendulum along the tangential direction, in the $\theta \to 0$ limit:




$$F_\theta = -mg\sin\theta - kl\frac{d\theta}{dt} \approx -mg\theta - kl\frac{d\theta}{dt}$$

$$m\frac{d^2r}{dt^2} \approx m\frac{d^2}{dt^2}(l\theta) = ml\frac{d^2\theta}{dt^2}$$

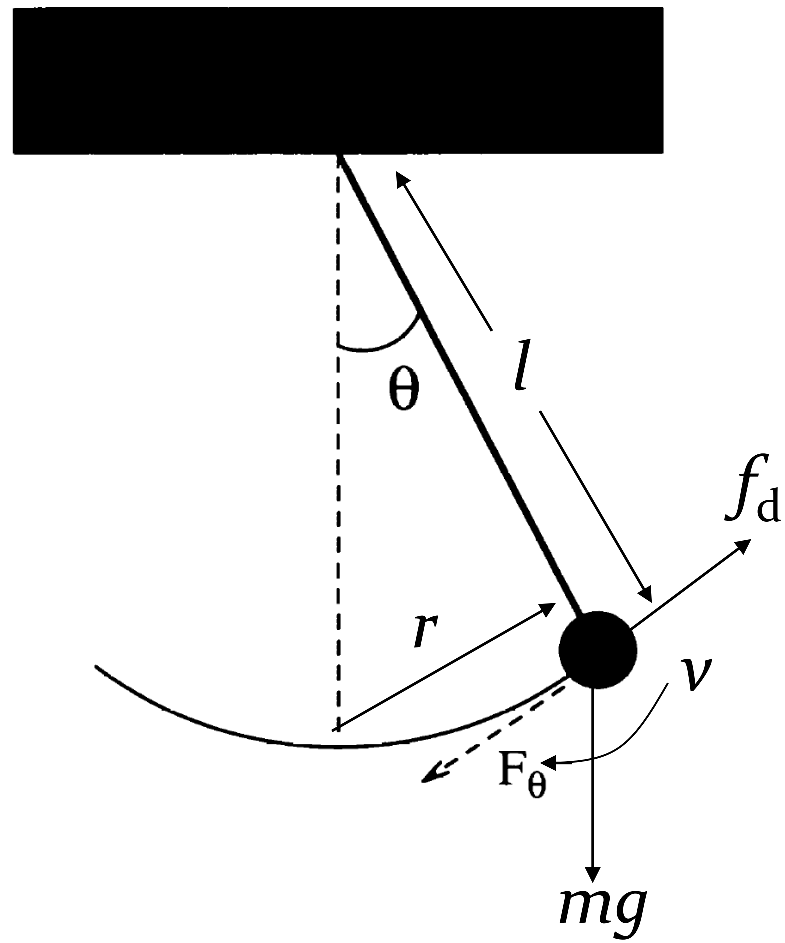
$$F_\theta = m\frac{d^2r}{dt^2} \Rightarrow \frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - \frac{k}{m}\frac{d\theta}{dt} \equiv -\frac{g}{l}\theta - q\frac{d\theta}{dt};\ q \equiv \frac{k}{m}$$

# Simple Harmonic pendulum with drag force as a special case of second order DE (cont.)

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)$$

$$x \equiv t$$
$$u(x) \equiv \theta(t)$$
$$a \equiv q$$
$$b \equiv \frac{g}{l}$$
$$n(x) \equiv 0$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt}; q = \frac{k}{m}$$



# Analytical solutions

1. Underdamped regime (small damping). Still oscillate, but amplitude decay slowly over many period before dying totally.

$$\theta(t) = \theta_0 e^{-qt/2} \sin\left(\varphi + t\sqrt{\Omega^2 - \frac{q^2}{4}}\right)$$

$$\Omega = \sqrt{\frac{g}{l}} \text{ the natural frequency of the system}$$

# Analytical solutions

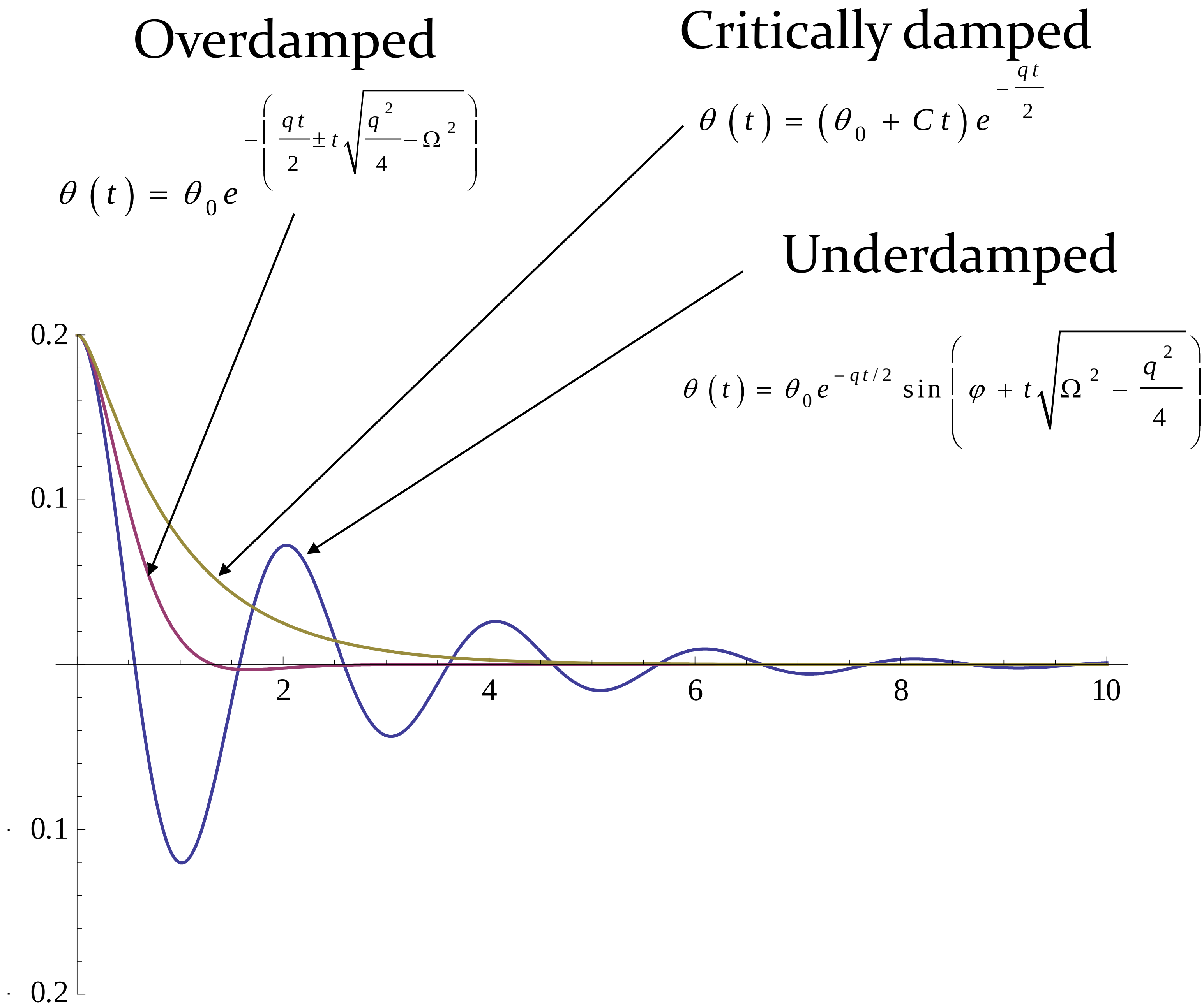
2. Overdamped regime (very large damping), decay slowly over several period before dying totally. $\theta$ is dominated by exponential term.

$$\theta(t) = \theta_0 e^{-\left(\frac{qt}{2} \pm t\sqrt{\frac{q^2}{4} - \Omega^2}\right)}$$

# Analytical solutions

3. Critically damped regime, intermediate between under- and overdamping case.

$$\theta(t) = (\theta_0 + Ct)\, e^{-\frac{qt}{2}}$$



Overdamped
$\theta(t) = \theta_0 e^{-\left(\frac{qt}{2} \pm t\sqrt{\frac{q^2}{4} - \Omega^2}\right)}$
Critically damped
$\theta(t) = (\theta_0 + Ct)\, e^{-\frac{qt}{2}}$
Underdamped
$\theta(t) = \theta_0 e^{-qt/2} \sin\left(\varphi + t\sqrt{\Omega^2 - \frac{q^2}{4}}\right)$
0.2
0.1
0.1
0.2
2
4
6
8
10

See C8_2ODE_Pendulum.nb where **Dsolve[]** solves the three cases of a damped pendulum analytically.

# Adding driving force to the damped oscillator: forced oscillator

$$F_\theta = -mg\sin\theta - kl\,(d\theta/dt) + F_D\sin(\Omega_D t)$$

*$\Omega_D$ frequency of the applied force*

$$F_\theta = -mg\sin\theta - kl\frac{d\theta}{dt} + F_D\sin(\Omega_D t) \approx -mg\theta - kl\frac{d\theta}{dt} + F_D\sin(\Omega_D t);$$

$$F_\theta = m\frac{d^2r}{dt^2} \approx m\frac{d^2}{dt^2}(l\theta) = ml\frac{d^2\theta}{dt^2};$$

$$F_\theta = m\frac{d^2r}{dt^2} \approx ml\frac{d^2\theta}{dt^2} \approx -mg\theta - kl\frac{d\theta}{dt} + F_D\sin(\Omega_D t)$$

$$\frac{d^2\theta}{dt^2} \approx -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D\sin(\Omega_D t)}{ml};\ q = \frac{k}{m}$$

# Analytical solution

$$\theta(t) = \theta_0 \sin\left(\Omega_D t + \phi\right)$$

$$\theta_0 = \frac{F_D / (ml)}{\sqrt{\left(\Omega^2 - \Omega_D^2\right)^2 + \left(q\Omega_D\right)^2}}$$

Resonance happens when $\Omega_D = \Omega = \sqrt{g/l}$

# Forced oscillator: An example of non homogeneous 2nd order DE

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)$$

$$x \equiv t$$

$$u(x) \equiv \theta(t)$$

$$a \equiv q$$

$$b \equiv \frac{g}{l}$$

$$n(x) \equiv \frac{F_D \sin(\Omega_D t)}{ml}$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{ml}$$

# Exercise: Forced oscillator

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{ml}$$

Use **DSolve**[] to solve the forced oscillator. Plot on the same graph the analytical solutions of $\theta(t)$ for $t$ from 0 to 10 $T$, where T= 2π/Ω, $\Omega = \sqrt{g/l}$, for $\Omega_D = 0.01\Omega$, $0.5\Omega$, $0.99\Omega$, $1.5\Omega$, $4\Omega$.

Assume the boundary conditions $\theta(t=0)=0$; $d\theta/dt(t=0)=0$; $m=l=F_D=1$; $q=0$.

Sample code: C8_2ODE_forcedPendulum.nb

# Second order Runge-Kutta (RK2) method

Consider a generic second order differential equation.

$$\frac{d^2 u(x)}{dx^2} = G(u)$$

It can be numerically solved using second order Runge-Kutta method.

Split the second order DE into two first order parts:

$$v(x) = \frac{du(x)}{dx} \qquad \frac{dv(x)}{dx} = G(u)$$

# Algorithm

Set boundary conditions: $u(x{=}x_o){=}u_o$, $u'(x{=}x_o){=}v(x{=}x_o){=}v_o$.

calculate $$\tilde{u} = u_i + \frac{1}{2} v_i \Delta x$$

calculate $$\tilde{v} = v_i + \frac{1}{2} G(\tilde{u}) \Delta x$$

calculate $$u_{i+1} = u_i + \tilde{v} \Delta x$$

calculate $$v_{i+1} = v_i + G(\tilde{u}) \Delta x$$

# Translating the SK2 algorithm into the case of simple pendulum

$$\frac{d^2 u(x)}{dx^2} = G(u)$$

$$G(u) \equiv -\frac{g}{l}\theta(t)$$

$$\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l}$$

$$v(x) = \frac{du(x)}{dx}$$

$$\omega(t) = \frac{d\theta(t)}{dt}$$

$$\frac{dv(x)}{dx} = G(u)$$

$$\frac{d\omega(t)}{dt} = -\frac{g\theta(t)}{l}$$

Set boundary conditions:
$u(x=x_o)=u_o$, $u'(x=x_o)=v(x=x_o)=v_o$

Set boundary conditions:
$\theta(t=t_o)= \theta_o$, $\theta'(t=t_o)=\omega(t=t_o)=\omega_o$

$$\tilde{u} = u_i + \frac{1}{2} v_i \Delta x$$

$$\tilde{v} = v_i + \frac{1}{2} G(\tilde{u}) \Delta x$$

$$u_{i+1} = u_i + \tilde{v}\Delta x$$

$$v_{i+1} = v_i + G(\tilde{u})\Delta x$$

$$\tilde{\theta} = \theta_i + \frac{1}{2}\omega_i \Delta t$$

$$\tilde{\omega} = \omega_i + \frac{1}{2}\left(-\frac{g\tilde{\theta}}{l}\right)\Delta t$$

$$\theta_{i+1} = \theta_i + \tilde{\omega}\Delta t$$

$$\omega_{i+1} = \omega_i + \left(-\frac{g\tilde{\theta}}{l}\right)\Delta t$$

Exercise: Develop a code to implement SK2 for the case of the simple pendulum.
Boundary conditions: $\omega(0) = \sqrt{\frac{g}{l}}; \theta(0) = 0$

See C8_pendulum_RK2.nb

# Translating the SK2 algorithm into the case of damped pendulum

$$\frac{d^2u(x)}{dx^2} = G(u)$$

$$\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l} - q\frac{d\theta}{dt}$$

$$v(x) = \frac{du(x)}{dx}$$

$$\omega(t) = \frac{d\theta(t)}{dt}$$

$$\frac{dv(x)}{dx} = G(u)$$

$$G(u) \equiv -\frac{g}{l}\theta(t) - q\omega(t)$$

$$\frac{d\omega(t)}{dt} = -\frac{g\theta}{l} - q\omega(t)$$

Set boundary conditions:
$u(x=x_o)=u_o$, $u'(x=x_o)=v(x=x_o)=v_o$

Set boundary conditions:
$\theta(t=t_o)=\theta_o$, $\theta'(t=t_o)=\omega(t=t_o)=\omega_o$

$$\tilde{u} = u_i + \frac{1}{2}v_i\Delta x$$

$$\tilde{v} = v_i + \frac{1}{2}G(\tilde{u})\Delta x$$

$$u_{i+1} = u_i + \tilde{v}\Delta x$$

$$v_{i+1} = v_i + G(\tilde{u})\Delta x$$

$$\tilde{\theta} = \theta_i + \frac{1}{2}\omega_i\Delta t$$

$$\tilde{\omega} = \omega_i + \frac{1}{2}\left(-\frac{g}{l}\tilde{\theta}(t) - q\tilde{\omega}\right)\Delta t \Rightarrow \tilde{\omega} = \frac{\left(\omega_i - \frac{g}{2l}\tilde{\theta}(t)\Delta t\right)}{\left(1 + \frac{1}{2}q\Delta t\right)}$$

$$\theta_{i+1} = \theta_i + \tilde{\omega}\Delta t$$

$$\omega_{i+1} = \omega_i + \left(-\frac{g\tilde{\theta}}{l} - q\tilde{\omega}\right)\Delta t$$

**Exercise**:
Develop a code to implement SK2 for the case of a pendulum experiencing a drag force, with damping coefficient $q$= 0.1* (4$\Omega$), $\Omega$= $\sqrt{g/l}, l = 1.0$ m.
Boundary conditions: $\theta(0) = 0.2; \omega(t = 0) = 0;$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt}$$

Sample code: C8_dampedpendulum_RK2.nb

# Exercise:

Develop a code to implement SK2 for the case of a forced pendulum experiencing no drag force but a driving force$F_D \sin(\Omega_\mathrm{D} t)$, $\Omega = \sqrt{g/l}, l = 1.0$ m, $m$=1kg; $F_\mathrm{D} = 1\mathrm{N}$; $\Omega_\mathrm{D}$=0.99 $\Omega$;
Boundary conditions: $\theta(0) = 0.0; \omega(t = 0) = 0;$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{ml}$$

# Exercise: Stability of the total energy a SHO in RK2.

$\omega = \frac{d\theta}{dt}$, angular velocity. $m$=1 kg; $l$=1 m.

The total energy of the SHO in can be calculated as

$$E_{i+1} = K_{i+1} + U_{i+1} = \frac{1}{2}m\left(l\omega_{i+1}\right)^2 + mgl\left(1 - \cos\theta_{i+1}\right)$$

$$\approx \frac{1}{2}ml^2\omega_{i+1}{}^2 + mgl\left[1 - \left(1 - \frac{\theta_{i+1}^2}{2}\right)\right]$$

$$= \frac{1}{2}ml^2\omega_{i+1}{}^2 + \frac{1}{2}mgl\theta_{i+1}^2$$

User your RK2 code to track the total energy for $t$ running from $t$=0 till t=25T; T=$\sqrt{g/l}$. Boundary conditions: $\omega(0) = \sqrt{\frac{g}{l}}; \theta(0) = 0$

$E_i$ should remain constant throughout all $t_i$.