Fourier series representation of a function defined on the general interval [*a*,*b*]

- For a function defined on the interval of [*a*,*b*] the Fourier series representation on [*a*,*b*] is actually

$$a_0 + \sum_{k=1}^{n} a_k \cos\frac{2\pi kx}{L} + b_k \sin\frac{2\pi kx}{L} x$$

$$a_0 = \frac{1}{L}\int_a^b f(x)dx$$

$$a_m = \frac{2}{L}\int_a^b f(x)\cos\frac{2\pi mx}{L}dx$$

$$b_m = \frac{2}{L}\int_a^b f(x)\sin\frac{2\pi mx}{L}dx, m \text{ positive integer}$$

- *L*=*b* - *a*

Derivation of a_0

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cos\frac{2\pi kx}{L} + b_k \sin\frac{2\pi kx}{L} x$$

$$\int_a^b f(x)dx = \int_a^b a_0 dx + \sum_{k=1}^{n} \int_a^b a_k \cos\frac{2\pi kx}{L} dx + b_k \sin\frac{2\pi kx}{L} dx =$$

$$\int_a^b a_0 dx + \sum_{k=1}^{n} a_k \int_a^b \cos\frac{2\pi kx}{L} dx + \sum_{k=1}^{n} b_k \int_a^b \sin\frac{2\pi kx}{L} dx$$

$$= a_0 (b - a)$$

$$\Rightarrow a_0 = \frac{1}{b-a} \int_a^b f(x)dx = \frac{1}{L} \int_a^b f(x)dx$$

Derivation of a_k

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cos\frac{2\pi kx}{L} + b_k \sin\frac{2\pi kx}{L} x$$

$$\int_a^b f(x)\cos\frac{2\pi mx}{L} dx$$

$$= \int_a^b a_0 \cos\frac{2\pi mx}{L} dx + \sum_{k=1}^{n} \int_a^b \left(a_k \cos\frac{2\pi kx}{L} \cos\frac{2\pi mx}{L} dx + b_k \sin\frac{2\pi kx}{L} \cos\frac{2\pi mx}{L} \right) dx$$

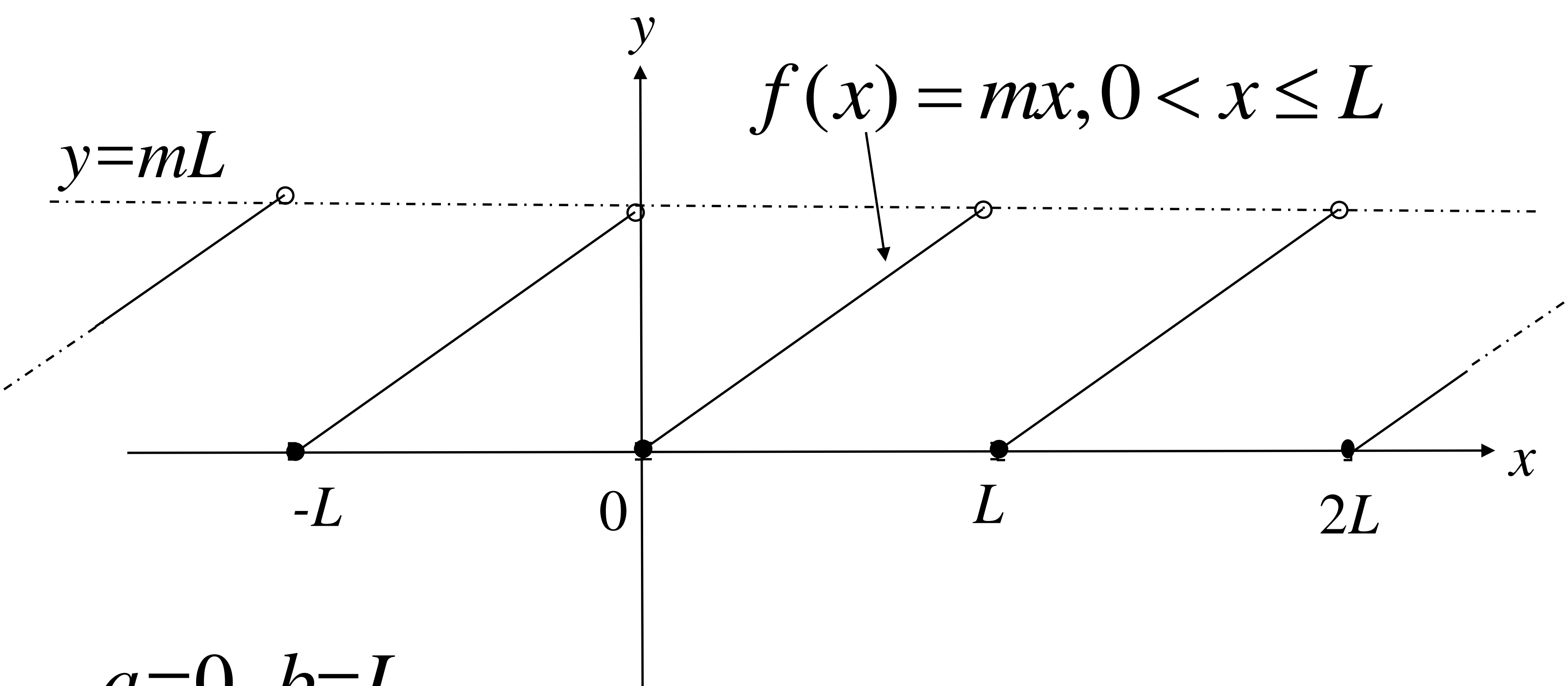
$$= 0 + a_m \int_a^b \cos^2\frac{2\pi mx}{L} dx + 0 = a_m \frac{L}{2}$$

$$\Rightarrow a_m = \frac{2}{L} \int_a^b f(x)\cos\frac{2\pi mx}{L} dx$$

Similarly,

$$b_m = \frac{2}{L} \int_a^b f(x)\sin\frac{2\pi mx}{L} dx$$

Example:



a=0, b=L

$$a_0 = \frac{1}{L}\int_a^b f(x)dx = \frac{1}{L}\int_a^b mxdx = \frac{m}{2L}\left(b^2 - a^2\right) = \frac{mL}{2}$$

$$a_k = \frac{2}{L}\int_a^b mx\cos\frac{2\pi kx}{L}dx = \frac{2m}{L}\int_a^b x\cos\frac{2\pi kx}{L}dx = \frac{2m}{L}\frac{L^2\left(\cos 2k\pi - 1\right)}{4k^2\pi^2} = 0;$$

$$b_k = \frac{2}{L}\int_a^b f(x)\sin\frac{2\pi kx}{L}dx = \frac{2m}{L}\int_0^L x\sin\frac{2\pi kx}{L}dx$$

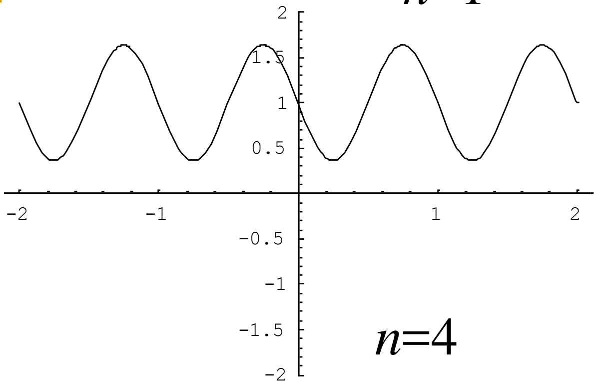
$$= \frac{2m}{L}\cdot L^2\left(\frac{-2k\pi\cos(2k\pi) + \sin 2k\pi}{4k^2\pi^2}\right) = \frac{-mL}{k\pi};$$

$$f(x) = mx = \frac{mL}{2} - \frac{mL}{\pi}\sum_{k=1}^{n}\frac{\sin 2\pi kx}{k}$$

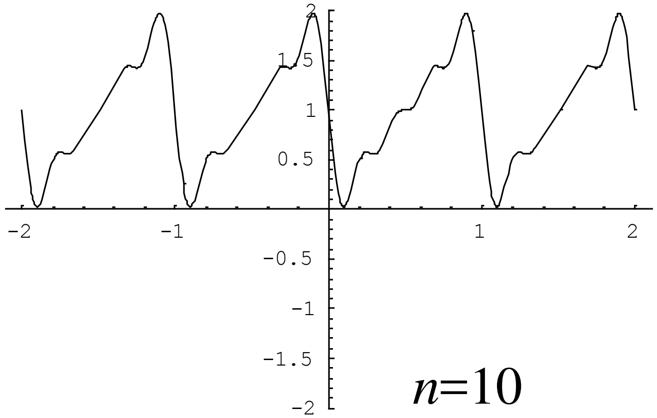
$$= mL\left(\frac{1}{2} - \frac{\sin 2\pi x}{\pi} - \frac{\sin 4\pi x}{2\pi} - \frac{\sin 6\pi x}{3\pi} - \dots - \frac{\sin 2n\pi x}{n\pi} + \dots\right)$$

$m=2$, $L = 1$

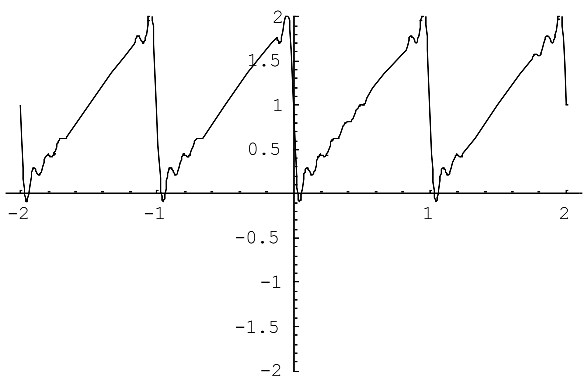
$n=1$

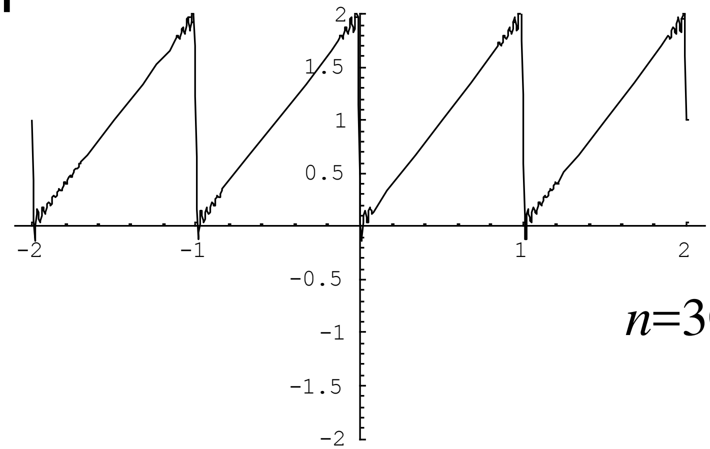


$n=4$

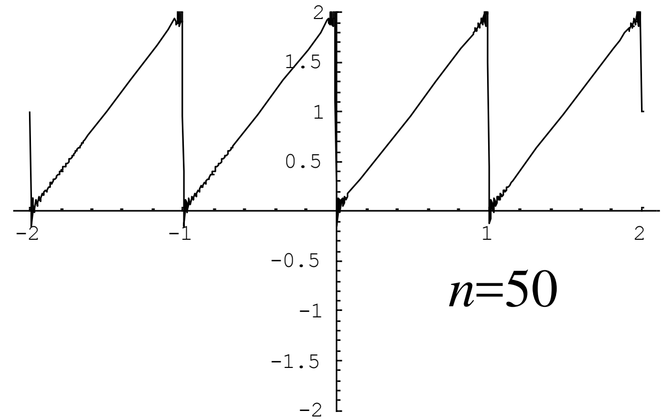


$n=10$





$n=30$



$n=50$

- **mathematica simulation*