Lecture 2
Mathematica for Series

Use Mathematica to find the convergence of a series

1. $$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{1(1+1)}+\frac{1}{2(2+1)}+\frac{1}{1(1+1)}+\ldots$$

2. $$\sum n^2=1+2+4+16+\ldots n^2+\ldots$$

3. $$\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}+\ldots$$

4. $$\sum_{n=1}^{\infty}\frac{5}{5n-1}=\frac{5}{5-1}+\frac{5}{5(2)-1}++\frac{5}{5(3)-1}\ldots$$

5. $$\sum_{n=1}^{\infty}\frac{1}{n!}=\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}\ldots$$

6. **(a)** $\displaystyle\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{n!n!}$

7. **(a)** $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^2}$ **(c)** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{1+n}\right)^n$

8. $$\sum_{n=1}^{\infty} 1\left(-\frac{1}{2}\right)^{n-1} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$$

9. $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

10. $$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$$

Use Mathematica to generate a power series. Compare it to the function it converges to by plotting both on the same plot. Deduce the interval of convergence from the plot.

DEFINITIONS Power Series, Center, Coefficients

A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \tag{1}$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \tag{2}$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

EXAMPLE 1 A Geometric Series

Taking all the coefficients to be 1 in Equation (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This is the geometric series with first term 1 and ratio x. It converges to $1/(1-x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1. \tag{3}$$

EXAMPLE 3 Testing for Convergence Using the Ratio Test

For what values of x do the following power series converge?

(a) $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

(b) $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

(c) $$\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(d) $$\sum_{n=0}^{\infty}n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$$

Series representation of $\tan^{-1}x$ (alternative approach)

- $\tan^{-1}x = \int \frac{1}{1+x^2}dx$ (from integral table)
- Identify $\frac{1}{1+x^2} \equiv \frac{a}{1-r} \Rightarrow a \equiv 1, r \equiv -x^2$, $|x| < 1$.
- Construct the geometry series:
- $s = ar^0 + ar^1 + ar^2 + ar^3 + ar^4 + \cdots$
  $= 1 \quad - x^2 + x^4 - x^6 \ + x^8 + \cdots$
- $\Rightarrow \frac{1}{1+x^2} = 1 \quad - x^2 + x^4 - x^6 \ + x^8 + \cdots$
- $\Rightarrow \int \frac{1}{1+x^2}dx = \int dx(1 - x^2 + x^4 - x^6 \ + x^8 + \cdots)$

$$\Rightarrow \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \frac{x^9}{9} + \cdots$$

EXAMPLE 6 A Series for $\ln(1 + x)$, $-1 < x \leq 1$

The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\ln(1 + x) = \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x \qquad \text{Theorem 20}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x < 1.$$

It can also be shown that the series converges at $x = 1$ to the number $\ln 2$, but that was not guaranteed by the theorem.

Use Mathematica to generate a Taylor series of a function $f(x)$ at a center $x = a$. Compare it to the generating function $f(x)$ by plotting both on the same plot. Deduce the interval of convergence from the plot.

DEFINITIONS **Taylor Series, Maclaurin Series**

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

Note: Maclaurin series is effectively a special case of Taylor series with $a = 0$.

Taylor series for $f(x)=1/x$ expanded at the center $x = 2$

- $f(x) = x^{-1}$; $f'(x) = -x^{-2}$; $f^{(n)}(x) = (-1)^n\, n!\, x^{-(n+1)}$
- The Taylor series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k = \sum_{k=0}^{\infty} \left.\frac{(-1)^k\, k!\, x^{-(k+1)}}{k!}\right|_{x=2} (x-2)^k$$

$$= \sum_{k=0}^{\infty} a_k (x-2)^k,$$

$$a_k = \left.\frac{(-1)^k\, k!\, x^{-(k+1)}}{k!}\right|_{x=2}$$

Taylor series for e^x at $x = 0$

$$f(x) = e^x \to f^{(n)}(x) = e^x$$

$$f(x) = \sum_{k=0} \frac{f^{(k)}(x)}{k!}\bigg|_{x=0} x^k = \frac{e^0}{0!}x^0 + \frac{e^0}{1!}x^1 + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \ldots \frac{e^0}{n!}x^n + \ldots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!}$$

Taylor series for cos x at $x = 0$

Solution The cosine and its derivatives are

$$\begin{aligned} f(x) &= \cos x, & f'(x) &= -\sin x, \\ f''(x) &= -\cos x, & f^{(3)}(x) &= \sin x, \\ &\vdots & &\vdots \\ f^{(2n)}(x) &= (-1)^n \cos x, & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x. \end{aligned}$$

At $x = 0$, the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0.$$

The Taylor series generated by f at 0 is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ = 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}. \end{aligned}$$

Taylor series for ln x at $x = 1$

- $f(x)=\ln x$; $f'(x) = x^{-1}$;
- $f''(x) = (-1)(1)x^{-2}$; $f'''(x) = (-1)^2(2)(1)x^{-3}$ …
- $f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$;

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(x)}{n!}\bigg|_{x=1}(x-1)^n = \frac{f^{(0)}(x)}{0!}\bigg|_{x=1}(x-1)^0 + \sum_{n=1}^{\infty}\frac{f^{(n)}(x)}{n!}\bigg|_{x=1}(x-1)^n$$

$$= \frac{\ln 1}{0!} + \sum_{n=1}^{\infty}\frac{(-1)^{(n-1)}(n-1)!x^{-n}}{n!}\bigg|_{x=1}(x-1)^n = 0 + \sum_{n=1}^{\infty}\frac{(-1)^{(n-1)}(1)^{-n}}{n}(x-1)^n$$

$$= \frac{(-1)^0}{1}(x-1)^1 + \frac{(-1)^1}{2}(x-1)^2 + \frac{(-1)^2}{3}(x-1)^3 + \ldots$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots + (-1)^n\frac{1}{n}(x-1)^n + \ldots$$

Binomial series

Consider the Taylor series generated by $f(x) = (1+x)^m$, where m is a constant:

$$f(x) = (1+x)^m$$

$$f'(x) = m(1+x)^{m-1},\ f''(x) = m(m-1)(1+x)^{m-2},$$

$$f'''(x) = m(m-1)(m-2)(1+x)^{m-3},$$

$$\vdots$$

$$f^{(k)}(x) = m(m-1)(m-2)\dots(m-k+1)(1+x)^{m-k};$$

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(0)x^k$$

$$= \sum_{k=0}^{\infty} \frac{m(m-1)(m-2)\dots(m-k+1)}{k!}x^k$$

$$= 1 + mx + m(m-1)x^2 + m(m-1)(m-2)x^3 + \dots$$

$$+ \frac{m(m-1)(m-2)\dots(m-k+1)}{k!}x^k + \dots$$

The Taylor series of $f(x) = (1+x)^m$, is called the binomial series

$$f(x) = (1+x)^m$$

$$= \sum_{k=0}^{\infty} \frac{m(m-1)(m-2)...(m-k+1)}{k!} x^k$$

- This series is called the binomial series, converges absolutely for $|x| < 1$.

The Taylor series of $f(x) = (1+x)^m$, is called the binomial series

The Binomial Series

For $-1 < x < 1$,

$$(1 + x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k \geq 3.$$

EXAMPLE 2 Using the Binomial Series

We know from Section 3.8, Example 1, that $\sqrt{1+x} \approx 1 + (x/2)$ for $|x|$ small. With $m = 1/2$, the binomial series gives quadratic and higher-order approximations as well, along with error estimates that come from the Alternating Series Estimation Theorem:

$$(1+x)^{1/2} = 1 + \frac{x}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots.$$

Substitution for x gives still other approximations. For example,

$$\sqrt{1-x^2} \approx 1 - \frac{x^2}{2} - \frac{x^4}{8} \qquad \text{for } |x^2| \text{ small}$$

$$\sqrt{1-\frac{1}{x}} \approx 1 - \frac{1}{2x} - \frac{1}{8x^2} \qquad \text{for } \left|\frac{1}{x}\right| \text{ small, that is, } |x| \text{ large.}$$

Generate the coefficients of the binomial series for $(1+x)^m$ using Mathematica.

Compare it to the generating function $(1+x)^m$ by plotting both on the same plot.

Deduce the interval of convergence from the plot.

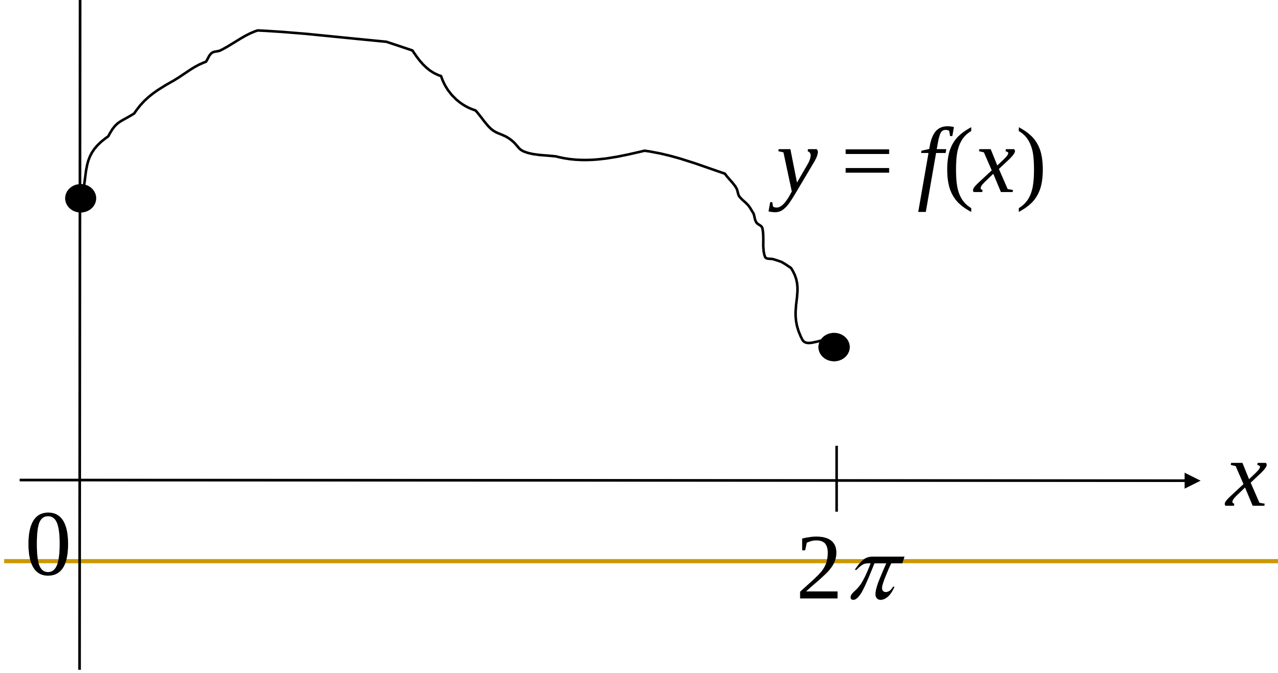
Fourier series

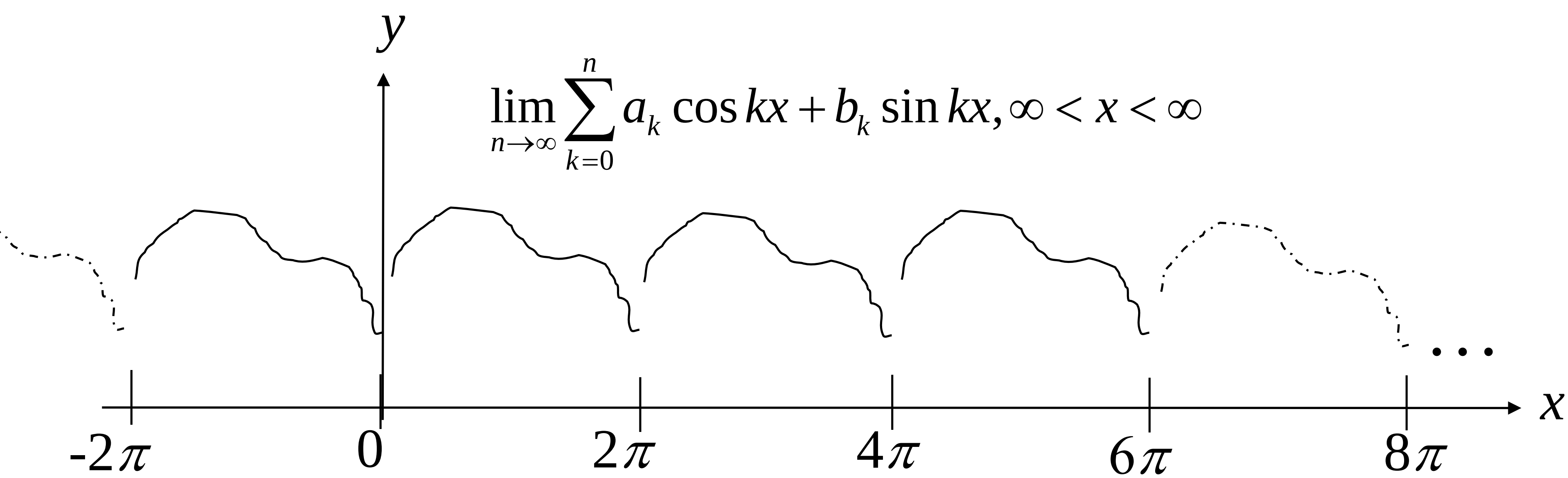
A function $f(x)$ defined on $[0, 2\pi]$ can be represented by a Fourier series

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} f_k(x) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k \cos kx + b_k \sin kx$$

$$= a_0 + \lim_{n\to\infty} \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx, \quad 0 \le x \le 2\pi.$$

← Fourier series representation of $f(x)$





If -∞<$x < \infty$, the Fourier series $\lim_{n\to\infty}\sum_{k=0}^{n} f_k(x) = \lim_{n\to\infty}\sum_{k=0}^{n} a_k \cos kx + b_k \sin kx$

acutally represents a periodic function $f(x)$ of a period of $L = 2\pi$,

Fourier series representation of a function defined on the general interval [*a*,*b*]

- For a function defined on the interval of [*a*,*b*] the Fourier series representation on [*a*,*b*] is actually

$$a_0 + \sum_{k=1}^{n} a_k \cos\frac{2\pi kx}{L} + b_k \sin\frac{2\pi kx}{L}x$$

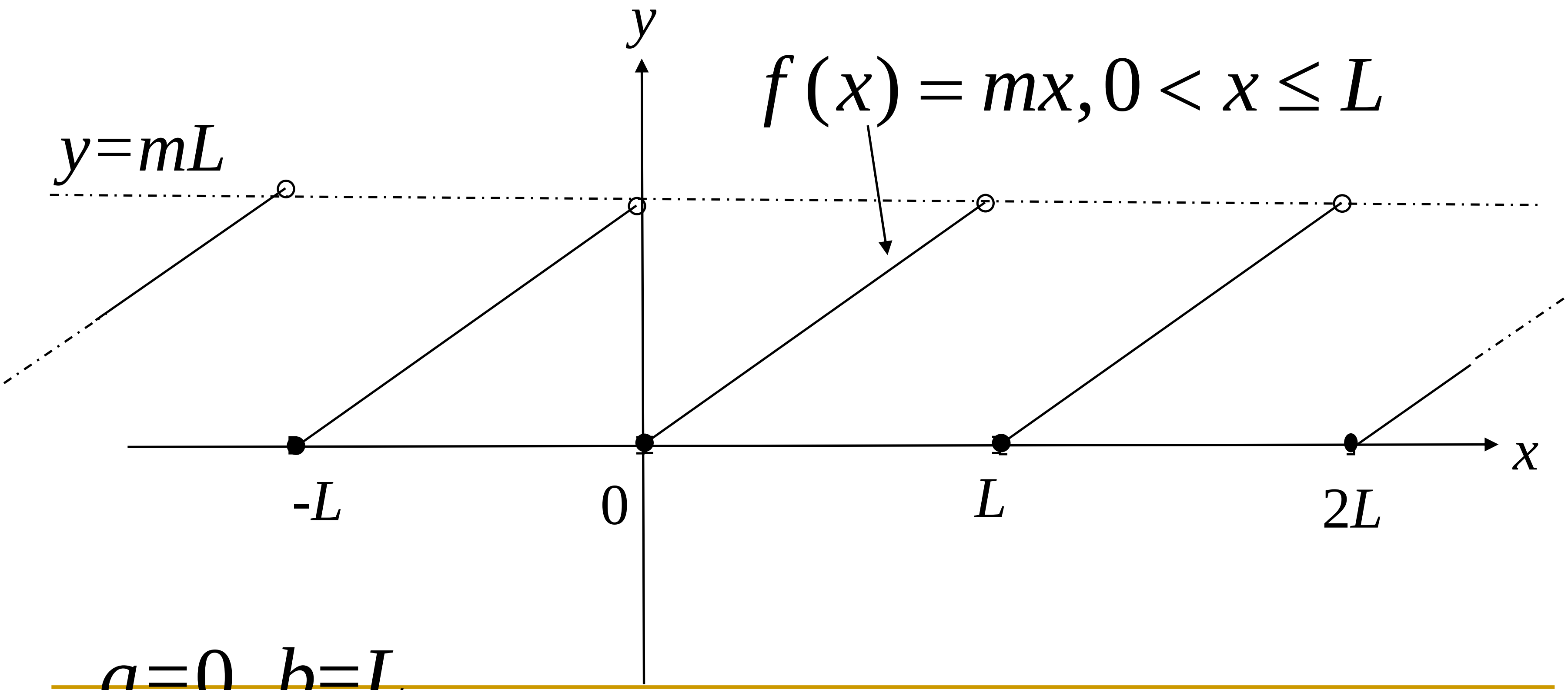
$$a_0 = \frac{1}{L}\int_a^b f(x)dx$$

$$a_m = \frac{2}{L}\int_a^b f(x)\cos\frac{2\pi mx}{L}dx$$

$$b_m = \frac{2}{L}\int_a^b f(x)\sin\frac{2\pi mx}{L}dx, m \text{ positive integer}$$

- *L*=*b* - *a*

Example:



a=0, b=L

$$a_0 = \frac{1}{L}\int_a^b f(x)dx = \frac{1}{L}\int_a^b mxdx = \frac{m}{2L}\left(b^2 - a^2\right) = \frac{mL}{2}$$

$$a_k = \frac{2}{L}\int_a^b mx\cos\frac{2\pi kx}{L}dx = \frac{2m}{L}\int_a^b x\cos\frac{2\pi kx}{L}dx = \frac{2m}{L}\,\frac{L^2\left(\cos 2k\pi - 1\right)}{4k^2\pi^2} = 0;$$

$$b_k = \frac{2}{L}\int_a^b f(x)\sin\frac{2\pi kx}{L}dx = \frac{2m}{L}\int_0^L x\sin\frac{2\pi kx}{L}dx$$

$$= \frac{2m}{L}\cdot L^2\left(\frac{-2k\pi\cos(2k\pi) + \sin 2k\pi}{4k^2\pi^2}\right) = \frac{-mL}{k\pi};$$

$$f(x) = mx = \frac{mL}{2} - \frac{mL}{\pi}\sum_{k=1}^{n}\frac{\sin 2\pi kx}{k}$$

$$= mL\left(\frac{1}{2} - \frac{\sin 2\pi x}{\pi} - \frac{\sin 4\pi x}{2\pi} - \frac{\sin 6\pi x}{3\pi} - \ldots - \frac{\sin 2n\pi x}{n\pi} + \ldots\right)$$

Saw-tooth function

Consider the function $f(x) = mx$, where m is the slope of the function, defined for $x \in \{0, L\}, L > 0$. The Fourier coefficients are given by

$$a_0 = \frac{mL}{2}, a_k = 0, b_k = -\frac{mL}{k\pi}$$

Plot the Fourier series with $n = 20$ terms for $x \in \{-2L, 2L\}$, and overlap it with the function $f(x)$ on the same plot. Assume $L = m = 1$.

Fourier series of a step function

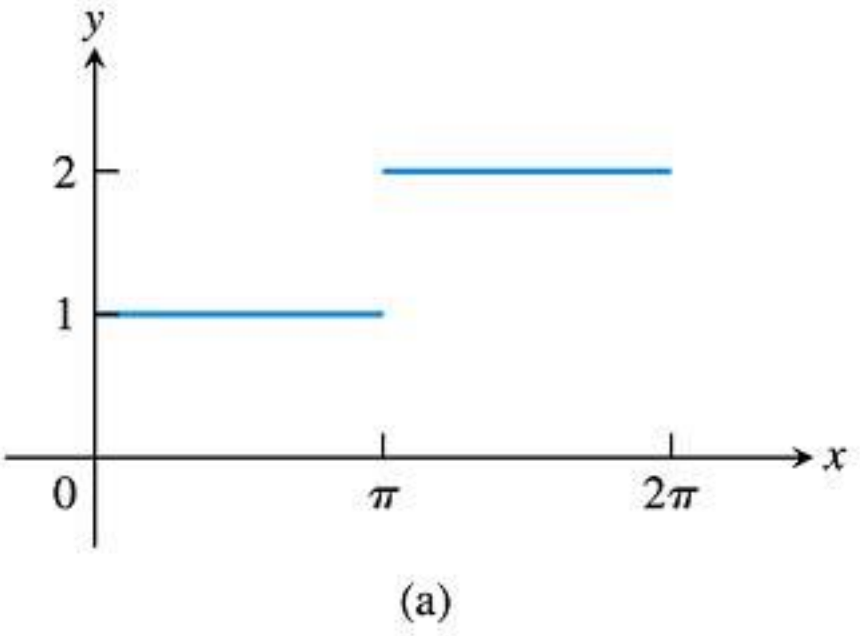


FIGURE 11.16 (a) The step function

$$f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 2, & \pi < x \le 2\pi \end{cases}$$

So

$$a_0 = \frac{3}{2}, \quad a_1 = a_2 = \cdots = 0,$$

and

$$b_1 = -\frac{2}{\pi}, \quad b_2 = 0, \quad b_3 = -\frac{2}{3\pi}, \quad b_4 = 0, \quad b_5 = -\frac{2}{5\pi}, \quad b_6 = 0, \ldots$$

The Fourier series is

$$\frac{3}{2} - \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

Plot the Fourier series of the step function using Mathematica

Plot the Fourier series of the step function using Mathematica

Taylor series revisited

- Taylor series representation for an arbitrary function $f(x)$ at the center $x = a$ up to n-th order

$$P_n(x) = \sum_{k=0}^{n} a_k (x-a)^k,$$

$$a_k = \frac{1}{k!} \frac{d^{(n)} f(x)}{dx^n}\bigg|_{x=a}$$

Examples

$$P_n(x) = \sum_{k=0}^{n} a_k (x-a)^k,$$

$$a_k = \frac{1}{k!} \frac{d^{(n)} f(x)}{dx^n}\bigg|_{x=a}$$

$$f(x) = e^x, a = 0 \rightarrow P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!}$$

$$f(x) = \ln x, a = 1 \rightarrow P_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots + (-1)^n \frac{1}{n}(x-1)^n + \ldots$$

$$f(x) = (1+x)^m, a = 0 \rightarrow P_n(x) = \sum_{k=0}^{n} \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!} x^k$$

A generic question to be solved

- Given any arbitrary function $f(x)$, what is the analytical expression of the i-th coefficient in the Taylor series $P_n(x)$ for $f(x)$ at $x = a$?
- Use Mathematica to for the explicit expression of $P_n(x)$ for $f(x)$ at $x = a$.
- Plot $P_n(x)$ and $f(x)$ for a few selected values of n, covering a range of x that includes $x = a$
- Check the correctness of your answer using the command **Series**[].

Try on these few functions

1. $f(x) = \tan^{-1} x$ at $x = 1$.
2. $f(x) = \sinh^{-1} x$ at $x = 0$
3. $f(x) = \frac{1}{\sqrt{1-x^2}}$ at $x = 0$.