Lecture 3
Mathematica for Differential Equations

Use Mathematica to find the analytical solutions to first order ordinary differential equations

Example of first order differential equation commonly encountered in physics

$$\frac{dv_y}{dt} = -g; \frac{dy}{dt} = v_y$$

$$m\frac{dv}{dt} = -mg - \eta v$$

$$\frac{dN}{dx} = -\lambda N; \frac{dN}{dt} = -\frac{N}{\tau}$$

$$m\frac{dv}{dx} = -kx$$

- Do you recognize these equations?

Analytical solution of $\frac{dv_y}{dt} = -g$

- ZCA 101 mechanics, kinematic equation for a free fall object $\frac{dv_y}{dt} = -g$

- What is the solution, i.e., $v_y = v_y(t)$?

$$\frac{dv_y}{dt} = -g$$

$$\Rightarrow \int \frac{dv_y}{dt} dt = -\int g \, dt$$

$$\int dv_y = v_y = -\int g \, dt = -gt + c$$

$$\Rightarrow v_y = v_y(t) = -gt + c$$

Analytical solution of $\frac{dv_y}{dt} = -g$

- To completely solve this first order differential equation, i.e., to determine v_y as a function of t, and the arbitrary constant c, a boundary value or initial value of v_y at a given time t is necessary. Usually (but not necessarily) $v_y(0)$, i.e., the value of v_y at t=0 has to be assumed.

$$\frac{dv_y}{dt} = -g$$

$$\int_{v_y(0)}^{v_y(t)} dv = -\int_0^t g\, dt = -gt$$

$$\Rightarrow v_y = v_y(t) = -gt + v_y(0)$$

Analytical solution of $\frac{dy}{dt} = v_y$

- Assume $v_y = v_y(t)$ a known function of t.
- To completely solve the equation so that we can know what is the function $y(t)$, we need to know the value of $y(0)$.

$$\int_{y(0)}^{y(t)} dy = \int_0^t v_y \, dt$$

$$\Rightarrow y(t) - y(0) = \int_0^t v_y \, dt$$

Analytical solution of $\frac{dy}{dt} = v_y$

- In free fall without drag force, $v_y(t) = v_y(0) - gt$.
- The complete solution takes the form

$$y(t) - y(0) = \int_0^t v_y \, dt = \int_0^t \left(v_y(0) - gt\right) dt$$

$$y(t) = y(0) + v_y(0)t - \frac{1}{2}gt^2$$

Boundary condition

- In general, to completely solve a first order differential equation for a function with single variable, a boundary condition value must be provided.
- Generalising such argument, two boundary condition values must be supplied in order to completely solve a second order differential equation.
- *n* boundary condition values must be supplied in order to completely solve a *n*-th order differential equation.
- Hence, supplying boundary condition values are necessary when numerically solving a differential equation.

Analytical solution of a free fall object in a viscous medium

$$m\frac{dv}{dt} = -mg - \eta v$$

- Boundary condition: v=0 at t = 0.

$$\int_{v(0)}^{v(t)} \frac{dv}{dt}dt = \int_{v(0)}^{v(t)} dv = \int_0^t \left(-g - \frac{\eta}{m}v\right)dt$$

$$\Rightarrow \int_{v(0)=0}^{v(t)} \frac{dv}{\left(-g - \frac{\eta}{m}v\right)} = \int_0^t dt$$

$$\Rightarrow v(t) = -\left(\frac{mg}{\eta}\right)\left[1 - \exp\left(\frac{-\eta t}{m}\right)\right]$$

Number of beta particles penetrating a medium (recall your first year lab experiments)

$$\frac{dN}{dx} = -\lambda N$$

$$\int_{N_0}^{N} \frac{dN}{N} = -\int_{0}^{x} \lambda\, dx$$

$$N(x) = N_0 \exp[-\lambda x]$$

Number of radioactive particle remained after time *t* (recall your first year lab experiments. τ: half-life)

$$\frac{dN}{dt} = -\frac{N}{\tau}$$

- $\int_{N_0}^{N} \frac{dN}{N} = -\int_0^t \frac{1}{\tau} dt$

$$N(t) = N_0 \exp\left[-\frac{t}{\tau}\right]$$

Relation of speed vs. displacement in SHM

$$E = K + P = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$\Rightarrow \frac{dE}{dx} = \frac{d}{dx}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = 0$$

$$\Rightarrow m\frac{dv}{dx} = -kx$$

The solution is

$$v(x) = v_0 + \frac{kx_0^2}{2}m - \frac{k}{2m}x^2$$

bondary condition: $v = v_0\ at\ x = x_0$

DSolve[ ]

- Now, we would learn how to solve these first order differential equations
  - **DSolve[ ]** (symbolically)

EXAMPLE 3 Solving a Separable Equation

Solve the differential equation

$$\frac{dy}{dx} = (1 + y^2)e^x.$$

Solution Since $1 + y^2$ is never zero, we can solve the equation by separating the variables.

$$\frac{dy}{dx} = (1 + y^2)e^x$$

$$dy = (1 + y^2)e^x\, dx$$

Treat dy/dx as a quotient of differentials and multiply both sides by dx.

$$\frac{dy}{1 + y^2} = e^x\, dx$$

Divide by $(1 + y^2)$.

$$\int \frac{dy}{1 + y^2} = \int e^x\, dx$$

Integrate both sides.

$$\tan^{-1} y = e^x + C$$

C represents the combined constants of integration.

The equation $\tan^{-1} y = e^x + C$ gives y as an implicit function of x. When $-\pi/2 < e^x + C < \pi/2$, we can solve for y as an explicit function of x by taking the tangent of both sides:

$$\tan(\tan^{-1} y) = \tan(e^x + C)$$

$$y = \tan(e^x + C).$$

■

$$\frac{dy}{dx} = y - x, \qquad y(0) = \frac{2}{3}.$$

Solution The equation

$$\frac{dy}{dx} = y - x$$

is a first-order differential equation with $f(x, y) = y - x$.

On the left:

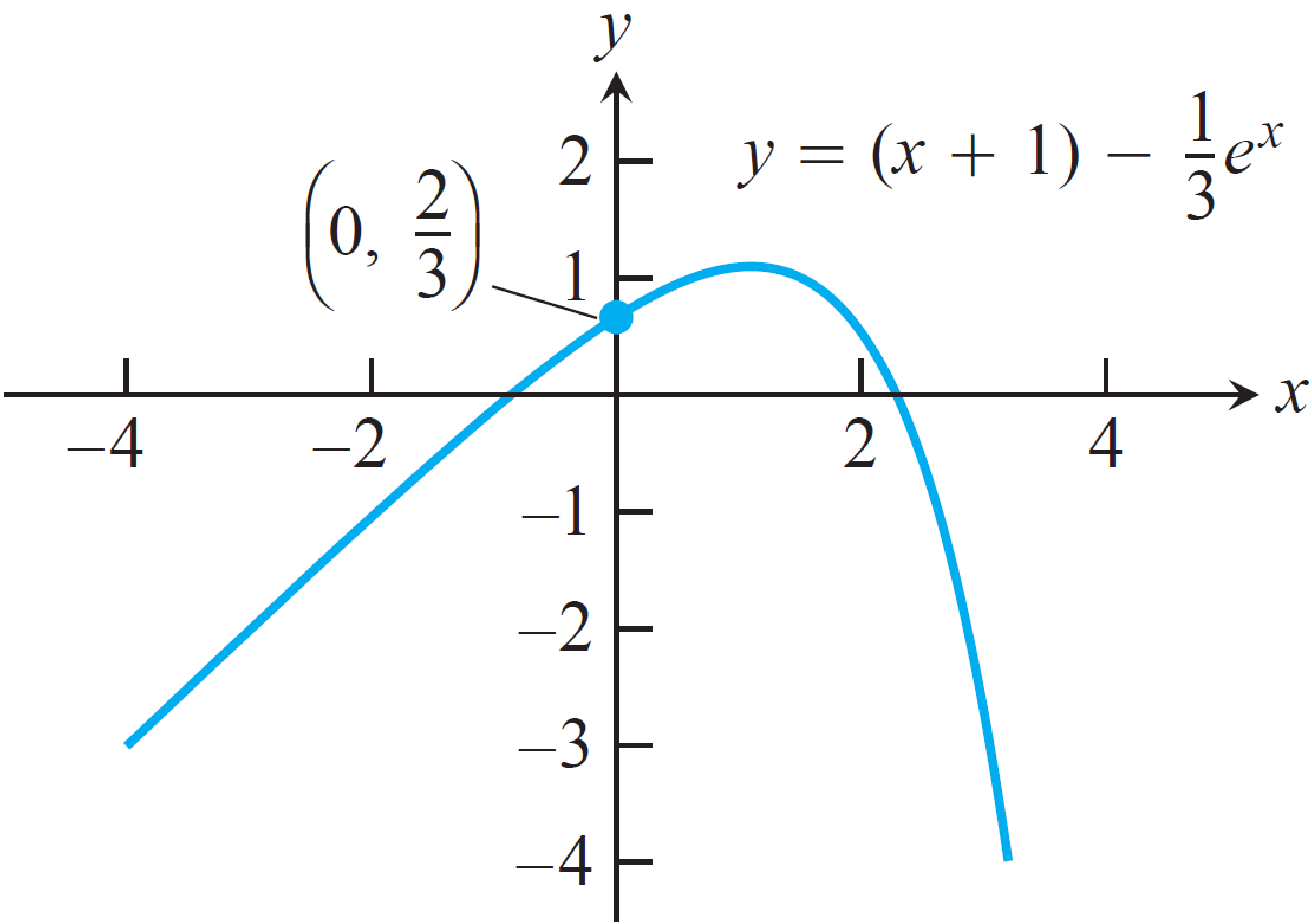
$$\frac{dy}{dx} = \frac{d}{dx}\left(x + 1 - \frac{1}{3}e^x\right) = 1 - \frac{1}{3}e^x.$$

On the right:

$$y - x = (x + 1) - \frac{1}{3}e^x - x = 1 - \frac{1}{3}e^x.$$

The function satisfies the initial condition because

$$y(0) = \left[(x + 1) - \frac{1}{3}e^x\right]_{x=0} = 1 - \frac{1}{3} = \frac{2}{3}.$$



y
x
$y = (x + 1) - \frac{1}{3}e^x$
$\left(0, \frac{2}{3}\right)$
2
1
−1
−2
−3
−4
−4
−2
2
4

EXAMPLE 5 Draining a Tank

A right circular cylindrical tank with radius 5 ft and height 16 ft that was initially full of water is being drained at the rate of $0.5\sqrt{x}\ \text{ft}^3/\text{min}$. Find a formula for the depth and the amount of water in the tank at any time t. How long will it take to empty the tank?

Solution The volume of a right circular cylinder with radius r and height h is $V = \pi r^2 h$, so the volume of water in the tank (Figure 9.4) is

$$V = \pi r^2 h = \pi(5)^2 x = 25\pi x.$$

Diffentiation leads to

$$\frac{dV}{dt} = 25\pi \frac{dx}{dt}$$ Negative because V is decreasing and $dx/dt < 0$

$$-0.5\sqrt{x} = 25\pi \frac{dx}{dt}$$ Torricelli's Law

Thus we have the initial value problem

$$\frac{dx}{dt} = -\frac{\sqrt{x}}{50\pi},$$

$$x(0) = 16$$ The water is 16 ft deep when $t = 0$.

$$x = \left(4 - \frac{t}{100\pi}\right)^2 \quad \text{and} \quad V = 25\pi x = 25\pi\left(4 - \frac{t}{100\pi}\right)^2.$$

EXAMPLE 2 Solving a First-Order Linear Differential Equation

Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution First we put the equation in standard form (Example 1):

$$\frac{dy}{dx} - \frac{3}{x}y = x,$$

so $P(x) = -3/x$ is identified.

The integrating factor is

$$\begin{aligned} v(x) = e^{\int P(x)\,dx} &= e^{\int(-3/x)\,dx} \\ &= e^{-3\ln|x|} \\ &= e^{-3\ln x} \\ &= e^{\ln x^{-3}} = \frac{1}{x^3}. \end{aligned}$$

Constant of integration is 0, so v is as simple as possible.

$x > 0$

Next we multiply both sides of the standard form by $v(x)$ and integrate:

$$\frac{1}{x^3} \cdot \left(\frac{dy}{dx} - \frac{3}{x}y\right) = \frac{1}{x^3} \cdot x$$

$$\frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y = \frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{x^3}y\right) = \frac{1}{x^2}$$

Left side is $\frac{d}{dx}(v \cdot y)$.

$$\frac{1}{x^3}y = \int \frac{1}{x^2}\,dx$$

Integrate both sides.

$$\frac{1}{x^3}y = -\frac{1}{x} + C.$$

Solving this last equation for y gives the general solution:

$$y = x^3\left(-\frac{1}{x} + C\right) = -x^2 + Cx^3, \qquad x > 0.$$

RL Circuits

The diagram in Figure 9.5 represents an electrical circuit whose total resistance is a constant R ohms and whose self-inductance, shown as a coil, is L henries, also a constant. There is a switch whose terminals at a and b can be closed to connect a constant electrical source of V volts.

Ohm's Law, $V = RI$, has to be modified for such a circuit. The modified form is

$$L \frac{di}{dt} + Ri = V, \tag{5}$$

where i is the intensity of the current in amperes and t is the time in seconds. By solving this equation, we can predict how the current will flow after the switch is closed.

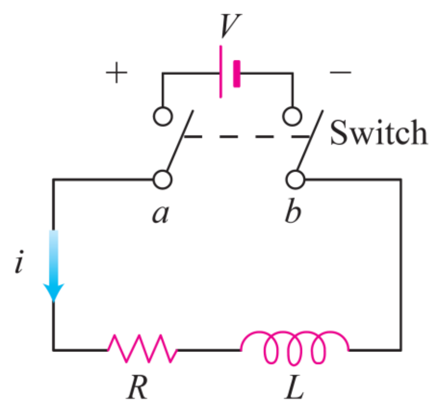


FIGURE 9.5 The *RL* circuit in Example 5.

EXAMPLE 5 Electric Current Flow

The switch in the RL circuit in Figure 9.5 is closed at time $t = 0$. How will the current flow as a function of time?

Solution Equation (5) is a first-order linear differential equation for i as a function of t. Its standard form is

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}, \tag{6}$$

and the corresponding solution, given that $i = 0$ when $t = 0$, is

$$i = \frac{V}{R} - \frac{V}{R}e^{-(R/L)t} \tag{7}$$

(Exercise 32). Since R and L are positive, $-(R/L)$ is negative and $e^{-(R/L)t} \to 0$ as $t \to \infty$. Thus,

$$\lim_{t\to\infty} i = \lim_{t\to\infty}\left(\frac{V}{R} - \frac{V}{R}e^{-(R/L)t}\right) = \frac{V}{R} - \frac{V}{R}\cdot 0 = \frac{V}{R}.$$

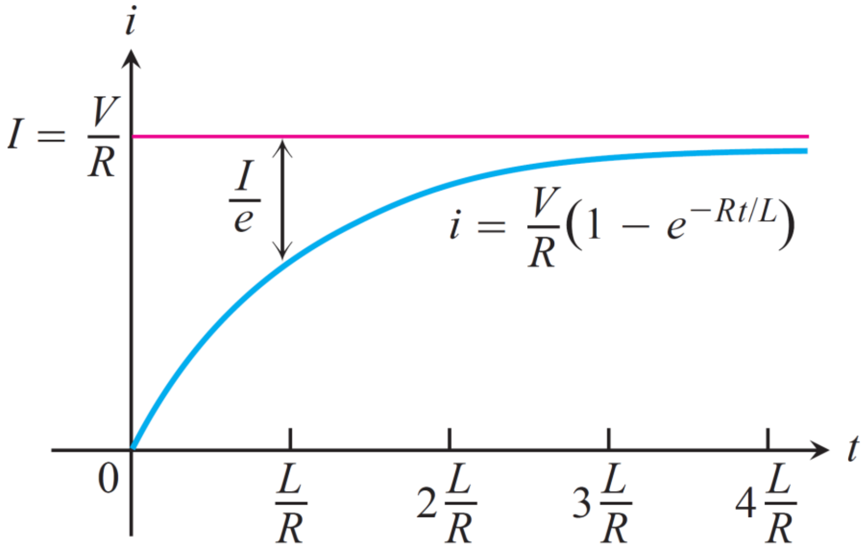


FIGURE 9.6 The growth of the current in the *RL* circuit in Example 5. I is the current's steady-state value. The number $t = L/R$ is the time constant of the circuit. The current gets to within 5% of its steady-state value in 3 time constants (Exercise 31).

Second order differential equations

2nd Order Homogeneous DEs

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0$$

with TWO given conditions

The solutions

Case 1: $a^2 - 4b > 0$:

$$u(x) = e^{-\frac{ax}{2}}\left(c_1 e^{\sqrt{a^2 - 4b}\ x/2} + c_2 e^{-\sqrt{a^2 - 4b}\ x/2}\right)$$

Case 2: $a^2 - 4b < 0$:

$$u(x) = e^{-\frac{ax}{2}}\left[A\,Sin\left(\frac{1}{2}\sqrt{4b - a^2}\right)x + B\,Cos\left(\frac{1}{2}\sqrt{4b - a^2}\right)x\right]$$

Case 3: $a^2 - 4b = 0$: — A special case

$$u(x) = c_1 e^{-\frac{ax}{2}} + c_2 x e^{-\frac{ax}{2}} = (c_1 + c_2 x) e^{-\frac{ax}{2}} \qquad (4.12)$$

where c_1, c_2, A and B are arbitrary constants to be determined by given conditions

Example 4.1 Solve the following differential equation

$$\frac{d^2u(x)}{dx^2} + 5\frac{du(x)}{dx} + 6u(x) = 0$$

$$u(x) = e^{-5x/2}\left(c_1 e^{x/2} + c_2 e^{-x/2}\right) = c_1 e^{-2x} + c_2 e^{-3x}$$

where c_1 and c_2 are arbitrary constants to be determined by given conditions

Example 4.2

$$\frac{d^2u(x)}{dx^2} + 6\frac{du(x)}{dx} + 9u(x) = 0$$

u(0) = 2

$$\left.\frac{du(x)}{dx}\right|_{x=0} = 0$$

$$u(x) = 2(1+3x)e^{-3x}$$

Typical second order, non-homogeneous ordinary differential

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x) \qquad (4.25)$$

Non-homogeneous term

Solution of Equation (4.25) consists **TWO** components:

Solution u(x) = Complementary solution $u_h(x)$ + Particular solution $u_p(x)$

$$u(x) = u_h(x) + u_p(x)$$

Typical second order, non-homogeneous ordinary differential

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x) \qquad (4.25)$$

Non-homogeneous term

$u(x) = u_h(x) + u_p(x)$

$$\frac{d^2u_h(x)}{dx^2} + a\frac{du_h(x)}{dx} + bu_h(x) = 0$$

There is **NO** fixed rule for deriving $u_p(x)$

Example 4.6

$$\frac{d^2 y(x)}{dx^2} - \frac{dy(x)}{dx} - 2y(x) = Sin\,2x$$

$$y(x) = y_h(x) + y_p(x)$$

$$\frac{d^2 y_h(x)}{dx^2} - \frac{dy_h(x)}{dx} - 2y_h(x) = 0$$

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x}$$

Guess: $y_p(x) = A\ Sin\ 2x + B\ Cos\ 2x$

After some algebra

$$y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 e^{2x} + \left(-\frac{3}{20} Sin\,2x + \frac{1}{20} Cos\,2x \right)$$

Example 4.8

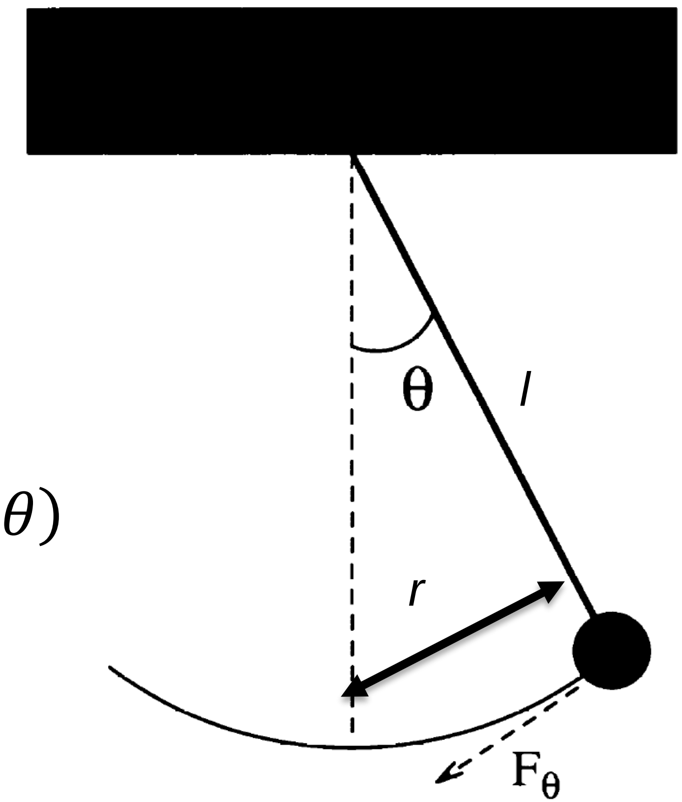
$$\frac{d^2u(x)}{dx^2} + 4u(x) = 2\,Sin\,2x$$

$$u(x) = u_h(x) + u_p(x) = c_1\,Cos\,2x + c_2\,Sin\,2x - \frac{x}{2}\,Cos\,2x$$

- Use **Dsolve**[ ] to solve examples 4.1, 4.2, 4.6 and 4.8 discussed in previous slides.
- Print out the analytical expression of the general solutions.
- For those examples without specified boundary conditions, impose your own choice of boundary conditions, and then plot the solutions.

SIMPLE HARMONIC PENDULUM AS A SPECIAL CASE OF SECOND ORDER DE

Force on the pendulum $F_\theta = -m\,g\,\sin\theta$



for small oscillation, $\sin\theta \approx \theta$.

Equation of motion (EoM)

$$F_\theta = ma_\theta$$

$$-mgsin\theta = m\frac{dv_\theta}{dt} = m\frac{d}{dt}\left(\frac{dr}{dt}\right) \approx m\frac{d^2}{dt^2}(l\theta)$$

$$\frac{d^2\theta}{dt^2} \approx -\frac{g\theta}{l}$$

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)$$

The period of the SHO is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

SIMPLE HARMONIC PENDULUM AS A SPECIAL CASE OF SECOND ORDER DE

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)$$

$$x \equiv t$$
$$u(x) \equiv \theta(t)$$
$$a \equiv 0$$
$$b \equiv \frac{g}{l}$$
$$n(x) \equiv 0$$

$$\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l}$$

SIMPLE HARMONIC PENDULUM AS A SPECIAL CASE OF SECOND ORDER DE (CONT.)

$$\frac{d^2\theta(t)}{dt^2} = -\frac{g\theta}{l}$$

Analytical solution:

$$\theta = \theta_0 \sin(\Omega t + \phi)$$

$\Omega = \sqrt{g/l}$ natural frequency of the pendulum;
θ_0 and ϕ are constant determined by boundary conditions

Simple Harmonic pendulum with drag force as a special case of second order DE

Drag force on a moving object, $f_d = - kv$
For a pendulum, instantaneous velocity

$v = \omega l = l\,(d\theta/dt)$

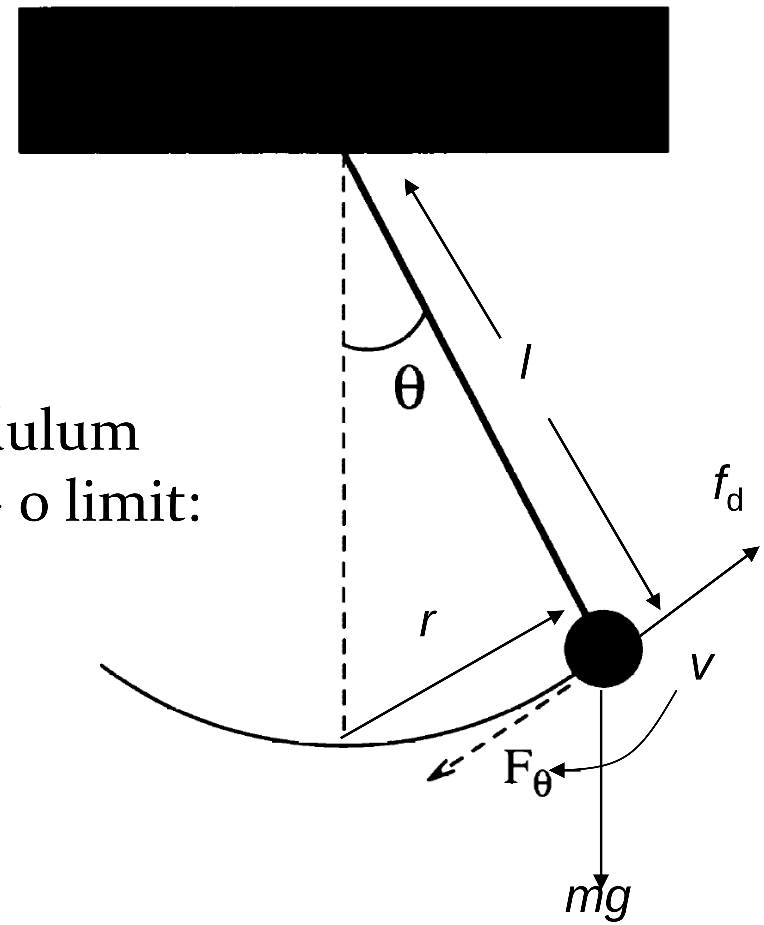
Hence, $f_d = - kl\,(d\theta/dt)$.

Consider the net force on the forced pendulum along the tangential direction, in the $\theta \to 0$ limit:

$$F_\theta = -mg\sin\theta - kl\frac{d\theta}{dt} \approx -mg\theta - kl\frac{d\theta}{dt}$$

$$m\frac{d^2r}{dt^2} \approx m\frac{d^2}{dt^2}(l\theta) = ml\frac{d^2\theta}{dt^2}$$

$$F_\theta = m\frac{d^2r}{dt^2} \Rightarrow \frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - \frac{k}{m}\frac{d\theta}{dt} \equiv -\frac{g}{l}\theta - q\frac{d\theta}{dt}; q \equiv \frac{k}{m}$$

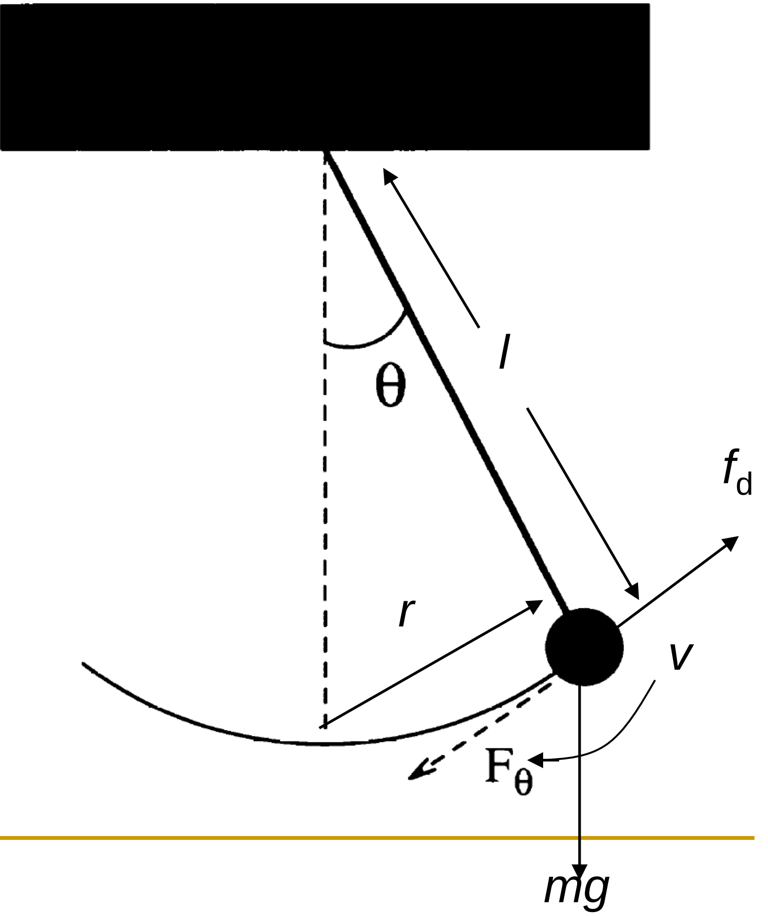


Simple Harmonic pendulum with drag force as a special case of second order DE (cont.)

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)$$

$$x \equiv t$$
$$u(x) \equiv \theta(t)$$
$$a \equiv q$$
$$b \equiv \frac{g}{l}$$
$$n(x) \equiv 0$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt}; q = \frac{k}{m}$$



Analytical solutions

1. Underdamped regime (small damping). Still oscillate, but amplitude decay slowly over many period before dying totally.

$$\theta(t) = \theta_0 e^{-qt/2} \sin\left(\varphi + t\sqrt{\Omega^2 - \frac{q^2}{4}}\right)$$

$$\Omega = \sqrt{\frac{g}{l}} \text{ the natural frequency of the system}$$

Analytical solutions

2. Overdamped regime (very large damping), decay slowly over several period before dying totally. θ is dominated by exponential term.

$$\theta(t) = \theta_0 e^{-\left(\frac{qt}{2} \pm t\sqrt{\frac{q^2}{4} - \Omega^2}\right)}$$

Analytical solutions

3. Critically damped regime, intermediate between under- and overdamping case.

$$\theta(t) = (\theta_0 + Ct)e^{-\frac{qt}{2}}$$

Overdamped

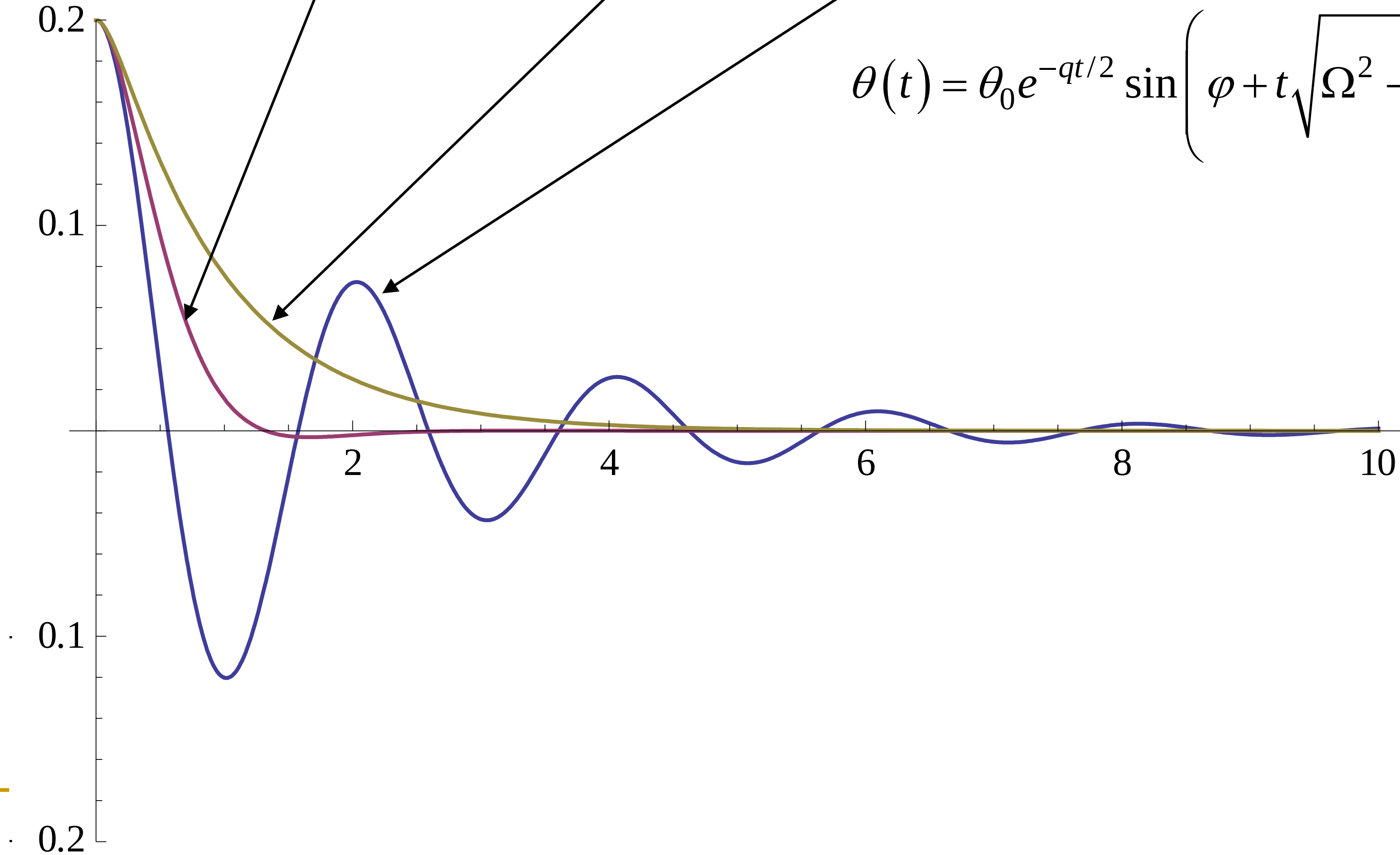
$$\theta(t)=\theta_0 e^{-\left(\frac{qt}{2}\pm t\sqrt{\frac{q^2}{4}-\Omega^2}\right)}$$

Critically damped

$$\theta(t)=\left(\theta_0+Ct\right)e^{-\frac{qt}{2}}$$

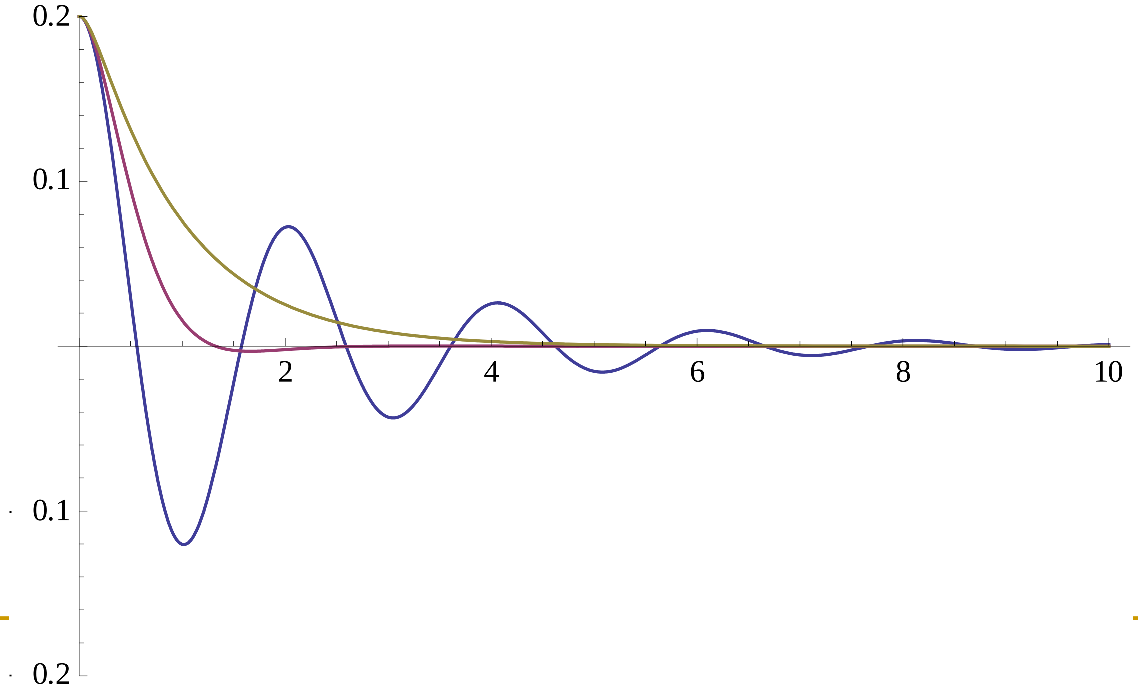
Underdamped

$$\theta(t)=\theta_0 e^{-qt/2}\sin\left(\varphi+t\sqrt{\Omega^2-\frac{q^2}{4}}\right)$$



Assignment

- Reproduce the overdamped, underdamed and critically damped oscillators. To thid end, you need to impose your own choice of boundary conditions.



ADDING DRIVING FORCE TO THE DAMPED OSCILLATOR: FORCED OSCILLATOR

$F_\theta = -mg\sin\theta$ - *kl* (*dθ/dt*) + F_D sin($\Omega_D t$)

Ω_D frequency of the applied force

$$F_\theta = -mg\sin\theta - kl\frac{d\theta}{dt} + F_D\sin(\Omega_D t) \approx -mg\theta - kl\frac{d\theta}{dt} + F_D\sin(\Omega_D t);$$

$$F_\theta = m\frac{d^2r}{dt^2} \approx m\frac{d^2}{dt^2}(l\theta) = ml\frac{d^2\theta}{dt^2};$$

$$F_\theta = m\frac{d^2r}{dt^2} \approx ml\frac{d^2\theta}{dt^2} \approx -mg\theta - kl\frac{d\theta}{dt} + F_D\sin(\Omega_D t)$$

$$\frac{d^2\theta}{dt^2} \approx -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D\sin(\Omega_D t)}{ml}; q = \frac{k}{m}$$

FORCED OSCILLATOR: AN EXAMPLE OF NON HOMOGENEOUS 2ND ORDER DE

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = n(x)$$

$$x \equiv t$$

$$u(x) \equiv \theta(t)$$

$$a \equiv q$$

$$b \equiv \frac{g}{l}$$

$$n(x) \equiv \frac{F_D \sin(\Omega_D t)}{ml}$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{ml}$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + \frac{F_D \sin(\Omega_D t)}{ml}$$

$$\frac{d^2\theta}{dt^2} = -\omega_0^2\theta - 2\xi\omega_0 \frac{d\theta}{dt} + \frac{F_0 \sin(\Omega_D t)}{m};$$

$$\omega_0^2 = \frac{g}{l}, \xi = \frac{q}{2\omega_0}, F_0 = \frac{F_D}{l}$$

ξ is known as the damping ratio

SOLUTION TO FORCED OSCILLATOR

$$\frac{d^2\theta}{dt^2} = -\omega_0^2\theta - 2\xi\omega_0\frac{d\theta}{dt} + \frac{F_0\sin(\Omega_D t)}{m}$$

https://en.wikipedia.org/wiki/Harmonic_oscillator

$$\theta(t) = \frac{F_0}{mZ_m\Omega_D}\sin(\omega t + \phi),$$

$$Z_m = \sqrt{(2\omega_0\xi)^2 + \frac{1}{\omega_D^2}(\omega_0^2 - \Omega_D^2)^2},$$

$$\phi = \tan^{-1}\left(\frac{2\Omega_D\omega_0\xi}{\Omega_D^2 - \omega_0^2}\right) + n\pi.$$

Resonance occurs at $\Omega_D = \omega_r = \omega_0\sqrt{1 - 2\xi^2}$

Assignment

Assume the following conditions:

$$\theta(t=0) = 0, \frac{d\theta}{dt}(t=0) = 0, F_0 = m = l = 1. g = 9.81,$$
$$\xi = 0$$

(i) Plot the solutions $\theta(t)$ for a forced, damped oscillator on the same graph for t running from 0 to $10T$, where $T = 2\pi\omega_0$, for $\Omega_\mathrm{D} = 0.01\omega_0, 0.5\omega_0, 0.99\omega_0, 1.5\omega_0, 4\omega_0.$

(ii) Repeat (i) for $\xi = 1/\sqrt{2}$

Assignment

For a freely falling object subjected to a frictional coefficient η, the equation of motion is

$$m\frac{d^2y}{dt^2} = -mg - \eta\frac{dy}{dt}.$$

Solve this second order DE using $\mathbf{DSolve}[\ ]$, assume

$$y(t=0)=0, v_y(t=0)=0, m=1, g=9.81.$$

Plot the solutions $y(t)$ for $\eta = 0.1, 0.2, 0.5$ on the same graph. Your plots should be adjusted such that terminal velocities in the solutions can be clearly displayed.