Lecture 4
Visualization and Animation

Use Mathematica to visualize the motion of physical systems

Simple examples

Animate the following:

- A freely moving particle bounded to a finite line: 1D motion
- A freely moving particle bounded to a square box: 2D motion
- A freely moving particle bounded to a square rectangular box: 3D motion
- N freely moving particle bounded to a square rectangular box: 3D motion

Parametric equations

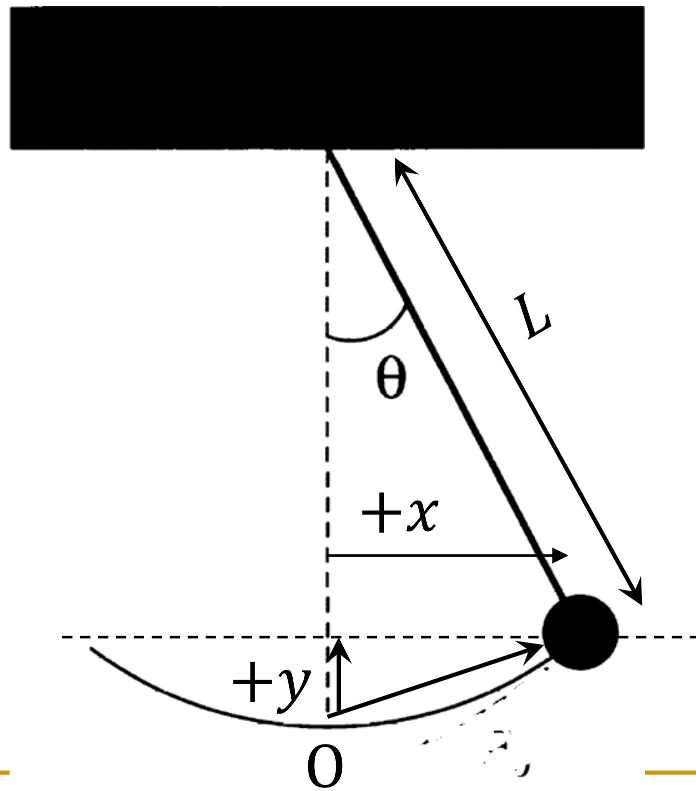
- Given a set of parametric equation describing the motion of a particle in space as a function of time,
$$x = f(t), y = g(t)$$
- one can easily visualize the motion using the command **Manipulate**[ ]
- To this end you may have to also invoke a For loop for generating the time-dependent coordinate variables before visualizing them.
- **ParametricPlot[]** is another useful command for this purpose.

Examples of parametric equations

- SHO

$$\theta(t) = \theta_0 \sin(\omega t + \phi)\,;$$

$$x = L\sin(\theta(t))\ ,\ y = L - L\cos(\theta(t))$$



$$\omega = \sqrt{\frac{g}{l}}$$

Derivation of the equation of motion for SHO

Force on the pendulum $F_\theta = -m\,g\,\sin\theta$

for small oscillation, $\sin\theta \approx \theta$.

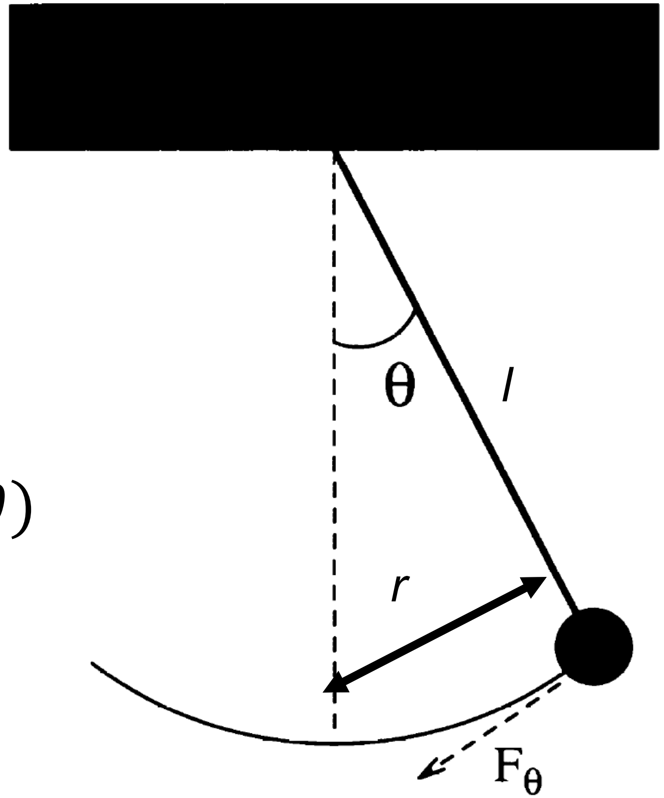
Equation of motion (EoM)

$$F_\theta = m a_\theta$$

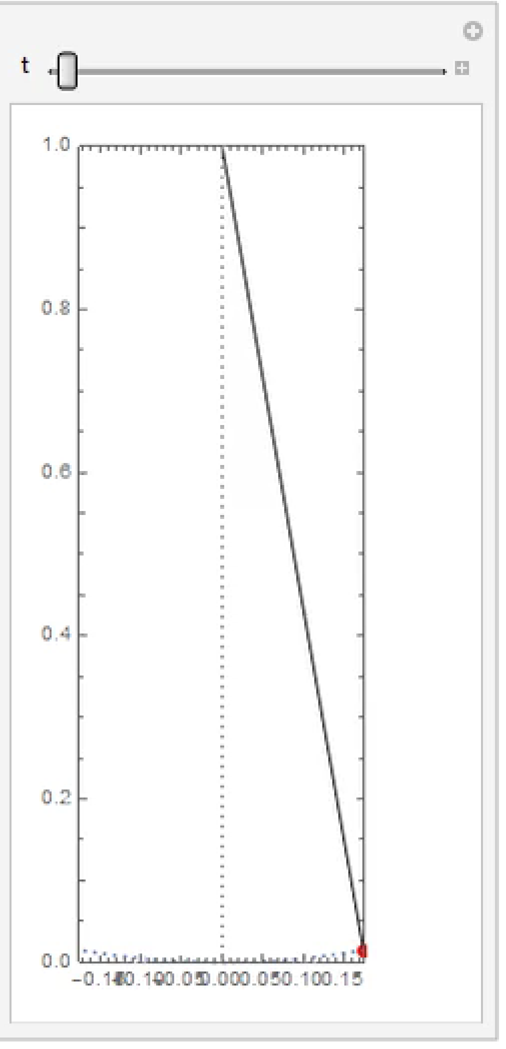
$$-mg\sin\theta = m\frac{dv_\theta}{dt} = m\frac{d}{dt}\left(\frac{dr}{dt}\right) \approx m\frac{d^2}{dt^2}(l\theta)$$

$$\frac{d^2\theta}{dt^2} \approx -\frac{g\theta}{l} = -\omega^2\theta$$

$$\theta(t) = \theta_0 \sin(\omega t + \phi) \qquad \omega = \sqrt{\frac{g}{l}};\ T = \frac{2\pi}{\omega}$$



Animation of simple pendulum



Examples of parametric equations: 2D projectile motion

- The trajectory of a 2D projectile with initial location (x_0, y_0), speed v_0 and launching angle θ are given by the equations:

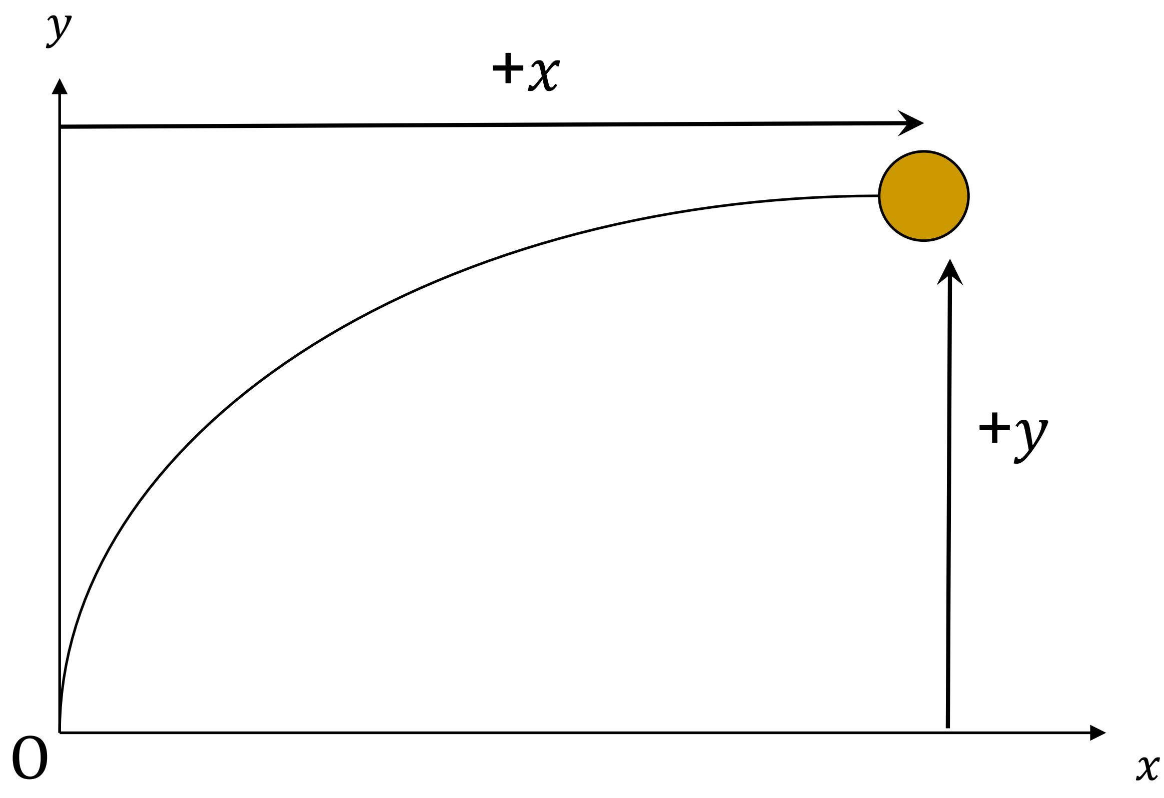
$$x(t) = x_0 + v_0 t \cos\theta;$$

$$y(t) = y_0 + v_0 t \sin\theta + \frac{g}{2} t^2$$

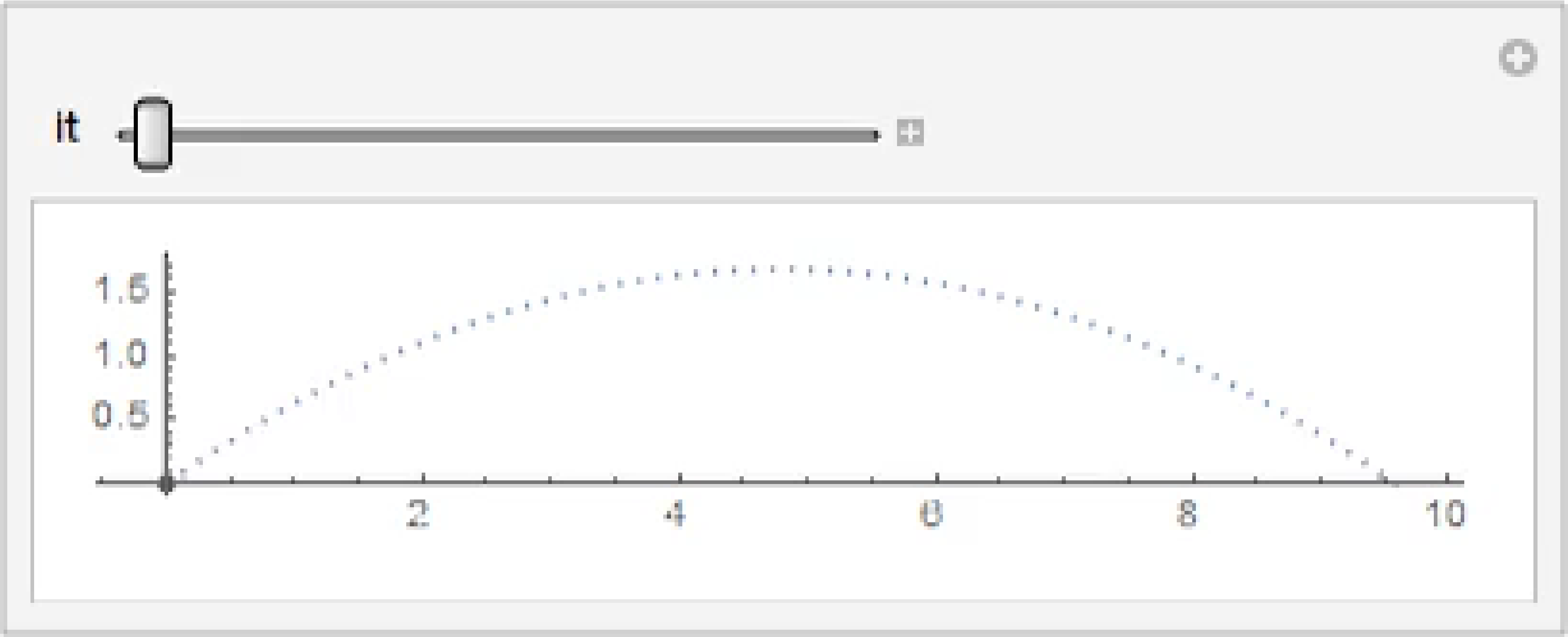
for *t* from 0 till *T*, defined as the time of flight,

$$T = -2(y_0 + v_0 \sin\theta)/g.$$

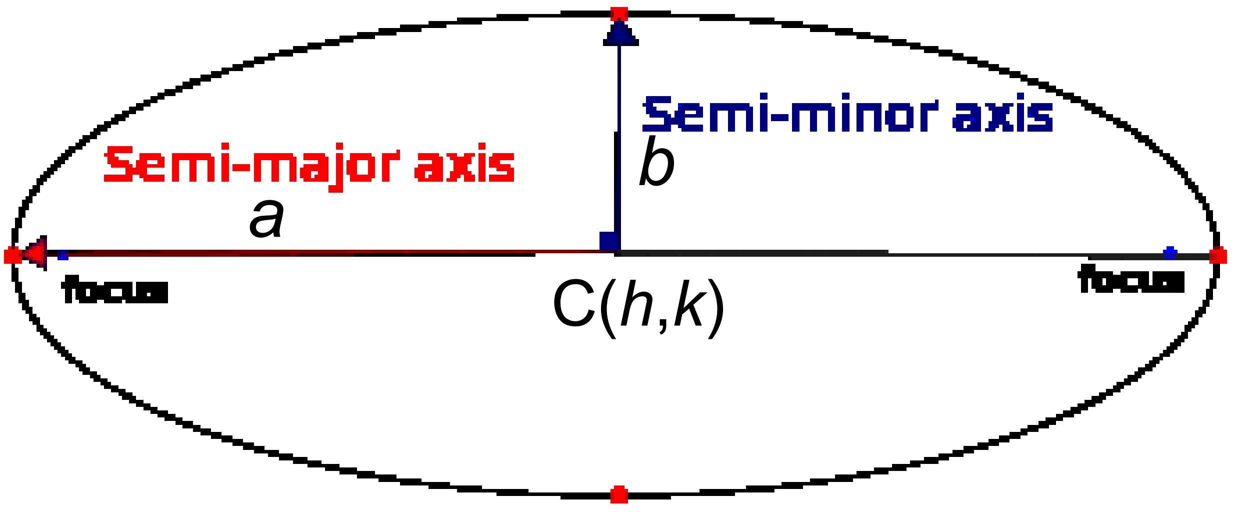
- $g = -9.81;$


y
+x
+y
O
x

Animation of 2D projectile



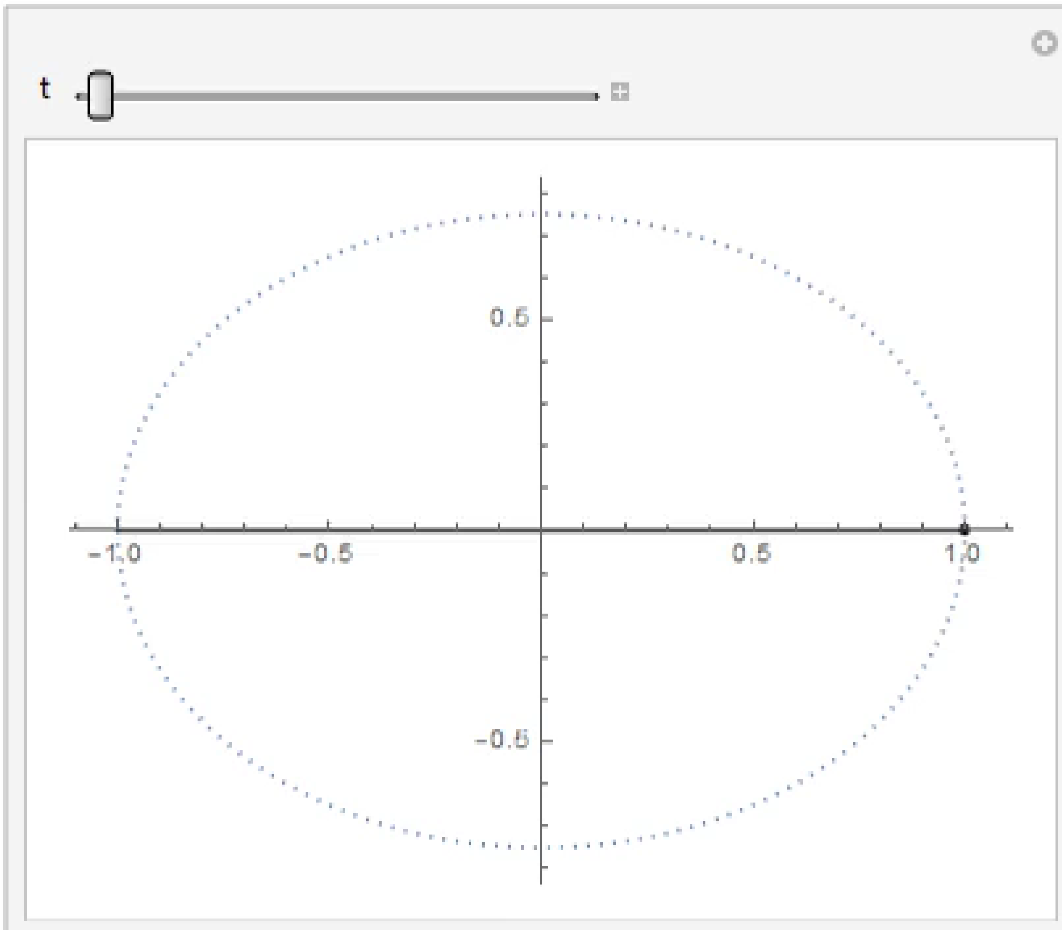
Examples of parametric equations: 2-body Planetary motion



- Consider a planet orbiting the Sun which is located at one of the foci of the ellipse.
- The coordinates of the planet at time *t* can be expressed in parametrised form:

$$x(t) = h + a\cos(\omega_0 t),\ y(t) = k + b\sin(\omega_0 t)$$

Animation of planetary orbital motion



Assignments

By using the corresponding parametric equations for the (x, y) coordinates,

1. Animate a SHO (w/o drag force and driving force)
2. Animate 2D projectile motion (w/o drag force and driving force)
3. Animate 2-body planetary motion

1D sinusoidal wave

- A sinusoidal wave is fundamentally characterized by two quantities: wave number (equivalent to wave length) and angular frequency (frequency)
- $\{k_x, \omega\}$ or $\{\lambda, f\}$.
- Wave number $k_x = \frac{2\pi}{\lambda}$; λ wave length
- Angular frequency $\omega = 2\pi f$; f frequency

1D sinusoidal wave

- $\psi(x,t) = A\sin\theta(x,t)$;
- $\theta(x,t) = k_x x - \omega t + \phi$
- Phase velocity of the wave can be obtained by imposing the condition:

$$\frac{\partial\theta(x,t)}{\partial t} = 0$$

$$\Rightarrow k_x \frac{\partial x}{\partial t} - \omega = 0$$

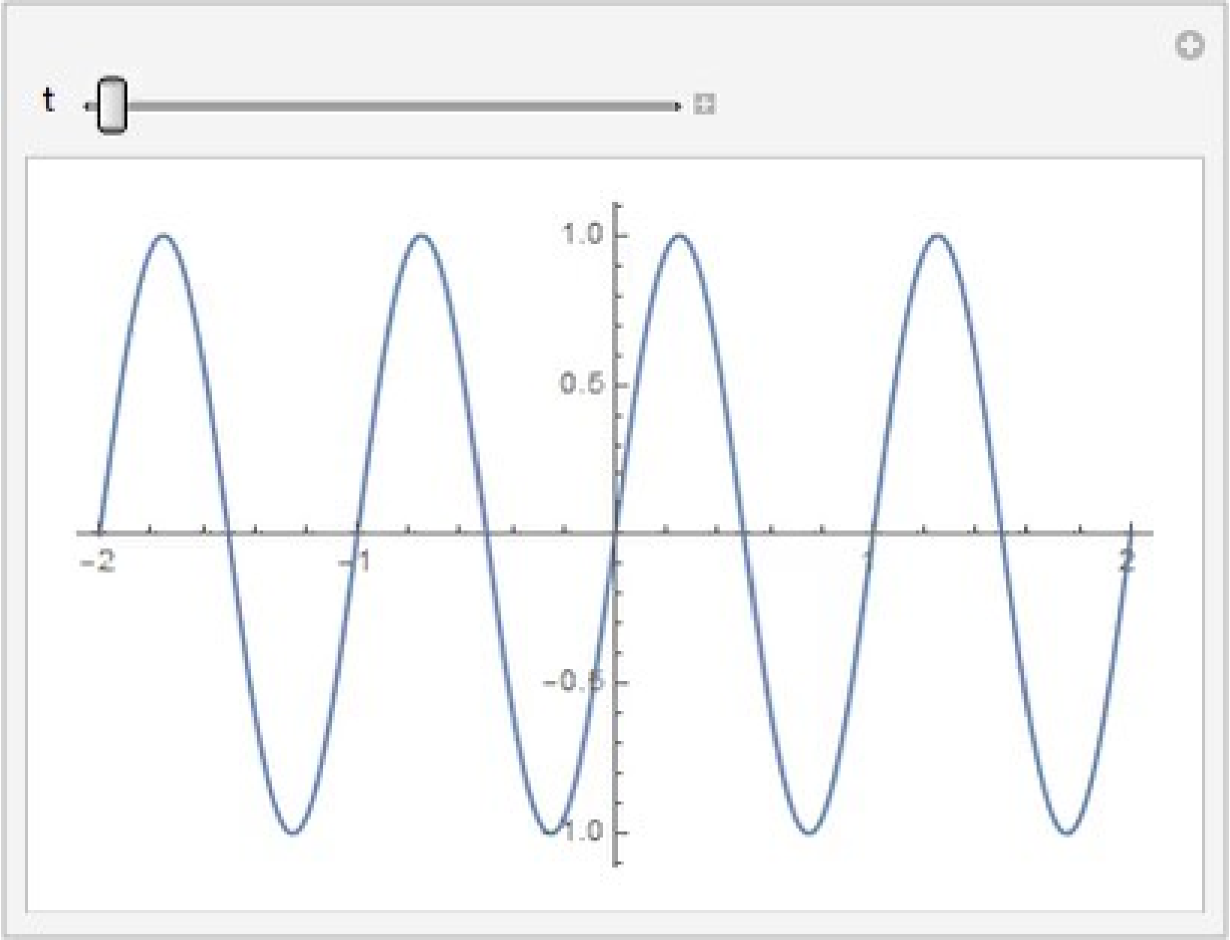
$$\Rightarrow v_x = \frac{\partial x}{\partial t} = \frac{\omega}{k_x} = f\lambda$$

$v_x = \frac{\omega}{k_x}$ implies the wave is moving to the $+x$ direction.

1D sinusoidal wave

- If $\psi(x,t) = A\sin\theta(x,t)$, with
- $\theta(x,t) = k_x x + \omega t + \phi$,
- $\Rightarrow v_x = -\frac{\omega}{k_x}$
- This wave is moving to the -x direction.

1D sinusoidal wave moving in $+x$ direction



Adding two 1D sinusoidal waves

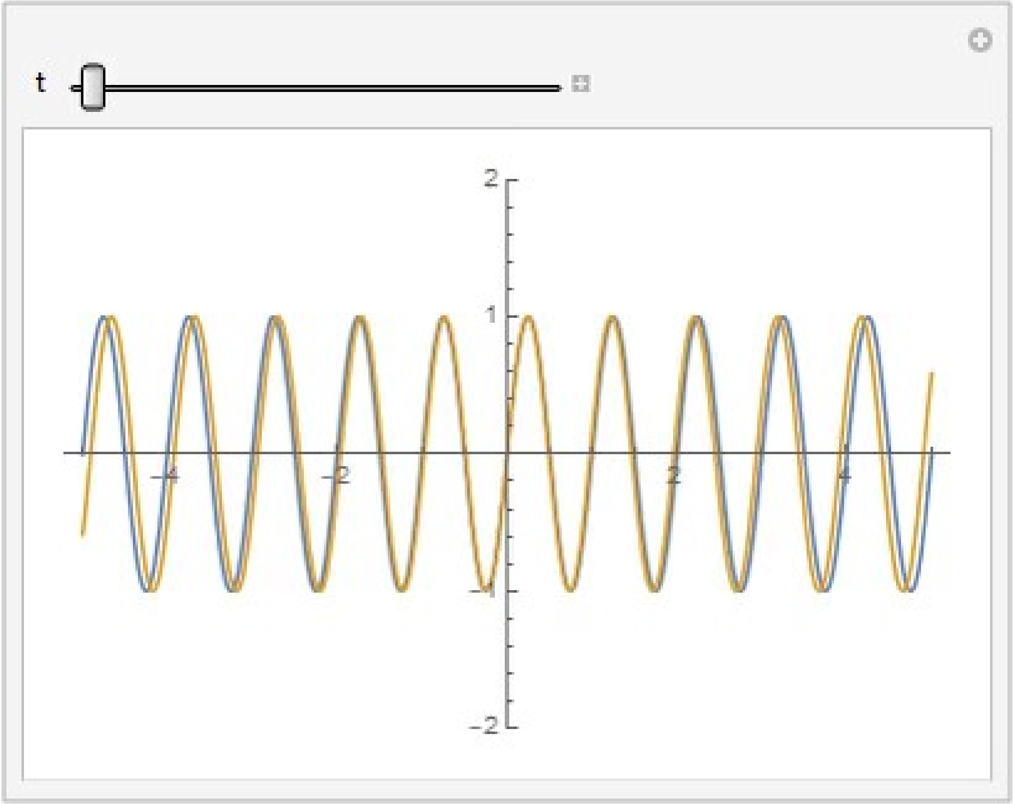
- Same amplitude, same frequency; To ignore all phases ϕ (by setting $\phi = 0$).
- Same / opposite directions
- Same / different wavenumbers
- Same / different angular frequencies
- $\psi_0(x,t) = A \sin \theta_0(x,t)$; $\psi_1(x) = A \sin \theta_1(x,t)$;
- $\theta_0(x,t) = k_{x,0}x \pm \omega_0 t;$
- $\theta_1(x,t) = k_{x,1}x \pm \omega_1 t$
$$= (k_{x,0}+\Delta k_{x,1})x \pm (\omega_0 + \Delta\omega_1)t$$
- $\Delta k_{x,i} = k_{x,i} - k_{x,i-1}; k_{x,i} = k_{x,0} + i\Delta k_{x,i}$
- $\Delta\omega_i = \omega_i - \omega_{i-1}; \omega_i = \omega_0 + i\Delta\omega_i;$
- Usually, $\Delta\omega_i = \Delta\omega$, $\Delta k_{x,i} = \Delta k_x$.

Animation exercises

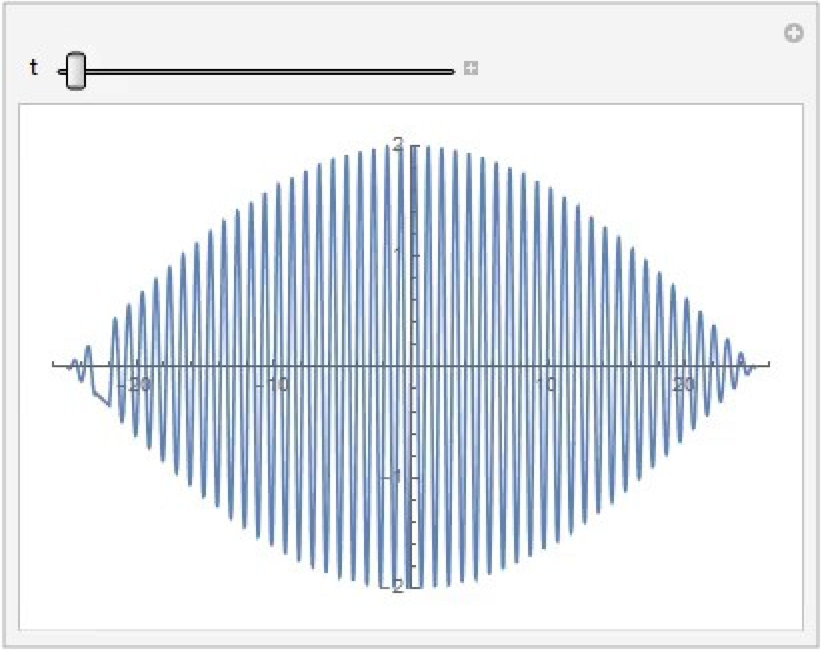
1. Based on the simple equation of a 1D sinusoidal wave:

i. Animate two 1D waves, one in the +x and another in -x. Display both on the same graph, without interfering each other.
ii. Repeat 1 with both in the same direction
iii. Repeat 1 with both waves are added to interfere
iv. Repeat 2 with both waves are added to interfere

Two sinusoidal waves, moving in the same direction, without interference, with a difference in wavenumber and angular frequency of $5\Delta k$ and $5\Delta\omega$; $k_0 = 1; \omega_0 = 1;$ $\Delta k=\Delta\omega = 1/50;$

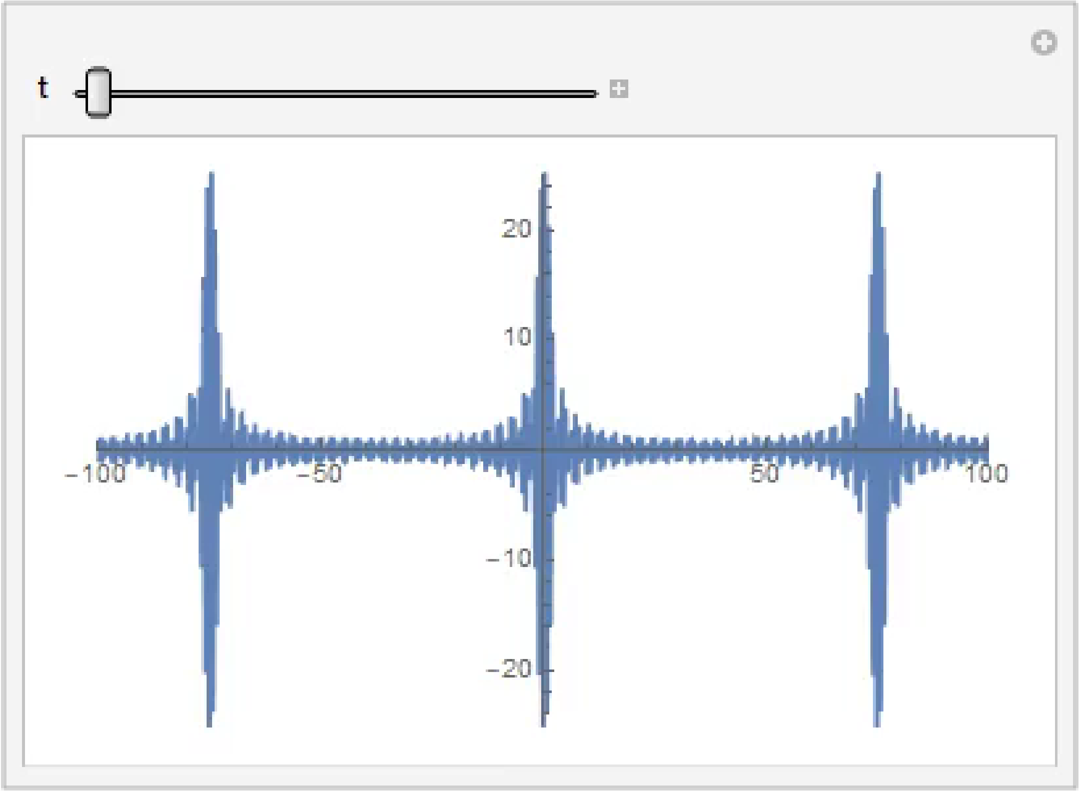


Superposition of two sinusoidal waves, moving in the same direction, with a difference in wavenumber and angular frequency of $5\Delta k$ and $5\Delta\omega$; $k_0 = 1$; $\omega_0 = 1$; $\Delta k = \Delta\omega = 1/50$;



Superpositioning N 1D sinusoidal waves

- Simulate the motion of the resultant wave form obtained from the superposition of N waves with the following conditions:
- All waves have the same amplitude A and moving in the same direction; Ignore all phases ϕ (set all $\phi = 0$);
- Each wave has a different wavenumber and angular frequency:
- $\psi_i(x) = A \sin \theta_i(x,t)$;
- $\theta_i(x,t) = k_{x,i} x \pm \omega t$;
- $k_{x,i} = k_{x,0} + i\Delta k_{x,i}$; $\omega_i = \omega_0 + i\Delta\omega_i$;
- Fix initial values: $A = 1, \omega_0 = 1, k_{x,0} = 1,\ N = 25, \Delta k_x = \frac{1}{75}, \Delta\omega = \frac{1}{75}$.
- Simulate for a total duration of 500 $T\left(T = \frac{2\pi}{\omega_0}\right)$; Width of the simulation box set to $[-100\lambda_0\,, +100\lambda_0\,]$


t
20
10
−10
−20
−100
−50
50
100

Animation exercises

1. Animate an outgoing 2D sinusoidal wave

2. Animate two outgoing 2D sinusoidal waves from two different origins that display interference.

More complicated examples

- Damped, forced SHO
- 2D projectile motion with drag force from the air
- Three-body planetary motion