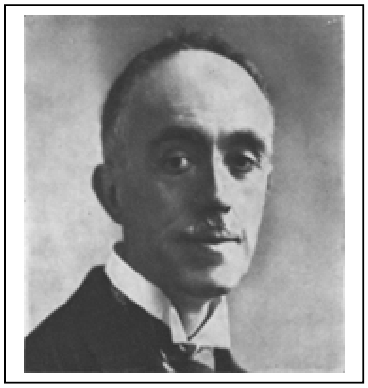
Wavelike properties of particles



Prince de Broglie, 1892-1987

In 1923, while still a graduate student at the University of Paris, Louis de Broglie published a brief note in the journal *Comptes rendus* containing an idea that was to revolutionize our understanding of the physical world at the most fundamental level

http://www.davis-inc.com/physics/index.shtml

de Broglie's postulate (1924)

The postulate: there should be a symmetry between matter and wave. The wave aspect of matter is related to its particle aspect in exactly the same quantitative manner that is in the case for radiation. The total energy E and momentum p of an entity, for both matter and wave alike, is related to the frequency ν of the wave associated with its motion via by Planck constant

$$E = h\nu; \; p = h/\lambda$$

$\lambda = h/p$ is the de Broglie relation predicting the wave length of the matter wave λ associated with the motion of a material particle with momentum p

(essentially, this means that a physical entity should possess both aspects of particle and wave in a complimentary manner)

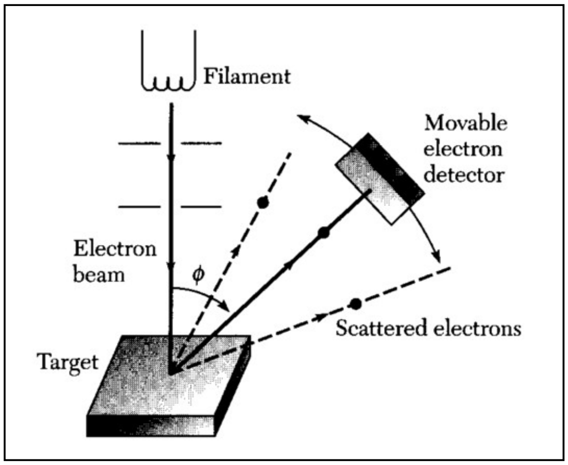
BUT why is the wave nature of material particle not observed?

(**ANS:** because the scale of observing such behaviour is too small, see example below)

To observe diffraction pattern of wave, the scale involved is $\theta = \lambda/a$, where a characterises the length dimension of apparatus used to display diffraction (e.g. $\sim 10^{-10}$ m, which is very small compared to the daily life scale of e.g. $\sim 10^{-3}$ m)
Wave properties of electron with kinetic energy 100 eV (equivalent to 1.2 Angstrom) becomes manifest when it is diffracted in plane of atoms in solid ($a \sim 10^{-10}$ Angstrom)

Davisson and Gremer experiment

- confirmed the wave nature of electron:



Electrons are produced by the hot filament, accelerated and focused onto the target. Electrons are scattered at an angle ϕ into a detector, which is movable. The distribution of electrons is measured as a function of ϕ. The entire apparatus is located in vacuum.

- Strong scattered e- beam is detected at ϕ = 50 degree for V = 54 V
- Explained as (first order, n= 1) constructive interference of wave scattered by the atoms in the crystalline lattice (similar to Bragg's diffraction except that here, instead of x-ray, it's the e' beam got diffracted)

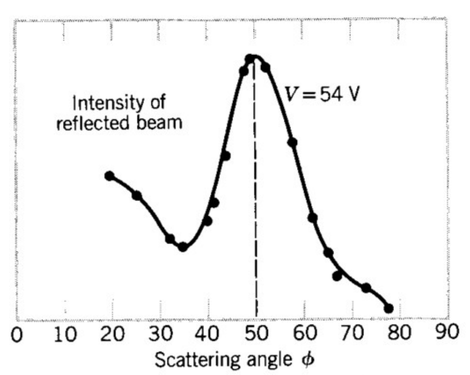


FIGURE 4.8 Results of Davisson and Germer. Constructive interference causes the intensity of the reflected beam to reach a maximum at $\phi = 50°$ for $V = 54$ V.

- interference here is an interference between different parts of the wave associated with a single electron that have been scattered form various regions of the crystal (it's observed even with 1 e-)

- If de Broglie's is right, the wavelength of the electron as inferred by the DG experiment should match that predicted by $\lambda = h/p$
- Let's see what is the wavelength of the electron by treating it as though it is a ``wave'' that undergo Bragg's diffraction:

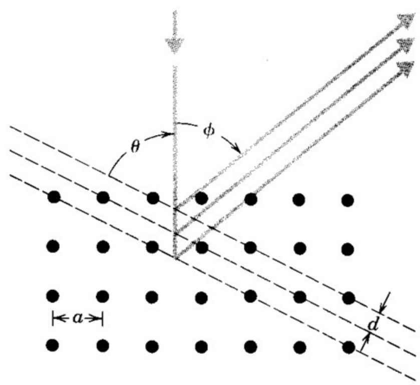


FIGURE 4.7 Detail of scattering from crystal planes. The atoms in the crystal are separated by a distance *a*, and at the Bragg angle *θ* the spacing of atomic planes is *d*. Constructive interference of the scattered beams occurs when the Bragg condition is satisfied.

Bragg's scattering:
$2d\sin\theta = n\lambda$, n = 0,1,2,..
(constructive interference)

$\theta = 90^0 - \phi/2$
d = spacing of atomic plane
a = interatomic spacing in the crystal

(d and a is related by simple geometry: $d = a \sin(\phi/2)$)

d (measured using Bragg's x-ray diffraction) = 0.91 Angstrom

Bragg's diffraction formula says that the equivalent wavelength of the scattered electron is $2d\sin(90^0 - \phi/2)$ = $2 \times 0.91\overset{0}{A} \times \sin(65)^0 = 1.65\overset{0}{A}$ (inferred from experiment)

According to de Broglie, the wavelength of an electron accelerated to kinetic energy of $K = p^2/2m_e$ = 54 eV has a equivalent matter wave wavelength $\lambda = h/p = h/\sqrt{2mK} =$

$1.67\overset{0}{A}$. The result of DG measurement agrees almost perfectly with the de Broglie's prediction.

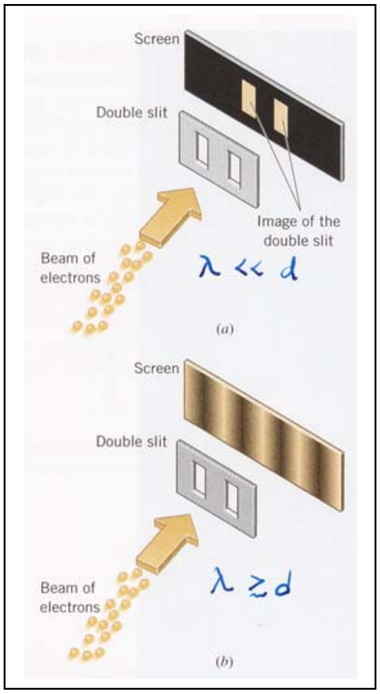
-In fact, wave nature of microscopic particles are observed not only in e- but also in other particles (e.g. neutron, proton, molecules etc. – most strikingly Bose-Einstein condensate)

- electron is also shown to display interference pattern in Double-slit experiment

-The smallness of h in the relation $\lambda = h/p$ makes wave characteristic of particles hard to be observed (as $h \to 0$,

$\lambda \to 0$: as wavelength bcoms vanishingly small the wave nature will bcom effectively ``shut-off''). Only when p is at the same order with $h \sim 10^{-34}$ Js will the wave nature of particles show up

- h characterised the scale at which quantum nature of particles starts to take over from macroscopic physics
- Below the scale h, particle behaves like wave, above the scale h particle behave like particle.



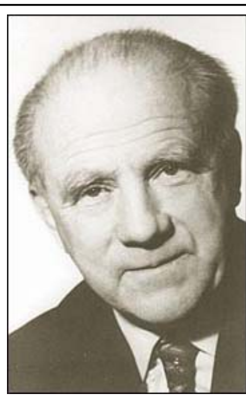
If asked: is electron wave or particle? They are both. In any experiment (or empirical observation) only one aspect of either wave or particle, but not both can be observed simultaneously. It's like a coin with two faces. But one can only see one side of the coin but not the other at any instance. This is the so-called wave-particle duality.

According to Principle of Complementarity, the complete description of a physical entity such as proton or electron cannot be done in terms of particles or wave exclusively, but that both aspect must be considered. The aspect of the behaviour of the system that we observe depends on the kind of experiment we are performing (e.g. in Double slit experiment we see only the wave nature of electron, but in Milikan's oil drop experiment we observe electron as a particle).

Application of electrons as wave: scanning electron microscope:

Figure 29.12 **CONCEPTS AT A GLANCE** Both a moving particle and a wave possess energy, momentum, and a wavelength. The wavelength of a particle is known as its de Broglie wavelength. This photograph shows a highly magnified view of a spruce aphid, made with a scanning electron microscope. This microscope uses electrons instead of light. The resolution of the fine detail is exceptional because the wavelength of an electron can be made much smaller than that of light. (© S. Lowry/Univ. Ulster/Stone/Getty Images)

Heisenberg's uncertainty principle



WERNER HEISENBERG (1901 - 1976)

was one of the greatest physicists of the twentieth century. He is best known as a founder of quantum mechanics, the new physics of the atomic world, and especially for the uncertainty principle in quantum theory. He is also known for his controversial role as a leader of Germany's nuclear fission research during World War II. After the war he was active in elementary particle physics and West German science policy.

http://www.aip.org/history/heisenberg/p01.htm

Since we experimentally confirmed that particles are wave in nature at the quantum scale h (matter wave) we now got to describe particles in term of waves.
Since particle is localised in space (not extending over an infinite extent in space), the wave representation of a particle has to be in the form of wave packet/wave pulse

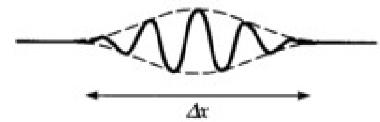
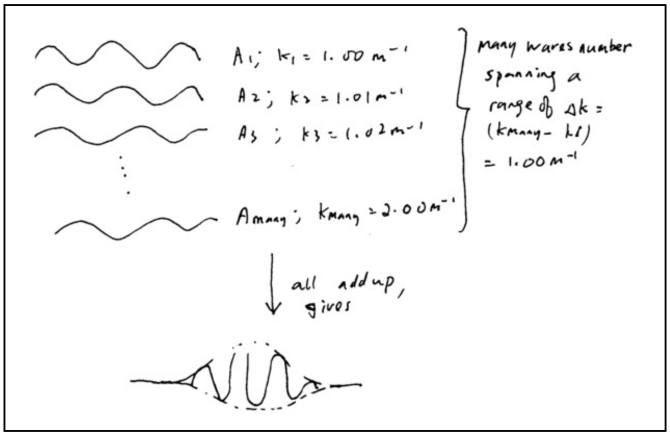


FIGURE 6.14 An idealized wave packet localized in space over a region Δx is the perposition of many waves of different amplitudes and frequencies.

wavepulse/wave packet is formed by adding many waves of different amplitudes and with the wave numbers spanning a range of Δk.



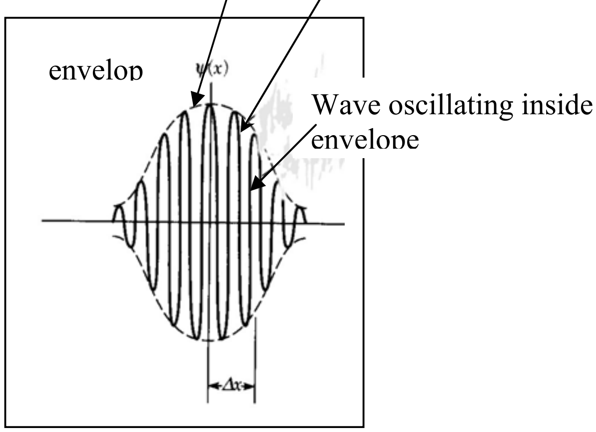
Mathematically, the wavepulse is formed by taking the Fourier integral in the form

$$\Psi(x,t) = \int \widetilde{A}(k)\cos(kx - \omega t)dk$$

At a given time say, t=0,

$$\Psi(x,0) = \psi(x) = A\exp(-k^2x^2)\cos(k_0 x)$$

k_0 mean value of k;
A the amplitude of the wavepacket determined by evaluating the Fourier integration



As discussed earlier, for a wave packet,

$$\Delta\lambda\Delta x \gtrsim \lambda^2 \equiv \Delta k\Delta x \gtrsim 2\pi \text{ , and}$$

$$\Delta\nu\Delta t \gtrsim 1$$

These relations are in fact only approximate. A more rigorous mathematical treatment gives the exact relation

$$\Delta\lambda\Delta x \geq \frac{\lambda^2}{4\pi} \equiv \Delta k\Delta x \geq 1/2 \text{ and}$$

$$\Delta\nu\Delta t \geq \frac{1}{4\pi}$$

(UTC)

Hence, to describe a particle with wave packet that is localised over a small region Δx requires a large range of wave number; that is, Δk is large. Conversely, a small range of wave number cannot produce a wave packet localised within a small distance.

For matter waves, for which their momentum (energy) and wavelength (frequency) are related by

$$p = h/\lambda$$
$$(E = h\nu)$$

The uncertainty relationship of Eq. (UTC) is translated into

$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \text{ and}$$

$$\Delta E\Delta t \geq \frac{\hbar}{2}$$

(HUP)

where $\hbar = \dfrac{h}{2\pi}$. The product of the uncertainty in momentum (energy) and in position (time) is at least as large as Planck's constant.

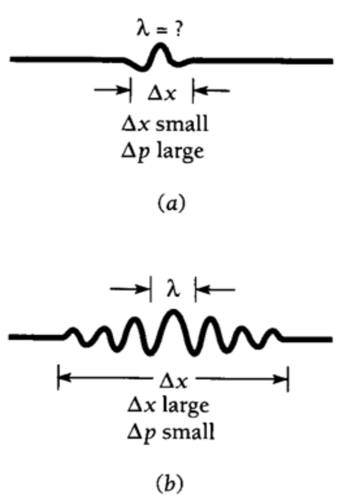


Figure 3.12 (*a*) A narrow de Broglie wave group. The position of the particle can be precisely determined, but the wavelength (and hence the particle's momentum) cannot be established because there are not enough waves to measure accurately. (*b*) A wide wave group. Now the wavelength can be precisely determined but not the position of the particle.

Heisenberg's Gedanken experiment: -the measurement of an electron's position by means of alight microscope (Fig. 4.15).

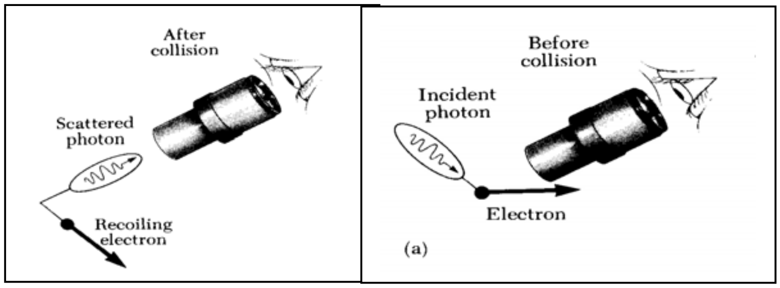


Figure 4.15 A thought experiment for viewing an electron with powerful microscope (a) The electron is shown before colliding wit the photon. (b) The electron re coils (is disturbed) as a result of the collision with the photon.

- consider the scattering of one single light quantum that will scatter from and perturb the electron in the process of measuring the position of electron
- In order to be collected by the lens, the photon may be scattered through any angle ranging from $-\theta$ t0 $+\theta$, which consequently imparts to the electron a momentum value ranging from + $(h\sin\theta)/\lambda$ to - $(h\sin\theta)/\lambda$.

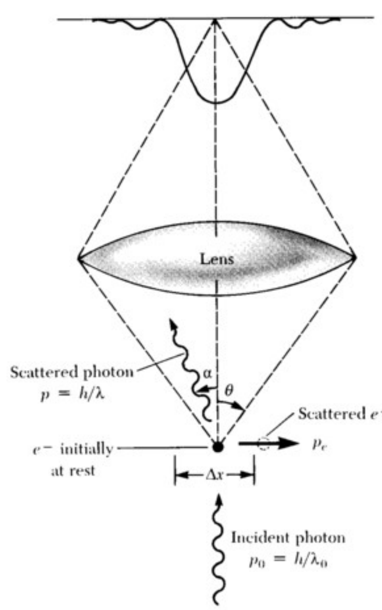


Figure 4.16 The Heisenberg microscope. Conservation of momentum requires $p_{e^-} = (h \sin \alpha)/\lambda$. Because of diffraction by the lens opening, the electron may be anywhere in the region Δx.

- Thus the uncertainty in the electron's momentum is $\Delta p = (2h \sin\theta)/\lambda$.

- The photon, after passing through the lens, forms diffraction image. The resolution, i.e. Δx between two points in an object that will produce separated images in a microscope is given by $\Delta x = \lambda/(2\sin\theta)$ (recall physical optics, on resolution power of a lens – the Rayleigh criteria)
- 2θ = angle subtended by the objective lens (see figure)

Hence,

$$\Delta p \Delta x \approx [(2h\sin\theta)/\lambda][\lambda/(2\sin\theta)] = h$$

Eq.(HUP) is the famous Heisenberg's uncertainty principle – giving the intrinsic lowest possible limits on the uncertainties in knowing the values of p_x and x, no matter how good an experiments is made. It is impossible to specify simultaneously and with infinite precision the linear momentum and the corresponding position of a particle.

If a system is known to exist in a state of energy E over a limited period Δt, then this energy is uncertain by at least an amount $\hbar/2\Delta t$; therefore, the energy of an object or system can be measured with infinite precision ($\Delta E=0$) only if the object of system exists for an infinite time ($\Delta t \to \infty$)

$\{p_x, x\}$ is called *conjugate variables. The conjugate variables cannot in principle be measured (or known) to infinite precision simultaneously* (so is $\{E, t\}$).

Minimal energy of a particle in a box according to the uncertainty principle

One of the most dramatic consequence of the uncertainty principle is that a particle confined in a small region of width *a* cannot be exactly at rest, since if it were, its momentum would be precisely zero, which would in turn violate the uncertainty principle. Hence, such a particle must has a minimal kinetic energy whose magnitude can be estimated (ignore the factor 2) from the uncertainty principle itself.

Uncertainty principle requires that $\Delta p \gtrsim \frac{\hbar}{a}$. Hence, the magnitude of p must be, on average, at least of the same order as Δp. Thus the kinetic energy, whether it has a definite value or not, must on average have the magnitude

$K_{av} = \left(\frac{p^2}{2m}\right)_{av} \gtrsim \frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{2ma^2}$. This is the zero-point energy, the minimal possible kinetic energy for a quantum particle confined in a region of width a. We will formally re-derived this result again when solving for the Schrodinger equation of this system (see later).

Example
The speed of an electron is measured to have a value of 5.00×10^3 m/s to an accuracy of 0.003%. Find the uncertainty in determining the position of this electron.

ANS
$p = m_e v = 4.56\times10^{-27}$ Ns; $\Delta p = 0.003\% \times p = 1.37\times10^{-27}$ Ns
Hence, $\Delta x \geq \hbar/2\Delta p = 0.38$ mm

Example
A charged π meson has rest energy of 140 MeV and a lifetime of 26 ns. Find the energy uncertainty of the π meson, expressed in MeV and also as a function of its rest energy.

ANS
Given $E = m_\pi c^2 = 140$ MeV, $\Delta\tau = 26$ ns.
$\Delta E \geq \hbar/2\Delta\tau = 2.03\times10^{-27}$J $= 1.27\times10^{-14}$ MeV;
$\Delta E/E \geq 1.27\times10^{-14}$ MeV/140 MeV $= 9\times10^{-17}$

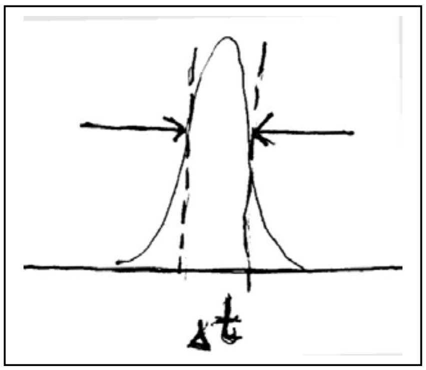
Example
Estimate the minimum uncertainty velocity of a billard ball (m ~ 100 g) confined to a billard table of dimension 1 m.

ANS
For Δx ~ 1 m, we have $\Delta p \geq \hbar/2\,\Delta x = 5.3\times10^{-35}$ Ns,
So $\Delta v = (\Delta p)/m \geq 5.3\times10^{-34}$ m/s.

One can consider $\Delta v \gtrsim 5.3\times10^{-34}$ m/s (extremely tiny) is the speed of the billard ball at anytime caused by quantum effects. In any case, such a motion can only sweep through a distance of ~ 1/100 the diameter of an atomic nucleus in the age of the universe (12 billion years). In quantum theory, no particle is absolutely at rest due to the Uncertainty Principle.

Example
An atom in an excited state normally remains in that state for a very short time (~10^{-8} s) before emitting a photon and returning to a lower energy state. The lifetime of the excited state can be regarded as an uncertainty in the time Δt = associated with a measurement of the energy of the state. This, in turn, implies an 'energy width' – namely, the corresponding energy uncertainty ΔE. Calculate the characteristic ``energy width'' of such a state.

ANS
$\Delta E \sim \hbar/2\Delta t = 3.2\times10^{-8}$ eV.



The `energy width' concept is also applicable to the creation (from `nothing') of a ``resonant'' particle at an excited state with a mass equivalent to $m = \Delta E \sim \hbar/\Delta t$ in particle collision experiments. The `energy width' observed in the collision experiment is equivalent to the existence of a particle state with the mass equivalent to $m = \Delta E \sim \hbar/\Delta t$ for a finite lifetime Δt. The creation of a particle from nothing violates the conservation of energy-momentum. However, as long as the lifetime of the resonant particle Δt conforms to the uncertainty limit, this is still allowed.

To recap:
Measurement necessarily involves interactions between observer and the observed system. Matter and radiation are the entities available to us for such measurements. The relations $p = h/\lambda$ and $E = h\nu$ are applicable to both matter and to radiation because of the intrinsic nature of wave-particle duality. When combining these relations with the universal waves properties, we obtain the Heisenberg uncertainty relations. Hence the uncertainty principle is a necessary consequence of particle-wave duality.

If a professor ever asks you where that assignment is that you didn't do, just say that you know it's momentum so precisely that it could be almost anywhere in the universe.

"But you can't go through life applying Heisenberg's Uncertainty Principle to *everything*."

Particle as wave packet (= wavepulse)

A free particle that is not confined in space is represented by a single de Broglie wave of definite frequency and wavelength – this is just a plain wave. For a free particle, it is not localised, for which $\Delta x = \infty$.

However, in contrast, a particle that is localised in a region of space of finite dimension Δx (such a electron in a atom, or atom trapped in an experimental setup), cannot be represented by a plain wave but must be represented by a group wave. A particle confined to move in a certain region of space is described by a wave packet, a superposition of de Broglie waves.

Now consider a particle, assumed to be represented by a wave packet, is moving through a generally dispersive medium.

For each component wave comprising the wave packet, the following relations hold:

$$E = h\nu = \hbar\omega; \quad p = h/\lambda = \hbar k.$$

By definition, the group velocity of a wave packet is given by (recall the lecture on ``preliminaries'')

$$v_g = \frac{d\omega}{dk} = \left(\frac{d\omega}{dE}\right)\left(\frac{dE}{dp}\right)\left(\frac{dp}{dk}\right) = \left(\frac{1}{\hbar}\right)\left(\frac{dE}{dp}\right)(\hbar) = \frac{dE}{dp}$$

But for a particle with its mechanical energy

$E = K$ + potential energy which is not p-independent,

$\mathrm{d}E/\mathrm{d}p = \mathrm{d}K/\mathrm{d}p = p/m = v$ = velocity of the particle

⇒ The velocity of a material particle is identified to be equal to the group velocity of the corresponding wave packet

To summarise, a particle confined to move in a certain region of space is described by a wave packet. The wave packet moves with a speed equal to the group velocity of the medium, which is equal to the speed of the particle.

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