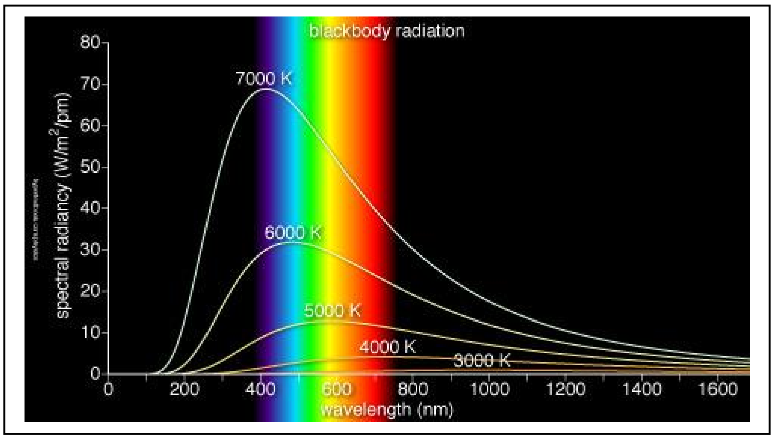
ZCT 104/3E Modern Physics
Semester II, Sessi 2006/07
Open Book Quiz 2 (5 Feb 2007)
Duration: 30 min

Name:

Matrics No:

INSTRUCTION: Answer the following question. Write your answer at the back of this question paper.
(5 + 5 = 10 marks.)

A typical spectral distribution of radiation energy of a blackbody for several temperatures is as shown.



The shift of the peak of the curve was found to obey the empirical relationship,

$$l_p T = \text{constant}, \ (\textit{Wein's displacement law})$$

where the symbol l_p refers to the value of the wavelength corresponding to the peak of the curve. The *total power radiated per unit area* (i.e. its intensity) of a blackbody is found to be empirically related to its absolute temperature by

$$I(T) = s\,T^4, \ (\textit{the Stefan-Boltzman law})$$

where s = 5.6699 ×10^{-8} watts m^{-2} degree^{-4} (Stefan constant). The radiance, $R(l,T)$ is the power radiated per unit area per unit wavelength interval at a given wavelength l and a given temperature *T*. $I(T)$ and $R(l,T)$ are related by the integral $I(T) = \int_0^\infty R(l,T)\,\mathrm{d}l$. Wein proposed an empirical form for the radiance $R(l,T)$ by constructing a mathematical function to fit the experimental blackbody curve, knows as the *Wein's law*:

$$R(l,T) = \frac{a e^{-b/lT}}{l^5} \ \text{(Wein's law)}$$

The quantities ***a*** and ***b*** are not derived but are simply curve-fitting parameters.

Question:
From Wien's law as given above,
(i) derive the constant in the displacement law, and (5 marks)
(ii) derive the Stefan constant (5 marks)
in terms of ***a*** and ***b***.

Hint: You may need $\int_0^\infty \frac{e^{-\frac{1}{x}}}{x^5}\mathrm{d}x = 6$ or $\int_0^\infty x^3 e^{-x}\mathrm{d}x = 6$.

Solution

(i) $R(l,T)=\dfrac{ae^{-b/lT}}{l^5}$

$$\frac{\mathrm{d}}{\mathrm{d}l}R(l,T)=a\left[\frac{b}{T}\left(\frac{1}{l^2}\right)l^5e^{-b/lT}-5l^4e^{-b/lT}\right]/l^{10} \qquad \text{(Eq. 1)}$$

Minimize $R(l,T)$ with respect to l, we will get l_{p}:

$$\left.\frac{\mathrm{d}}{\mathrm{d}l}R(l,T)\right|_{l_{\mathrm{p}}}=0$$ *(2 marks)*

Setting Eq. 1 to zero, we get

$$\frac{b}{T}l_{\mathrm{p}}^3e^{-b/l_{\mathrm{p}}T}=5l_{\mathrm{p}}^4e^{-b/l_{\mathrm{p}}T}$$ *(show working, 2 marks)*

$$\Rightarrow l_{\mathrm{p}}T=b/5$$ *(correct answer: 1 marks)*

(ii) Stafan-Boltzman law: $I(T)=sT^4$

Substitute Wein's law, $R(l,T)=\dfrac{ae^{-b/lT}}{l^5}$, into the definition of $I(T)=\int_0^\infty R(l,T)\mathrm{d}l$, we get

$$I(T)=\int_0^\infty R(l,T)\mathrm{d}l=\int_0^\infty \frac{ae^{-b/lT}}{l^5}\mathrm{d}l\ .$$ *(1 marks)*

Define $x=\dfrac{b}{lT}\Rightarrow \mathrm{d}x=-\dfrac{b}{T}\dfrac{1}{l^2}\mathrm{d}l$

$$I(T)=a\left(\frac{T}{b}\right)^4\int_0^\infty x^3e^{-x}\mathrm{d}x=6a\left(\frac{T}{b}\right)^4$$ (*show working, 3 marks*)

$$\Rightarrow I(T)=6a\left(\frac{T}{b}\right)^4\equiv sT^4\Rightarrow s\equiv\frac{6a}{b^4}$$ (*correct answer, 1 mark*)