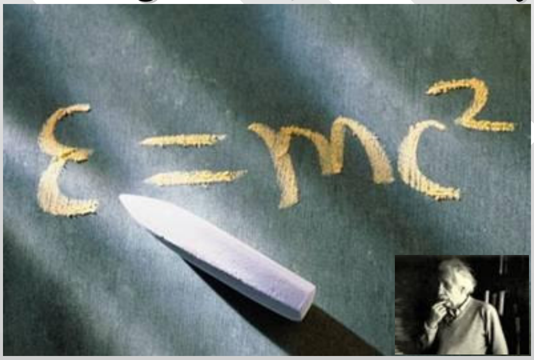
Special theory of Relativity

Notes based on

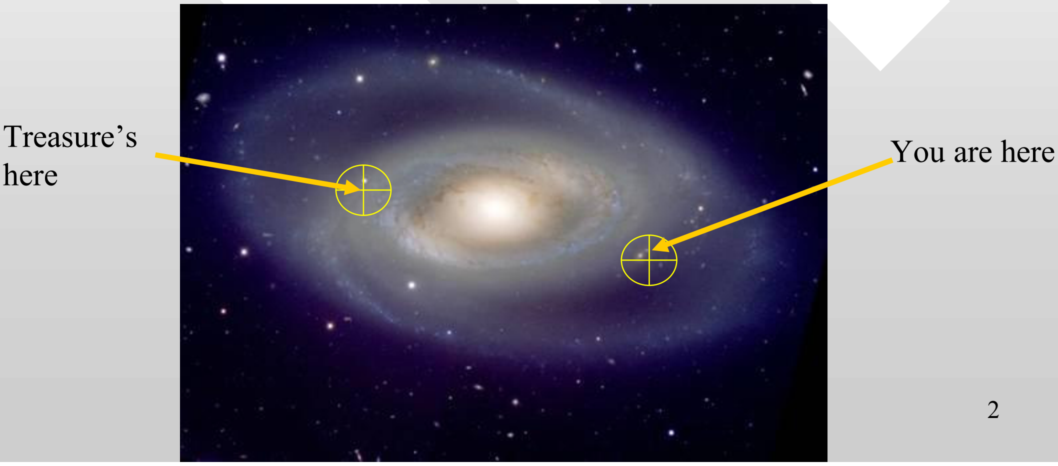
Understanding Physics

by Karen Cummings et al., John Wiley & Sons



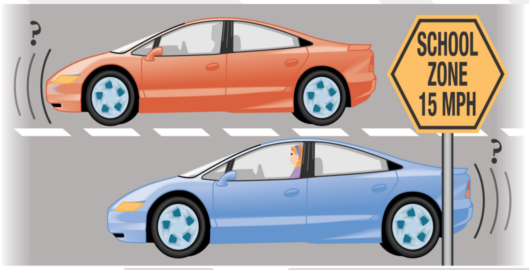
An open-chapter question

- Let say you have found a map revealing a huge galactic treasure at the opposite edge of the Galaxy 200 ly away.
- Is there any chance for you to travel such a distance from Earth and arrive at the treasure site by traveling on a rocket within your lifetime of say, 60 years, given the constraint that the rocket cannot possibly travel faster than the light speed?



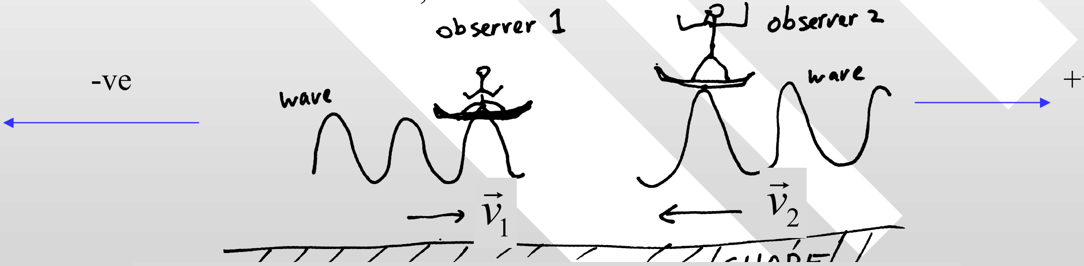
Relative motions at ordinary speed

- Relative motion in ordinary life is commonplace
- E.g. the relative motions of two cars (material objects) along a road
- When you observe another car from within your car, can you tell whether you are at rest or in motion if the other car is seen to be "moving?"



Relative motion of wave

- Another example: wave motion
- Speed of wave measured by Observer 1 on wave 2 depends on the speed of wave 1 wrp (with respect) to the shore: $\vec{v}_{2,1} = \vec{v}_2 - \vec{v}_1$



$$\vec{v}_1 = v_1\hat{x}; \qquad \vec{v}_2 = -v_2\hat{x}$$

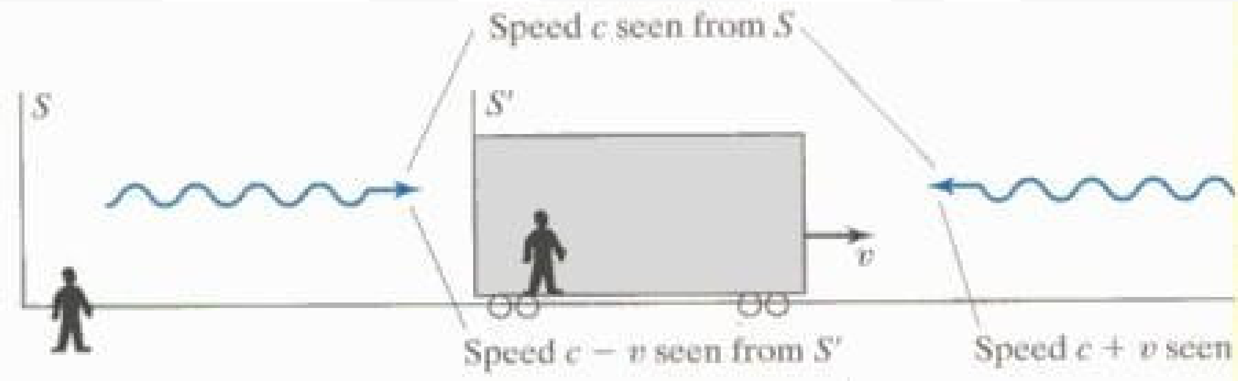
$$\vec{v}_{1,2} = \vec{v}_1 - \vec{v}_2 = v_1\hat{x} - (-v_2\hat{x}) = (v_1 + v_2)\hat{x};$$

$$\vec{v}_{2,1} = \vec{v}_2 - \vec{v}_1 = (-v_2\hat{x}) - (v_1\hat{x}) = -(v_1 + v_2)\hat{x}$$

Query: can we surf light waves?

- Light is known to be wave
- If either or both wave 1 and wave 2 in the previous picture are light wave, do they follow the addition of velocity rule too?
- Can you surf light wave ? (if so light shall appear *at rest* to you then)

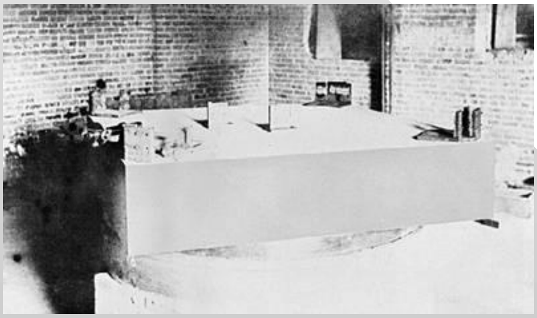
In other word, does light wave follows Galilean law of addition of velocity?

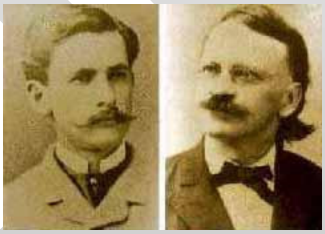


Frame S' travels with velocity *v* relative to S. If light waves obey Galilean laws of addition velocity, the speeds of the two opposite light waves would be different as seen by S'. But does light really obey Galilean law of addition of velocity?

The negative result of Michelson-Morley experiment on Ether

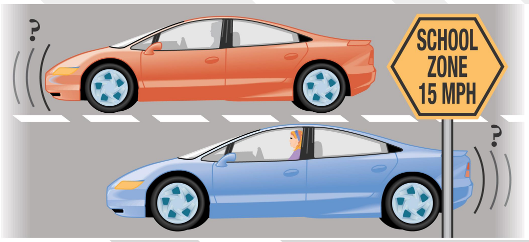
- In the pre-relativity era, light is thought to be propagating in a medium called ether -
- an direct analogy to mechanical wave propagating in elastic medium such as sound wave in air
- If exist, ether could render measurable effect in the apparent speed of light in various direction
- However Michelson-Morley experiment only find negative result on such effect
- A great puzzlement to the contemporary physicist: what does light wave move relative to?





How could we know whether we are at rest or moving?

- Can we cover the windows of our car and carry out experiments inside to tell whether we are at rest or in motion?
- NO



In a “covered” reference frame, we can’t tell whether we are moving or at rest

- Without referring to an external reference object (such as a STOP sign or a lamp post), whatever experiments we conduct in a constantly moving frame of reference (such as a car at rest or a car at constant speed) could not tell us the state of our motion (whether the reference frame is at rest or is moving at constant velocity)

《尚書經．考靈曜》

- 「地恒動不止，而人不知，譬如人在大舟中，閉牖而坐，舟行而不覺也。」
- “The Earth is at constant state of motion yet men are unaware of it, as in a simile: if one sits in a boat with its windows closed, he would not aware if the boat is moving” in “*Shangshu jing*”, 200 B.C

Physical laws must be invariant in any reference frame

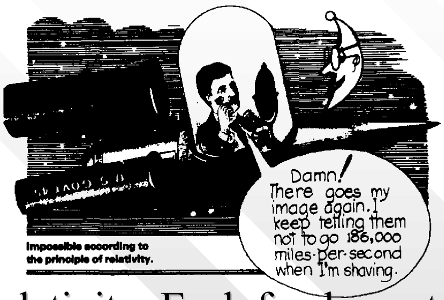
- Such an inability to deduce the state of motion is a consequence of a more general principle:
- There must be no any difference in the physical laws in any reference frame with constant velocity
- (which would otherwise enable one to differentiate the state of motion from experiment conducted in these reference frame)
- Note that a reference frame at rest is a special case of reference frame moving at constant velocity ($v = 0 =$ constant)

The Principle of Relativity

- All the laws of Physics are the same in every reference frame

Einstein's Puzzler about running fast while holding a mirror

Space travel at the speed of light

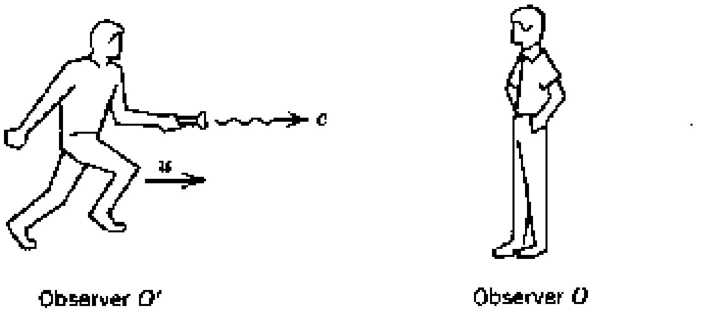


- Says Principle of Relativity: Each fundamental constants must have the same numerical value when measured in any reference frame (*c, h, e,* m_e*, etc*)
- (Otherwise the laws of physics would predict inconsistent experimental results in different frame of reference – which must not be according to the Principle)
- Light always moves past you with the same speed *c*, no matter how fast you run
- Hence: you will not observe light waves to slow down as you move faster

c, one of the fundamental constants of Nature

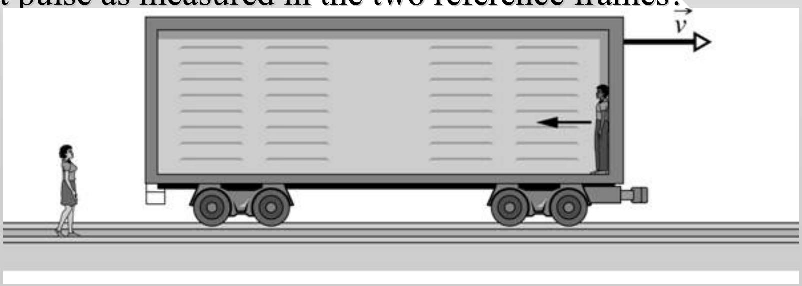
Constancy of the speed of light

Figure 1-2
Two observers in relative motion. O is at rest and O' moves toward O at constant speed u. O and O' agree on the speed of light coming from the source carried by O'.



Reading Exercise (RE) 38-2

- While standing beside a railroad track, we are startled by a boxcar traveling past us at half the speed of light. A passenger (shown in the figure) standing at the front of the boxcar fires a laser pulse toward the rear of the boxcar. The pulse is absorbed at the back of the box car. While standing beside the track we measure the speed of the pulse through the open side door.
- (a) Is our measured value of the speed of the pulse greater than, equal to, or less than its speed measured by the rider?
- (b) Is our measurement of the distance between emission and absorption of the light pulse great than, equal to, or less than the distance between emission and absorption measured by the rider?
- (c) What conclusion can you draw about the relation between the times of flight of the light pulse as measured in the two reference frames?



Touchstone Example 38-1: Communication storm!

- A sunspot emits a tremendous burst of particles that travels toward the Earth. An astronomer on the Earth sees the emission through a solar telescope and issues a warning. The astronomer knows that when the particle pulse arrives it will wreak havoc with broadcast radio transmission. Communications systems require ten minutes to switch from over-the-air broadcast to underground cable transmission. What is the maximum speed of the particle pulse emitted by the Sun such that the switch can occur in time, between warning and arrival of the pulse? Take the sun to be 500 light-seconds distant from the Earth.

Solution

- It takes 500 seconds for the warning light flash to travel the distance of 500 light-seconds between the Sun and the Earth and enter the astronomer's telescope. If the particle pulse moves at half the speed of light, it will take twice as long as light to reach the Earth. If the pulse moves at one-quarter the speed of light, it will take four times as long to make the trip. We generalize this by saying that if the pulse moves with speed v/c, it will take time to make the trip given by the expression:

  $\Delta t_{\text{pulse}} = 500\text{ s}/ (v_{\text{pulse}}/c)$
- How long a warning time does the Earth astronomer have between arrival of the light flash carrying information about the pulse the arrival of the pulse itself? It takes 500 seconds for the light to arrive. Therefore the warning time is the difference between pulse transit time and the transit time of light:

  $\Delta t_{\text{warning}} = \Delta t_{\text{pulse}} - 500\text{ s}.$
- But we know that the minimum possible warning time is 10 min = 600 s.
- Therefore we have
- $600\text{ s} = 500\text{ s} / (v_{\text{pulse}}/c) - 500\text{ s},$
- which gives the maximum value for v_{puls} if there is to he sufficient time for warning:

  $$v_{\text{puls}} = 0.455\ c. \qquad \text{(Answer)}$$
- Observation reveals that pulses of particles emitted from the sun travel much slower than this maximum value. So we would have much longer warning time than calculated here.

Relating Events is science

- Science: trying to relate one event to another event
- E.g. how the radiation is related to occurrence of cancer; how lightning is related to electrical activities in the atmosphere etc.
- Since observation of events can be made from different frames of reference (e.g. from an stationary observatory or from a constantly moving train), we must also need to know how to predict events observed in one reference frame will look to an observer in another frame

Some examples

- How is the time interval measured between two events observed in one frame related to the time interval measured in another frame for the same two events?
- How is the velocity of a moving object measured by a stationary observer and that by a moving observer related?

Defining events

- So, before one can work out the relations between two events, one must first precisely define what an event is

Locating Events

- ***An event is an occurrence that happens at a unique place and time***
- Example: a collision, and explosion, emission of a light flash
- An event must be sufficiently localised in space and time
- e.g. your birthday: you are born in the General Hospital PP at year 1986 1st April 12.00 am)

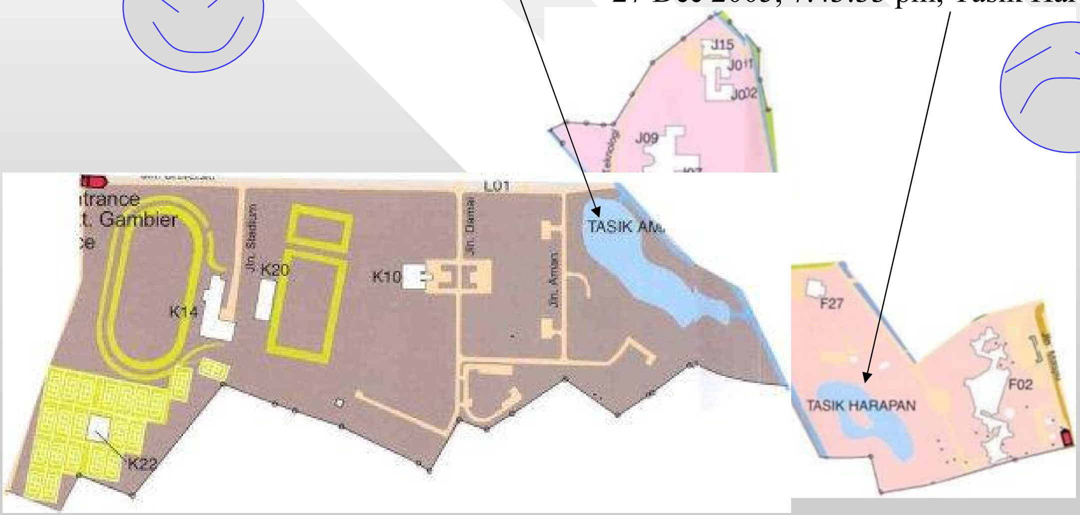
Example of two real-life events

Event 1: She said “I love you”

July 1Dec, 12.01:12 am, Tasik Aman

Event 2: She said “Let’s break up-lah”

27 Dec 2005, 7.43:33 pm, Tasik Harapan



Subtle effect to locate an event: delay due to finiteness of light speed

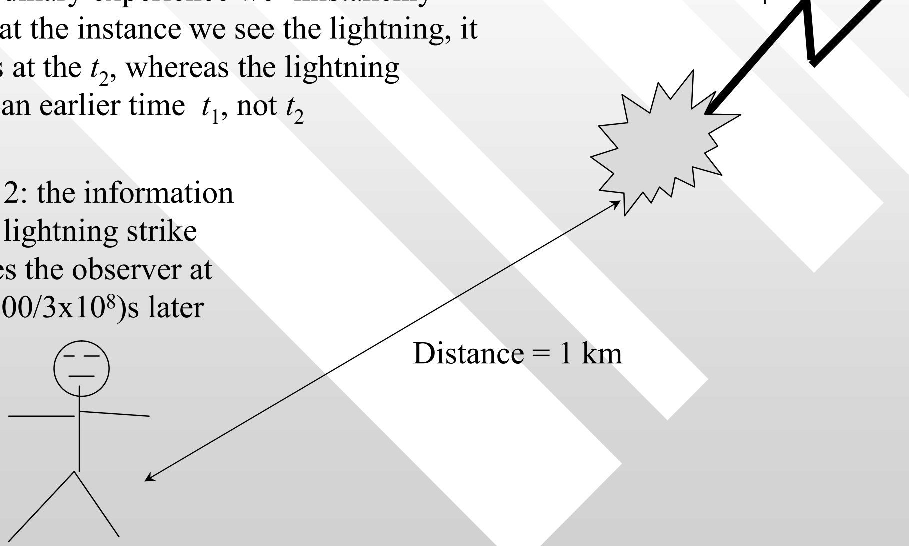
- In our (erroneous) “common sense” information are assumed to reach us instantaneously as though it is an immediate action through a distance without any delay
- In fact, since light takes finite time to travel, locating events is not always as simple it might seems at first

An illustrative example of delay while measuring an event far away

t_2 is very short due to the very fast speed of light c. In our ordinary experience we ‘mistakenly’ think that, at the instance we see the lightning, it also occurs at the t_2, whereas the lightning actually at an earlier time t_1, not t_2

Event 1: Lightning strikes at t_1 =0.00am

Event 2: the information of the lightning strike reaches the observer at t_2=(1000/3x10^8)s later



Reading Exercise 38-4

- When the pulse of protons passes through detector A (next to us), we start our clock from the time t = 0 microseconds. The light flash from detector B (at distance L=30 m away) arrives back at detector A at a time t = 0.225 microsecond later.
- (a) At what time did the pulse arrive at detector B?
- (b) Use the result from part (a) to find the speed at which the proton pulse moved, as a fraction of the speed of light.

Answer

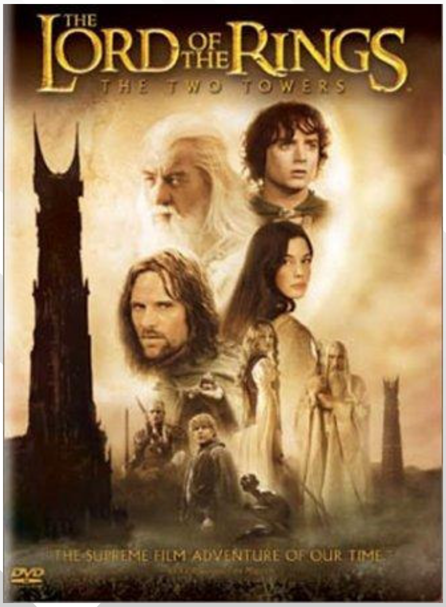
- The time taken for light pulse to travel from B to A is $L/c = 10^{-7}$ s $=0.1\ \mu$s
- Therefore the proton pulse arrived detector B $0.225 - 0.1\ \mu$s $= 0.125\ \mu$s after it passed us at detector A.
- (b) The protons left detector A at $t = 0$ and, according to part (a), arrived at detector B at $t = 0.125\ \mu$s. Therefore its speed from A to B is $L/0.125\ \mu$s $= \ldots = 0.8c$

Redefining Simultaneity

- Hence to locate an event accurately we must take into account the factor of such time delay
- An intelligent observer is an observer who, in an attempt to register the time and spatial location of an event far away, takes into account the effect of the delay factor
- (In our ordinary daily life we are more of an unintelligent observer)
- For an intelligent observer, he have to redefine the notion of "simultaneity" (example 38-2)

Example 38-2: Simultaneity of the Two Towers

- Frodo is an intelligent observer standing next to Tower A, which emits a flash of light every l0 s. 100 km distant from him is the tower B, stationary with respect to him, that also emits a light flash every 10 s. Frodo wants to know whether or not each flash is emitted from remote tower B simultaneous with (at the same time as) the flash from Frodo's own Tower A. Explain how to do this with out leaving Frodo position next to Tower A. Be specific and use numerical values.



Solution

- Frodo is an intelligent observer, which means that he know how to take into account the speed of light in determining the time of a remote event, in this case the time of emission of a flash by the distant Tower B. He measures the time lapse between emission of a flash by his Tower A and his reception of flash from Tower B.
- If this time lapse is just that required for light move from Tower B to Tower A, then the two emissions occur the same time.
- The two Towers are 100 km apart. Call this distance L. Then the time t for a light flash to move from B to A is
- $t = L/c = 10^5$ m$/3 \times 10^8$ m/s $= 0.333$ ms. (ANS)
- If this is the time Frodo records between the flash nearby Tower A and reception of the flash from distant tower then he is justified in saying that the two Towers emit flashes simultaneously in his frame.

One same event can be considered in any frame of reference

- One same event, in principle, can be measured by many separate observers in different (inertial) frames of reference (reference frames that are moving at a constant velocity with respect to each other)
- Example: On the table of a moving train, a cracked pot is dripping water
- The rate of the dripping water can be measured by (1) Ali, who is also in the train, or by (2) Baba who is an observer standing on the ground. Furthermore, you too can imagine (3) ET is also performing the same measurement on the dripping water from Planet Mars. (4) By Darth Veda from Dead Star…!

No 'superior' (or preferred) frame

- In other words, any event can be considered in infinitely many different frames of references.
- No particular reference frame is 'superior' than any other
- In the previous example, Ali's frame is in no way superior than Baba's frame, nor ET's frame, despite the fact that the water pot is stationary with respect to Ali.

Transformation laws

- Measurements done by any observers from all frame of reference are equally valid, and are all equivalent.
- Transformation laws such as Lorentz transformation can be used to relate the measurements done in one frame to another.
- In other words, once you know the values of a measurement in one frame, you can calculate the equivalent values as would be measured in other frames.
- In practice, the choice of frame to analyse any event is a matter of convenience.

Example 1

- In the previous example, obviously, the pot is stationary with respect to Ali, but is moving with respect to Baba.
- Ali, who is in the frame of the moving train, measures that the water is dripping at a rate of, say, r_A.
- Baba, who is on the ground, also measures the rate of dripping water, say r_B.
- Both of the rates measured by Ali and that measured by Baba have equal status – you can't say any one of the measurements is 'superior' than the other
- One can use Lorentz transformation to relate r_A with r_B. In reality, we would find that $r_B = r_A/\gamma$ where
- $1/\gamma^2 = 1 - (v/c)^2$, with v the speed of the train with respect to ground, and c the speed of light in vacuum.
- Note: r_B is not equal to r_A (would this contradict your expectation?)

Against *conventional wisdom*?

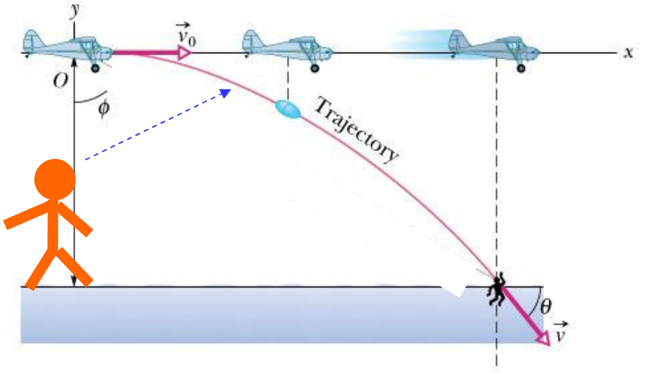
- According to SR, r_A and r_B are different in general.
- This should come as a surprise as your *conventional wisdom* (as according to Newtonian view point) may tell you that both r_A and r_B should be equal in their numerical value.
- However, as you will see later, such an assumption is false in the light of SR since the rate of time flow in two frames in relative motion are different
- Both rates, r_A and r_B, despite being different, are correct in their own right.

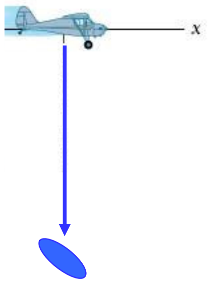
Example 2

- Consider a stone is thrown into the air making a projectile motion.
- If the trajectory of the stone is considered in the frame of Earth (the so-called Lab frame, in which the ground is made as a stationary reference), the trajectory of the stone is a parabolic curve.
- The trajectory of the stone can also be analysed in a moving frame traveling at velocity v_x along the same horizontal direction as the stone. In this frame (the so-called rocket frame), the trajectory of the stone is not a parabolic curve but a vertical line.

View from different frames

- In the Lab frame, the observer on the ground sees a parabolic trajectory
- In the Rocket frame, the pilot sees the projectile to follow a vertically straight line downwards



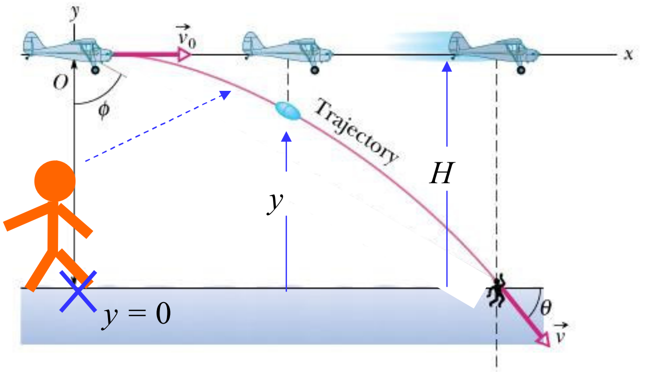


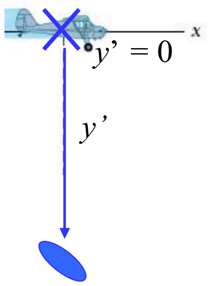
Transformation law for the coordinates

- In Lab frame
- $y = -(gt^2)/2 + H$
- $x = v_x t$

- In rocket frame
- $y' = -(gt^2)/2$
- $x' = 0$

- Transformation law relating the coordinates of projectile in both frames is
- $y' = y - H$, $x' = x - v_x t$





Time dilation as direct consequence of constancy of light speed

- According to the Principle of Relativity, the speed of light is invariant (i.e. it has the same value) in every reference frame (constancy of light speed)
- A direct consequence of the constancy of the speed of light is time stretching
- Also called time dilation
- Time between two events can have different values as measured in lab frame and rocket frames in relative motion
- "Moving clock runs slow"

Experimental verification of time stretching with pions

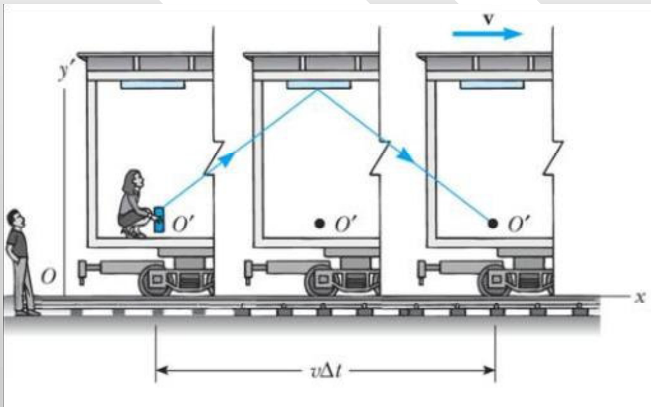
- Pion's half life $t_{½}$ is 18 ns.
- Meaning: If N_0 of them is at rest in the beginning, after 18 ns, $N_0/2$ will decay
- Hence, by measuring the number of pion as a function of time allows us to deduce its half life
- Consider now N_0 of them travel at roughly the speed of light c, the distance these pions travel after $t_{½}$=18 ns would be $ct_{½} \approx$5.4 m.
- Hence, if we measure the number of these pions at a distance 5.4 m away, we expect that $N_0/2$ of them will survive
- However, experimentally, the number survived at 5.4m is much greater than expected
- The flying poins travel tens of meters before half of them decay
- How do you explain this? the half life of these pions seems to have been stretched to a larger value!
- Conclusion: in our lab frame the time for half of the pions to decay is much greater than it is in the rest frame of the pions!

RE 38-5

- Suppose that a beam of pions moves so fast that at 25 meters from the target in the laboratory frame exactly half of the original number remain undecayed. As an experimenter, you want to put more distance between the target and your detectors. You are satisfied to have one-eighth of the initial number of pions remaining when they reach your detectors. How far can you place your detectors from the target?
- ANS: 75 m

A Gedanken Experiment

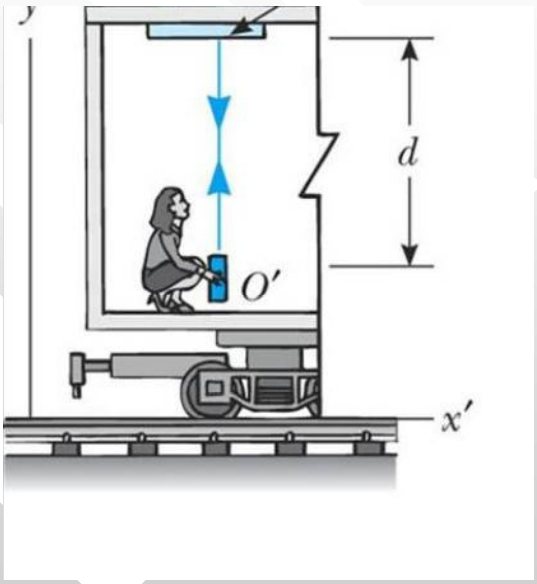
- Since light speed *c* is invariant (i.e. the same in all frames), it is suitable to be used as a clock to measure time and space
- Use light and mirror as clock – light clock
- A mirror is fixed to a moving vehicle, and a light pulse leaves O′ at rest in the vehicle. O′ is the rocket frame.
- Relative to a lab frame observer on Earth, the mirror and O′ move with a speed *v*.



(b)

In the rocket frame

- The light pulse is observed to be moving in the vertical direction only
- The distance the light pulse traversed is $2d$
- The total time travel by the light pulse to the top, get reflected and then return to the source is $\Delta\tau = 2d/c$

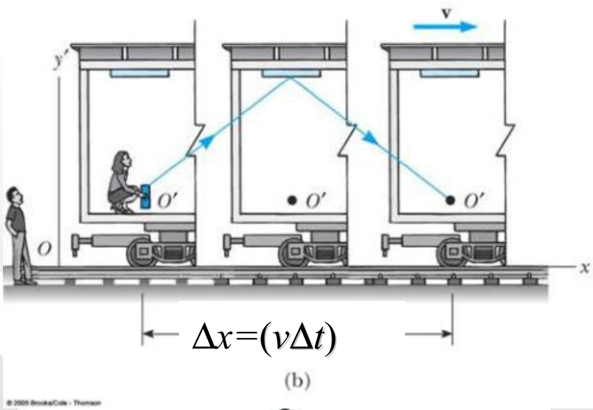


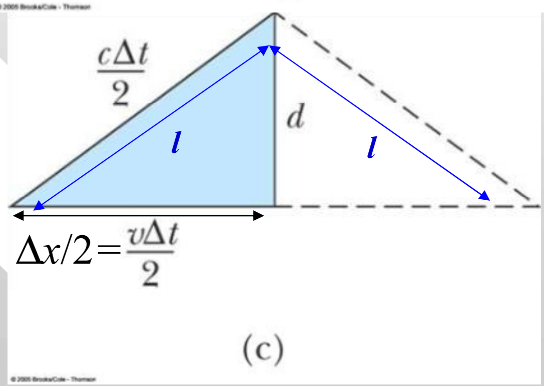
In the lab frame

- However, O in the lab frame observes a different path taken by the light pulse – it's a triangle instead of a vertical straight line
- The total light path is longer $= 2l$
- $l^2 = (c\Delta t/2)^2$

$$= d^2 + (\Delta x/2)^2$$

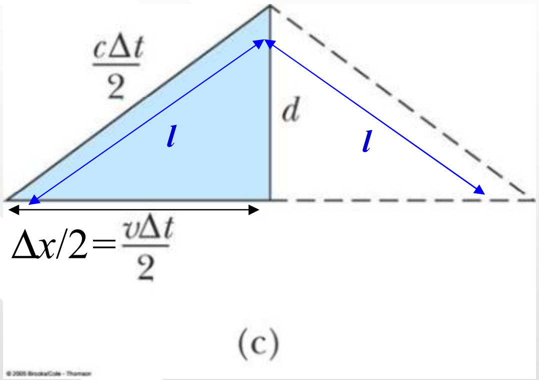
$$= d^2 + (v\Delta t/2)^2$$





Light triangle

- We can calculate the relationship between Δt, $\Delta\tau$ and v:
- $l^2=(c\Delta t/2)^2=d^2+(v\Delta t/2)^2$ (lab frame)



- $\Delta\tau = 2d/c$ (Rocket frame)
- Eliminating d,

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \gamma\Delta\tau \qquad \gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \geq 1$$

Time dilation equation

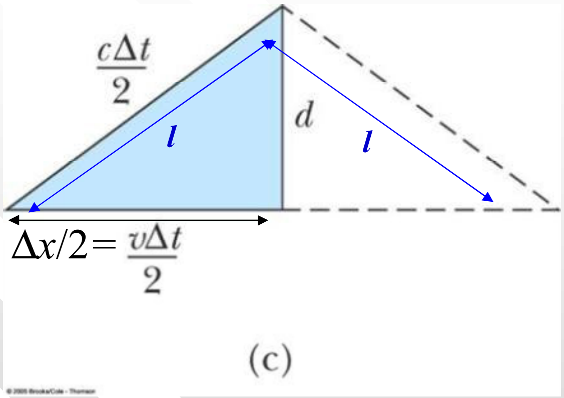
- Time dilation equation $\Delta t = \frac{\Delta\tau}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \gamma\Delta\tau$
- Gives the value of time $\Delta\tau$ between two events occur at time Δt apart in some reference frame
- Lorentz factor $\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \geq 1$
- Note that as $v \ll c$, $\gamma \approx 1$; as $v \rightarrow c$, $\gamma \rightarrow \infty$
- Appears frequently in SR as a measure of relativistic effect: $\gamma \approx 1$ means little SR effect; $\gamma \gg 1$ is the ultra-relativistic regime where SR is most pronounce

RE 38-6

- A set of clocks is assembled in a stationary boxcar. They include a quartz wristwatch, a balance wheel alarm clock, a pendulum grandfather clock, a cesium atomic clock, fruit flies with average individual lifetimes of 2.3 days. A clock based on radioactive decay of nuclei, and a clock timed by marbles rolling down a track. The clocks are adjusted to run at the same rate as one another. The boxcar is then gently accelerated along a smooth horizontal track to a final velocity of 300 km/hr. At this constant final speed, which clocks will run at a different rate from the others as measured in that moving boxcar?

The Metric Equation

- From the light triangle in lab frame and the vertical light pulse in the rocket frame:



- $l^2 = (c\Delta t/2)^2 = d^2 + (\Delta x/2)^2$;
- $d = c\Delta\tau/2$

$\Rightarrow (c\Delta t/2)^2 = (c\Delta\tau/2)^2 + (\Delta x/2)^2$

- Putting the terms that refer to the lab frame are on the right: $(c\Delta\tau)^2 = (c\Delta t)^2 - (\Delta x)^2$

"the *invariant space-time interval*"

- We call the RHS, $s^2 \equiv (c\Delta t)^2-(\Delta x)^2$ "*invariant space-time interval squared*" (or sometimes simply "*the space-time interval*")
- In words, the space-time interval reads:
- s^2 = ($c\times$time interval between two events as observed in the frame)^2 - (distance interval between the two events as observed in the frame)^2
- We can always calculate the space-time intervals for any pairs of events
- The interval squared s^2 is said to be an *invariant* because it has the same value as calculated by all observers (take the simile of the mass-to-high^2 ratio)
- Obviously, in the light-clock gadanken experiment, the space-time interval of the two light pulse events $s^2 \equiv (c\Delta t)^2-(\Delta x)^2 = (c\Delta\tau)^2$ is positive because $(c\Delta\tau)^2 > 0$
- The space-time interval for such two events being positive is deeply related to the fact that such pair of events are causally related
- The space-time interval of such event pairs is said to be '*time-like*' (because the time component in the interval is larger in magnitude than the spatial component)
- Not all pairs of events has a positive space-time interval
- Pairs of events with a negative value of space-time interval is said to be "space-like", and these pairs of event cannot be related via any causal relation

RE 38-8

- Points on the surfaces of the Earth and the Moon that face each other are separated by a distance of 3.76×10^8 m.
- (a) How long does it take light to travel between these points?
- A firecraker explodes at each of these two points; the time between these explosion is one second.
- (b) What is the invariant space-time interval for these two events?
- Is it possible that one of these explosions caused the other explosion?

Solution

(a) Time taken is

$t = L / c = 3.76\times10^8$ m/ 3×10^8 m/s = 1.25 s

(b) $s^2 = (ct)^2 - L^2$

$= (3\times10^8 \text{ m/s} \times 1.25 \text{ s})^2 - (3.76\times10^8 \text{ m})^2 = -7.51 \text{ m}^2$

(space-like interval)

(c) It is known that the two events are separated by only 1 s. Since it takes 1.25 s for light to travel between these point, it is impossible that one explosion is caused by the other, given that no information can travel fast than the speed of light.

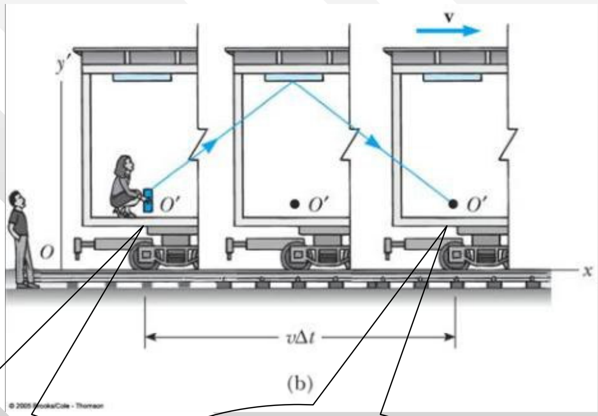
Alternatively, from (b), these events are separated by a space-like space-time interval. Hence it is impossible that the two explosions have any causal relation

Proper time

- Imagine you are in the rocket frame, O', observing two events taking place at the same spot, separated by a time interval $\Delta\tau$ (such as the emission of the light pulse from source (EV1), and re-absorption of it by the source again, (EV2))
- Since both events are measured on the same spot, they appeared at rest wrp to you
- The time lapse $\Delta\tau$ between the events measured on the clock at rest is called the proper time or wristwatch time (one's own time)

Improper time

- In contrast, the time lapse measured by an observer between two events not at the same spot, i.e. $\Delta x \neq 0$, are termed *improper time*
- E.g., the time lapse, Δt, measured by the observer O observing the two events of light pulse emission and absorption in the train is improper time since both events appear to occur at different spatial location according to him.



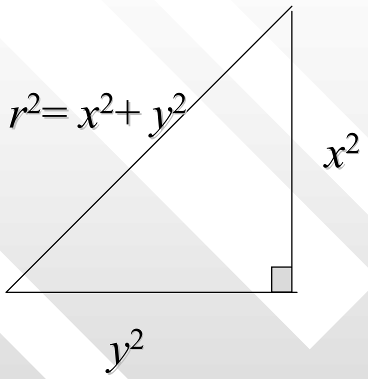
Event 1 occurs here at $x = 0$ (according to O)

Event 2 occurs here at $x = v\Delta t$ (according to O)

Space and time are combined by the metric equation: Space-time

$$s^2 \equiv (c\Delta t)^2 - (\Delta x)^2 = \text{invariant} = (\Delta\tau)^2$$

- The metric equation says $(c\Delta t)^2 - (\Delta x)^2$ = invariant $= (c\Delta\tau)^2$ in all frames
- *It combines space and time in a single expression on the RHS!!*
- Meaning: Time and space are interwoven in a fabric of space-time, and is not independent from each other anymore (we used to think so in Newton′s absolute space and absolute time system)



The space-time invariant is the 1+1 dimension Minkowsky space-time

analogous to the 3-D Pythagoras theorem with the hypotenuse $r^2 = x^2 + y^2$. However, in the Minkowsky space-time metric, the space and time components differ by an relative minus sign

s^2 relates two different measures of time between the *same* two events

$$s^2 \equiv (c\Delta t)^2 - (\Delta x)^2 = \text{invariant} = (c\Delta\tau)^2$$

- (1) the time recorded on clocks in the reference frame in which the events occur at different places (improper time, Δt), and
- (2) the wristwatch time read on the clock carried by a traveler who records the two events as occurring a the same place (proper time, $\Delta\tau$)
- In different frames, Δt and Δx measured for the same two events will yield different values in general. However, the interval squared, $(c\Delta t)^2-(\Delta x)^2$ will always give the same value, see example that ensues

Example of calculation of space-time interval squared

- In the light-clock gedanken experiment: For O′, he observes the proper time interval of the two light pulse events to be $\Delta\tau$. For him, $\Delta x' = 0$ since these events occur at the same place
- Hence, for O′,
- $s'^2 = (c\times\text{time interval observed in the frame})^2 - (\text{distance interval observed in the frame})^2$
- $= (c\Delta\tau)^2 - \cancel{(\Delta x')^2} = (c\Delta\tau)^2$
- For O, the time-like interval for the two events is $s^2=(c\Delta t)^2-(\Delta x)^2 = (c\gamma\Delta\tau)^2-(v\Delta t)^2 = (c\gamma\Delta\tau)^2-(v\gamma\Delta\tau)^2 = \gamma^2 (c^2-v^2)\Delta\tau^2 = c^2\Delta\tau^2 = s'^2$

What happens at high and low speed

$$\Delta t = \gamma \Delta \tau, \qquad \gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \geq 1$$

- At low speed, $v << c$, $\gamma \approx 1$, and $\Delta\tau \approx \Delta t$, not much different, and we can't feel their difference in practice
- However, at high speed, proper time interval ($\Delta\tau$) becomes much SMALLER than improper time interval (Δt) in comparison, i.e. $\Delta\tau = \Delta t/\gamma << \Delta t$
- Imagine this: to an observer on the Earth frame, the person in a rocket frame traveling near the light speed appears to be in a 'slow motion' mode. This is because, according to the Earth observer, the rate of time flow in the rocket frame appear to be slower as compared to the Earth's frame rate of time flow.
- A journey that takes, say, 10 years to complete, according to a traveler on board (this is his proper time), looks like as if they take 10γ yr according to Earth observers.

Space travel with time-dilation

- A spaceship traveling at speed $v = 0.995c$ is sent to planet 100 light-year away from Earth
- How long will it takes, according to a Earth's observer?
- $\Delta t = 100 \text{ ly}/0.995c = 100.05\text{yr}$
- But, due to time-dilation effect, according to the traveler on board, the time taken is only

  $\Delta\tau = \Delta t/\gamma = \Delta t\sqrt{(1-0.995^2)} = 9.992$ yr, not 100.05 yr as the Earthlings think
- So it is still possible to travel a very far distance within one's lifetime ($\Delta\tau \approx 50$ yr) as long as γ (or equivalently, v) is large enough

Nature’s Speed Limit

- Imagine one in the lab measures the speed of a rocket v to be larger than c.
- As a consequence, according to $\Delta\tau = \Delta t\sqrt{1-\left(\frac{v}{c}\right)^2}$
- The proper time interval measurement $\Delta\tau$ in the rocket frame would be proportional to an imaginary number, $i = \sqrt{(-1)}$
- This is unphysical (and impossible) as no real time can be proportional to an imaginary number
- Conclusion: no object can be accelerated to a speed greater than the speed of light in vacuum, c
- Or more generally, no information can propagate faster than the light speed in vacuum, c
- Such limit is the consequence required by the logical consistency of SR

Time dilation in ancient legend

- 天上方一日，人间已十年
- One day in the heaven, ten years in the human plane

RE 38-7

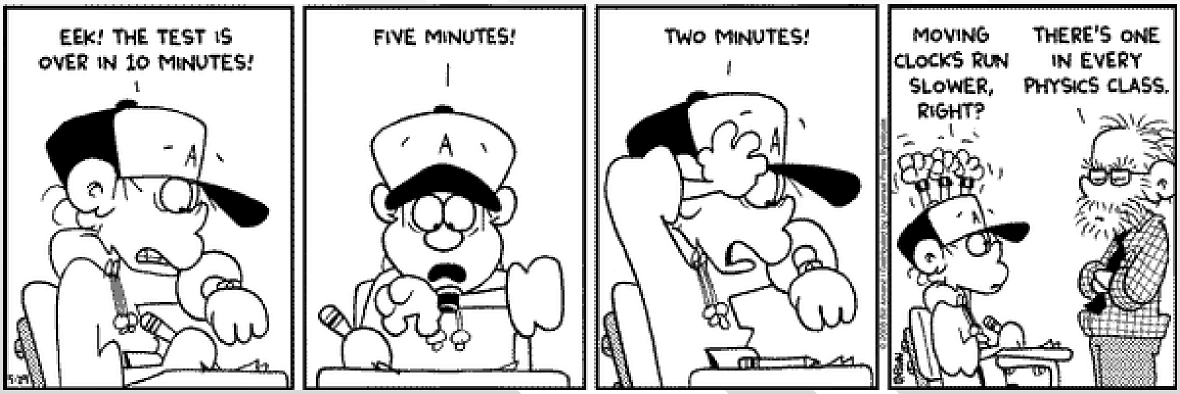
- Find the rocket speed v at which the time $\Delta\tau$ between ticks on the rocket is recorded by the lab clock as $\Delta t = 1.01\Delta\tau$
- *Ans:* $\gamma = 1.01$, i.e. $(v/c)^2 = 1 - 1/\gamma^2 = \ldots$
- Solve for v in terms of c: $v = \ldots$

Satellite Clock Runs Slow?

- An Earth satellite in circular orbit just above the atmosphere circles the Earth once every $T = 90$ min. Take the radius of this orbit to be $r = 6500$ kilometers from the center of the Earth. How long a time will elapse before the reading on the satellite clock and the reading on a clock on the Earth's surface differ by one microsecond?
- For purposes of this approximate analysis, assume that the Earth does not rotate and ignore gravitational effects due the difference in altitude between the two clocks (gravitational effects described by general relativity).

Solution

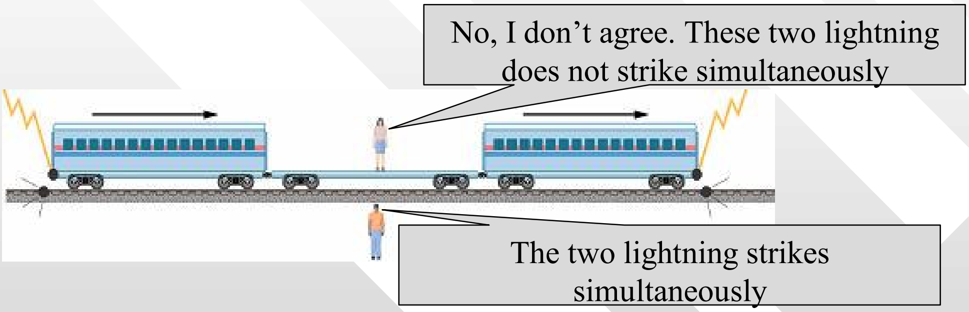
- First we need to know the speed of the satellite in orbit. From the radius of the orbit we compute the circumference and divide by the time needed to cover that circumference:
- $v = 2\pi r/T = (2\pi\times 6500\text{ km})/(90\times 60\text{ s}) = 7.56\text{ km/s}$
- Light speed is almost exactly $c = 3\times 10^5$ km/s. so the satellite moves at the fraction of the speed of light given by
- $(v/c)^2 = [(7.56\text{ km/s})/(3\times 10^5\text{ km/s})]^2 = (2.52\times 10^5)^2 = 6.35\times 10^{-10}$.
- The relation between the time lapse $\Delta\tau$ recorded on the satellite clock and the time lapse Δt on the clock on Earth (ignoring the Earth's rotation and gravitational effects) is given by
- $\Delta\tau = (1-(v/c)^2)^{1/2}\,\Delta t$
- We want to know the difference between Δt and $\Delta\tau$ i.e. $\Delta t - \Delta\tau$.
- We are asked to find the elapsed time for which the satellite clock and the Earth clock differ in their reading by one microsecond, i.e. $\Delta t - \Delta\tau = 1\mu\text{s}$
- Rearrange the above equation to read $\Delta t^2 - \Delta\tau^2 = (\Delta t+\Delta\tau)(\Delta t-\Delta\tau)$, one shall arrive at the relation that $\Delta t = [1+(1-(v/c)^2)^{1/2}](1\mu\text{s}) / (v/c)^2 \approx 3140\text{s}$
- This is approximately one hour. A difference of one microsecond between atomic clock is easily detectable.



Disagreement on simultaneity

Two events that are simultaneous in one frame are not necessarily simultaneous in a second frame in uniform relative motion

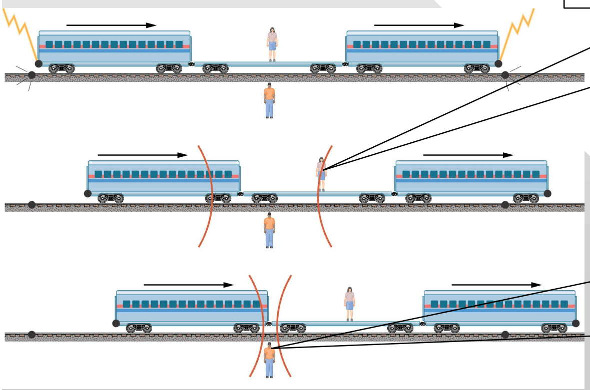
Example



Einstein Train illustration

- Lightning strikes the front and back of a moving train, leaving char marks on both track and train. Each emitted flash spreads out in all directions.

I am equidistant from the front and back char marks on the train. Light has the standard speed in my frame, and equal speed in both direction. The flash from the front of the train arrived first, therefore the flash must have left the front of the train first. The front lightning bolt fell first before the rear light bolt fell. I conclude that the two strokes are not simultaneous.



I stand by the tracks halfway between the char marks on the track. The flashes from the strokes reach me a the same time and I am equidistance from the char marks on the track. I conclude that two events were simultaneous

Why?

- This is due to the invariance of the space-time invariant in all frames, (i.e. the invariant must have the same value for all frames)

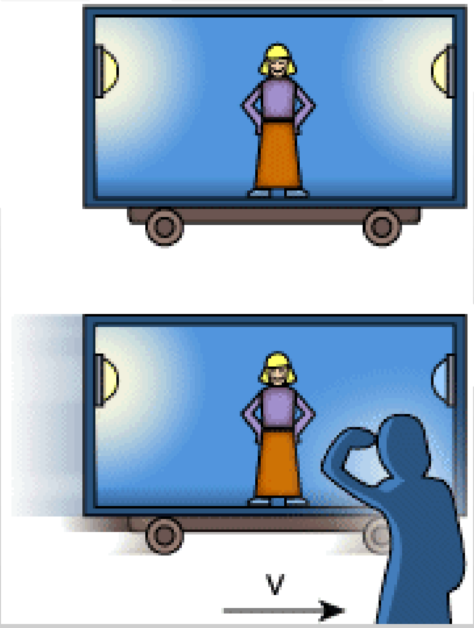
How invariance of space-time interval explains disagreement on simultaneity by two observers

- Consider a pair of events with space-time interval

$$s^2=(c\Delta t)^2-(\Delta x)^2=(c\Delta t')^2-(\Delta x')^2$$

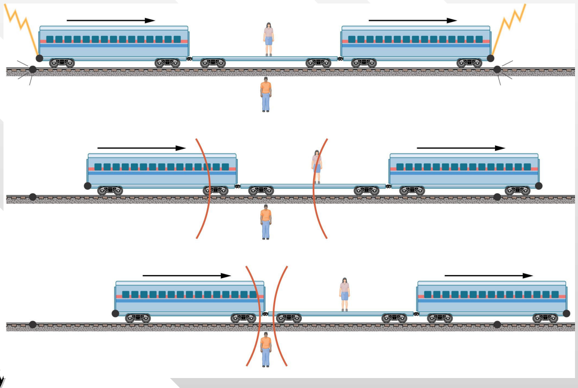
- where the primed and un-primed notation refer to space and time coordinates of two frames at relative motion (lets call them O and O′)
- Say O observes two simultaneous event in his frame (i.e. $\Delta t = 0$) and are separate by a distance of (Δx), hence the space-time interval is $s^2 = -(\Delta x)^2$
- The space-time interval for the same two events observed in another frame, O′, $s^2 = (c\Delta t')^2 - (\Delta x')^2$ must read the same, as $-(\Delta x)^2$
- Hence, $(c\Delta t')^2 = (\Delta x')^2 - (\Delta x)^2$ which may not be zero on the RHS. i.e. $\Delta t'$ is generally not zero. This means in frame O′, these events are not observed to be occurring simultaneously

Simulation of disagreement on simultaneity by two observers (be aware that this simulation simulates the scenario in which the lady in the moving car sees simultaneity whereas the observer on the ground disagrees)



RE 38-9

- Susan, the rider on the train pictured in the figure is carrying an audio tape player. When she received the light flash from the front of the train she switches on the tape player, which plays very loud music. When she receives the light flash from the back end of the train, Susan switches off the tape player. Will Sam, the observers on the ground be able to hear the music?
- Later Susan and Sam meet for coffee and examine the tape player. Will they agree that some tape has been wound from one spool to the other?
- The answer is: …



Solution

- Music has been emitted from the tape player. This is a fact that must be true in both frames of reference. Hence Sam on the ground will be able to hear the music (albeit with some distortion).
- When the meet for coffee, they will both agree that some tape has been wound from one spool to the other in the tape recorder.

Touchstone Example 38-5: Principle of Relativity Applied

- Divide the following items into two lists, On one list, labeled SAME, place items that name properties and laws that are always the same in every frame. On the second list, labeled MAY BE DIF FERENT. place items that name properties that can be different in different frames:
- a. the time between two given events
- b. the distance between two given events
- c. the numerical value of Planck's constant h
- d. the numerical value of the speed of light c
- e. the numerical value of the charge e on the electron
- f. the mass of an electron (measured at rest)
- g. the elapsed time on the wristwatch of a person moving between two given events
- h. the order ot elements in the periodic table
- i. Newton's First Law of Motion ("A particle initially at rest remains at rest, and …")
- j. Maxwell's equations that describe electromagnetic fields in a vacuum
- k. the distance between two simultaneous events

Solution

THE SAME IN ALL FRAMES

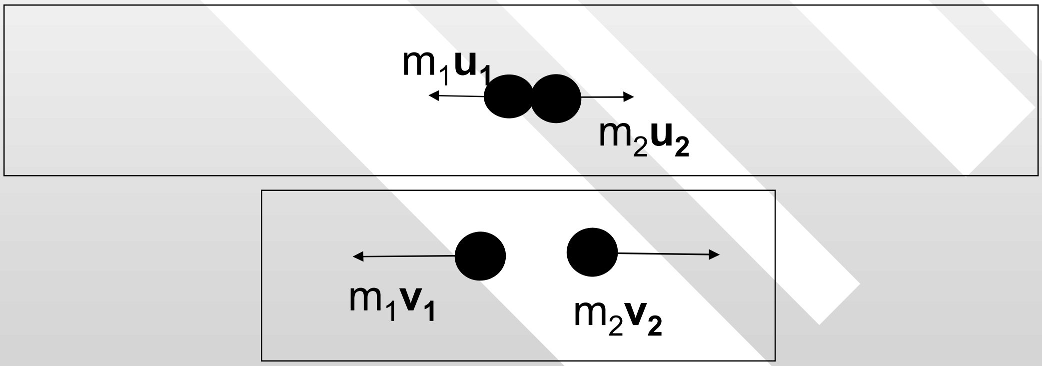
- c. numerical value of *h*
- d. numerical value of *c*
- e. numerical value of *e*
- f. mass of electron (at rest)
- g. wristwatch time between two event (this is the proper time interval between two event)
- h. order of elements in the periodic table
- i. Newton's First Law of Motion
- j. Maxwell's equations
- k. distance between two simultaneous events

MAY BE DIFFERENT IN DIFFERENT FRAMES

- a. time between two given events
- b. distance between two give events

Relativistic Dynamics

- Where does $E=mc^2$ comes from?
- By Einstein′s postulate, the observational law of linear momentum must also hold true in all frames of reference



Conservation of linear momentum classically means

$m_1\mathbf{u_1} + m_2\mathbf{u_2} = m_1\mathbf{v_1} + m_2\mathbf{v_2}$

Classical definition of linear momentum

- Classically, one defines linear momentum as mass × velocity
- Consider, in a particular reference frame where the object with a mass m_0 is moving with velocity v, then the momentum is defined (according to classical mechanics) as
  - $p = m_0 v$.
- If $v = 0$, the mass m_0 is called the rest mass.
- Similarly, in the other frame, (say the O' frame), $p' = m'v'$
- According to Newton's mechanics, the mass m' (as seen in frame O′) is the same as the mass m_0 (as seen in O frame). There is no distinction between the two.
- In particular, there is no distinction between the rest mass and the 'moving mass'

Modification of expression of linear momentum

- However, simple analysis will reveal that in order to preserve the consistency between conservation of momentum and the Lorentz Transformation (to be discussed later), the definition of momentum has to be modified to
  - $momentum = \gamma m_0 v$
- where m_0 is the rest mass (an invariant quantity).
- A popular interpretation of the above re-definition of linear momentum holds that the mass of an moving object, m, is different from its value when it's at rest, m_0, by a factor of γ, i.e
  - $m = \gamma m_0$
- (however some physicists argue that this is actually not a correct interpretation. For more details, see the article by Okun posted on the course webpage. In any case, for pedagogical reason, we will stick to the "popular interpretation" of the "relativistic mass")

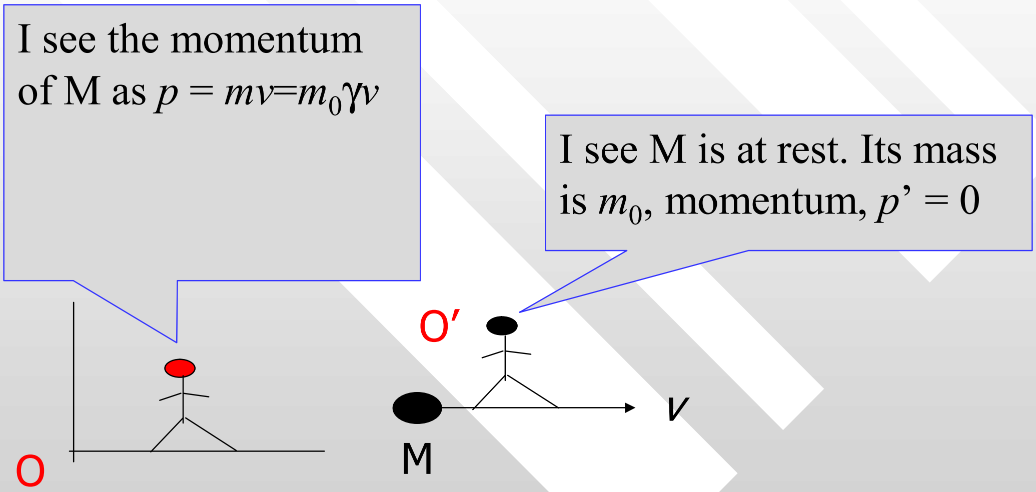
In other words…

- In order to preserve the consistency between Lorentz transformation of velocity and conservation of linear momentum, the definition of 1-D linear momentum, classically defined as
  - $p_{classical} = rest\ mass \times velocity,$
- has to be modified to

$$p_{classical} \rightarrow \quad p_{sr} = \text{"}relativistic\ mass\text{"} \times velocity$$
$$= mv = \gamma m_0 v$$

- where the relativisitic mass $m = \gamma m_0$ is not the same the rest mass, m_0
- Read up the text for a more rigorous illustration why the definition of classical momentum is inconsistent with LT

Pictorially…



Two kinds of mass

- Differentiate two kinds of mass: rest mass and relativistic mass
- m_0 = rest mass = the mass measured in a frame where the object is at rest. The rest mass of an object must be the same in all frames (not only in its rest frame).
- Relativistic mass $m = \gamma m_0$. The relativistic mass is speed-dependent

Behaviour of p_{SR} as compared to $p_{classic}$

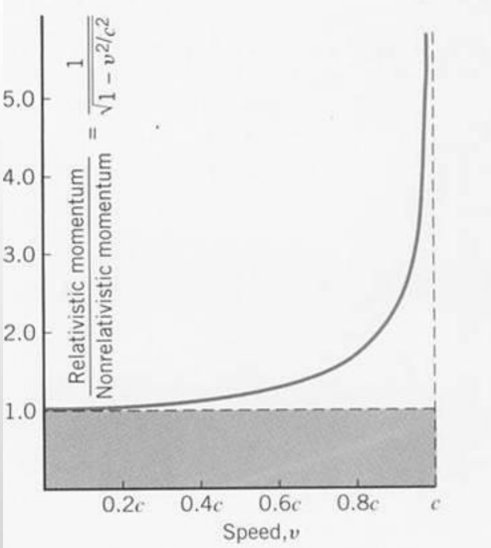


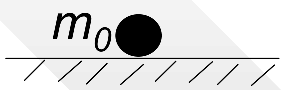
Figure 28.7 This graph shows how the ratio of the magnitude of the relativistic momentum to the magnitude of the nonrelativistic momentum increases as the speed of an object approaches the speed of light.

- Classical momentum is constant in mass, $p_{classic} = m_0 v$
- Relativistic momentum is $p_{SR} = m_0 \gamma v$
- $p_{SR} / p_{classic} = \gamma \rightarrow \infty$ as $v \rightarrow c$
- In the other limit, $v << c$

  $p_{SR} / p_{classic} = 1$

Example

The rest mass of an electron is $m_0 = 9.11 \times 10^{-31}$kg.



If it moves with $v = 0.75\ c$, what is its relativistic momentum?

$$p = m_0 \gamma u$$

Compare the relativistic p with that calculated with classical definition

Solution

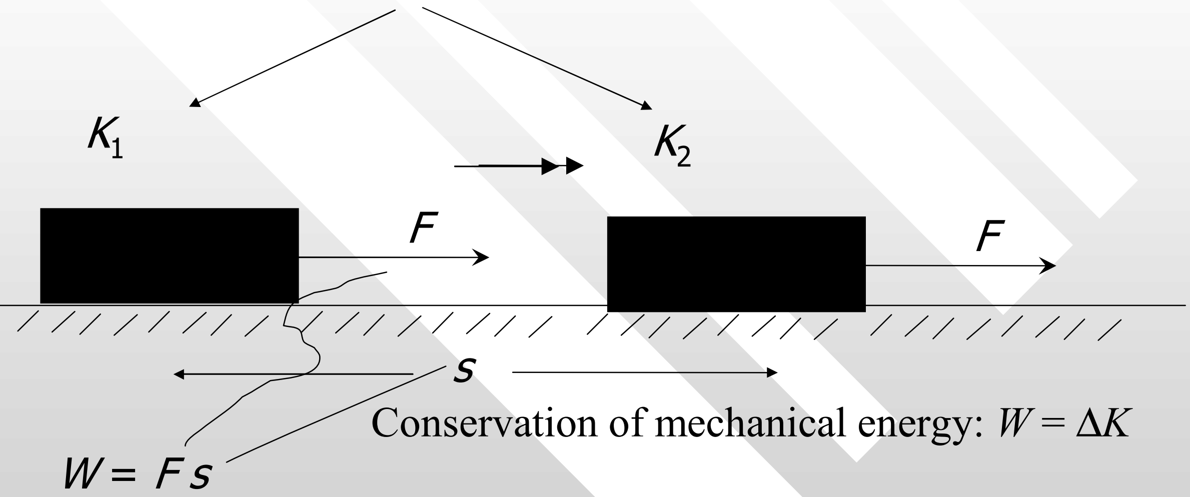
- The Lorentz factor is
  $\gamma = [1-(v/c)^2]^{-1/2} = [1-(0.75c/c)^2]^{-1/2} = 1.51$
- Hence the relativistic momentum is simply
  $p = \gamma\, m_0 \times 0.75c$
  $= 1.51 \times 9.11 \times 10^{-31}\text{kg} \times 0.75 \times 3 \times 10^8$ m/s
  $= 3.1 \times 10^{-22}$ Ns
- In comparison, classical momentum gives $p_{\text{classical}} = m_0 \times 0.75c = 2.5 \times 10^{-22}$ Ns – about 34% lesser than the relativistic value

Work-Kinetic energy theorem

- Recall the law of conservation of mechanical energy:

  Work done by external force on a system, W = the change in kinetic energy of the system, ΔK

$$\Delta K = K_2 - K_1$$



Conservation of mechanical energy: $W = \Delta K$

The total energy of the object, $E = K + U$. Ignoring potential energy, E of the object is solely in the form of kinetic energy. If $K_1 = 0$, then $E = K_2$. But in general, U also needs to be taken into account for E.

- In classical mechanics, mechanical energy (kinetic + potential) of an object is closely related to its momentum and mass
- Since in SR we have redefined the classical mass and momentum to that of relativistic version
- $m_{\text{class}}(\text{cosnt}, = m_0) \rightarrow m_{\text{SR}} = m_0\gamma$
- $p_{\text{class}} = m_{\text{class}} v \rightarrow p_{\text{SR}} = (m_0\gamma)v$
- we must also modify the relation btw work and energy so that the law conservation of energy is consistent with SR
- E.g, in classical mechanics, $K_{class} = p^2/2m = mv^2/2$. However, this relationship has to be supplanted by the relativistic version

$$K_{class} = mv^2/2 \rightarrow K_{SR} = E - m_0c^2 = \gamma m_0 c^2 - m_0 c^2$$

- We shall derive K in SR in the following slides

Force, work and kinetic energy

- When a force is acting on an object with rest mass m_0, it will get accelerated (say from rest) to some speed (say v) and increase in kinetic energy from 0 to K

K as a function of v can be derived from first principle based on the definition of:

Force, $F = \mathrm{d}p/\mathrm{d}t$,

work done, $W = \int F \, \mathrm{d}x$,

and conservation of mechanical energy, $\Delta K = W$

Derivation of relativistic kinetic energy

Force = rate change of momentum

$$W = \int_{x_1=0}^{x_2} F\,dx = \int_{x_1=0}^{x_2} \frac{dp}{dt}dx = \int_{x_1=0}^{x_2} \left(\frac{dp}{dx}\frac{dx}{dt}\right)dx$$

Chain rule in calculus

$$= \int_{x_1=0}^{x_2} \frac{dp}{dx}v\,dx = \int_{x_1=0}^{x_2} \left(\frac{dp}{dv}\frac{dv}{dx}\right)v\,dx = \int_0^v \frac{dp}{dv}v\,dv$$

where, by definition, $v = \dfrac{dx}{dt}$ is the velocity of the object

Explicitly, $p = \gamma m_0 v$,

Hence, $dp/dv = d/dv\,(\gamma m_0 v)$

$$= m_0 [v\,(d\gamma/dv) + \gamma]$$

$$= m_0 [\gamma + (v^2/c^2)\,\gamma^3] = m_0\,(1-v^2/c^2)^{-3/2}$$

in which we have inserted the relation

$$\frac{d\gamma}{dv} = \frac{d}{dv}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{v}{c^2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} = \frac{v}{c^2}\gamma^3$$

integrate

$$W = m_0 \int_0^v v\left(1-\frac{v^2}{c^2}\right)^{-3/2} dv$$

$$\boxed{\Rightarrow K = W = m_0\gamma c^2 - m_0 c^2 = mc^2 - m_0 c^2}$$

$$\boxed{K = m_0\gamma c^2 - m_0 c^2 = mc^2 - m_0 c^2}$$

- The relativistic kinetic energy of an object of rest mass m_0 traveling at speed v
- $E = mc^2$ is the total relativistic energy of an moving object
- $E_0 = m_0 c^2$ is called the rest energy of the object.
- Any object has non-zero rest mass contains energy $E_0 = m_0 c^2$
- One can imagine that masses are 'frozen energies in the form of masses' as per $E_0 = m_0 c^2$
- The rest energy (or rest mass) is an invariant

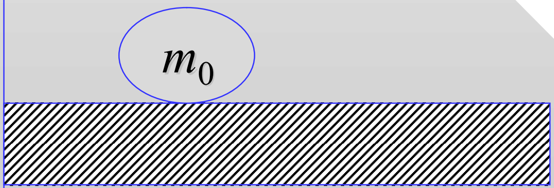
- Or in other words, the total relativistic energy of a moving object is the sum of its rest energy and its relativistic kinetic energy

$$E = mc^2 = m_0c^2 + K$$

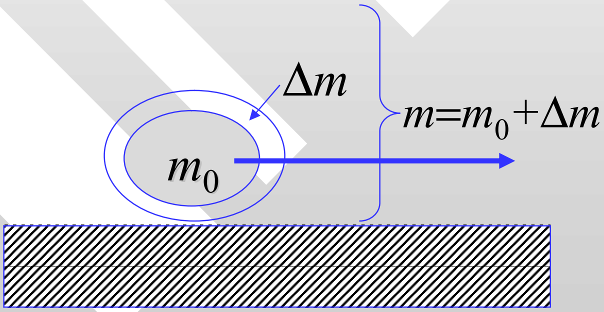
- The (relativistic) mass of an moving object *m* is larger than its rest mass m_0 due to the contribution from its relativistic kinetic energy – this is a pure relativistic effect not possible in classical mechanics

Pictorially

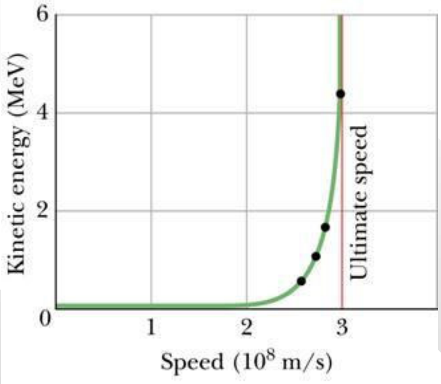
- Object at rest
- Total relativistic energy = rest energy only (no kinetic energy)
- $E=E_0=m_0c^2$



- A moving object
- Total relativistic energy = kinetic energy + rest energy
- $E=mc^2=K+E_0$
- $K=mc^2 - E_0 = \Delta mc^2$



Relativistic Kinetic Energy of an electron



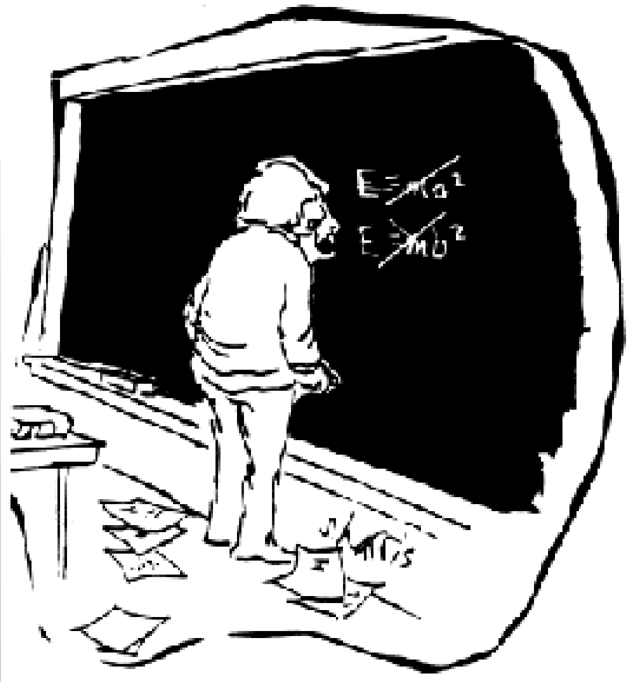
- The kinetic energy increases without limit as the particle speed v approaches the speed of light
- In principle we can add as much kinetic energy as we want to a moving particle in order to increase the kinetic energy of a particle without limit
- What is the kinetic energy required to accelerate an electron to the speed of light?
- Exercise: compare the classical kinetic energy of an object, $K_{çlas}=m_0v^2/2$ to the relativistic kinetic energy, $K_{sr}=(\gamma-1)m_0c^2$. What are their difference?

Mass energy equivalence, $E = mc^2$

- $E = mc^2$ relates the relativistic mass of an object to the total energy released when the object is converted into pure energy

  Example, 10 kg of mass, if converted into pure energy, it will be equivalent to $E = mc^2 = 10 \times (3 \times 10^8)^2$ J $= 9 \times 10^{17}$J - equivalent to a few tons of TNT explosive

So, now you know how $E=mc^2$ comes about…



Example 38-6: Energy of Fast Particle

- A particle of rest mass m_0 moves so fast that its total (relativistic) energy is equal to 1.1 times its rest energy.
- (a) What is the speed v of the particle?
- (b) What is the kinetic energy of the particle?

Solution

(a)

- Rest energy $E_0 = m_0c^2$
- We are looking for a speed such that the energy is 1.1 times the rest energy.
- We know how the relativistic energy is related to the rest energy via
- $E = \gamma E_0 = 1.1E_0$
- $\Rightarrow 1/\gamma^2 = 1/1.1^2 = 1/1.21 = 0.8264$
- $1 - v^2/c^2 = 0.8264$
- $\Rightarrow v^2/c^2 = 1 - 0.8264 = 0.1736$
- $\Rightarrow v = 0.4166\ 2c$

(b) Kinetic energy is $K = E - E_0 = 1.1E_0 - E_0 = 0.1E_0 = 0.1\, m_0c^2$

Reduction of relativistic kinetic energy to the classical limit

- The expression of the relativistic kinetic energy

$$K = m_0\gamma c^2 - m_0c^2$$

must reduce to that of classical one in the limit $v/c \rightarrow 0$, i.e.

$$\lim_{v<<c} K_{relativistic} = \frac{p^2{}_{classical}}{2m_0} \left(= \frac{m_0v^2}{2}\right)$$

Expand γ with binomial expansion

- For $v << c$, we can always expand γ in terms of $(v/c)^2$ as

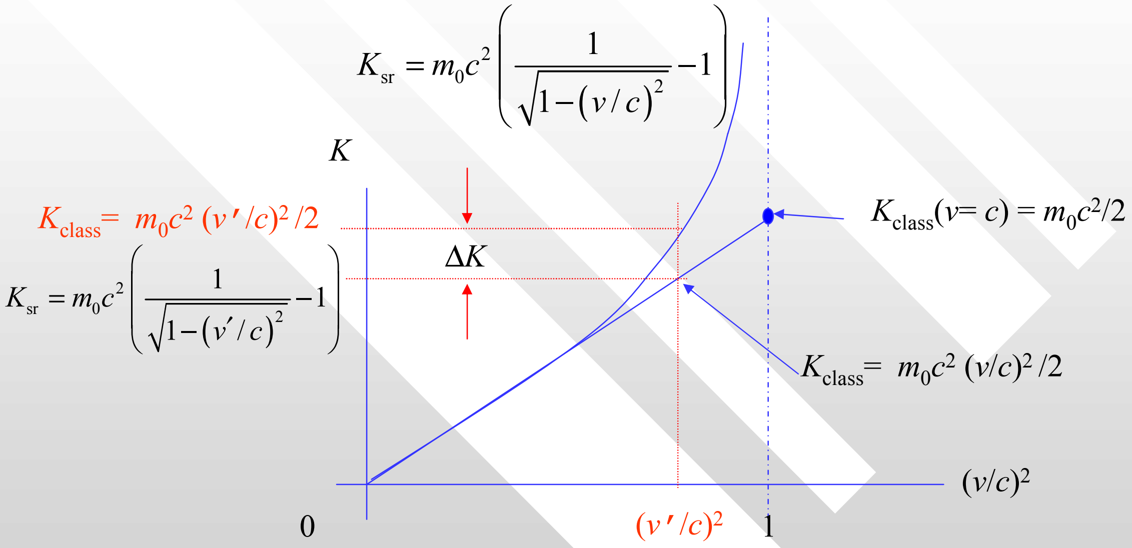
$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{v^2}{2c^2} + \text{terms of order } \frac{v^4}{c^4} \text{ and higher}$$

$$\begin{aligned} K &= mc^2 - m_0c^2 = m_0c^2(\gamma - 1) \\ &= m_0c^2\left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\right) - 1\right] \approx \frac{m_0v^2}{2} \end{aligned}$$

i.e., the relativistic kinetic energy reduces to classical expression in the $v << c$ limit

Exercise

- Plot K_{class} and K_{sr} vs $(v/c)^2$ on the same graph for $(v/c)^2$ between 0 and 1.
- Ask: In general, for a given velocity, does the classical kinetic energy of an moving object larger or smaller compared to its relativistic kinetic energy?
- In general does the discrepancy between the classical KE and relativistic KE increase or decrease as v gets closer to c?



Note that ΔK gets larger as $v \rightarrow c$

Recap

- Important formula for total energy, kinetic energy and rest energy as predicted by SR:

E = total relativisitic energy;

m_0 = rest mass;

m = relativistic mass;

E_0 = rest energy ;

p = relativistic momentum,

K = relativistic momentum;

$$m = \gamma m_0; p = \gamma m_0 v \;; K = \gamma m_0 c^2 - m_0 c^2; E_0 = m_0 c^2; E = \gamma m_0 c^2;$$

Example

- A microscopic particle such as a proton can be accelerated to extremely high speed of $v = 0.85c$ in the Tevatron at Fermi National Accelerator Laboratory, US.
- Find its total energy and kinetic energy in eV.

Solution

Due to mass-energy equivalence, sometimes we express the mass of an object in unit of energy

- Proton has rest mass $m_p = 1.67 \times 10^{-27}$kg
- The rest mass of the proton can be expressed as energy equivalent, via
- $m_p c^2 = 1.67\times10^{-31}\text{kg} \times (3 \times 10^8 \text{m/s})^2$
- $= 1.5\times10^{-10}$ J
- $= 1.5 \times 10^{-10} \times (1.6\text{x}10^{-19})^{-1}$ eV
- $= 939{,}375{,}000$ eV $= 939$ MeV

Solution

- First, find the Lorentz factor, $\gamma = 1.89$
- The rest mass of proton, m_0c^2, is 939 MeV
- Hence the total energy is

  $E = mc^2 = \gamma\,(m_0c^2) = 1.89 \times 939$ MeV $= 1774$ MeV
- Kinetic energy is the difference between the total relativistic energy and the rest mass,

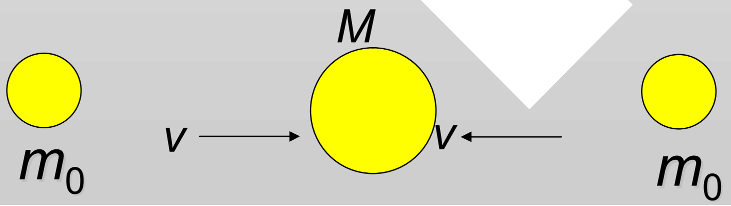
  $K = E - m_0c^2 = (1774 - 939)$ MeV $= 835$ MeV

Exercise

- Show that the rest mass of an electron is equivalent to 0.51 MeV

Conservation of Kinetic energy in relativistic collision

- Calculate (i) the kinetic energy of the system and (ii) mass increase for a completely inelastic head-on of two balls (with rest mass m_0 each) moving toward the other at speed $v/c = 1.5\times10^{-6}$ (the speed of a jet plane). M is the resultant mass after collision, assumed at rest.



Solution

- (i) $K = 2mc^2 - 2m_0c^2 = 2(\gamma-1)m_0c^2$
- (ii) $E_{\text{before}} = E_{\text{after}} \Rightarrow 2\gamma m_0c^2 = Mc^2 \Rightarrow M = 2\gamma m_0$
- Mass increase $\Delta M = M - 2m_0 = 2(\gamma-1)m_0$
- Approximation: $v/c = ...=1.5\text{x}10^{-6} \Rightarrow \gamma\approx 1 + ½\, v^2/c^2$ (binomail expansion) $\Rightarrow M \approx 2(1 + ½\, v^2/c^2)m_0$
- Mass increase $\Delta M = M - 2m_0$
- $\approx (v^2/c^2)m_0 = (1.5\text{x}10^{-6})^2 m_0$
- Comparing K with ΔMc^2: the kinetic energy is not lost in relativistic inelastic collision but is converted into the mass of the final composite object, i.e. kinetic energy is conserved
- In contrast, in classical mechanics, momentum is conserved but kinetic energy is not in an inelastic collision

Relativistic momentum and relativistic Energy

In terms of relativistic momentum, the relativistic total energy can be expressed as followed

$$E^2 = \gamma^2 m_0^2 c^4; p^2 = \gamma^2 m_0^2 v^2 \Rightarrow \frac{v^2}{c^2} = \frac{c^2 p^2}{E^2}$$

$$\Rightarrow E^2 = \gamma^2 m_0^2 c^4 = \frac{m_0^2 c^4}{1 - \frac{v^2}{c^2}} = \left(\frac{m_0^2 c^4 E^2}{E^2 - c^2 p^2} \right)$$

$$\boxed{E^2 = p^2 c^2 + m_0^2 c^4}$$

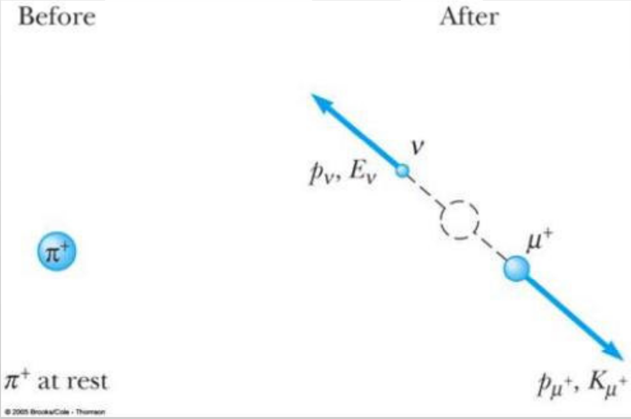
Conservation of energy-momentum

Invariance in relativistic dynamics

- Note that $E^2 - p^2c^2$ is an invariant, numerically equal to $(m_0c^2)^2$
- i.e., in any dynamical process, the difference between the total energy squared and total momentum squared of a given system must remain unchanged
- In additional, when observed in other frames of reference, the total relativistic energy and total relativistic momentum may have different values, but their difference, $E^2 - p^2c^2$, must remain invariant
- Such invariance greatly simplify the calculations in relativistic dynamics

Example: measuring pion mass using conservation of momentum-energy

- pi meson decays into a muon + massless neutrino
- If the mass of the muon is known to be 106 MeV/c^2, and the kinetik energy of the muon is measured to be 4.6 MeV, find the mass of the pion



Solution

Relationship between Kinetic energy and momentum:

$$E_\mu^2 = p_\mu^2 c^2 + m_\mu^2 c^4$$

Conservation of relativistic energy: $E_\pi = E_\mu + E_\nu$

$$\Rightarrow m_\pi c^2 = \sqrt{m_\mu^2 c^4 + c^2 p_\mu{}^2} + \sqrt{\cancel{m_\nu}^2 c^4 + c^2 p_\nu{}^2}$$

$$\Rightarrow m_\pi c = \sqrt{m_\mu^2 c^2 + p_\mu{}^2} + p_\mu$$

Momentum conservation: $p_\mu = p_\nu$

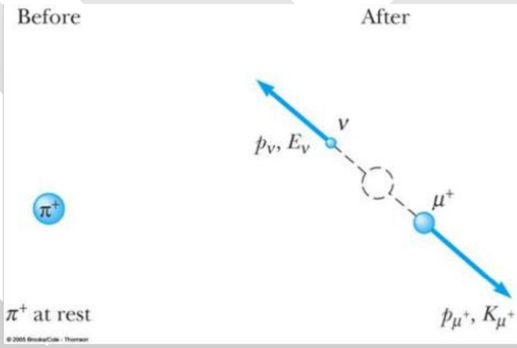
Also, total energy = K.E. + rest energy

$$E_\mu = K_\mu + m_\mu c^2 \Rightarrow E_\mu^2 = \left(K_\mu + m_\mu c^2\right)^2$$

But $E_\mu^2 = p_\mu^2 c^2 + m_\mu^2 c^4$

$$\Rightarrow E_\mu^2 = p_\mu^2 c^2 + m_\mu^2 c^4 = \left(K_\mu + m_\mu c^2\right)^2;$$

$$p_\mu c = \sqrt{\left(K_\mu + m_\mu c^2\right)^2 - m_\mu^2 c^4}$$



Plug $p_\mu^2 c^2 = \left(K_\mu + m_\mu c^2\right)^2 - m_\mu^2 c^4$ into

$$m_\pi c^2 = \sqrt{m_\mu^2 c^4 + c^2 {p_\mu}^2} + c p_\mu$$

$$= \sqrt{{m_\mu}^2 c^4 + \left[\left(K_\mu + m_\mu c^2\right)^2 - m_\mu^2 c^4\right]} + \sqrt{\left(K_\mu + m_\mu c^2\right)^2 - m_\mu^2 c^4}$$

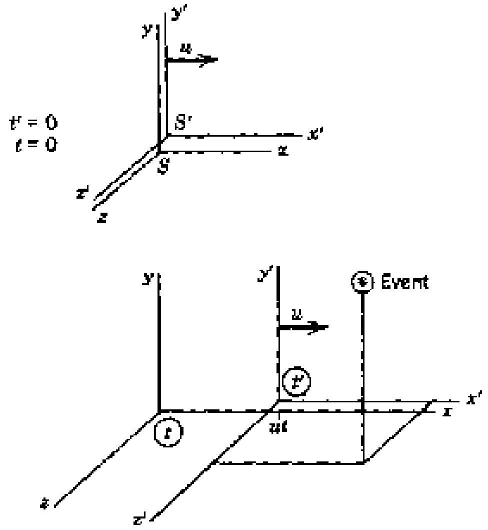
$$= \left(K_\mu + m_\mu c^2\right) + \sqrt{\left(K_\mu^2 + 2K_\mu m_\mu c^2\right)}$$

$$= \left(4.6\text{MeV} + \frac{106\text{MeV}}{c^2}c^2\right) + \sqrt{(4.6\text{MeV})^2 + 2(4.6\text{MeV})\left(\frac{106\text{MeV}}{c^2}\right)c^2}$$

$$= 111\text{MeV} + \sqrt{996}\text{MeV} = 143\text{MeV}$$

Observing an event from lab frame and rocket frame

Figure 1-13
Reference systems S and S' in relative motion. An event occurs at (x, y, z, t) in S and (x', y', z', t') in S'. In this view, S' is moving through S.



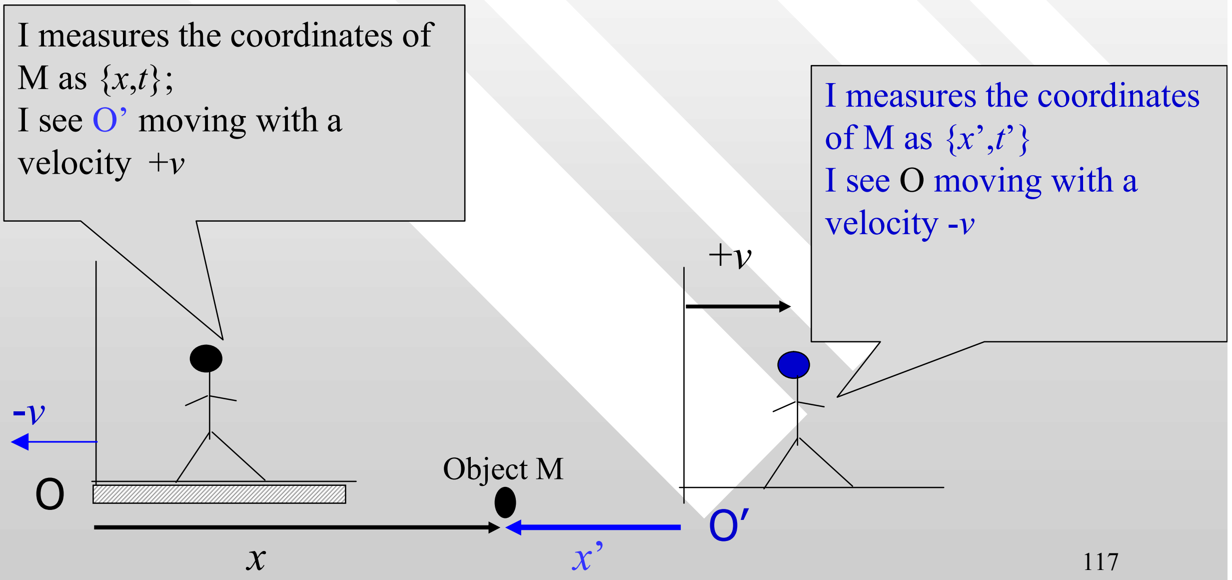
Lorentz Transformation

- All inertial frames are equivalent
- Hence all physical processes analysed in one frame can also be analysed in other inertial frame and yield consistent results
- Any event observed in two frames of reference must yield consistent results related by transformation laws
- Specifically such a transformation law is required to related the space and time coordinates of an event observed in one frame to that observed from the other

Different frame uses different notation for coordinates

- O' frame uses $\{x',y',z';t'\}$ to denote the coordinates of an event, whereas O frame uses $\{x,y,z;t\}$
- How to related $\{x',y',z',t'\}$ to $\{x,y,z;t\}$?
- In Newtonian mechanics, we use Galilean transformation

Two observers in two inertial frames with relative motion use different notation

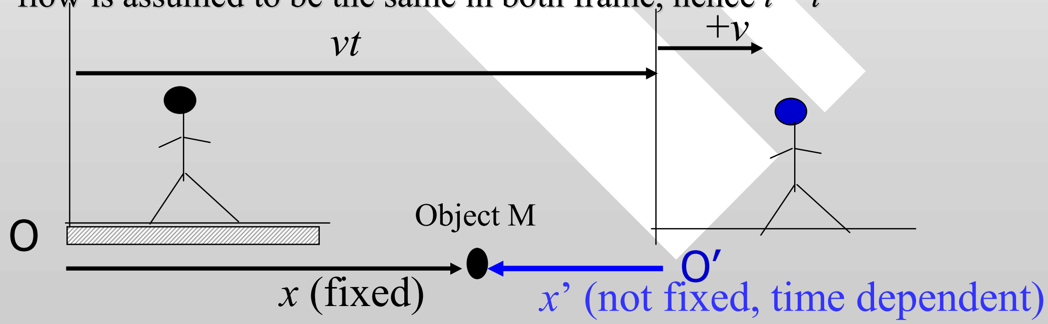


Galilean transformation (applicable only for $v<<c$)

- For example, the spatial coordinate of the object M as observed in O is x and is being observed at a time t, whereas according to O', the coordinate for the space and time coordinates are x' and t'. At low speed $v<<c$, the transformation that relates $\{x',t'\}$ to $\{x,t\}$ is given by Galilean transformation
- $\{x'=x-vt,\ t'=t\}$ (x' and t' in terms of x,t)
- $\{x=x'+vt,\ t=t'\}$ (x and t in terms of x',t')

Illustration on Galilean transformation of $\{x' = x - vt,\ t' = t\}$

- Assume object M is at rest in the O frame, hence the coordinate of the object M in O frame is fixed at x
- Initially, when $t = t' = 0$, O and O' overlap
- O' is moving away from O at velocity $+v$
- The distance of the origin of O' is increasing with time. At time t (in O frame), the origin of O' frame is at an instantaneous distance of $+vt$ away from O
- In the O' frame the object M is moving away with a velocity $-v$ (to the left)
- Obviously, in O' frame, the coordinate of the object M, denoted by x', is time-dependent, being $x' = x - vt$
- In addition, under current assumption (i.e. classical viewpoint) the rate of time flow is assumed to be the same in both frame, hence $t = t'$



However, GT contradicts the SR postulate when v approaches the speed of light, hence it has to be supplanted by a relativistic version of transformation law when near-to-light speeds are involved:

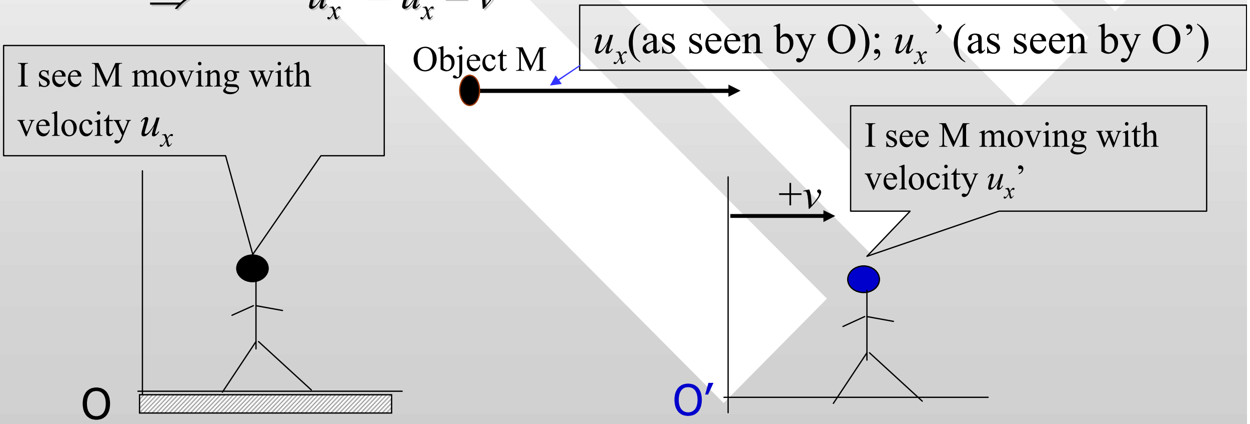
Lorentz transformation

(The contradiction becomes more obvious if GT on velocities, rather than on space and time, is considered)

Galilean transformation of velocity (applicable only for $u_x, v << c$)

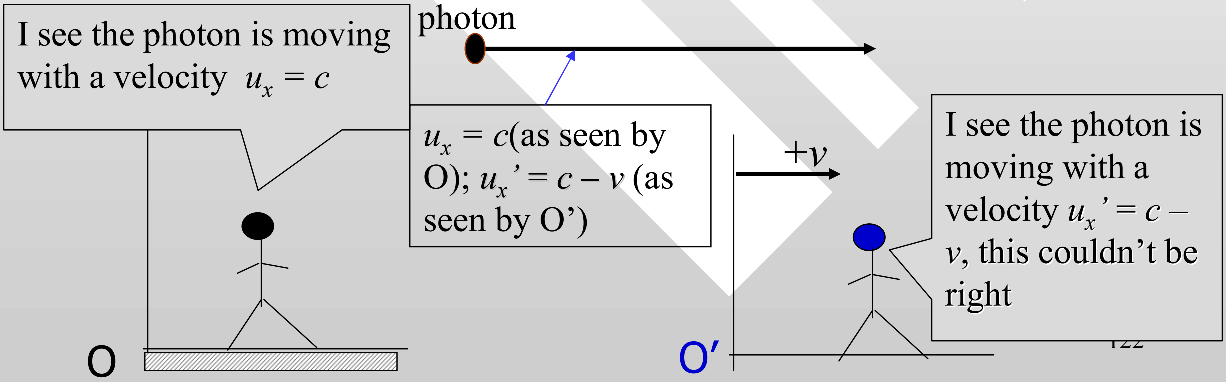
- Now, say object M is moving as a velocity of v wrp to the lab frame O
- What is the velocity of M as measured by O'?
- Differentiate $x' = x - vt$ wrp to t ($=t'$), we obtain
- $$d(x')/dt' = d(x - vt)/dt = d(x)/dt - v$$

$$\Rightarrow \quad u_x' = u_x - v$$



If applied to light Galilean transformation of velocity contradicts the SR Postulate

- Consider another case, now, a photon (particle of light) observed from different frames
- A photon. being a massless particle of light must move at a speed $u_x = c$ when observed in O frame
- However Galilean velocity addition law $u_x' = u_x - v$, if applied to the photon, says that in O' frame, the photon shall move at a lower speed of $u_x' = u_x - v = c - v$
- This is a contradiction to the constancy of light speed in SR



Conclusion

- GT (either for spatial, temporal or velocity) cannot be applicable when dealing with object moving near or at the speed of light
- It has to be supplanted by a more general form of transformation – Lorentz transformation, LT
- LT must reduce to GT when $v << c$.

Derivation of Lorentz transformation

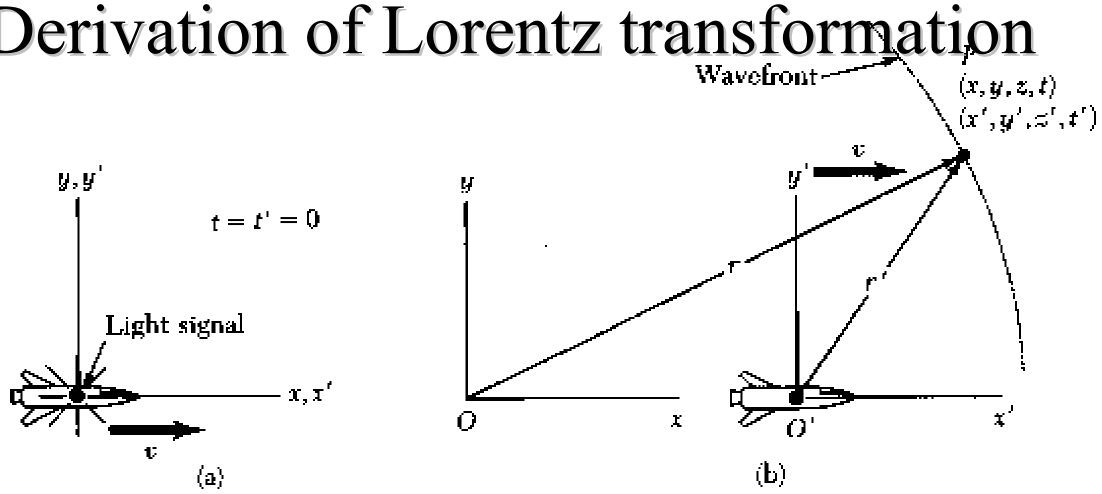


Figure 1.13 A rocket moves with a speed v along the xx' axes. (a) A pulse of light is sent out from the rocket at $t = t' = 0$ when the two systems coincide. (b) Coordinates of some point P on an expanding spherical wavefront as measured by observers in both inertial systems. (This figure is entirely schematic, and you should not be misled by the geometry.)

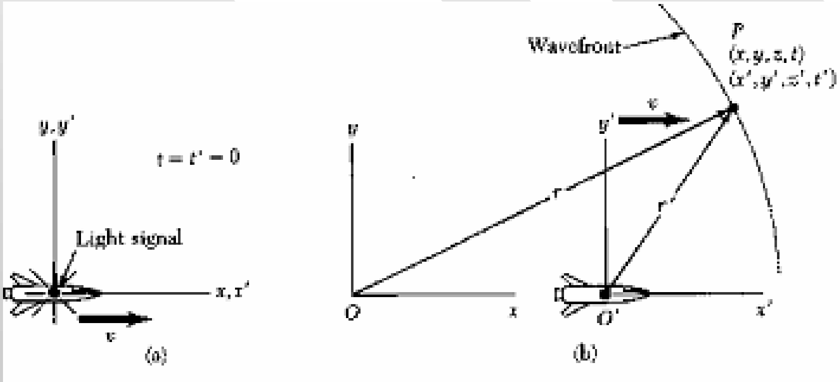
Our purpose is to find the general transformation that relates $\{x,t\}$ to $\{x',t'\}$ (applicable not only for $v<<c$ but also for $v \approx c$)

Read derivation of LT from the texts

- Brehm and Mullin
- Krane
- Serway, Mayer and Mosses

Derivation of Lorentz transformation

- Consider a rocket moving with a speed v (O' frame) along the xx' direction wrp to the stationary O frame
- A light pulse is emitted at the instant $t' = t = 0$ when the two origins of the two reference frames coincide
- The light signal travels as a spherical wave at a constant speed c in both frames
- After in time interval of t, the origin of the wave centred at O has a radius $r = ct$, where $r^2 = x^2 + y^2 + z^2$



Arguments

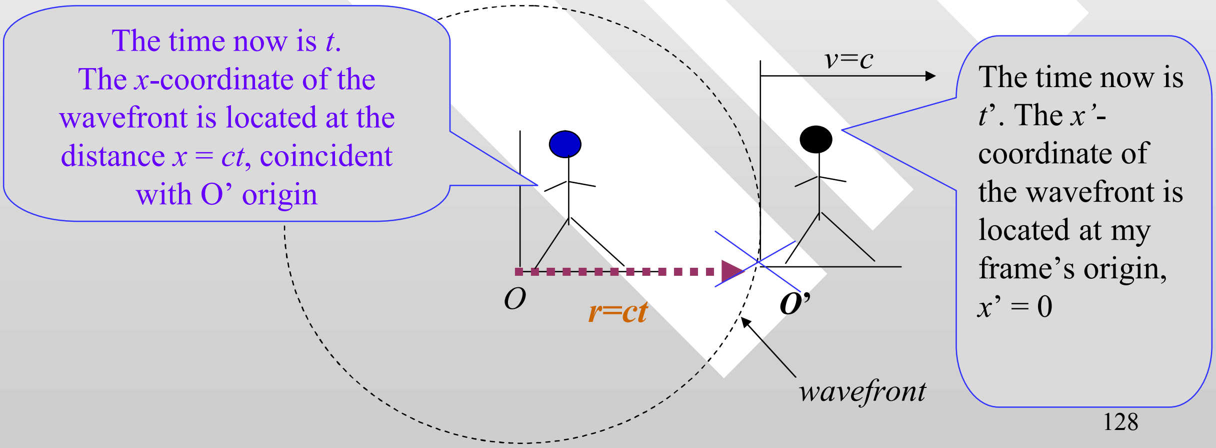
- From the view point of O', after an interval t' the origin of the wave, centred at O' has a radius:

$$r' = ct', \; (r')^2 = (x')^2 + (y')^2 + (z')^2$$

- $y'=y$, $z'=z$ (because the motion of O' is along the xx') axis – no change for y,z coordinates (*condition A*)
- The transformation from x to x' (and vice versa) must be linear, i.e. $x' \propto x$ (*condition B*)
- Boundary condition (1): If $v = c$, from the viewpoint of O, the origin of O′ is located on the wavefront (to the right of O)
- $\Rightarrow x' = 0$ must correspond to $x = ct$
- Boundary condition (2): In the same limit, from the viewpoint of O′, the origin of O is located on the wavefront (to the left of O′)
- $\Rightarrow x = 0$ corresponds to $x' = -ct'$
- Putting everything together we assume the transformation that relates x' to $\{x, t\}$ takes the form $x' = k(x - ct)$ as this will fulfill all the conditions (B) and boundary condition (1) ; (k some proportional constant to be determined)
- Likewise, we assume the form $x = k(x' + ct')$ to relate x to $\{x', t'\}$ as this is the form that fulfill all the conditions (B) and boundary condition (2)

Illustration of Boundary condition (1)

- $x = ct$ ($x' = ct'$) is defined as the x-coordinate (x'-coordinate) of the wavefront in the O (O') frame
- Now, we choose O as the rest frame, O' as the rocket frame. Furthermore, assume O' is moving away to the right from O with light speed, i.e. $v = +c$
- Since $u = c$, this means that the wavefront and the origin of O' coincides all the time
- For O, the x-coordinate of the wavefront is moving away from O at light speed; this is tantamount to the statement that $x = ct$
- From O' point of view, the x'-coordinate of the wavefront is at the origin of it's frame; this is tantamount to the statement that $x' = 0$
- Hence, in our yet-to-be-derived transformation, $x' = 0$ must correspond to $x = ct$

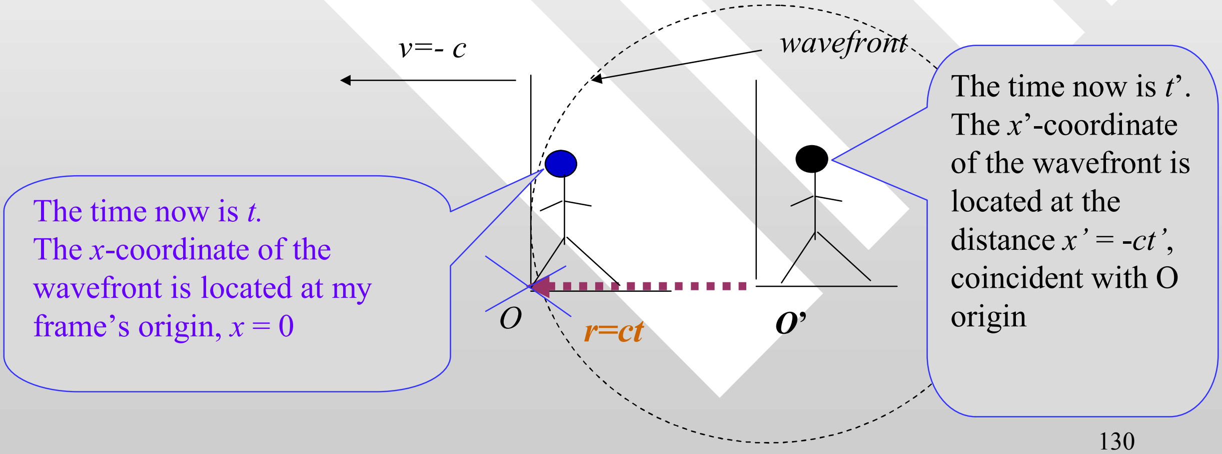


Permuting frames

- Since all frames are equivalent, physics analyzed in O' frame moving to the right with velocity $+v$ is equivalent to the physics analyzed in O frame moving to the left with velocity $-v$
- Previously we choose O frame as the lab frame and O' frame the rocket frame moving to the right (with velocity $+v$ wrp to O)
- Alternatively, we can also fix O' as the lab frame and let O frame becomes the rocket frame moving to the left (with velocity $-v$ wrp to O')

Illustration of Boundary condition (2)

- Now, we choose O' as the rest frame, O as the rocket frame. From O' point of view, O is moving to the left with a relative velocity $v = -c$
- From O' point of view, the wavefront and the origin of O coincides. The x'-coordinate of the wavefront is moving away from O' at light speed to the left; this is tantamount to the statement that $x' = -ct'$
- From O point of view, the x-coordinate of the wavefront is at the origin of it's frame; this is tantamount to the statement that $x = 0$
- Hence, in our yet-to-be-derived transformation, $x = 0$ must correspond to $x' = -ct'$



Finally, the transformation obtained

- We now have
- $r = ct$, $r^2 = x^2 + y^2 + z^2$; $y' = y$, $z' = z$; $x = k(x' + ct')$;
- $r' = ct'$, $r'^2 = x'^2 + y'^2 + z'^2$; $x' = k(x - ct)$;
- With some algebra, we can solve for $\{x', t'\}$ in terms of $\{x, t\}$ to obtain the desired transformation law (do it as an exercise)
- The constant k turns out to be identified as the Lorentz factor, γ

$$x' = \frac{x - vt}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma(x - vt) \qquad t' = \frac{t - \left(v/c^2\right)x}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma\left[t - (v/c^2)x\right]$$

(x' and t' in terms of x, t)

Space and time now becomes state-of-motion dependent (via γ)

- Note that, now, the length and time interval measured become dependent of the state of motion (in terms of γ) − in contrast to Newton's classical viewpoint
- **Lorentz transformation reduces to Galilean transformation when $v << c$ (show this yourself)**
- **i.e. LT → GT in the limit $v << c$**

How to express $\{x, t\}$ in terms of $\{x', t'\}$?

- We have expressed $\{x', t'\}$ in terms of $\{x, t\}$ as per

$$x' = \frac{x - vt}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma(x - vt) \qquad t' = \frac{t - \left(v/c^2\right)x}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma\left[t - (v/c^2)x\right]$$

- Now, how do we express $\{x, t\}$ in terms of $\{x', t'\}$?

Simply permute the role of x and x' and reverse the sign of v

$$x \leftrightarrow x', v \rightarrow -v$$

$$x' = \gamma(x - vt) \rightarrow x = \gamma(x' + vt')$$

$$t' = \gamma\left[t - (v/c^2)x\right] \rightarrow t = \gamma\left[t' + (v/c^2)x'\right]$$

The two transformations above are equivalent; use which is appropriate in a given question

Length contraction

- Consider the rest length of a ruler as measured in frame O′ is
- $L' = \Delta x' = x'_2 - x'_1$ (proper length) measured at the same instance in that frame ($t'_2 = t'_1$)
- What is the length of the rule as measured by O?
- The length in O, L, according the LT can be deduced as followed:

  $L' = \Delta x' = x'_2 - x'_1 = \gamma[(x_2 - x_1) - v(t_2 - t_1)]$
- The length of the ruler in O, L, is simply the distance btw x_2 and x_1 measured at the same instance in that frame ($t_2 = t_1$). Hence we have
  - $L' = \gamma L$
- where L = is the improper length.
- As a consequence, we obtain the relation between the proper length measured by the observer at rest wrp to the ruler and that measured by an observer who is at a relative motion wrp to the ruler:

$$L' = \gamma L$$

Moving rulers appear shorter

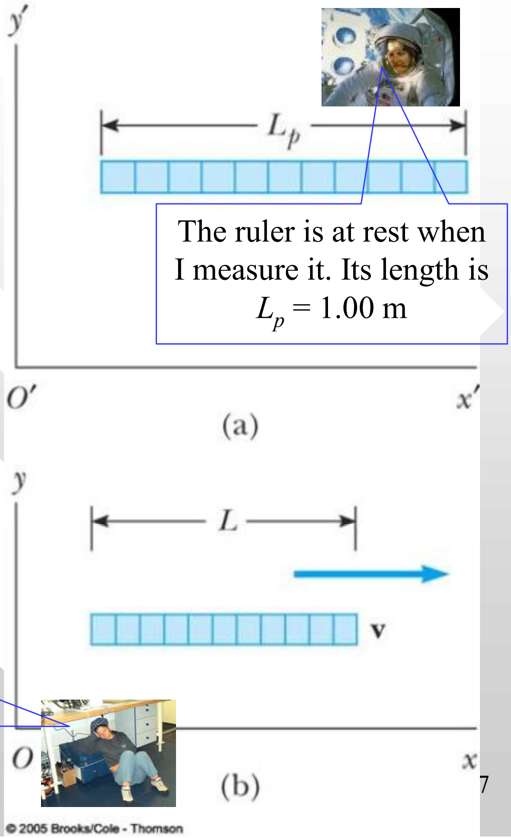
$$L' = \gamma L$$

- L' is defined as the proper length = length of and object measured in the frame in which the object is at rest
- L is the length measured in a frame which is moving wrp to the ruler
- If an observer at rest wrp to an object measures its length to be L', an observer moving with a relative speed u wrp to the object will find the object to be shorter than its rest length by a factor $1 / \gamma$
- i.e., the length of a moving object is measured to be shorter than the proper length – hence "length contraction"
- In other words, a moving rule will appear shorter!!

Example of a moving ruler

Consider a meter rule is carried on board in a rocket (call the rocket frame O′)

- An astronaut in the rocket measure the length of the ruler. Since the ruler is at rest wrp to the astronaut in O′, the length measured by the astronaut is the proper length, L_p = 1.00 m, see (a)
- Now consider an observer on the lab frame on Earth. The ruler appears moving when viewed by the lab observer. If the lab observer attempts to measure the ruler, the ruler would appear shorter than 1.00m



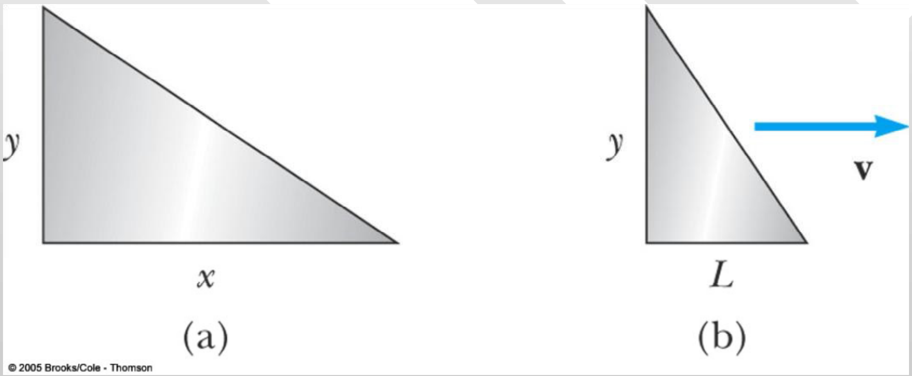
The ruler is at moving at a speed v when I measure it. Its length is L = 0.999 m

RE 38-11

- What is the speed v of a passing rocket in the case that we measure the length of the rocket to be half its length as measured in a frame in which the rocket is at rest?

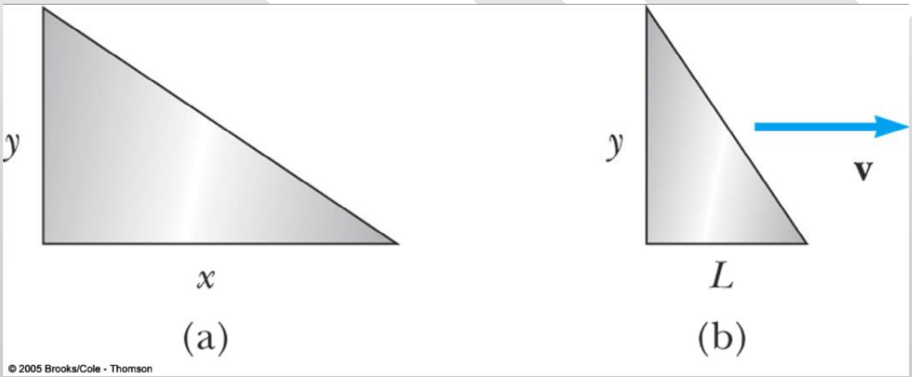
Length contraction only happens along the direction of motion

Example: A spaceship in the form of a triangle flies by an oberver at rest wrp to the ship (see fig (a)), the distance x and y are found to be 50.0 m and 25.0 m respectively. What is the shape of the ship as seen by an observer who sees the ship in motion along the direction shown in fig (b)?



Solution

- The observer sees the horizontal length of the ship to be contracted to a length of
- $L = L_p/\gamma = 50 \text{ m} \sqrt{(1 - 0.950^2)} = 15.6 \text{ m}$
- The 25 m vertical height is unchanged because it is perpendicular to the direction of relative motion between the observer and the spaceship.



Similarly, one could also derive time dilation from the LT

Do it as homework

- Mr. Thompkins's in ….

What is the velocities of the ejected stone?

- Imagine you ride on a rocket moving ¾ c wrp to the lab. From your rocket you launch a stone forward at ½ c, as measured in your rocket frame. What is the speed of the stone observed by the lab observer?

¾ ***c*,** wrp to lab

½ c, wrp to the rocket

The speed of the stone as I measure it is…

Adding relativistic velocities using Galilean transformation

- According to GT of velocity (which is valid at low speed regime $v << c$),
- the lab observer would measure a velocity of $u_x = u_x' + v = ½\, c + ¾\, c$
- $= 1.25\, c$ for the ejected stone.
- However, in SR, c is the ultimate speed and no object can ever exceed this ultimate speed limit
- So something is no right here…Galilean addition law is no more valid to handle addition of relativistic velocities (i.e. at speed near to c)

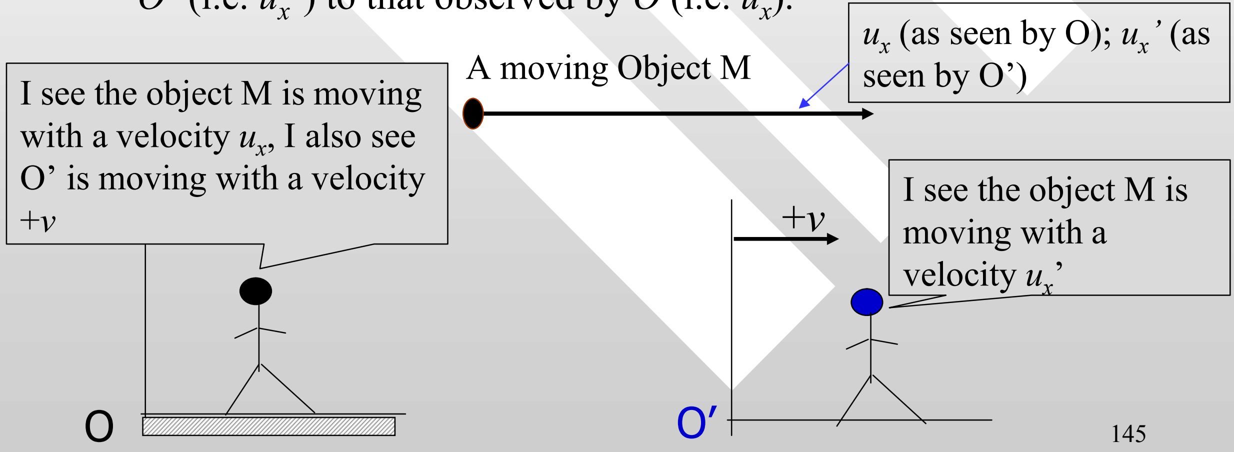
¾ ***c*,** wrp to lab

½ c, wrp to the rocket

If I use GT, the speed of the stone as is 1.25 c!!! It couldn't be right

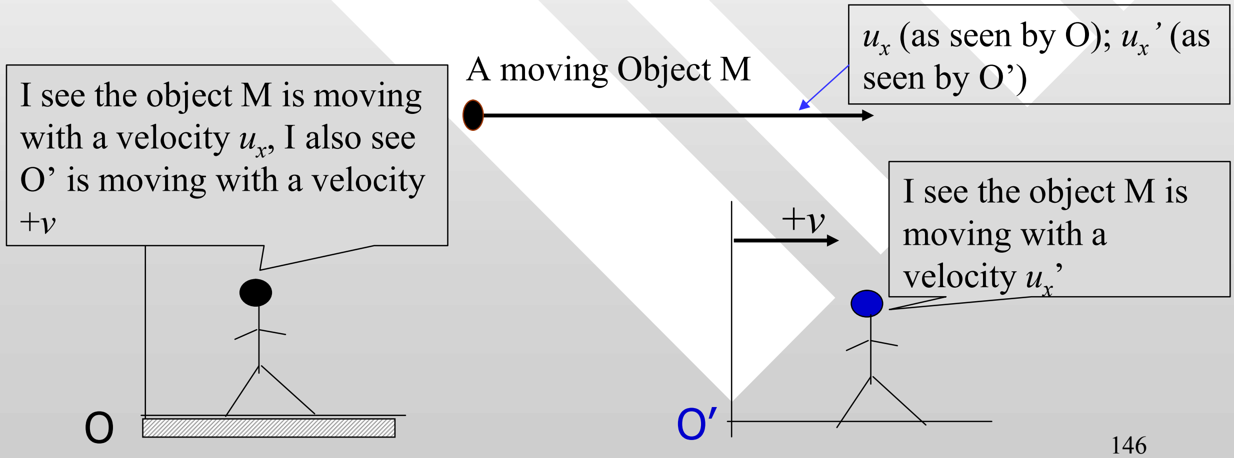
Relativity of velocities

- The generalised transformation law of velocity used for addition of relativistic velocities is called Lorentz transformation of velocities, derived from the Lorentz transformation of space-time
- Our task is to relate the velocity of the object M as observed by O' (i.e. u_x') to that observed by O (i.e. u_x).



Relativity of velocities

- Consider an moving object being observed by two observers, one in the lab frame and the other in the rocket frame
- We could derive the Lorentz transformation of velocities by taking time derivative wrp to the LT for space-time, see next slide



Derivation of Lorentz transformation of velocities

- By definition, $u_x = dx/dt$, $u'_x = dx'/dt'$
- The velocity in the O′ frame can be obtained by taking the differentials of the Lorentz transformation

$$x' = \gamma(x - vt) \quad t' = \gamma\left[t - (v/c^2)x\right]$$

$$dx' = \gamma(dx - vdt),\ dt' = \gamma(dt - \frac{v}{c^2}dx)$$

Combining

$$u'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{dt\left(\frac{dx}{dt} - v\frac{dt}{dt}\right)}{dt\left(\frac{dt}{dt} - \frac{v}{c^2}\frac{dx}{dt}\right)}$$

$$= \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

where we have made used of the definition $u_x = dx/dt$

Comparing the LT of velocity with that of GT

Lorentz transformation of velocity:

$$u'_x = \frac{dx'}{dt'} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

Galilean transformation of velocity:

$$u'_x = u_x - v$$

LT reduces to GT in the limit $u_x v << c^2$

- *Please try to make a clear distinction among the definitions of various velocities, i.e. u_x, u'_x, v so that you wont get confused*

LT is consistent with the constancy of speed of light

- In either O or O′ frame, the speed of light seen must be the same, *c*. LT is consistent with this requirement.
- Say object M is moving with speed of light as seen by O, i.e. $u_x = c$
- According to LT, the speed of M as seen by O′ is

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{c - v}{1 - \frac{cv}{c^2}} = \frac{c - u}{1 - \frac{v}{c}} = \frac{c - v}{\frac{1}{c}(c - v)} = c$$

- That is, in either frame, both observers agree that the speed of light they measure is the same, $c = 3 \times 10^8$m/s

How to express u_x in terms of u_x'?

- Simply permute v with $-v$ and change the role of u_x with that of u_x':

$$u_x \to u'_x, u'_x \to u_x, v \to -v$$

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \to u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}$$

Recap: Lorentz transformation relates $\{x', t'\} \leftrightarrow \{x, t\}$; $u'_x \leftrightarrow u_x$

$$x' = \gamma(x - vt) \qquad t' = \gamma\left[t - (v/c^2)x\right]$$

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$x = \gamma(x' + vt') \qquad t = \gamma\left[t' + (v/c^2)x'\right]$$

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}$$

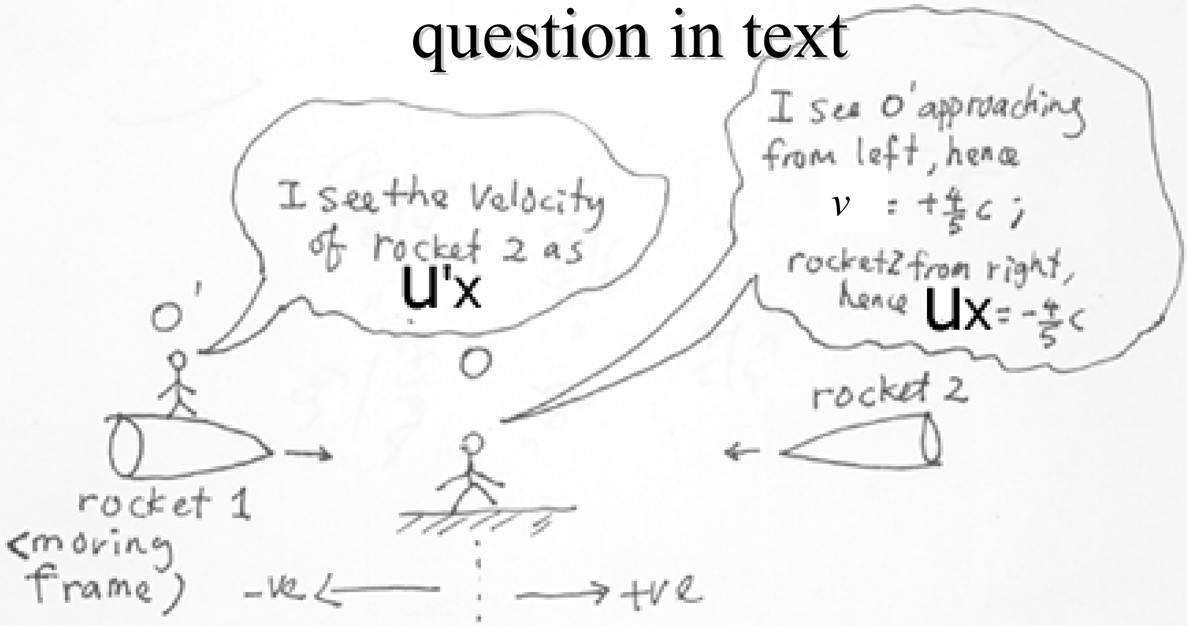
RE 38-12

- A rocket moves with speed 0.9*c* in our lab frame. A flash of light is sent toward from the front end of the rocket. Is the speed of that flash equal to 1.9 *c* as measured in our lab frame? If not, what is the speed of the light flash in our frame? Verify your answer using LT of velocity formula.

Example (relativistic velocity addition)

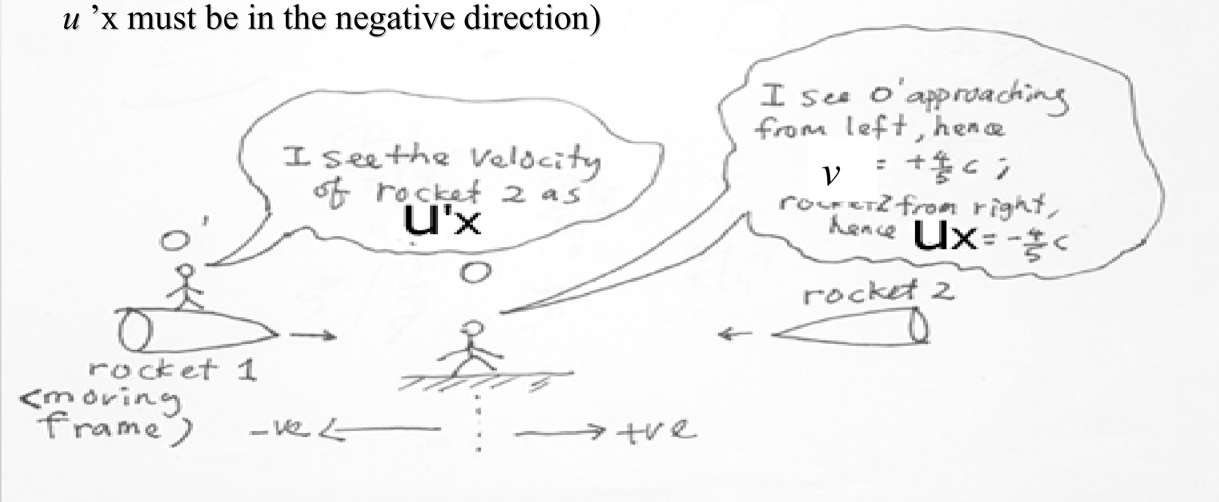
- Rocket 1 is approaching rocket 2 on a head-on collision course. Each is moving at velocity 4*c*/5 relative to an independent observer midway between the two. With what velocity does rocket 2 approaches rocket 1?

Diagramatical translation of the question in text

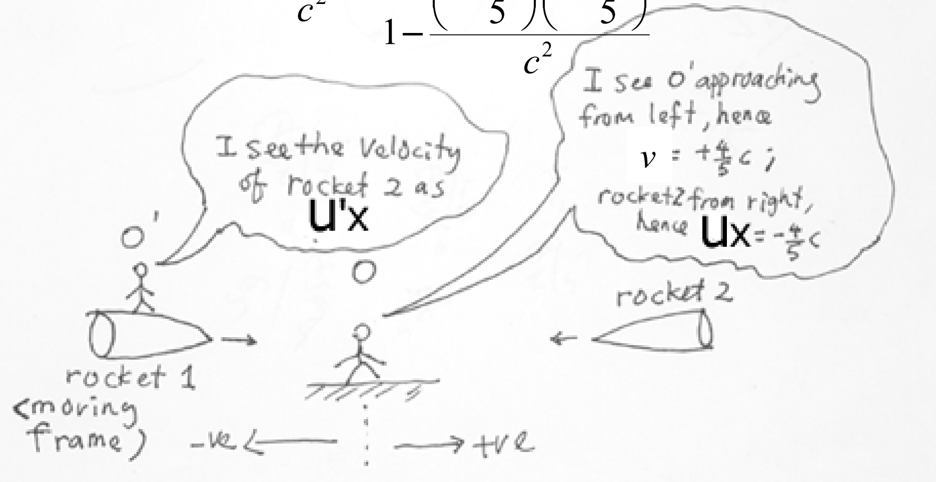


Note: c.f. in GT, their relative speed would just be 4c/5 + 4c/5 = 1.6 *c* which violates constancy of speed of light postulate. See how LT handle this situation:

- Choose the observer in the middle as in the stationary frame, O
- Choose rocket 1 as the moving frame O’
- Call the velocity of rocket 2 as seen from rocket 1 *u*’x. This is the quantity we are interested in
- Frame O' is moving in the +ve direction as seen in O, so $v = +4c/5$
- The velocity of rocket 2 as seen from O is in the
- -ve direction, so $ux = -4c/5$
- Now, what is the velocity of rocket 2 as seen from frame O', $u'x$ = ? (intuitively, u ’x must be in the negative direction)



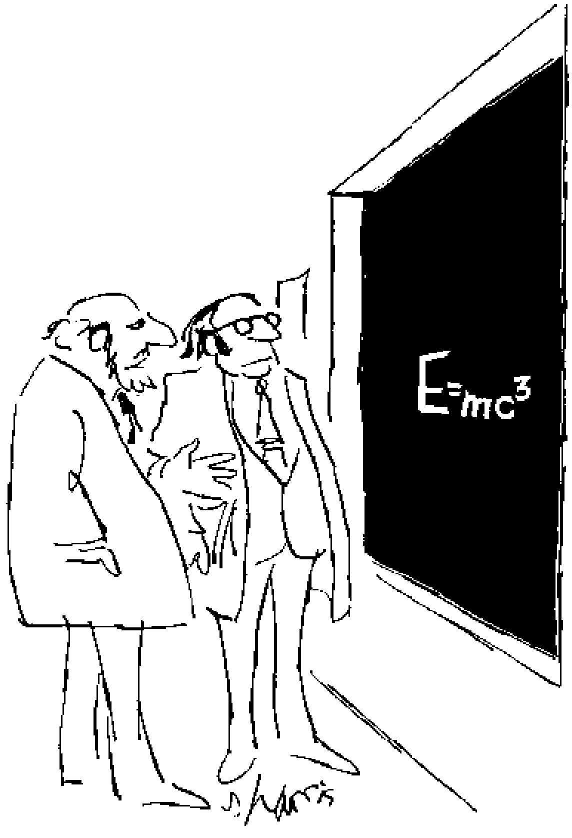
Using LT:
$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{\left(-\frac{4c}{5}\right) - \left(+\frac{4c}{5}\right)}{1 - \frac{\left(-\frac{4c}{5}\right)\left(+\frac{4c}{5}\right)}{c^2}} = -\frac{40}{41}c$$



i.e. the velocity of rocket 2 as seen from rocket 1 (the moving frame, O’) is $-40c/41$, which means that O’ sees rocket 2 moving in the –ve direction (to the left in the picture), as expected.

Doppler Shift

- R.I.Y



"These days *everything* is higher."