ZCT 104 (Class A)

MODERN PHYSICS

ACADEMIC SESSION 2008/09
(SECOND SEMESTER)

LECTURE NOTES,
TUOTRIAL PROBLEM SET
And
PAST YEAR QUESTIONS

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POWERPOINT
LECTURE NOTES

SESSI 2008/09

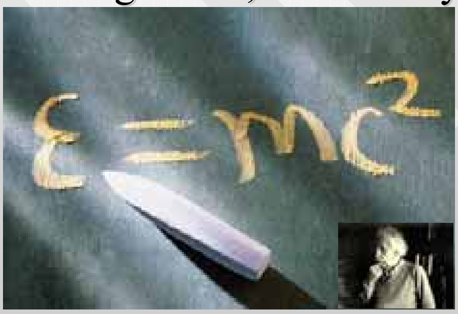
SEMESTER II

Special theory of Relativity

Notes based on

Understanding Physics

by Karen Cummings et al., John Wiley & Sons



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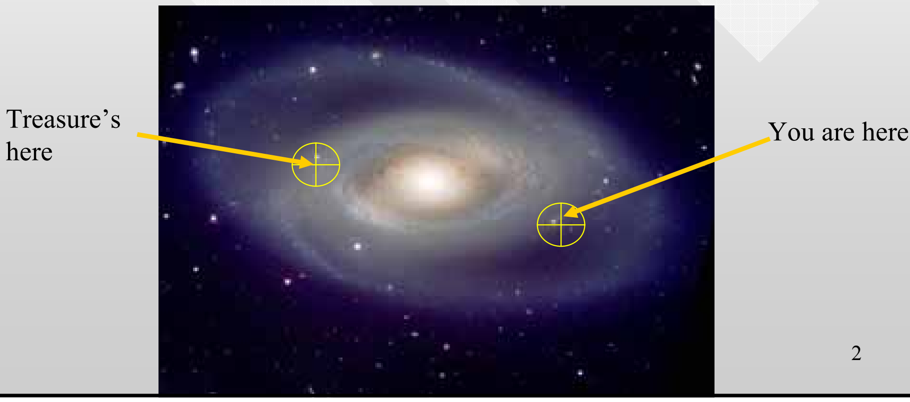
An open-chapter question

- Let say you have found a map revealing a huge galactic treasure at the opposite edge of the Galaxy 200 ly away.
- Is there any chance for you to travel such a distance from Earth and arrive at the treasure site by traveling on a rocket within your lifetime of say, 60 years, given the constraint that the rocket cannot possibly travel faster than the light speed?



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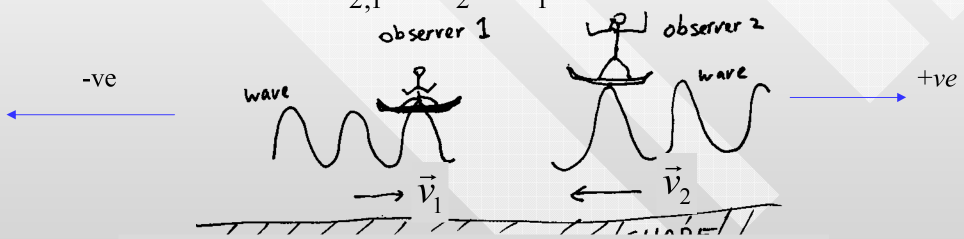
Relative motions at ordinary speed

- Relative motion in ordinary life is commonplace
- E.g. the relative motions of two cars (material objects) along a road
- When you observe another car from within your car, can you tell whether you are at rest or in motion if the other car is seen to be “moving?”

Relative motion of wave

- Another example: wave motion
- Speed of wave measured by Observer 1 on wave 2 depends on the speed of wave 1 wrp (with respect) to the shore: $\vec{v}_{2,1} = \vec{v}_2 - \vec{v}_1$



$$\vec{v}_1 = v_1\hat{x}; \qquad \vec{v}_2 = -v_2\hat{x}$$

$$\vec{v}_{1,2} = \vec{v}_1 - \vec{v}_2 = v_1\hat{x} - (-v_2\hat{x}) = (v_1 + v_2)\hat{x};$$

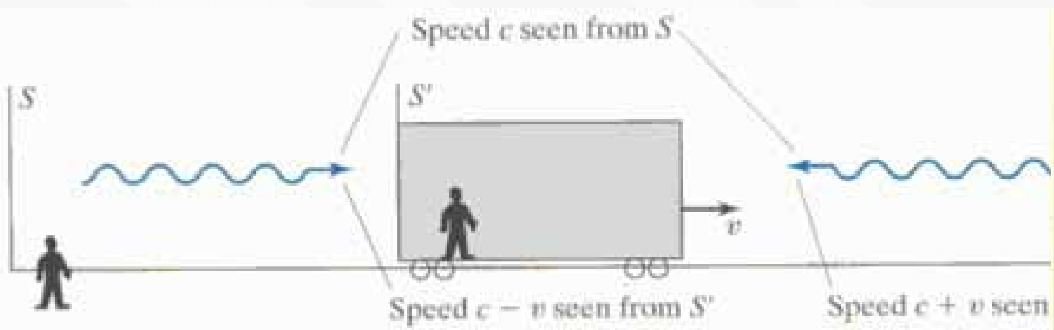
$$\vec{v}_{2,1} = \vec{v}_2 - \vec{v}_1 = (-v_2\hat{x}) - (v_1\hat{x}) = -(v_1 + v_2)\hat{x}$$

Query: can we surf light waves?

- Light is known to be wave
- If either or both wave 1 and wave 2 in the previous picture are light wave, do they follow the addition of velocity rule too?
- Can you surf light wave ? (if so light shall appear *at rest* to you then)

5

In other word, does light wave follows Galilean law of addition of velocity?

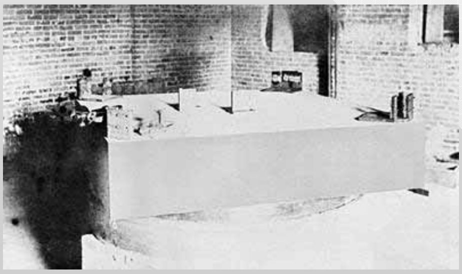


Frame S' travels with velocity *v* relative to S. If light waves obey Galilean laws of addition velocity, the speeds of the two opposite light waves would be different as seen by S'. But does light really obey Galilean law of addition of velocity?

6

The negative result of Michelson-Morley experiment on Ether

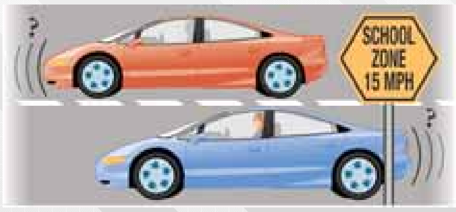
- In the pre-relativity era, light is thought to be propagating in a medium called ether -
- an direct analogy to mechanical wave propagating in elastic medium such as sound wave in air
- If exist, ether could render measurable effect in the apparent speed of light in various direction
- However Michelson-Morley experiment only find negative result on such effect
- A great puzzlement to the contemporary physicist: what does light wave move relative to?



7

How could we know whether we are at rest or moving?

- Can we cover the windows of our car and carry out experiments inside to tell whether we are at rest or in motion?
- NO



8

In a "covered" reference frame, we can't tell whether we are moving or at rest

- Without referring to an external reference object (such as a STOP sign or a lamp post), whatever experiments we conduct in a constantly moving frame of reference (such as a car at rest or a car at constant speed) could not tell us the state of our motion (whether the reference frame is at rest or is moving at constant velocity)

《尚書經．考靈曜》

- 「地恒動不止，而人不知，譬如人在大舟中，閉牖而坐，舟行而不覺也。」
- "The Earth is at constant state of motion yet men are unaware of it, as in a simile: if one sits in a boat with its windows closed, he would not aware if the boat is moving" in "*Shangshu jing*", 200 B.C

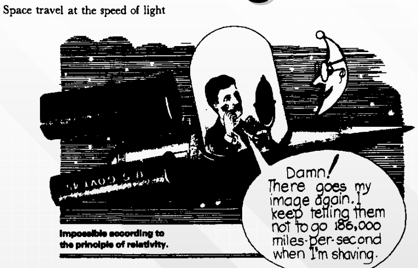
Physical laws must be invariant in any reference frame

- Such an inability to deduce the state of motion is a consequence of a more general principle:
- There must be no any difference in the physical laws in any reference frame with constant velocity
- (which would otherwise enable one to differentiate the state of motion from experiment conducted in these reference frame)
- Note that a reference frame at rest is a special case of reference frame moving at constant velocity ($v = 0$ = constant)

The Principle of Relativity

- All the laws of Physics are the same in every reference frame

Einstein's Puzzler about running fast while holding a mirror



- Says Principle of Relativity: Each fundamental constants must have the same numerical value when measured in any reference frame (*c, h, e,* m_e*, etc*)
- (Otherwise the laws of physics would predict inconsistent experimental results in different frame of reference – which must not be according to the Principle)
- Light always moves past you with the same speed *c*, no matter how fast you run
- Hence: you will not observe light waves to slow down as you move faster

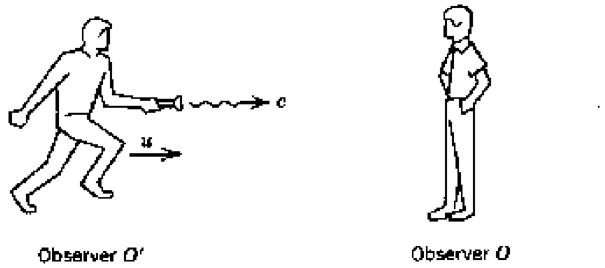
13

c, one of the fundamental constants of Nature

14

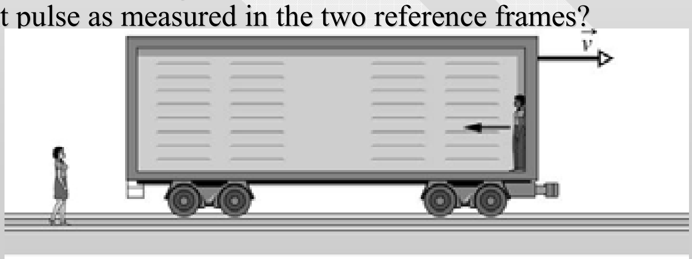
Constancy of the speed of light

Figure 1-2
Two observers in relative motion. O is at rest and O' moves toward O at constant speed u. O and O' agree on the speed of light coming from the source carried by O'.



Reading Exercise (RE) 38-2

- While standing beside a railroad track, we are startled by a boxcar traveling past us at half the speed of light. A passenger (shown in the figure) standing at the front of the boxcar fires a laser pulse toward the rear of the boxcar. The pulse is absorbed at the back of the box car. While standing beside the track we measure the speed of the pulse through the open side door.
- (a) Is our measured value of the speed of the pulse greater than, equal to, or less than its speed measured by the rider?
- (b) Is our measurement of the distance between emission and absorption of the light pulse great than, equal to, or less than the distance between emission and absorption measured by the rider?
- (c) What conclusion can you draw about the relation between the times of flight of the light pulse as measured in the two reference frames?



Touchstone Example 38-1: Communication storm!

- A sunspot emits a tremendous burst of particles that travels toward the Earth. An astronomer on the Earth sees the emission through a solar telescope and issues a warning. The astronomer knows that when the particle pulse arrives it will wreak havoc with broadcast radio transmission. Communications systems require ten minutes to switch from over-the-air broadcast to underground cable transmission. What is the maximum speed of the particle pulse emitted by the Sun such that the switch can occur in time, between warning and arrival of the pulse? Take the sun to be 500 light-seconds distant from the Earth.

17

Solution

- It takes 500 seconds for the warning light flash to travel the distance of 500 light-seconds between the Sun and the Earth and enter the astronomer's telescope. If the particle pulse moves at half the speed of light, it will take twice as long as light to reach the Earth. If the pulse moves at one-quarter the speed of light, it will take four times as long to make the trip. We generalize this by saying that if the pulse moves with speed v/c, it will take time to make the trip given by the expression:

  $\Delta t_{\text{pulse}} = 500\text{ s}/\ (v_{\text{pulse}}/c)$
- How long a warning time does the Earth astronomer have between arrival of the light flash carrying information about the pulse the arrival of the pulse itself? It takes 500 seconds for the light to arrive. Therefore the warning time is the difference between pulse transit time and the transit time of light:

  $\Delta t_{\text{warning}} = \Delta t_{\text{pulse}} - 500\text{ s}.$
- But we know that the minimum possible warning time is 10 min = 600 s.
- Therefore we have
- $600\text{ s} = 500\text{ s} / (v_{\text{pulse}}/c) - 500\text{ s},$
- which gives the maximum value for v_{puls} if there is to he sufficient time for warning:

  $v_{\text{puls}} = 0.455\, c.$ (Answer)
- Observation reveals that pulses of particles emitted from the sun travel much slower than this maximum value. So we would have much longer warning time than calculated here.

18

Relating Events is science

- Science: trying to relate one event to another event
- E.g. how the radiation is related to occurrence of cancer; how lightning is related to electrical activities in the atmosphere etc.
- Since observation of events can be made from different frames of reference (e.g. from an stationary observatory or from a constantly moving train), we must also need to know how to predict events observed in one reference frame will look to an observer in another frame

19

Some examples

- How is the time interval measured between two events observed in one frame related to the time interval measured in another frame for the same two events?
- How is the velocity of a moving object measured by a stationary observer and that by a moving observer related?

20

Defining events

- So, before one can work out the relations between two events, one must first precisely define what an event is

Locating Events

- ***An event is an occurrence that happens at a unique place and time***
- Example: a collision, and explosion, emission of a light flash
- An event must be sufficiently localised in space and time
- e.g. your birthday: you are born in the General Hospital PP at year 1986 1st April 12.00 am)

Example of two real-life events

Event 1: She said "I love you"

July 1Dec, 12.01:12 am, Tasik Aman

Event 2: She said "Let's break up-lah"

27 Dec 2005, 7.43:33 pm, Tasik Harapan

Subtle effect to locate an event: delay due to finiteness of light speed

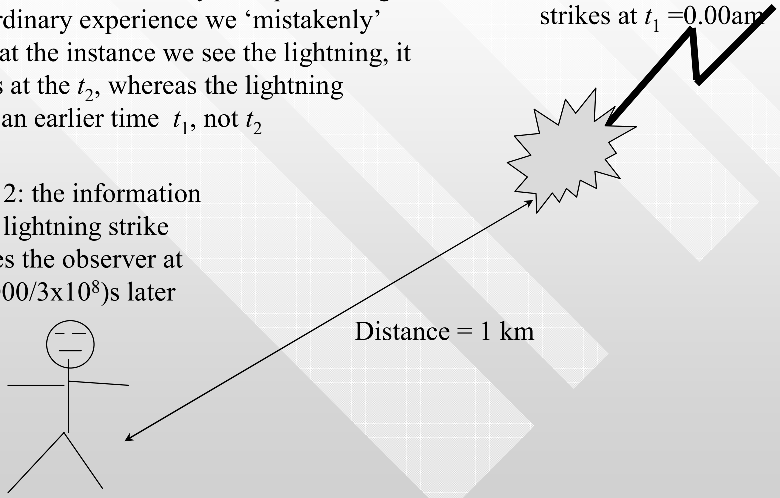
- In our (erroneous) "common sense" information are assumed to reach us instantaneously as though it is an immediate action through a distance without any delay
- In fact, since light takes finite time to travel, locating events is not always as simple it might seems at first

An illustrative example of delay while measuring an event far away

t_2 is very short due to the very fast speed of light c. In our ordinary experience we 'mistakenly' think that, at the instance we see the lightning, it also occurs at the t_2, whereas the lightning actually at an earlier time t_1, not t_2

Event 1: Lightning strikes at t_1 =0.00am

Event 2: the information of the lightning strike reaches the observer at t_2=(1000/3x10^8)s later



Reading Exercise 38-4

- When the pulse of protons passes through detector A (next to us), we start our clock from the time t = 0 microseconds. The light flash from detector B (at distance L=30 m away) arrives back at detector A at a time t = 0.225 microsecond later.
- (a) At what time did the pulse arrive at detector B?
- (b) Use the result from part (a) to find the speed at which the proton pulse moved, as a fraction of the speed of light.

Answer

- The time taken for light pulse to travel from B to A is $L/c = 10^{-7}$ s $=0.1\ \mu$s
- Therefore the proton pulse arrived detector B $0.225 - 0.1\ \mu\text{s} = 0.125\ \mu$s after it passed us at detector A.
- (b) The protons left detector A at t =0 and, according to part (a), arrived at detector B at t = 0.125 μs. Therefore its speed from A to B is $L/0.125\ \mu\text{s} = \ldots = 0.8c$

Redefining Simultaneity

- Hence to locate an event accurately we must take into account the factor of such time delay
- An intelligent observer is an observer who, in an attempt to register the time and spatial location of an event far away, takes into account the effect of the delay factor
- (In our ordinary daily life we are more of an unintelligent observer)
- For an intelligent observer, he have to redefine the notion of "simultaneity" (example 38-2)

Example 38-2: Simultaneity of the Two Towers

- Frodo is an intelligent observer standing next to Tower A, which emits a flash of light every l0 s. 100 km distant from him is the tower B, stationary with respect to him, that also emits a light flash every 10 s. Frodo wants to know whether or not each flash is emitted from remote tower B simultaneous with (at the same time as) the flash from Frodo's own Tower A. Explain how to do this with out leaving Frodo position next to Tower A. Be specific and use numerical values.

29

Solution

- Frodo is an intelligent observer, which means that he know how to take into account the speed of light in determining the time of a remote event, in this case the time of emission of a flash by the distant Tower B. He measures the time lapse between emission of a flash by his Tower A and his reception of flash from Tower B.
- If this time lapse is just that required for light move from Tower B to Tower A, then the two emissions occur the same time.
- The two Towers are 100 km apart. Call this distance L. Then the time t for a light flash to move from B to A is
- $t = L/c = 10^5$ m$/3 \times 10^8$ m/s $= 0.333$ ms. (ANS)
- If this is the time Frodo records between the flash nearby Tower A and reception of the flash from distant tower then he is justified in saying that the two Towers emit flashes simultaneously in his frame.

30

One same event can be considered in any frame of reference

- One same event, in principle, can be measured by many separate observers in different (inertial) frames of reference (reference frames that are moving at a constant velocity with respect to each other)
- Example: On the table of a moving train, a cracked pot is dripping water
- The rate of the dripping water can be measured by (1) Ali, who is also in the train, or by (2) Baba who is an observer standing on the ground. Furthermore, you too can imagine (3) ET is also performing the same measurement on the dripping water from Planet Mars. (4) By Darth Veda from Dead Star…

31

No 'superior' (or preferred) frame

- In other words, any event can be considered in infinitely many different frames of references.
- No particular reference frame is 'superior' than any other
- In the previous example, Ali's frame is in no way superior than Baba's frame, nor ET's frame, despite the fact that the water pot is stationary with respect to Ali.

32

Transformation laws

- Measurements done by any observers from all frame of reference are equally valid, and are all equivalent.
- Transformation laws such as Lorentz transformation can be used to relate the measurements done in one frame to another.
- In other words, once you know the values of a measurement in one frame, you can calculate the equivalent values as would be measured in other frames.
- In practice, the choice of frame to analyse any event is a matter of convenience.

Example 1

- In the previous example, obviously, the pot is stationary with respect to Ali, but is moving with respect to Baba.
- Ali, who is in the frame of the moving train, measures that the water is dripping at a rate of, say, r_A.
- Baba, who is on the ground, also measures the rate of dripping water, say r_B.
- Both of the rates measured by Ali and that measured by Baba have equal status – you can't say any one of the measurements is 'superior' than the other
- One can use Lorentz transformation to relate r_A with r_B. In reality, we would find that $r_B = r_A/\gamma$ where
- $1/\gamma^2 = 1 - (v/c)^2$, with v the speed of the train with respect to ground, and c the speed of light in vacuum.
- Note: r_B is not equal to r_A (would this contradict your expectation?)

Against *conventional wisdom*?

- According to SR, r_A and r_B are different in general.
- This should come as a surprise as your *conventional wisdom* (as according to Newtonian view point) may tell you that both r_A and r_B should be equal in their numerical value.
- However, as you will see later, such an assumption is false in the light of SR since the rate of time flow in two frames in relative motion are different
- Both rates, r_A and r_B, despite being different, are correct in their own right.

35

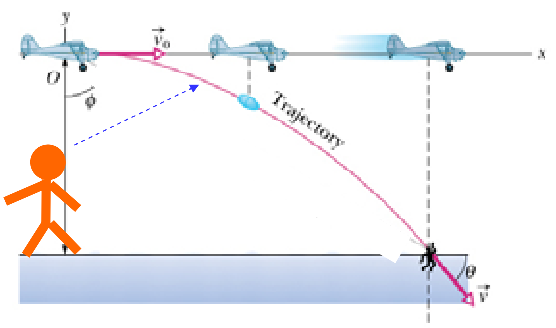
Example 2

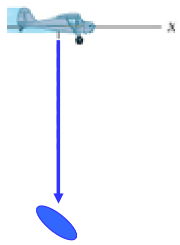
- Consider a stone is thrown into the air making a projectile motion.
- If the trajectory of the stone is considered in the frame of Earth (the so-called Lab frame, in which the ground is made as a stationary reference), the trajectory of the stone is a parabolic curve.
- The trajectory of the stone can also be analysed in a moving frame traveling at velocity v_x along the same horizontal direction as the stone. In this frame (the so-called rocket frame), the trajectory of the stone is not a parabolic curve but a vertical line.

36

View from different frames

- In the Lab frame, the observer on the ground sees a parabolic trajectory
- In the Rocket frame, the pilot sees the projectile to follow a vertically straight line downwards

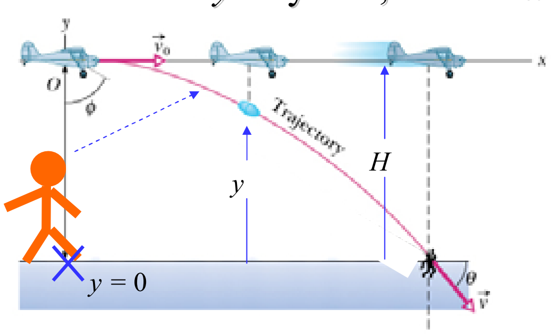


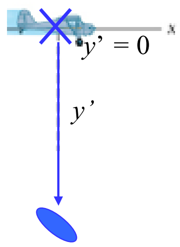


37

Transformation law for the coordinates

- In Lab frame
- $y = -(gt^2)/2 + H$
- $x = v_x t$
- In rocket frame
- $y' = -(gt^2)/2$
- $x' = 0$
- Transformation law relating the coordinates of projectile in both frames is
- $y' = y - H$, $x' = x - v_x t$





38

Time dilation as direct consequence of constancy of light speed

- According to the Principle of Relativity, the speed of light is invariant (i.e. it has the same value) in every reference frame (constancy of light speed)
- A direct consequence of the constancy of the speed of light is time stretching
- Also called time dilation
- Time between two events can have different values as measured in lab frame and rocket frames in relative motion
- "Moving clock runs slow"

39

Experimental verification of time stretching with pions

- Pion's half life $t_{½}$ is 18 ns.
- Meaning: If N_0 of them is at rest in the beginning, after 18 ns, $N_0/2$ will decay
- Hence, by measuring the number of pion as a function of time allows us to deduce its half life
- Consider now N_0 of them travel at roughly the speed of light c, the distance these pions travel after $t_{½}$=18 ns would be $ct_{½} \approx$5.4 m.
- Hence, if we measure the number of these pions at a distance 5.4 m away, we expect that $N_0/2$ of them will survive
- However, experimentally, the number survived at 5.4m is much greater than expected
- The flying poins travel tens of meters before half of them decay
- How do you explain this? the half life of these pions seems to have been stretched to a larger value!
- Conclusion: in our lab frame the time for half of the pions to decay is much greater than it is in the rest frame of the pions!

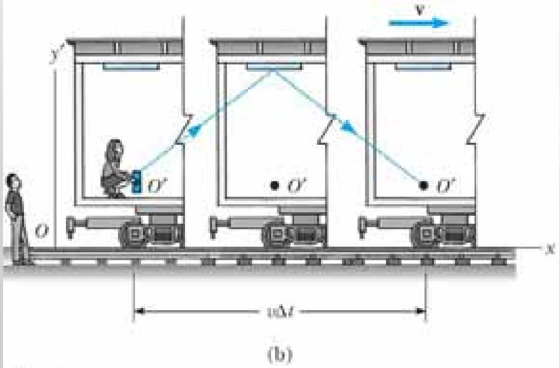
40

RE 38-5

- Suppose that a beam of pions moves so fast that at 25 meters from the target in the laboratory frame exactly half of the original number remain undecayed. As an experimenter, you want to put more distance between the target and your detectors. You are satisfied to have one-eighth of the initial number of pions remaining when they reach your detectors. How far can you place your detectors from the target?
- ANS: 75 m

A Gedanken Experiment

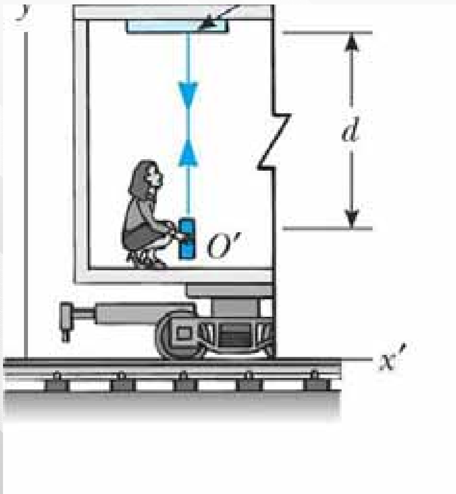
- Since light speed *c* is invariant (i.e. the same in all frames), it is suitable to be used as a clock to measure time and space
- Use light and mirror as clock – light clock
- A mirror is fixed to a moving vehicle, and a light pulse leaves O' at rest in the vehicle. O' is the rocket frame.
- Relative to a lab frame observer on Earth, the mirror and O' move with a speed *v*.



(b)

In the rocket frame

- The light pulse is observed to be moving in the vertical direction only
- The distance the light pulse traversed is 2d
- The total time travel by the light pulse to the top, get reflected and then return to the source is $\Delta\tau = 2d/c$



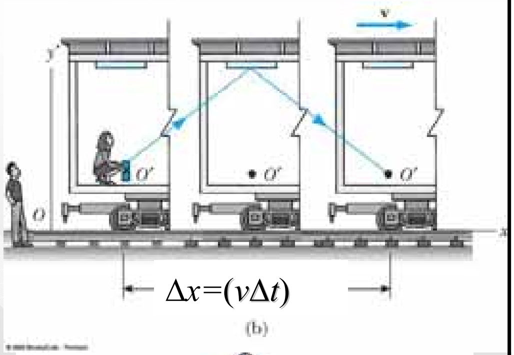
43

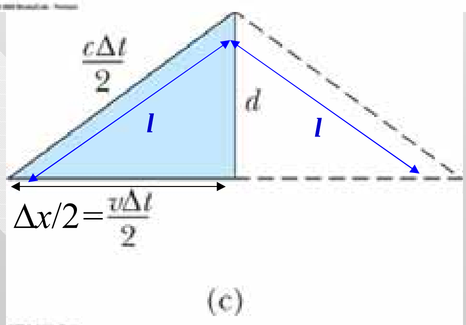
In the lab frame

- However, O in the lab frame observes a different path taken by the light pulse – it's a triangle instead of a vertical straight line
- The total light path is longer = 2l
- $l^2 = (c\Delta t/2)^2$

  $= d^2 + (\Delta x/2)^2$

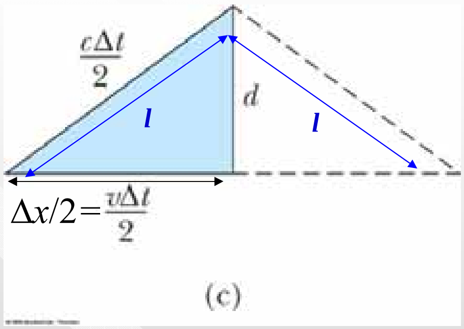
  $= d^2 + (v\Delta t/2)^2$





Light triangle

- We can calculate the relationship between Δt, $\Delta\tau$ and v:
- $l^2=(c\Delta t/2)^2=d^2+(v\Delta t/2)^2$ (lab frame)
- $\Delta\tau = 2d/c$ (Rocket frame)
- Eliminating d,



$$\Delta t = \frac{\Delta\tau}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \gamma\Delta\tau \qquad \gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \geq 1$$

45

Time dilation equation

- Time dilation equation $\Delta t = \frac{\Delta\tau}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \gamma\Delta\tau$
- Gives the value of time $\Delta\tau$ between two events occur at time Δt apart in some reference frame
- Lorentz factor $\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \geq 1$
- Note that as $v << c$, $\gamma \approx 1$; as $v \rightarrow c$, $\gamma \rightarrow \infty$
- Appears frequently in SR as a measure of relativistic effect: $\gamma \approx 1$ means little SR effect; $\gamma >> 1$ is the ultra-relativistic regime where SR is most pronounce

46

RE 38-6

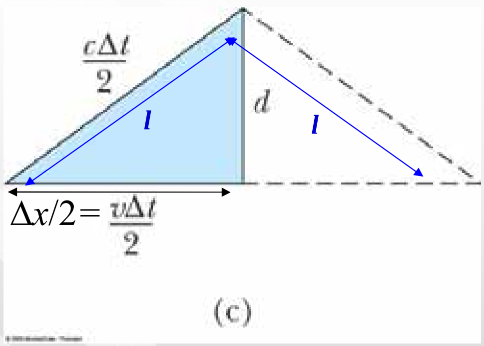
- A set of clocks is assembled in a stationary boxcar. They include a quartz wristwatch, a balance wheel alarm clock, a pendulum grandfather clock, a cesium atomic clock, fruit flies with average individual lifetimes of 2.3 days. A clock based on radioactive decay of nuclei, and a clock timed by marbles rolling down a track. The clocks are adjusted to run at the same rate as one another. The boxcar is then gently accelerated along a smooth horizontal track to a final velocity of 300 km/hr. At this constant final speed, which clocks will run at a different rate from the others as measured in that moving boxcar?

47

The Metric Equation

- From the light triangle in lab frame and the vertical light pulse in the rocket frame:
- $l^2 = (c\Delta t/2)^2 = d^2 + (\Delta x/2)^2$;
- $d = c\Delta\tau/2$

$\Rightarrow (c\Delta t/2)^2 = (c\Delta\tau/2)^2 + (\Delta x/2)^2$



- Putting the terms that refer to the lab frame are on the right: $(c\Delta\tau)^2 = (c\Delta t)^2 - (\Delta x)^2$

48

"the *invariant space-time interval*"

- We call the RHS, $s^2 \equiv (c\Delta t)^2 - (\Delta x)^2$ "*invariant space-time interval squared*" (or sometimes simply "*the space-time interval*")
- In words, the space-time interval reads:
- s^2 = ($c\times$time interval between two events as observed in the frame)^2 - (distance interval between the two events as observed in the frame)^2
- We can always calculate the space-time intervals for any pairs of events
- The interval squared s^2 is said to be an *invariant* because it has the same value as calculated by all observers (take the simile of the mass-to-high^2 ratio)
- Obviously, in the light-clock gadanken experiment, the space-time interval of the two light pulse events $s^2 \equiv (c\Delta t)^2 - (\Delta x)^2 = (c\Delta\tau)^2$ is positive because $(c\Delta\tau)^2 > 0$
- The space-time interval for such two events being positive is deeply related to the fact that such pair of events are causally related

49

Time-like, space-like and light-like

- The space-time interval of event pairs is said to be "*time-like*" (because the time component in the interval is larger in magnitude than the spatial component) if $s^2 > 0$.
- Not all pairs of events has a positive space-time interval.
- Pairs of events with a negative value of space-time interval is said to be "*space-like*", and these pairs of event cannot be related via any causal relation
- Pairs of events with $s^2 = 0$ is said to be "light like".

50

RE 38-8

- Points on the surfaces of the Earth and the Moon that face each other are separated by a distance of 3.76×10^8 m.
- (a) How long does it take light to travel between these points?
- A firecraker explodes at each of these two points; the time between these explosion is one second.
- (b) What is the invariant space-time interval for these two events?
- Is it possible that one of these explosions caused the other explosion?

51

Solution

(a) Time taken is

$t = L / c = 3.76\times10^8$ m/ 3×10^8 m/s = 1.25 s

(b) $s^2 = (ct)^2 - L^2$

$= (3\times10^8 \text{ m/s} \times 1.25 \text{ s})^2 - (3.76\times10^8 \text{ m})^2 = -7.51 \text{ m}^2$

(space-like interval)

(c) It is known that the two events are separated by only 1 s. Since it takes 1.25 s for light to travel between these point, it is impossible that one explosion is caused by the other, given that no information can travel fast than the speed of light.

Alternatively, from (b), these events are separated by a space-like space-time interval. Hence it is impossible that the two explosions have any causal relation

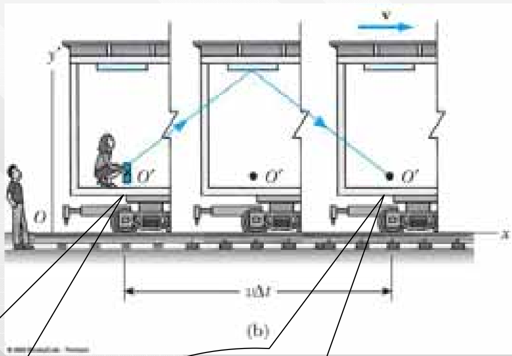
52

Proper time

- Imagine you are in the rocket frame, O', observing two events taking place at the same spot, separated by a time interval $\Delta\tau$ (such as the emission of the light pulse from source (EV1), and re-absorption of it by the source again, (EV2))
- Since both events are measured on the same spot, they appeared at rest wrp to you
- The time lapse $\Delta\tau$ between the events measured on the clock at rest is called the proper time or wristwatch time (one's own time)

Improper time

- In contrast, the time lapse measured by an observer between two events not at the same spot, i.e. $\Delta x \neq 0$, are termed *improper time*
- E.g., the time lapse, Δt, measured by the observer O observing the two events of light pulse emission and absorption in the train is improper time since both events appear to occur at different spatial location according to him.



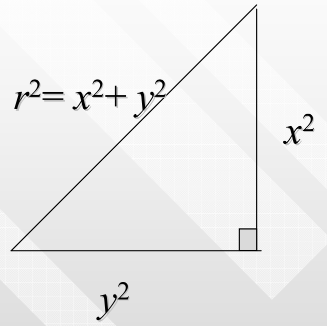
Event 1 occurs here at $x = 0$ (according to O)

Event 2 occurs here at $x = v\Delta t$ (according to O)

Space and time are combined by the metric equation: Space-time

$s^2 \equiv (c\Delta t)^2 - (\Delta x)^2 =$ invariant$=(\Delta\tau)^2$

- The metric equation says $(c\Delta t)^2 - (\Delta x)^2 =$ invariant $=(c\Delta\tau)^2$ in all frames
- *It combines space and time in a single expression on the RHS!!*
- Meaning: Time and space are interwoven in a fabric of space-time, and is not independent from each other anymore (we used to think so in Newton's absolute space and absolute time system)



The space-time invariant is the 1+1 dimension Minkowsky space-time

analogous to the 3-D Pythagoras theorem with the hypotenuse $r^2 = x^2 + y^2$. However, in the Minkowsky space-time metric, the space and time components differ by an relative minus sign

55

s^2 relates two different measures of time between the *same* two events

$$s^2 \equiv (c\Delta t)^2 - (\Delta x)^2 = \text{invariant} = (c\Delta\tau)^2$$

- (1) the time recorded on clocks in the reference frame in which the events occur at different places (improper time, Δt), and
- (2) the wristwatch time read on the clock carried by a traveler who records the two events as occurring a the same place (proper time, $\Delta\tau$)
- In different frames, Δt and Δx measured for the same two events will yield different values in general. However, the interval squared, $(c\Delta t)^2 - (\Delta x)^2$ will always give the same value, see example that ensues

56

Example of calculation of space-time interval squared

- In the light-clock gedanken experiment: For O', he observes the proper time interval of the two light pulse events to be $\Delta\tau$. For him, $\Delta x' = 0$ since these events occur at the same place
- Hence, for O',
- $s'^2 = (c\times$time interval observed in the frame$)^2$ - (distance interval observed in the frame$)^2$
- $= (c\Delta\tau)^2 - (\cancel{\Delta x'})^2 = (c\Delta\tau)^2$
- For O, the time-like interval for the two events is $s^2=(c\Delta t)^2-(\Delta x)^2 = (c\gamma\Delta\tau)^2-(v\Delta t)^2 = (c\gamma\Delta\tau)^2-(v\gamma\Delta\tau)^2 = \gamma^2 (c^2-v^2)\Delta\tau^2 = c^2\Delta\tau^2 = s'^2$

What happens at high and low speed

$$\Delta t = \gamma\Delta\tau, \qquad \gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \geq 1$$

- At low speed, $v \ll c$, $\gamma \approx 1$, and $\Delta\tau \approx \Delta t$, not much different, and we can't feel their difference in practice
- However, at high speed, proper time interval ($\Delta\tau$) becomes much SMALLER than improper time interval (Δt) in comparison, i.e. $\Delta\tau = \Delta t/\gamma \ll \Delta t$
- Imagine this: to an observer on the Earth frame, the person in a rocket frame traveling near the light speed appears to be in a 'slow motion' mode. This is because, according to the Earth observer, the rate of time flow in the rocket frame appear to be slower as compared to the Earth's frame rate of time flow.
- A journey that takes, say, 10 years to complete, according to a traveler on board (this is his proper time), looks like as if they take 10γ yr according to Earth observers.

Space travel with time-dilation

- A spaceship traveling at speed $v = 0.995c$ is sent to planet 100 light-year away from Earth
- How long will it takes, according to a Earth's observer?
- $\Delta t = 100\text{ ly}/0.995c = 100.05\text{yr}$
- But, due to time-dilation effect, according to the traveler on board, the time taken is only

  $\Delta\tau = \Delta t/\gamma = \Delta t\sqrt{(1-0.995^2)} = 9.992$ yr, not 100.05 yr as the Earthlings think
- So it is still possible to travel a very far distance within one's lifetime ($\Delta\tau \approx 50$ yr) as long as γ (or equivalently, v) is large enough

59

Nature's Speed Limit

- Imagine one in the lab measures the speed of a rocket v to be larger than c.
- As a consequence, according to $\Delta\tau = \Delta t\sqrt{1-\left(\frac{v}{c}\right)^2}$
- The proper time interval measurement $\Delta\tau$ in the rocket frame would be proportional to an imaginary number, $i = \sqrt{(-1)}$
- This is unphysical (and impossible) as no real time can be proportional to an imaginary number
- Conclusion: no object can be accelerated to a speed greater than the speed of light in vacuum, c
- Or more generally, no information can propagate faster than the light speed in vacuum, c
- Such limit is the consequence required by the logical consistency of SR

60

Time dilation in ancient legend

- 天上方一日，人间已十年
- One day in the heaven, ten years in the human plane

61

RE 38-7

- Find the rocket speed v at which the time $\Delta\tau$ between ticks on the rocket is recorded by the lab clock as $\Delta t = 1.01\Delta\tau$
- *Ans:* $\gamma = 1.01$, i.e. $(v/c)^2 = 1 - 1/\gamma^2 = \ldots$
- Solve for v in terms of c: $v = \ldots$

62

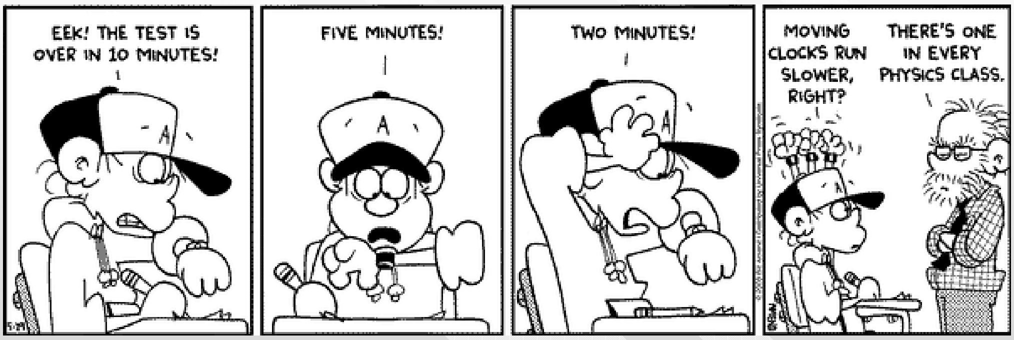
Satellite Clock Runs Slow?

- An Earth satellite in circular orbit just above the atmosphere circles the Earth once every $T = 90$ min. Take the radius of this orbit to be $r = 6500$ kilometers from the center of the Earth. How long a time will elapse before the reading on the satellite clock and the reading on a clock on the Earth's surface differ by one microsecond?
- For purposes of this approximate analysis, assume that the Earth does not rotate and ignore gravitational effects due the difference in altitude between the two clocks (gravitational effects described by general relativity).

dfds

Solution

- First we need to know the speed of the satellite in orbit. From the radius of the orbit we compute the circumference and divide by the time needed to cover that circumference:
- $v = 2\pi r/T = (2\pi\times 6500\text{ km})/(90\times 60\text{ s}) = 7.56$ km/s
- Light speed is almost exactly $c = 3\times 10^5$ km/s. so the satellite moves at the fraction of the speed of light given by
- $(v/c)^2 = [(7.56\text{ km/s})/(3\times 10^5\text{ km/s})]^2 = (2.52\times 10^5)^2 = 6.35\times 10^{-10}$.
- The relation between the time lapse $\Delta\tau$ recorded on the satellite clock and the time lapse Δt on the clock on Earth (ignoring the Earth's rotation and gravitational effects) is given by
- $\Delta\tau = (1-(v/c)^2)^{1/2}\,\Delta t$
- We want to know the difference between Δt and $\Delta\tau$ i.e. $\Delta t - \Delta\tau$.
- We are asked to find the elapsed time for which the satellite clock and the Earth clock differ in their reading by one microsecond, i.e. $\Delta t - \Delta\tau = 1\,\mu$s
- Rearrange the above equation to read $\Delta t^2 - \Delta\tau^2 = (\Delta t+\Delta\tau)(\Delta t-\Delta\tau)$, one shall arrive at the relation that $\Delta t = [1+(1-(v/c)^2)^{1/2}](1\,\mu\text{s}) / (v/c)^2 \approx 3140$s
- This is approximately one hour. A difference of one microsecond between atomic clock is easily detectable.



65

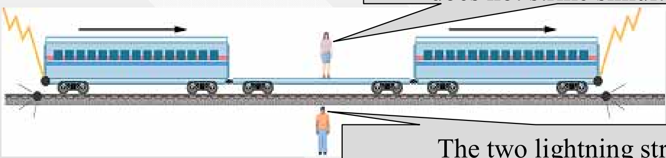
Disagreement on simultaneity

Two events that are simultaneous in one frame are not necessarily simultaneous in a second frame in uniform relative motion

66

Example

No, I don't agree. These two lightning does not strike simultaneously



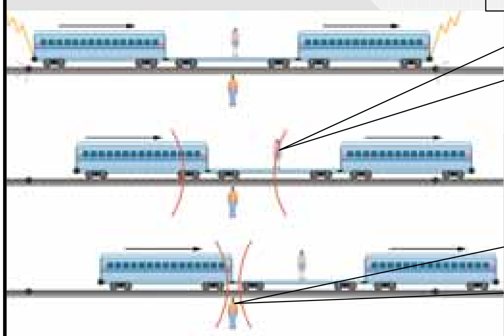
The two lightning strikes simultaneously

67

Einstein Train illustration

- Lightning strikes the front and back of a moving train, leaving char marks on both track and train. Each emitted flash spreads out in all directions.

I am equidistant from the front and back char marks on the train. Light has the standard speed in my frame, and equal speed in both direction. The flash from the front of the train arrived first, therefore the flash must have left the front of the train first. The front lightning bolt fell first before the rear light bolt fell. I conclude that the two strokes are not simultaneous.



I stand by the tracks halfway between the char marks on the track. The flashes from the strokes reach me a the same time and I am equidistance from the char marks on the track. I conclude that two events were simultaneous

68

Why?

- This is due to the invariance of the space-time invariant in all frames, (i.e. the invariant must have the same value for all frames)

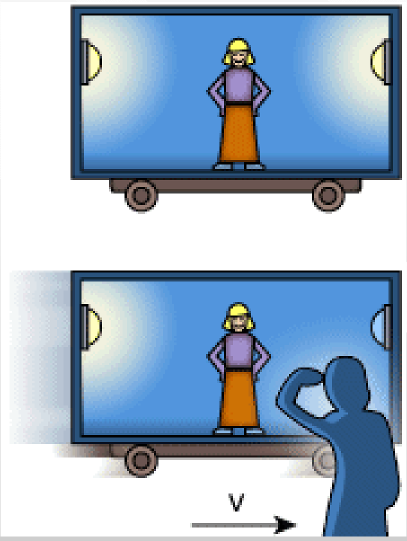
69

How invariance of space-time interval explains disagreement on simultaneity by two observers

- Consider a pair of events with space-time interval
  $$s^2=(c\Delta t)^2-(\Delta x)^2=(c\Delta t')^2-(\Delta x')^2$$
- where the primed and un-primed notation refer to space and time coordinates of two frames at relative motion (lets call them O and O')
- Say O observes two simultaneous event in his frame (i.e. $\Delta t = 0$) and are separate by a distance of (Δx), hence the space-time interval is $s^2 = -(\Delta x)^2$
- The space-time interval for the same two events observed in another frame, O', $s'^2 = (c\Delta t')^2 - (\Delta x')^2$ must read the same, as $-(\Delta x)^2$
- Hence, $(c\Delta t')^2 = (\Delta x')^2 - (\Delta x)^2$ which may not be zero on the RHS. i.e. $\Delta t'$ is generally not zero. This means in frame O', these events are not observed to be occurring simultaneously

70

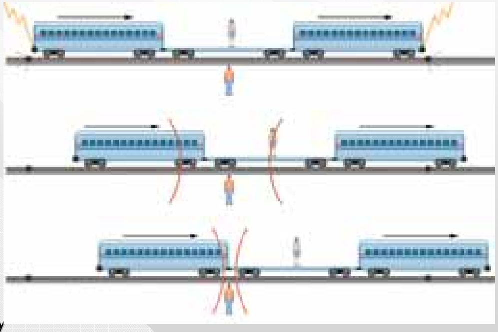
Simulation of disagreement on simultaneity by two observers (be aware that this simulation simulates the scenario in which the lady in the moving car sees simultaneity whereas the observer on the ground disagrees)



71

RE 38-9

- Susan, the rider on the train pictured in the figure is carrying an audio tape player. When she received the light flash from the front of the train she switches on the tape player, which plays very loud music. When she receives the light flash from the back end of the train, Susan switches off the tape player. Will Sam, the observers on the ground be able to hear the music?
- Later Susan and Sam meet for coffee and examine the tape player. Will they agree that some tape has been wound from one spool to the other?
- The answer is: …



72

Solution

- Music has been emitted from the tape player. This is a fact that must be true in both frames of reference. Hence Sam on the ground will be able to hear the music (albeit with some distortion).
- When the meet for coffee, they will both agree that some tape has been wound from one spool to the other in the tape recorder.

Touchstone Example 38-5: Principle of Relativity Applied

- Divide the following items into two lists, On one list, labeled SAME, place items that name properties and laws that are always the same in every frame. On the second list, labeled MAY BE DIF FERENT. place items that name properties that can be different in different frames:
- a. the time between two given events
- b. the distance between two given events
- c. the numerical value of Planck's constant h
- d. the numerical value of the speed of light c
- e. the numerical value of the charge e on the electron
- f. the mass of an electron (measured at rest)
- g. the elapsed time on the wristwatch of a person moving between two given events
- h. the order ot elements in the periodic table
- i. Newton's First Law of Motion ("A particle initially at rest remains at rest, and …")
- j. Maxwell's equations that describe electromagnetic fields in a vacuum
- k. the distance between two simultaneous events

Solution

THE SAME IN ALL FRAMES

- c. numerical value of *h*
- d. numerical value of *c*
- e. numerical value of *e*
- f. mass of electron (at rest)
- g. wristwatch time between two event (this is the proper time interval between two event)
- h. order of elements in the periodic table
- i. Newton's First Law of Motion
- j. Maxwell's equations
- k. distance between two simultaneous events

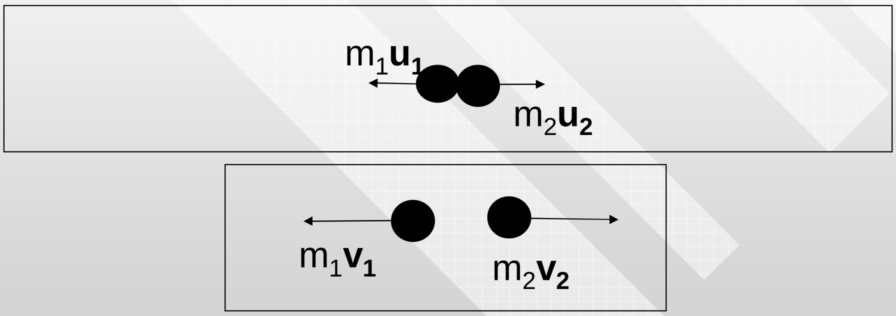
MAY BE DIFFERENT IN DIFFERENT FRAMES

- a. time between two given events
- b. distance between two give events

75

Relativistic Dynamics

- Where does $E=mc^2$ comes from?
- By Einstein's postulate, the observational law of linear momentum must also hold true in all frames of reference



Conservation of linear momentum classically means

$m_1\mathbf{u_1} + m_2\mathbf{u_2} = m_1\mathbf{v_1} + m_2\mathbf{v_2}$

76

Classical definition of linear momentum

- Classically, one defines linear momentum as mass × velocity
- Consider, in a particular reference frame where the object with a mass m_0 is moving with velocity v, then the momentum is defined (according to classical mechanics) as
  - $p = m_0 v$.
- If $v = 0$, the mass m_0 is called the rest mass.
- Similarly, in the other frame, (say the O' frame), $p' = m' v'$
- According to Newton's mechanics, the mass m' (as seen in frame O') is the same as the mass m_0 (as seen in O frame). There is no distinction between the two.
- In particular, there is no distinction between the rest mass and the 'moving mass'

77

Modification of expression of linear momentum

- However, simple analysis will reveal that in order to preserve the consistency between conservation of momentum and the Lorentz Transformation (to be discussed later), the definition of momentum has to be modified to
  - $momentum = \gamma m_0 v$
- where m_0 is the rest mass (an invariant quantity).
- A popular interpretation of the above re-definition of linear momentum holds that the mass of an moving object, m, is different from its value when it's at rest, m_0, by a factor of γ, i.e
  - $m = \gamma m_0$
- (however some physicists argue that this is actually not a correct interpretation. For more details, see the article by Okun posted on the course webpage. In any case, for pedagogical reason, we will stick to the "popular interpretation" of the "relativistic mass")

78

In other words…

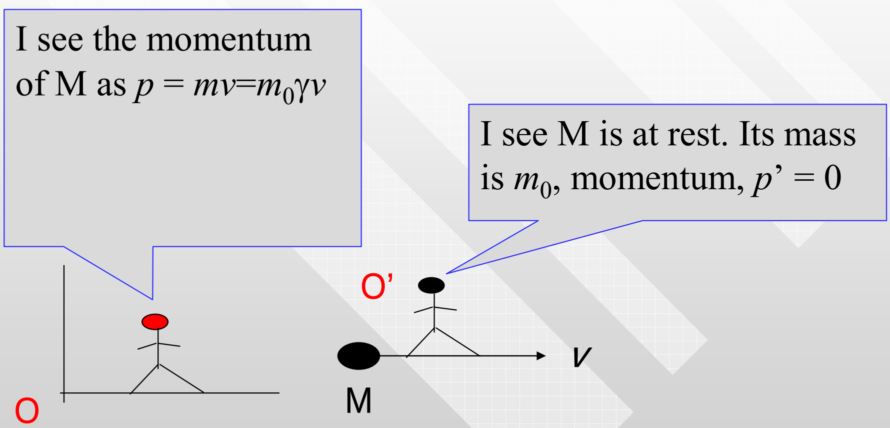
- In order to preserve the consistency between Lorentz transformation of velocity and conservation of linear momentum, the definition of 1-D linear momentum, classically defined as
    - $p_{classical} = \textit{rest mass} \times \textit{velocity},$
- has to be modified to

$$p_{classical} \rightarrow \quad p_{sr} = \text{“}\textit{relativistic mass}\text{”} \times \textit{velocity}$$
$$= mv = \gamma m_0 v$$

- where the relativisitic mass $m = \gamma m_0$ is not the same the rest mass, m_0
- Read up the text for a more rigorous illustration why the definition of classical momentum is inconsistent with LT

79

Pictorially…



80

Two kinds of mass

- Differentiate two kinds of mass: rest mass and relativistic mass
- m_0 = rest mass = the mass measured in a frame where the object is at rest. The rest mass of an object must be the same in all frames (not only in its rest frame).
- Relativistic mass $m = \gamma m_0$. The relativistic mass is speed-dependent

81

Behaviour of p_{SR} as compared to $p_{classic}$

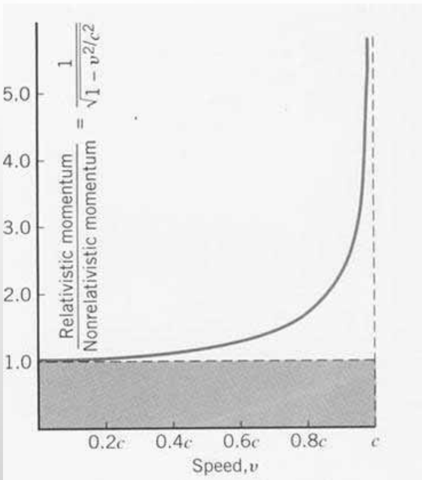


Figure 28.7 This graph shows how the ratio of the magnitude of the relativistic momentum to the magnitude of the nonrelativistic momentum increases as the speed of an object approaches the speed of light.

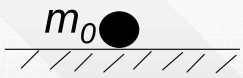
- Classical momentum is constant in mass, $p_{classic} = m_0 v$
- Relativistic momentum is $p_{SR} = m_0 \gamma v$
- $p_{SR} / p_{classic} = \gamma \rightarrow \infty$ as $v \rightarrow c$
- In the other limit, $v << c$

  $p_{SR} / p_{classic} = 1$

82

Example

The rest mass of an electron is $m_0 = 9.11 \times 10^{-31}$kg.



If it moves with $v = 0.75\,c$, what is its relativistic momentum?

$p = m_0 \gamma u$

Compare the relativistic p with that calculated with classical definition

Solution

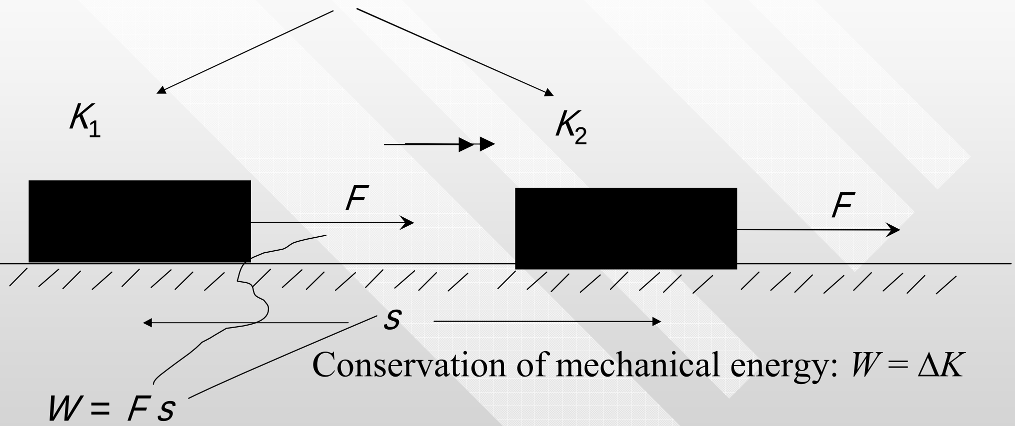
- The Lorentz factor is
  $\gamma = [1-(v/c)^2]^{-1/2} = [1-(0.75c/c)^2]^{-1/2}=1.51$
- Hence the relativistic momentum is simply
  $p = \gamma\, m_0 \times 0.75c$
  $= 1.51 \times 9.11 \times 10^{-31}\text{kg} \times 0.75 \times 3 \times 10^8$ m/s
  $= 3.1 \times 10^{-22}$ Ns
- In comparison, classical momentum gives $p_{\text{classical}} = m_0 \times 0.75c = 2.5 \times 10^{-22}$ Ns – about 34% lesser than the relativistic value

Work-Kinetic energy theorem

- Recall the law of conservation of mechanical energy:

  Work done by external force on a system, W = the change in kinetic energy of the system, ΔK

$$\Delta K = K_2 - K_1$$



Conservation of mechanical energy: $W = \Delta K$

The total energy of the object, $E = K + U$. Ignoring potential energy, E of the object is solely in the form of kinetic energy. If $K_1 = 0$, then $E = K_2$. But in general, U also needs to be taken into account for E.

◆In classical mechanics, mechanical energy (kinetic + potential) of an object is closely related to its momentum and mass

◆Since in SR we have redefined the classical mass and momentum to that of relativistic version

◆m_{class}(cosnt, $= m_0$) → $m_{SR} = m_0\gamma$

◆$p_{\text{class}} = m_{\text{class}}v$ → $p_{SR} = (m_0\gamma)v$

◆we must also modify the relation btw work and energy so that the law conservation of energy is consistent with SR

◆E.g, in classical mechanics, $K_{class} = p^2/2m = mv^2/2$. However, this relationship has to be supplanted by the relativistic version

$$K_{class} = mv^2/2 \rightarrow K_{SR} = E - m_0c^2 = \gamma m_0c^2 - m_0c^2$$

◆We shall derive K in SR in the following slides

87

Force, work and kinetic energy

- When a force is acting on an object with rest mass m_0, it will get accelerated (say from rest) to some speed (say v) and increase in kinetic energy from 0 to K

K as a function of v can be derived from first principle based on the definition of:

Force, $F = dp/dt$,

work done, $W = \int F\,dx$,

and conservation of mechanical energy, $\Delta K = W$

88

Derivation of relativistic kinetic energy

Force = rate change of momentum

$$W = \int_{x_1=0}^{x_2} F\,dx = \int_{x_1=0}^{x_2} \frac{dp}{dt}dx = \int_{x_1=0}^{x_2} \left(\frac{dp}{dx}\frac{dx}{dt}\right)dx$$

Chain rule in calculus

$$= \int_{x_1=0}^{x_2} \frac{dp}{dx}v dx = \int_{x_1=0}^{x_2} \left(\frac{dp}{dv}\frac{dv}{dx}\right)v dx = \int_0^v \frac{dp}{dv}v dv$$

where, by definition, $v = \frac{dx}{dt}$ is the velocity of the object

89

Explicitly, $p = \gamma m_0 v$,

Hence, $dp/dv = d/dv\,(\gamma m_0 v)$

$= m_0\,[v\,(d\gamma/dv) + \gamma]$

$= m_0\,[\gamma + (v^2/c^2)\,\gamma^3] = m_0\,(1-v^2/c^2)^{-3/2}$

in which we have inserted the relation

$$\frac{d\gamma}{dv} = \frac{d}{dv}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{v}{c^2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} = \frac{v}{c^2}\gamma^3$$

integrate

$$W = m_0\int_0^v v\left(1-\frac{v^2}{c^2}\right)^{-3/2} dv$$

$$\boxed{\Rightarrow K = W = m_0\gamma c^2 - m_0 c^2 = mc^2 - m_0 c^2}$$

90

$$K = m_0\gamma c^2 - m_0 c^2 = mc^2 - m_0 c^2$$

- The relativistic kinetic energy of an object of rest mass m_0 traveling at speed v
- $E = mc^2$ is the total relativistic energy of an moving object
- $E_0 = m_0c^2$ is called the rest energy of the object.
- Any object has non-zero rest mass contains energy $E_0 = m_0c^2$
- One can imagine that masses are 'frozen energies in the form of masses' as per $E_0 = m_0c^2$
- The rest energy (or rest mass) is an invariant

91

- Or in other words, the total relativistic energy of a moving object is the sum of its rest energy and its relativistic kinetic energy

$$E = mc^2 = m_0c^2 + K$$

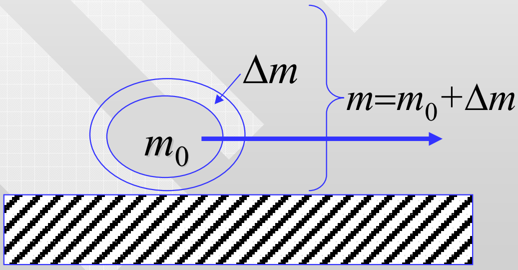
- The (relativistic) mass of an moving object m is larger than its rest mass m_0 due to the contribution from its relativistic kinetic energy – this is a pure relativistic effect not possible in classical mechanics

92

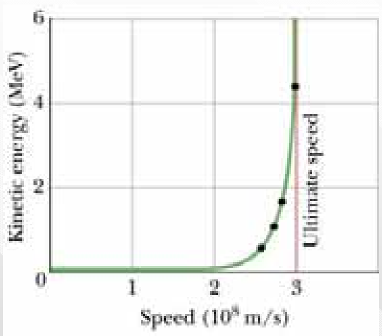
Pictorially

- Object at rest
- Total relativistic energy = rest energy only (no kinetic energy)
- $E=E_0=m_0c^2$

- A moving object
- Total relativistic energy = kinetic energy + rest energy
- $E=mc^2=K+E_0$
- $K=mc^2 - E_0 = \Delta mc^2$



Relativistic Kinetic Energy of an electron



- The kinetic energy increases without limit as the particle speed v approaches the speed of light
- In principle we can add as much kinetic energy as we want to a moving particle in order to increase the kinetic energy of a particle without limit
- What is the kinetic energy required to accelerate an electron to the speed of light?
- Exercise: compare the classical kinetic energy of an object, $K_{clas}=m_0v^2/2$ to the relativistic kinetic energy, $K_{sr}=(\gamma-1)m_0c^2$. What are their difference?

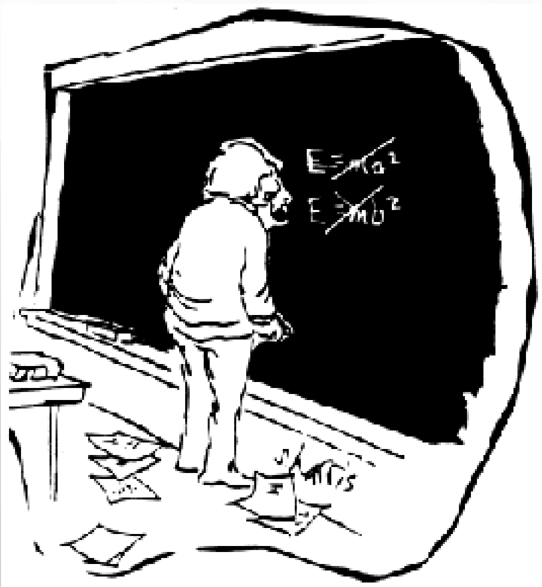
Mass energy equivalence, $E = mc^2$

- $E = mc^2$ relates the relativistic mass of an object to the total energy released when the object is converted into pure energy

  Example, 10 kg of mass, if converted into pure energy, it will be equivalent to $E = mc^2 = 10 \times (3 \times 10^8)^2$ J $= 9 \times 10^{17}$J - equivalent to a few tons of TNT explosive

95

So, now you know how $E=mc^2$ comes about…



96

Example 38-6: Energy of Fast Particle

- A particle of rest mass m_0 moves so fast that its total (relativistic) energy is equal to 1.1 times its rest energy.
- (a) What is the speed v of the particle?
- (b) What is the kinetic energy of the particle?

97

Solution

(a)

- Rest energy $E_0 = m_0c^2$
- We are looking for a speed such that the energy is 1.1 times the rest energy.
- We know how the relativistic energy is related to the rest energy via
- $E = \gamma E_0 = 1.1E_0$
- $\Rightarrow 1/\gamma^2 = 1/1.1^2 = 1/1.21 = 0.8264$
- $1 - v^2/c^2 = 0.8264$
- $\Rightarrow v^2/c^2 = 1 - 0.8264 = 0.1736$
- $\Rightarrow v = 0.4166\ 2c$

(b) Kinetic energy is $K = E - E_0 = 1.1E_0 - E_0 = 0.1E_0 = 0.1\ m_0c^2$

98

Reduction of relativistic kinetic energy to the classical limit

- The expression of the relativistic kinetic energy

$$K = m_0\gamma c^2 - m_0 c^2$$

must reduce to that of classical one in the limit v/c → 0, i.e.

$$\lim_{v<<c} K_{relativistic} = \frac{p^2{}_{classical}}{2m_0} \left(= \frac{m_0 v^2}{2}\right)$$

Expand γ with binomial expansion

- For $v << c$, we can always expand γ in terms of $(v/c)^2$ as

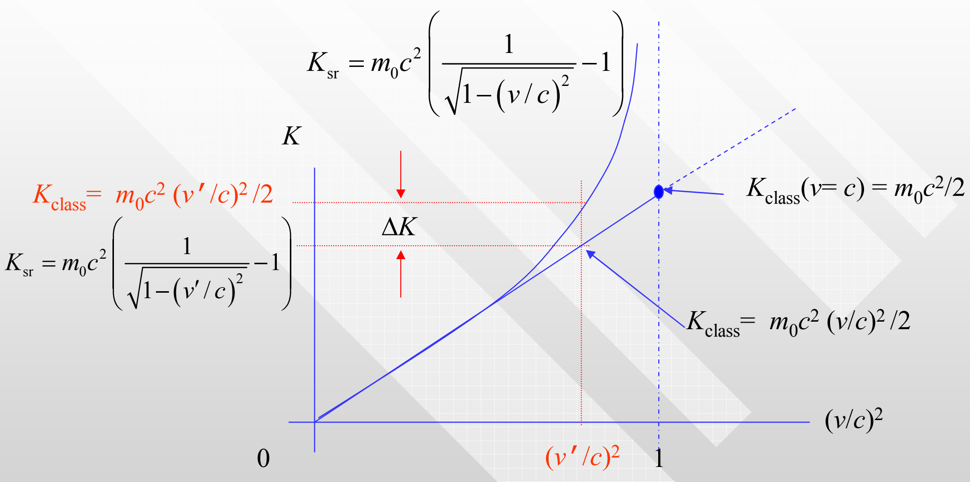
$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{v^2}{2c^2} + \text{terms of order } \frac{v^4}{c^4} \text{ and higher}$$

$$K = mc^2 - m_0c^2 = m_0c^2(\gamma - 1)$$
$$= m_0c^2\left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\right) - 1\right] \approx \frac{m_0 v^2}{2}$$

i.e., the relativistic kinetic energy reduces to classical expression in the $v << c$ limit

Exercise

- Plot K_{class} and K_{sr} vs $(v/c)^2$ on the same graph for $(v/c)^2$ between 0 and 1.
- Ask: In general, for a given velocity, does the classical kinetic energy of an moving object larger or smaller compared to its relativistic kinetic energy?
- In general does the discrepancy between the classical KE and relativistic KE increase or decrease as v gets closer to c?



Note that ΔK gets larger as $v \to c$

Recap

- Important formula for total energy, kinetic energy and rest energy as predicted by SR:

E = total relativisitic energy;

m_0 = rest mass;

m = relativistic mass;

E_0 = rest energy ;

p = relativistic momentum,

K = relativistic momentum;

$$m = \gamma m_0 ; p = \gamma m_0 v \;\; ; K = \gamma m_0 c^2 - m_0 c^2 ; E_0 = m_0 c^2 ; E = \gamma m_0 c^2 ;$$

Example

- A microscopic particle such as a proton can be accelerated to extremely high speed of $v = 0.85c$ in the Tevatron at Fermi National Accelerator Laboratory, US.
- Find its total energy and kinetic energy in eV.

Solution

Due to mass-energy equivalence, sometimes we express the mass of an object in unit of energy

- Proton has rest mass $m_p = 1.67 \times 10^{-27}$kg
- The rest mass of the proton can be expressed as energy equivalent, via
- $m_p c^2 = 1.67\times10^{-31}\text{kg} \times (3 \times 10^8\text{m/s})^2$
- $= 1.5\times10^{-10}$ J
- $= 1.5 \times 10^{-10} \times (1.6\text{x}10^{-19})^{-1}$ eV
- = 939,375,000 eV = 939 MeV

105

Solution

- First, find the Lorentz factor, $\gamma = 1.89$
- The rest mass of proton, m_0c^2, is 939 MeV
- Hence the total energy is

  $E = mc^2 = \gamma\,(m_0c^2) = 1.89 \times 939 \text{ MeV} = 1774 \text{ MeV}$
- Kinetic energy is the difference between the total relativistic energy and the rest mass,

  $K = E - m_0c^2 = (1774 - 939) \text{ MeV} = 835 \text{ MeV}$

106

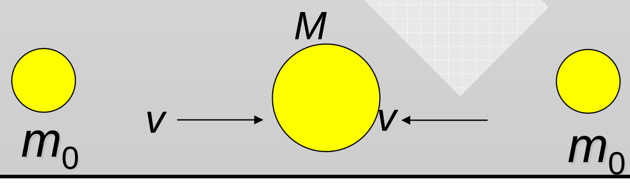
Exercise

- Show that the rest mass of an electron is equivalent to 0.51 MeV

107

Conservation of Kinetic energy in relativistic collision

- Calculate (i) the kinetic energy of the system and (ii) mass increase for a completely inelastic head-on of two balls (with rest mass m_0 each) moving toward the other at speed $v/c = 1.5\times10^{-6}$ (the speed of a jet plane). M is the resultant mass after collision, assumed at rest.



108

Solution

- (i) $K = 2mc^2 - 2m_0c^2 = 2(\gamma-1)m_0c^2$
- (ii) $E_{\text{before}} = E_{\text{after}} \Rightarrow 2\gamma m_0c^2 = Mc^2 \Rightarrow M = 2\gamma m_0$
- Mass increase $\Delta M = M - 2m_0 = 2(\gamma-1)m_0$
- Approximation: $v/c = ... = 1.5\text{x}10^{-6} \Rightarrow \gamma \approx 1 + ½\, v^2/c^2$ (binomail expansion) $\Rightarrow M \approx 2(1 + ½\, v^2/c^2)m_0$
- Mass increase $\Delta M = M - 2m_0$
- $\approx (v^2/c^2)m_0 = (1.5\text{x}10^{-6})^2 m_0$
- Comparing K with ΔMc^2: the kinetic energy is not lost in relativistic inelastic collision but is converted into the mass of the final composite object, i.e. kinetic energy is conserved
- In contrast, in classical mechanics, momentum is conserved but kinetic energy is not in an inelastic collision

109

Relativistic momentum and relativistic Energy

In terms of relativistic momentum, the relativistic total energy can be expressed as followed

$$E^2 = \gamma^2 m_0^2 c^4;\, p^2 = \gamma^2 m_0^2 v^2 \Rightarrow \frac{v^2}{c^2} = \frac{c^2p^2}{E^2}$$

$$\Rightarrow E^2 = \gamma^2 m_0^2 c^4 = \frac{m_0^2c^4}{1-\frac{v^2}{c^2}} = \left(\frac{m_0^2c^4E^2}{E^2 - c^2p^2}\right)$$

$$\boxed{E^2 = p^2c^2 + m_0^2c^4}$$

Energy-momentum invariance

110

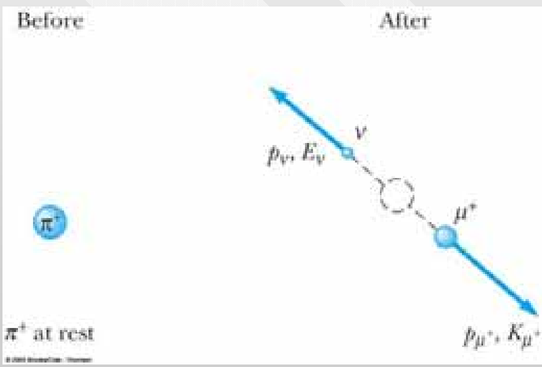
Invariance in relativistic dynamics

- Note that $E^2 - p^2c^2$ is an invariant, numerically equal to $(m_0c^2)^2$
- i.e., in any dynamical process, the difference between the total energy squared and total momentum squared of a given system must remain unchanged
- In additional, when observed in other frames of reference, the total relativistic energy and total relativistic momentum may have different values, but their difference, $E^2 - p^2c^2$, must remain invariant
- Such invariance greatly simplify the calculations in relativistic dynamics

111

Example: measuring pion mass using conservation of momentum-energy

- pi meson decays into a muon + massless neutrino
- If the mass of the muon is known to be 106 MeV/c^2, and the kinetik energy of the muon is measured to be 4.6 MeV, find the mass of the pion



112

Solution

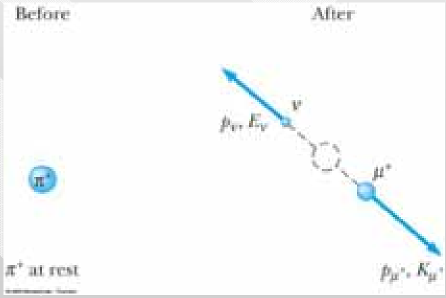
Relationship between Kinetic energy and momentum:

$$E_\mu^2 = p_\mu^2 c^2 + m_\mu^2 c^4$$

Conservation of relativistic energy: $E_\pi = E_\mu + E_\nu$

$$\Rightarrow m_\pi c^2 = \sqrt{m_\mu^2 c^4 + c^2 {p_\mu}^2} + \sqrt{\cancel{m_\nu^2} c^4 + c^2 {p_\nu}^2}$$

$$\Rightarrow m_\pi c = \sqrt{m_\mu^2 c^2 + {p_\mu}^2} + p_\mu$$



Momentum conservation: $p_\mu = p_\nu$

Also, total energy = K.E. + rest energy

$$E_\mu = K_\mu + m_\mu c^2 \Rightarrow E_\mu^2 = \left(K_\mu + m_\mu c^2\right)^2$$

But $E_\mu^2 = p_\mu^2 c^2 + m_\mu^2 c^4$

$$\Rightarrow E_\mu^2 = p_\mu^2 c^2 + m_\mu^2 c^4 = \left(K_\mu + m_\mu c^2\right)^2;$$

$$p_\mu c = \sqrt{\left(K_\mu + m_\mu c^2\right)^2 - m_\mu^2 c^4} = \sqrt{K_\mu^2 + 2K_\mu m_\mu c^2}$$

113

Plug $p_\mu^2 c^2 = \left(K_\mu + m_\mu c^2\right)^2 - m_\mu^2 c^4$ into

$$m_\pi c^2 = \sqrt{m_\mu^2 c^4 + c^2 {p_\mu}^2} + cp_\mu$$

$$= \sqrt{{m_\mu}^2 c^4 + \left[\left(K_\mu + m_\mu c^2\right)^2 - m_\mu^2 c^4\right]} + \sqrt{\left(K_\mu + m_\mu c^2\right)^2 - m_\mu^2 c^4}$$

$$= \left(K_\mu + m_\mu c^2\right) + \sqrt{\left(K_\mu^2 + 2K_\mu m_\mu c^2\right)}$$

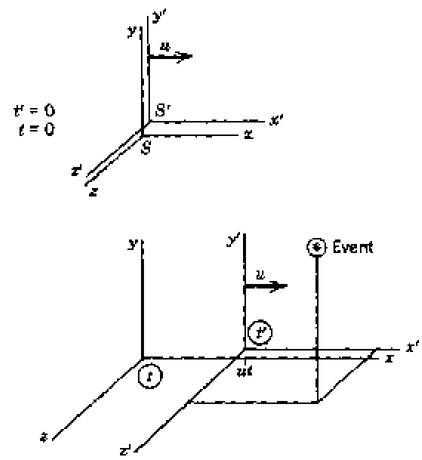
$$= \left(4.6\text{MeV} + \frac{106\text{MeV}}{c^2}c^2\right) + \sqrt{(4.6\text{MeV})^2 + 2(4.6\text{MeV})\left(\frac{106\text{MeV}}{c^2}\right)c^2}$$

$$= 111\text{MeV} + \sqrt{996}\text{MeV} = 143\text{MeV}$$

114

Observing an event from lab frame and rocket frame

Figure 1-13
Reference systems S and S' in relative motion. An event occurs at (x, y, z, t) in S and (x', y', z', t') in S'. In this view, S' is moving through S.



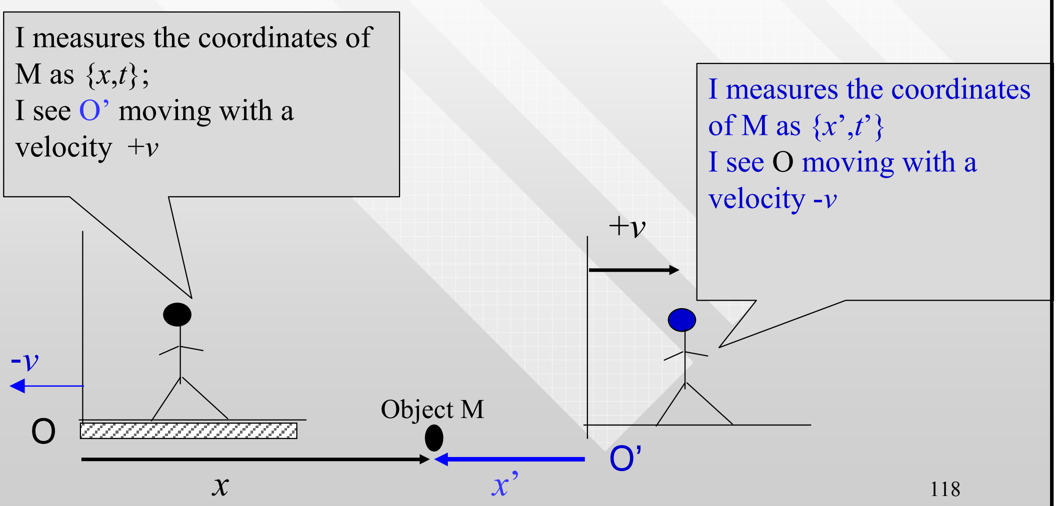
Lorentz Transformation

- All inertial frames are equivalent
- Hence all physical processes analysed in one frame can also be analysed in other inertial frame and yield consistent results
- Any event observed in two frames of reference must yield consistent results related by transformation laws
- Specifically such a transformation law is required to related the space and time coordinates of an event observed in one frame to that observed from the other

Different frame uses different notation for coordinates

- O' frame uses $\{x',y',z';t'\}$ to denote the coordinates of an event, whereas O frame uses $\{x,y,z;t\}$
- How to related $\{x',y',z',t'\}$ to $\{x,y,z;t\}$?
- In Newtonian mechanics, we use Galilean transformation

Two observers in two inertial frames with relative motion use different notation

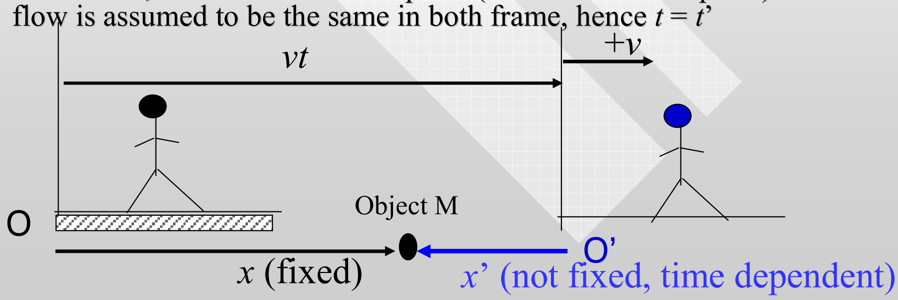


Galilean transformation (applicable only for $v \ll c$)

- For example, the spatial coordinate of the object M as observed in O is x and is being observed at a time t, whereas according to O', the coordinate for the space and time coordinates are x' and t'. At low speed $v \ll c$, the transformation that relates $\{x',t'\}$ to $\{x,t\}$ is given by Galilean transformation
- $\{x'=x-vt,\ t' = t\}$ (x' and t' in terms of x,t)
- $\{x = x' + vt,\ t = t'\}$ (x and t in terms of x',t')

Illustration on Galilean transformation of $\{x'=x-vt,\ t' = t\}$

- Assume object M is at rest in the O frame, hence the coordinate of the object M in O frame is fixed at x
- Initially, when $t = t' = 0$, O and O' overlap
- O' is moving away from O at velocity $+v$
- The distance of the origin of O' is increasing with time. At time t (in O frame), the origin of O' frame is at an instantaneous distance of $+vt$ away from O
- In the O' frame the object M is moving away with a velocity $-v$ (to the left)
- Obviously, in O' frame, the coordinate of the object M, denoted by x', is time-dependent, being $x' = x - vt$
- In addition, under current assumption (i.e. classical viewpoint) the rate of time flow is assumed to be the same in both frame, hence $t = t'$

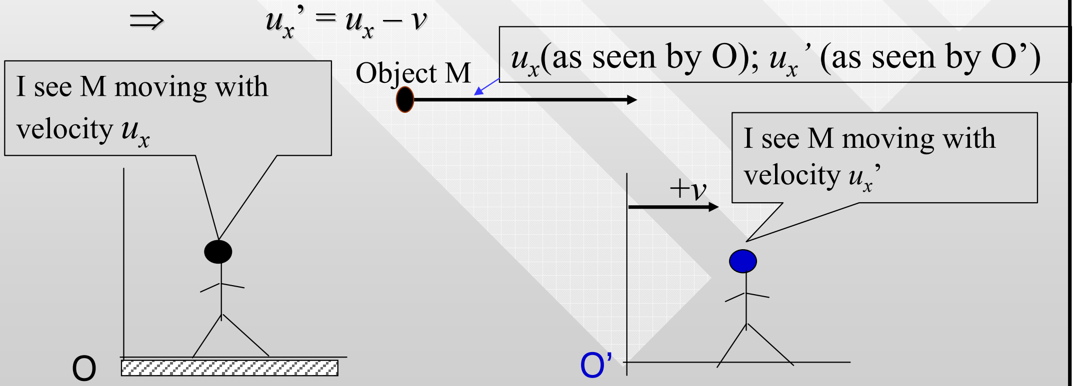


However, GT contradicts the SR postulate when v approaches the speed of light, hence it has to be supplanted by a relativistic version of transformation law when near-to-light speeds are involved:
Lorentz transformation
(The contradiction becomes more obvious if GT on velocities, rather than on space and time, is considered)

Galilean transformation of velocity (applicable only for $u_x, v \ll c$)

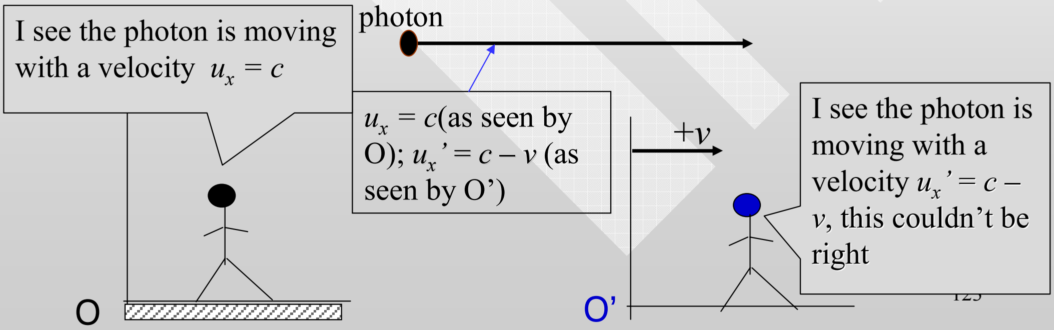
- Now, say object M is moving as a velocity of v wrp to the lab frame O
- What is the velocity of M as measured by O'?
- Differentiate $x' = x - vt$ wrp to t ($=t'$), we obtain
- $d(x')/dt' = d(x - vt)/dt = d(x)/dt - v$

$$\Rightarrow \quad u_x' = u_x - v$$



If applied to light Galilean transformation of velocity contradicts the SR Postulate

- Consider another case, now, a photon (particle of light) observed from different frames
- A photon. being a massless particle of light must move at a speed $u_x = c$ when observed in O frame
- However Galilean velocity addition law $u_x' = u_x - v$, if applied to the photon, says that in O' frame, the photon shall move at a lower speed of $u_x' = u_x - v = c - v$
- This is a contradiction to the constancy of light speed in SR



Conclusion

- GT (either for spatial, temporal or velocity) cannot be applicable when dealing with object moving near or at the speed of light
- It has to be supplanted by a more general form of transformation – Lorentz transformation, LT
- LT must reduce to GT when $v \ll c$.

Derivation of Lorentz transformation

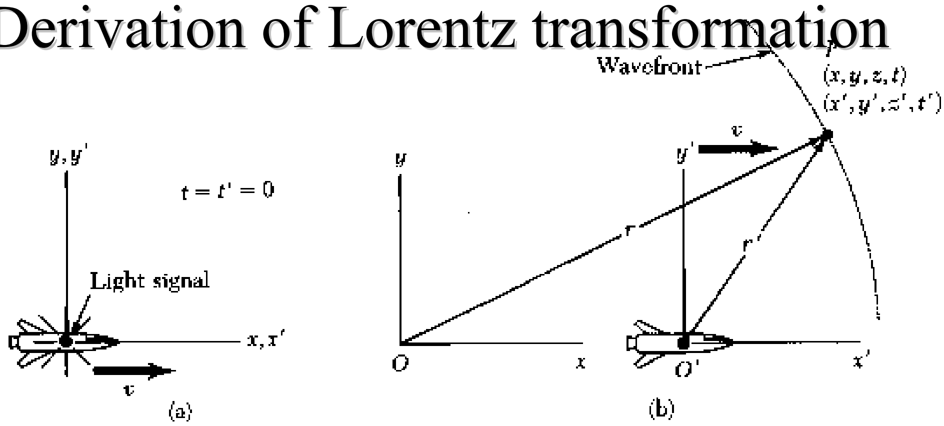


Figure 1.13 A rocket moves with a speed v along the xx' axes. (a) A pulse of light is sent out from the rocket at $t = t' = 0$ when the two systems coincide. (b) Coordinates of some point P on an expanding spherical wavefront as measured by observers in both inertial systems. (This figure is entirely schematic, and you should not be misled by the geometry.)

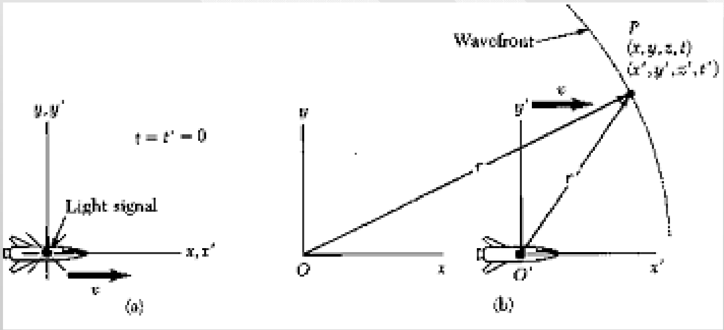
Our purpose is to find the general transformation that relates $\{x,t\}$ to $\{x',t'\}$ (applicable not only for v<<c but also for v≈c)

Read derivation of LT from the texts

- Brehm and Mullin
- Krane
- Serway, Mayer and Mosses

Derivation of Lorentz transformation

- Consider a rocket moving with a speed v (O' frame) along the xx' direction wrp to the stationary O frame
- A light pulse is emitted at the instant $t' = t = 0$ when the two origins of the two reference frames coincide
- The light signal travels as a spherical wave at a constant speed c in both frames
- After in time interval of t, the origin of the wave centred at O has a radius $r = ct$, where $r^2 = x^2 + y^2 + z^2$



Arguments

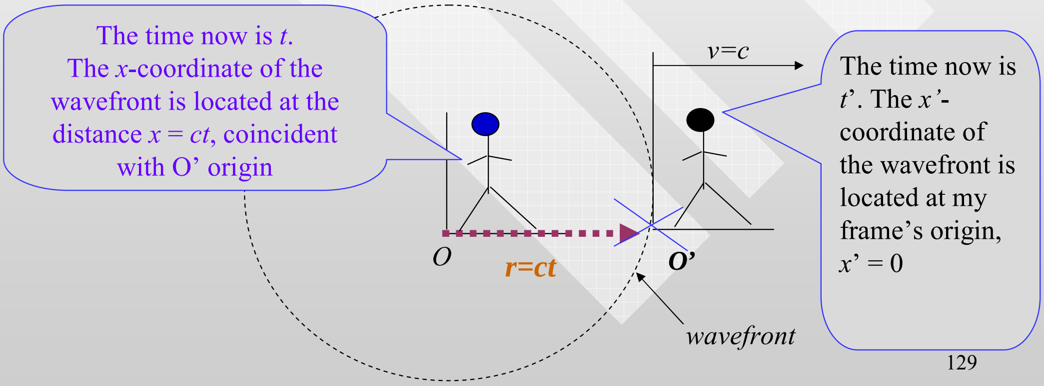
- From the view point of O', after an interval t' the origin of the wave, centred at O' has a radius:

$$r' = ct' , (r')^2 = (x')^2 + (y')^2 + (z')^2$$

- $y'=y$, $z' = z$ (because the motion of O' is along the xx') axis – no change for y,z coordinates (*condition A*)
- The transformation from x to x' (and vice versa) must be linear, i.e. $x' \propto x$ (*condition B*)
- Boundary condition (1): If $v = c$, from the viewpoint of O, the origin of O' is located on the wavefront (to the right of O)
- $\Rightarrow x' = 0$ must correspond to $x = ct$
- Boundary condition (2): In the same limit, from the viewpoint of O', the origin of O is located on the wavefront (to the left of O')
- $\Rightarrow x = 0$ corresponds to $x' = -ct'$
- Putting everything together we assume the transformation that relates x' to $\{x, t\}$ takes the form $x' = k(x - ct)$ as this will fulfill all the conditions (B) and boundary condition (1) ; (k some proportional constant to be determined)
- Likewise, we assume the form $x = k(x' + ct')$ to relate x to $\{x', t'\}$ as this is the form that fulfill all the conditions (B) and boundary condition (2)

Illustration of Boundary condition (1)

- $x = ct$ ($x' = ct'$) is defined as the x-coordinate (x'-coordinate) of the wavefront in the O (O') frame
- Now, we choose O as the rest frame, O' as the rocket frame. Furthermore, assume O' is moving away to the right from O with light speed, i.e. $v = +c$
- Since $u = c$, this means that the wavefront and the origin of O' coincides all the time
- For O, the x-coordinate of the wavefront is moving away from O at light speed; this is tantamount to the statement that $x = ct$
- From O' point of view, the x'-coordinate of the wavefront is at the origin of it's frame; this is tantamount to the statement that $x' = 0$
- Hence, in our yet-to-be-derived transformation, $x' = 0$ must correspond to $x = ct$

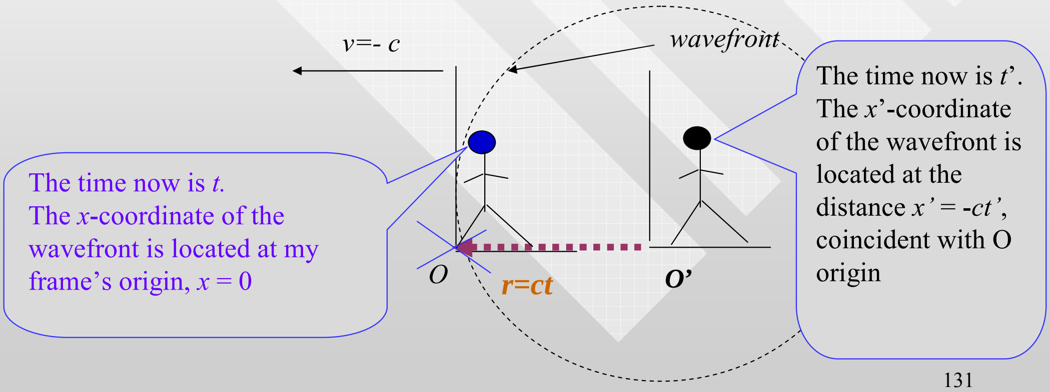


Permuting frames

- Since all frames are equivalent, physics analyzed in O' frame moving to the right with velocity $+v$ is equivalent to the physics analyzed in O frame moving to the left with velocity $-v$
- Previously we choose O frame as the lab frame and O' frame the rocket frame moving to the right (with velocity $+v$ wrp to O)
- Alternatively, we can also fix O' as the lab frame and let O frame becomes the rocket frame moving to the left (with velocity $-v$ wrp to O')

Illustration of Boundary condition (2)

- Now, we choose O' as the rest frame, O as the rocket frame. From O' point of view, O is moving to the left with a relative velocity $v = -c$
- From O' point of view, the wavefront and the origin of O coincides. The x'-coordinate of the wavefront is moving away from O' at light speed to the left; this is tantamount to the statement that $x' = -ct'$
- From O point of view, the x-coordinate of the wavefront is at the origin of it's frame; this is tantamount to the statement that $x = 0$
- Hence, in our yet-to-be-derived transformation, $x = 0$ must correspond to $x' = -ct'$



131

Finally, the transformation obtained

- We now have
- $r = ct$, $r^2 = x^2 + y^2 + z^2$; $y' = y$, $z' = z$; $x = k(x' + ct')$;
- $r' = ct'$, $r'^2 = x'^2 + y'^2 + z'^2$; $x' = k(x - ct)$;
- With some algebra, we can solve for $\{x',t'\}$ in terms of $\{x,t\}$ to obtain the desired transformation law (do it as an exercise)
- The constant k turns out to be identified as the Lorentz factor, γ

$$x' = \frac{x - vt}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma(x - vt) \qquad t' = \frac{t - \left(v/c^2\right)x}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma\left[t - (v/c^2)x\right]$$

(x' and t' in terms of x,t)

132

Space and time now becomes state-of-motion dependent (via γ)

- Note that, now, the length and time interval measured become dependent of the state of motion (in terms of γ) – in contrast to Newton's classical viewpoint
- **Lorentz transformation reduces to Galilean transformation when $v \ll c$ (show this yourself)**
- **i.e. LT → GT in the limit $v \ll c$**

133

How to express $\{x, t\}$ in terms of $\{x', t'\}$?

- We have expressed $\{x', t'\}$ in terms of $\{x, t\}$ as per

$$x' = \frac{x - vt}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma(x - vt) \qquad t' = \frac{t - \left(v/c^2\right)x}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma\left[t - (v/c^2)x\right]$$

- Now, how do we express $\{x, t\}$ in terms of $\{x', t'\}$?

134

Simply permute the role of *x* and *x*' and reverse the sign of *v*

$$t \leftrightarrow t', x \leftrightarrow x', v \rightarrow -v$$

$$x' = \gamma(x - vt) \rightarrow x = \gamma(x' + vt')$$

$$t' = \gamma\left[t - (v/c^2)x\right] \rightarrow t = \gamma\left[t' + (v/c^2)x'\right]$$

The two transformations above are equivalent; use which is appropriate in a given question

135

Length contraction

- Consider the rest length of a ruler as measured in frame O' is
- $L' = \Delta x' = x'_2 - x'_1$ (proper length) measured at the same instance in that frame ($t'_2 = t'_1$)
- What is the length of the rule as measured by O?
- The length in O, L, according the LT can be deduced as followed:

  $L' = \Delta x' = x'_2 - x'_1 = \gamma[(x_2 - x_1) - v(t_2 - t_1)]$
- The length of the ruler in O, L, is simply the distance btw x_2 and x_1 measured at the same instance in that frame ($t_2 = t_1$). Hence we have
  - $L' = \gamma L$
- where L = is the improper length.
- As a consequence, we obtain the relation between the proper length measured by the observer at rest wrp to the ruler and that measured by an observer who is at a relative motion wrp to the ruler:

$$L' = \gamma L$$

136

Moving rulers appear shorter

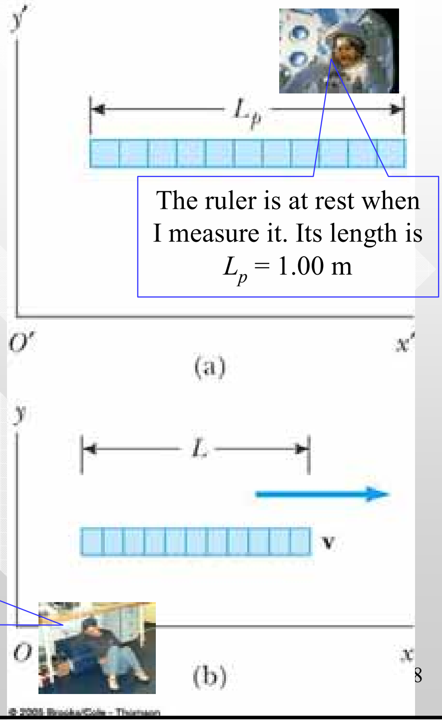
$$L' = \gamma L$$

- L' is defined as the proper length = length of and object measured in the frame in which the object is at rest
- L is the length measured in a frame which is moving wrp to the ruler
- If an observer at rest wrp to an object measures its length to be L', an observer moving with a relative speed u wrp to the object will find the object to be shorter than its rest length by a factor $1 / \gamma$
- i.e., the length of a moving object is measured to be shorter than the proper length – hence "length contraction"
- In other words, a moving rule will appear shorter!!

Example of a moving ruler

Consider a meter rule is carried on board in a rocket (call the rocket frame O')

- An astronaut in the rocket measure the length of the ruler. Since the ruler is at rest wrp to the astronaut in O', the length measured by the astronaut is the proper length, $L_p = 1.00$ m, see (a)
- Now consider an observer on the lab frame on Earth. The ruler appears moving when viewed by the lab observer. If the lab observer attempts to measure the ruler, the ruler would appear shorter than 1.00m



The ruler is at moving at a speed v when I measure it. Its length is $L = 0.999$ m

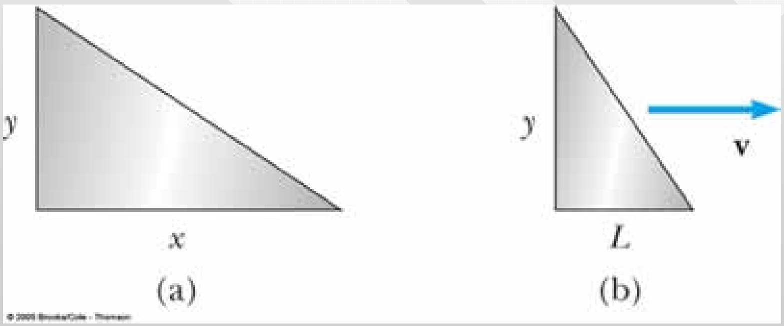
RE 38-11

- What is the speed v of a passing rocket in the case that we measure the length of the rocket to be half its length as measured in a frame in which the rocket is at rest?

139

Length contraction only happens along the direction of motion

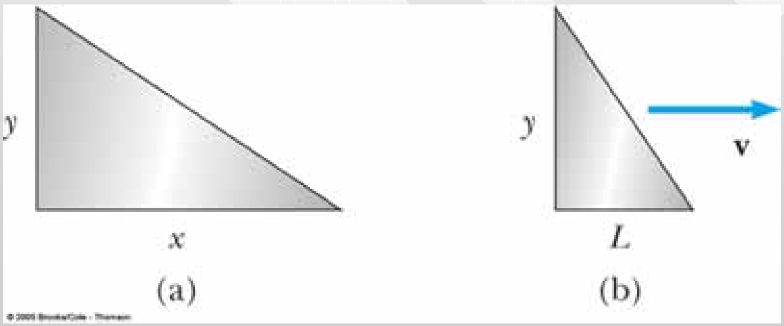
Example: A spaceship in the form of a triangle flies by an oberver at rest wrp to the ship (see fig (a)), the distance x and y are found to be 50.0 m and 25.0 m respectively. What is the shape of the ship as seen by an observer who sees the ship in motion along the direction shown in fig (b)?



140

Solution

- The observer sees the horizontal length of the ship to be contracted to a length of
- $L = L_p/\gamma = 50\text{ m}\ \sqrt{(1 - 0.950^2)} = 15.6\text{ m}$
- The 25 m vertical height is unchanged because it is perpendicular to the direction of relative motion between the observer and the spaceship.



Similarly, one could also derive time dilation from the LT

Do it as homework

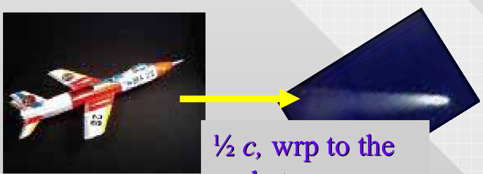
An interesting story book to read

- Mr. Thompkins's in paper back, by George Gamow.

What is the velocities of the ejected stone?

- Imagine you ride on a rocket moving ¾ *c* wrp to the lab. From your rocket you launch a stone forward at ½ *c*, as measured in your rocket frame. What is the speed of the stone observed by the lab observer?

¾ **c,** wrp to lab

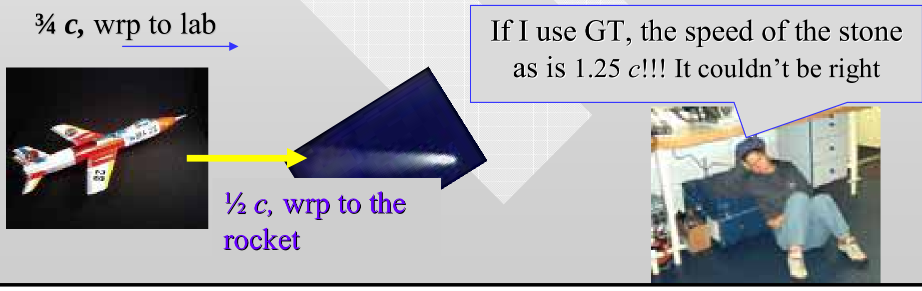


½ *c*, wrp to the rocket

The speed of the stone as I measure it is…

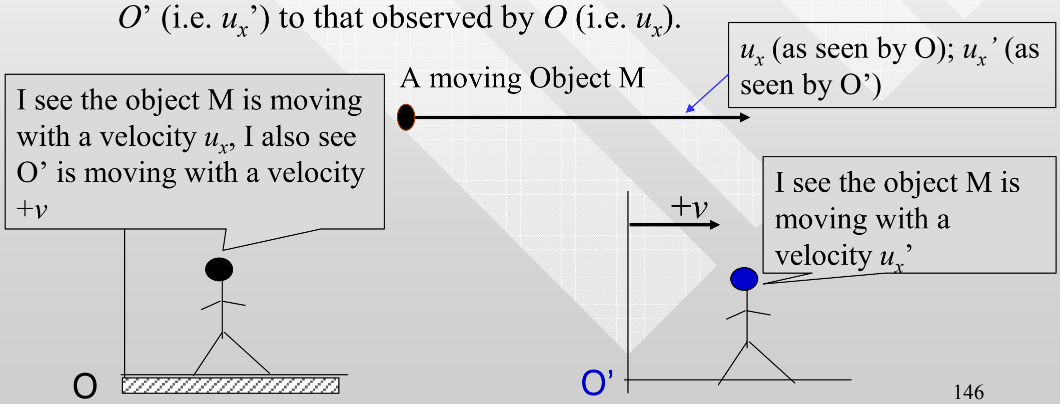
Adding relativistic velocities using Galilean transformation

- According to GT of velocity (which is valid at low speed regime v << c),
- the lab observer would measure a velocity of $u_x = u_x' + v$ = ½ c + ¾ c
- = 1.25 c for the ejected stone.
- However, in SR, c is the ultimate speed and no object can ever exceed this ultimate speed limit
- So something is no right here…Galilean addition law is no more valid to handle addition of relativistic velocities (i.e. at speed near to c)



Relativity of velocities

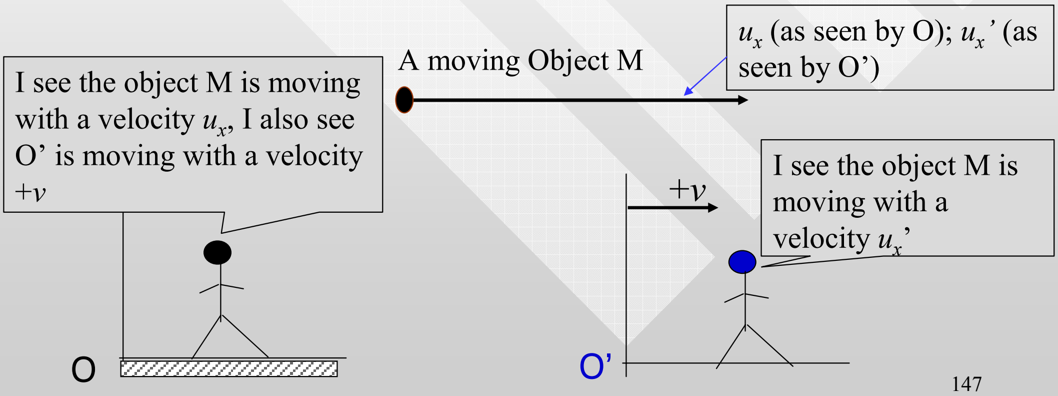
- The generalised transformation law of velocity used for addition of relativistic velocities is called Lorentz transformation of velocities, derived from the Lorentz transformation of space-time
- Our task is to relate the velocity of the object M as observed by O' (i.e. u_x') to that observed by O (i.e. u_x).



146

Relativity of velocities

- Consider an moving object being observed by two observers, one in the lab frame and the other in the rocket frame
- We could derive the Lorentz transformation of velocities by taking time derivative wrp to the LT for space-time, see next slide



147

Derivation of Lorentz transformation of velocities

- By definition, $u_x = dx/dt$, $u'_x = dx'/dt'$
- The velocity in the O' frame can be obtained by taking the differentials of the Lorentz transformation

$$x' = \gamma(x - vt) \quad t' = \gamma\left[t - (v/c^2)x\right]$$

$$dx' = \gamma(dx - vdt),\ dt' = \gamma(dt - \frac{v}{c^2}dx)$$

148

Combining

$$u'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{dt\left(\frac{dx}{dt} - v\frac{dt}{dt}\right)}{dt\left(\frac{dt}{dt} - \frac{v}{c^2}\frac{dx}{dt}\right)}$$

$$= \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

where we have made used of the definition $u_x = dx/dt$

149

Comparing the LT of velocity with that of GT

Lorentz transformation of velocity:

$$u'_x = \frac{dx'}{dt'} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

Galilean transformation of velocity:

$$u'_x = u_x - v$$

LT reduces to GT in the limit $u_x v << c^2$

150

- *Please try to make a clear distinction among the definitions of various velocities, i.e. u_x, u'_x, v so that you wont get confused*

151

LT is consistent with the constancy of speed of light

- In either O or O' frame, the speed of light seen must be the same, c. LT is consistent with this requirement.
- Say object M is moving with speed of light as seen by O, i.e. $u_x = c$
- According to LT, the speed of M as seen by O' is

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{c - v}{1 - \frac{cv}{c^2}} = \frac{c - u}{1 - \frac{v}{c}} = \frac{c - v}{\frac{1}{c}(c - v)} = c$$

- That is, in either frame, both observers agree that the speed of light they measure is the same, $c = 3 \times 10^8$m/s

152

How to express u_x in terms of u_x'?

- Simply permute v with $-v$ and change the role of u_x with that of u_x':

$$u_x \rightarrow u'_x, u'_x \rightarrow u_x, v \rightarrow -v$$

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \rightarrow u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}$$

153

Recap: Lorentz transformation relates $\{x', t'\} \leftrightarrow \{x, t\}$; $u'_x \leftrightarrow u_x$

$$x' = \gamma(x - vt) \qquad t' = \gamma\left[t - (v/c^2)x\right]$$

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$x = \gamma(x' + vt') \qquad t = \gamma\left[t' + (v/c^2)x'\right]$$

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}$$

154

RE 38-12

- A rocket moves with speed $0.9c$ in our lab frame. A flash of light is sent toward from the front end of the rocket. Is the speed of that flash equal to $1.9\ c$ as measured in our lab frame? If not, what is the speed of the light flash in our frame? Verify your answer using LT of velocity formula.

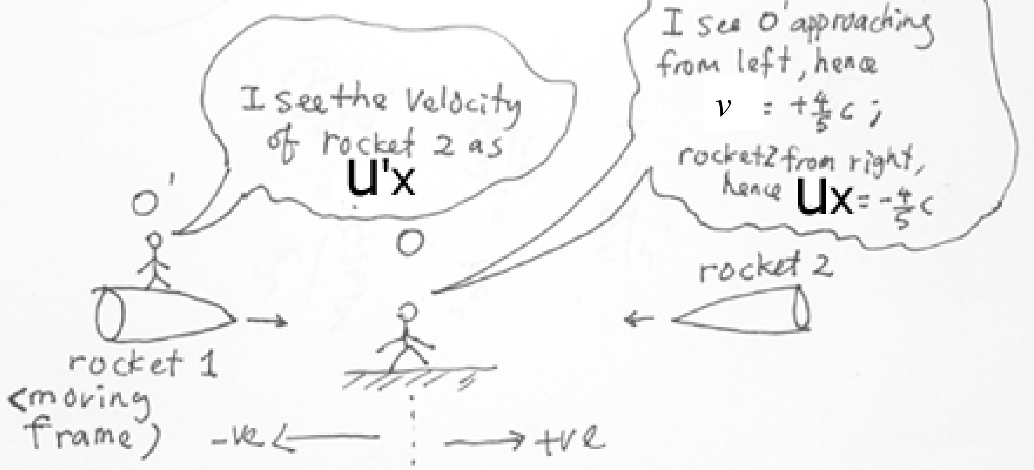
155

Example (relativistic velocity addition)

- Rocket 1 is approaching rocket 2 on a head-on collision course. Each is moving at velocity $4c/5$ relative to an independent observer midway between the two. With what velocity does rocket 2 approaches rocket 1?

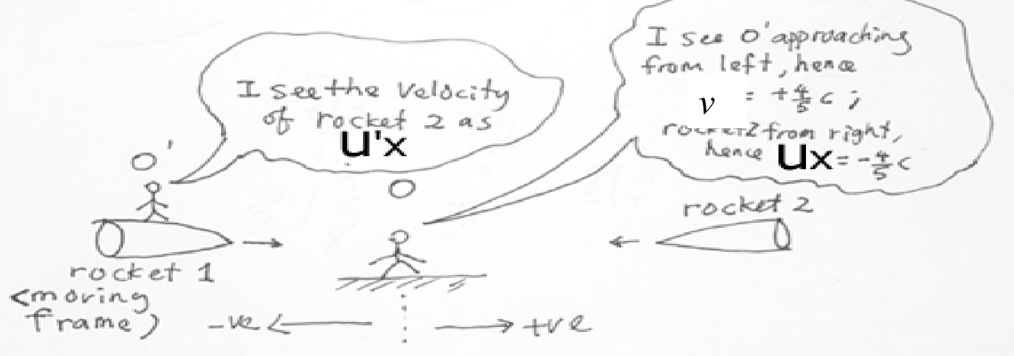
156

Diagramatical translation of the question in text

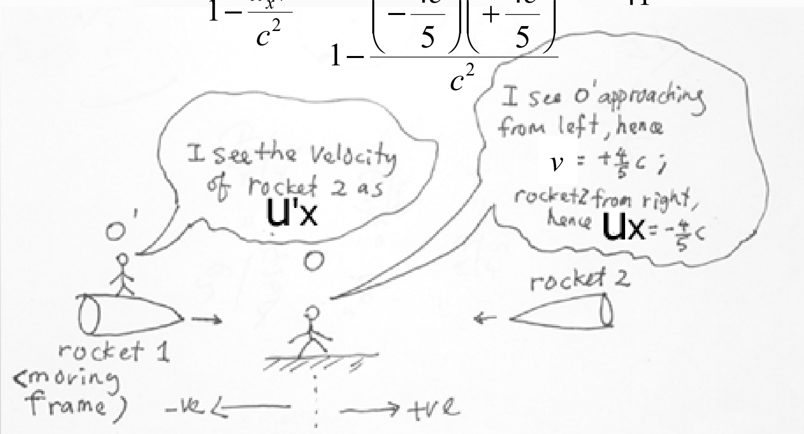


Note: c.f. in GT, their relative speed would just be 4c/5 + 4c/5 = 1.6 c which violates constancy of speed of light postulate. See how LT handle this situation:

- Choose the observer in the middle as in the stationary frame, O
- Choose rocket 1 as the moving frame O'
- Call the velocity of rocket 2 as seen from rocket 1 u'x. This is the quantity we are interested in
- Frame O' is moving in the +ve direction as seen in O, so $v = +4c/5$
- The velocity of rocket 2 as seen from O is in the
- -ve direction, so $ux = -4c/5$
- Now, what is the velocity of rocket 2 as seen from frame O', $u'x$ = ? (intuitively, u 'x must be in the negative direction)



Using LT: $$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{\left(-\frac{4c}{5}\right) - \left(+\frac{4c}{5}\right)}{1 - \frac{\left(-\frac{4c}{5}\right)\left(+\frac{4c}{5}\right)}{c^2}} = -\frac{40}{41}c$$



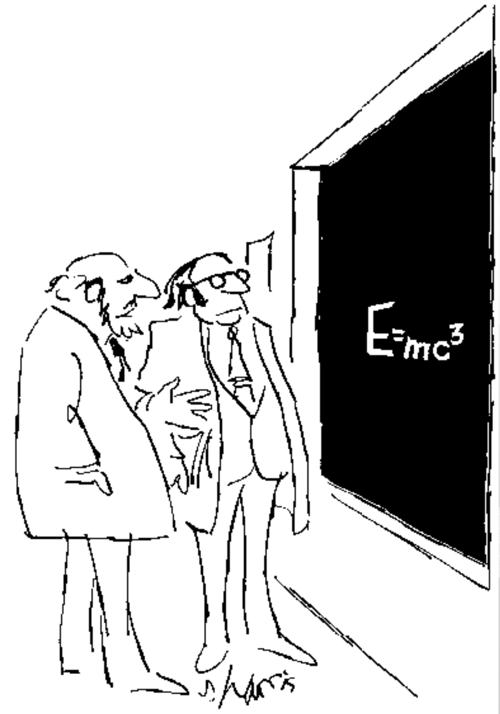
i.e. the velocity of rocket 2 as seen from rocket 1 (the moving frame, O') is –40c/41, which means that O' sees rocket 2 moving in the –ve direction (to the left in the picture), as expected.

159

Doppler Shift

- R.I.Y

160



"These days *everything* is higher."

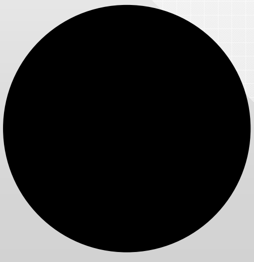
161

CHAPTER 2

- PROPERTIES OF WAVES AND MATTER
- BLACK BODY RADIATION

Matter, energy and interactions

- One can think that our universe is like a stage existing in the form of space-time as a background
- All existence in our universe is in the form of either matter or energy (Recall that matter and energy are `equivalent' as per the equation $E = mc^2$)



matter

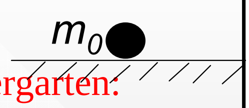
energy

Interactions

- Matter and energy exist in various forms, but they constantly transform from one to another according to the law of physics
- we call the process of transformation from one form of energy/matter to another energy/matter as ‘interactions’
- Physics attempts to elucidate the interactions between them
- But before we can study the basic physics of the matter-energy interactions, we must first have some general idea to differentiate between the two different modes of physical existence: matter and wave
- This is the main purpose of this lecture

Matter (particles)

- **Consider a particles with mass:**

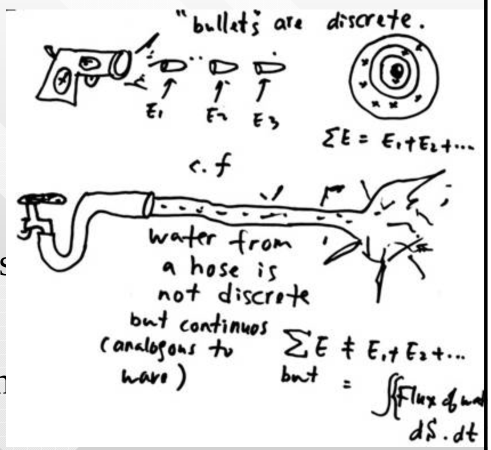


- you should know the following facts since kindergarten:
- A particle is discrete, or in another words, corpuscular, in nature.
- a particle can be localized completely, has mass and electric charge that can be determined with infinite precision (at least in principle)
- So is its momentum
- These are all implicitly assumed in Newtonian mechanics
- This is to be contrasted with energy exists in the forms of wave which is not corpuscular in nature (discuss later)

Energy in particle is corpuscular (discrete) i.e. not spread out all over the place like a continuum

- The energy carried by a particle is given by

$$E^2 = m_0^2 c^4 + p^2 c^2$$



- The energy of a particles is concentrated within the boundary of a particle (e,g. in the bullet)
- Hence we say "energy of a particle is corpuscular"
- This is in contrast to the energy carried by the water from the host, in which the energy is distributed spread all over the space in a continuous manner

Example of particles

- Example of `particles': bullet, billiard ball, you and me, stars, sands, etc...
- Atoms, electrons, molecules (or are they?)

What is not a `particle'?

- Waves - electromagnetic radiation (light is a form of electromagnetic radiation), mechanical waves and matter waves is classically thought to not have attributes of particles as mentioned

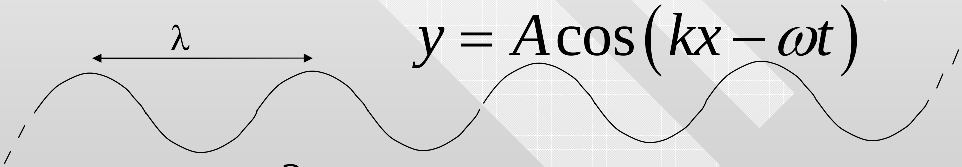
Analogy

- Imagine energy is like water
- A cup containing water is like a particle that carries some energy within it
- Water is contained within the cup as in energy is contained in a particle.
- The water is not to be found outside the cup because they are all retained inside it. Energy of a particle is corpuscular in the similar sense that they are all inside the carrier which size is a finite volume.
- In contrast, water that is not contained by any container will spill all over the place (such as water in the great ocean). This is the case of the energy carried by wave where energy is not concentrated within a finite volume but is spread throughout the space

Wave

- Three kinds of wave in Nature: mechanical, electromagnetical and matter waves
- The simplest type of wave is strictly sinusoidal and is characterised by a `sharp' frequency ν (= 1/*T*, *T* = the period of the wave), wavelength λ and its travelling speed *c*

$$y = A\cos\left(kx - \omega t\right)$$



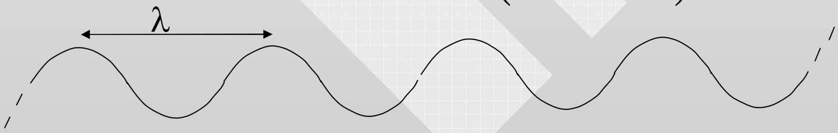
$$C = \lambda\nu; k = \frac{2\pi}{\lambda}$$

A `pure' (or 'plain') wave which has `sharp' wavelength and frequency

Quantities that characterise a pure wave

- The quantities that quantify a pure (or called a plane) wave:
  λ, wave length, equivalent to $k = 2\pi / \lambda$, the wave number
  $\nu = 1/T$, frequency, equivalent angular frequency,
  $\omega = 2\pi\nu$
- c speed of wave, related to the above quantities via
  $c = \lambda\nu = \omega / k$

$$y = A\cos\left(kx - \omega t\right)$$



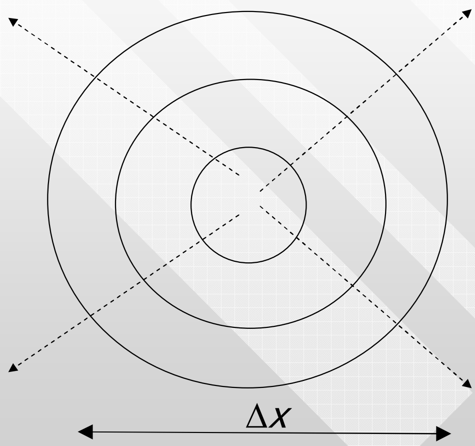
$$C = \lambda\nu;$$

Where is the wave?

- For the case of a particle we can locate its location and momentum precisely
- But how do we 'locate' a wave?
- Wave spreads out in a region of space and is not located in any specific point in space like the case of a particle
- To be more precise we says that a plain wave exists within some region in space, Δx
- For a particle, Δx is just the 'size' of its dimension, e.g. Δx for an apple is 5 cm, located exactly in the middle of a square table, x = 0.5 m from the edges. In principle, we can determine the position of x to infinity
- But for a wave, Δx could be infinity

In fact, for the `pure' (or 'plain') wave which has `sharp' wavelength and frequency mentioned in previous slide, the Δx is infinity

For example, a ripple



the ripple exists within the region Δx

A pure wave has $\Delta x \rightarrow$ infinity

- If we know the wavelength and frequency of a pure wave with infinite precision (= the statement that the wave number and frequency are 'sharp'), one can shows that :
- The wave cannot be confined to any restricted region of space but must have an infinite extension along the direction in which it is propagates
- In other words, the wave is 'everywhere' when its wavelength is 'sharp'
- This is what it means by the mathematical statement that "Δx is infinity"

13

More quantitatively,

$$\Delta x \Delta \lambda \geq \lambda^2$$

- This is the uncertainty relationships for classical waves

  $\Delta\lambda$ is the uncertainty in the wavelength.
- When the wavelength `sharp' (that we knows its value precisely), this would mean $\Delta\lambda = 0$.
- In other words, $\Delta\lambda \rightarrow$ infinity means we are totally ignorant of what the value of the wavelength of the wave is.

14

Δx is the uncertainty in the location of the wave (or equivalently, the region where the wave exists)

- $\Delta x = 0$ means that we know exactly where the wave is located, whereas $\Delta x \rightarrow$ infinity means the wave is spread to all the region and we cannot tell where is it's `location'

$\Delta\lambda\ \Delta x \geqslant \lambda^2$ means the more we knows about x, the less we knows about λ as Δx is inversely proportional to $\Delta\lambda$

Other equivalent form

- $\Delta x \Delta \lambda \geq \lambda^2$ can also be expressed in an equivalence form

$$\Delta t \Delta \nu \geq 1$$

via the relationship $c = \nu\lambda$ and $\Delta x = c\Delta t$

- Where Δt is the time required to measure the frequency of the wave
- The more we know about the value of the frequency of the wave, the longer the time taken to measure it
- If u want to know exactly the precise value of the frequency, the required time is Δt = infinity
- We will encounter more of this when we study the Heisenberg uncertainty relation in quantum physics

- The classical wave uncertain relationship

$$\Delta x \Delta \lambda \geq \lambda^2$$

- can also be expressed in an equivalence form

$$\Delta t \Delta \nu \geq 1$$

via the relationship $c = \nu\lambda$ and $\Delta x = c\Delta t$

- Where Δt is the time required to measure the frequency of the wave
- The more we know about the value of the frequency of the wave, the longer the time taken to measure it
- If u want to know exactly the precise value of the frequency, the required time is Δt = infinity
- We will encounter more of this when we study the Heisenberg uncertainty relation in quantum physics

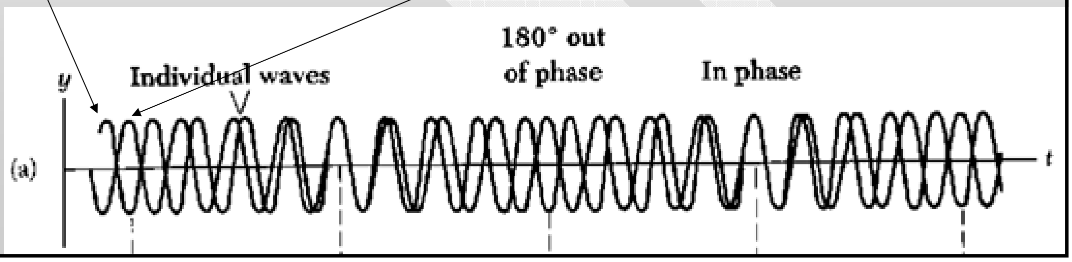
Wave can be made more ``localised"

- We have already shown that the 1-D plain wave is infinite in extent and can't be properly localised (because for this wave, $\Delta x \rightarrow$infinity)
- However, we can construct a relatively localised wave (i.e., with smaller Δx) by :
- adding up two plain waves of slightly different wavelengths (or equivalently, frequencies)

Constructing wave groups

- Two pure waves with slight difference in frequency and wave number $\Delta\omega = \omega_1 - \omega_2$, $\Delta k = k_1 - k_2$, are superimposed

$$y_1 = A\cos(k_1 x - \omega_1 t); \quad y_2 = A\cos(k_2 x - \omega_2 t)$$

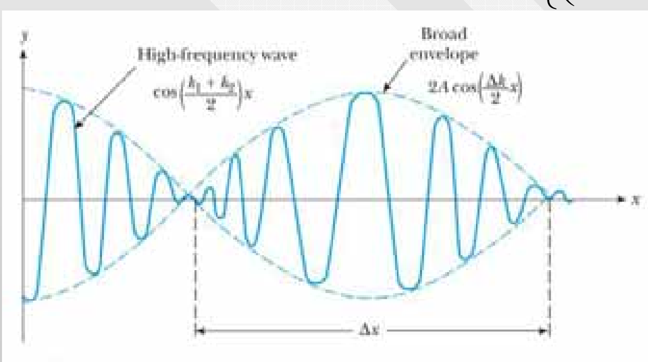


Envelop wave and phase wave

The resultant wave is a 'wave group' comprise of an `envelop' (or the group wave) and a phase waves

$$y = y_1 + y_2$$

$$= 2A\cos\frac{1}{2}\left(\{k_2 - k_1\}x - \{\omega_2 - \omega_1\}t\right)\cdot\cos\left\{\left(\frac{k_2 + k_1}{2}\right)x - \left(\frac{\omega_2 + \omega_1}{2}\right)t\right\}$$



- As a comparison to a plain waves, a group wave is more 'localised' (due to the existence of the wave envelop. In comparison, a plain wave has no `envelop' but only 'phase wave')

- It comprises of the slow envelop wave

$$2A\cos\frac{1}{2}\left(\{k_2-k_1\}x-\{\omega_2-\omega_1\}t\right)=2A\cos\frac{1}{2}\left(\Delta kx-\Delta\omega t\right)$$

that moves at group velocity $v_g = \Delta\omega/\Delta k$

- and the phase waves (individual waves oscillating inside the envelop)

$$\cos\left\{\left(\frac{k_2+k_1}{2}\right)x-\left(\frac{\omega_2+\omega_1}{2}\right)t\right\}=\cos\left\{k_p x-\omega_p t\right\}$$

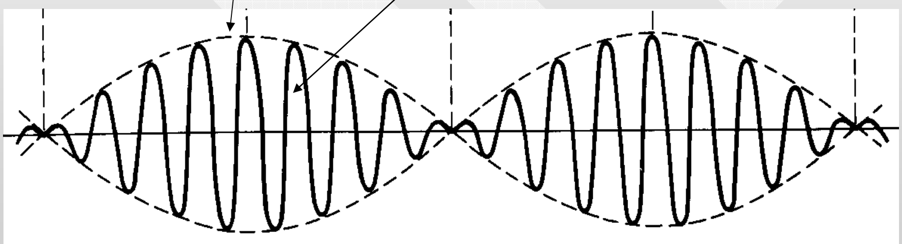
moving at phase velocity $v_p = \omega_p/k_p$

In general, $v_g = \Delta\omega/\Delta k << v_p = (\omega_1+\omega_2)/(k_1 + k_2)$ because $\omega_2 \approx \omega_1$, $k_1 \approx k_2$

$$y = y_1 + y_2 = \left\{2A\cos\frac{1}{2}\left(\Delta kx-\Delta\omega t\right)\right\}\cdot\cos\left\{k_p x-\omega_p t\right\}$$

Phase waves

`envelop' (group waves). Sometimes it's called 'modulation'



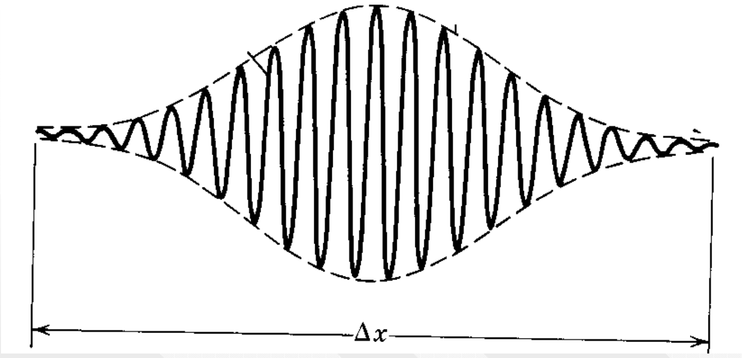
Energy is carried at the speed of the group wave

- The energy carried by the group wave is concentrated in regions in which the amplitude of the envelope is large
- The speed with which the waves' energy is transported through the medium is the speed with which the envelope advances, not the phase wave
- In this sense, the envelop wave is of more 'physical' relevance in comparison to the individual phase waves (as far as energy transportation is concerned)

Wave pulse – an even more `localised' wave

- In the previous example, we add up only two slightly different wave to form a train of wave group
- An even more `localised' group wave – what we call a "*wavepulse*" can be constructed by adding more sine waves of different numbers k_i *and* possibly different amplitudes so that they interfere constructively over a small region Δx and outside this region they interfere destructively so that the resultant field approach zero
- Mathematically,

$$y_{\text{wave pulse}} = \sum_i^{\infty} A_i \cos(k_i x - \omega_i t)$$



A wavepulse – the wave is well localised within Δx. This is done by adding a lot of waves with with their wave parameters $\{A_i, k_i, \omega_i\}$ slightly differ from each other (i = 1, 2, 3….as many as it can)

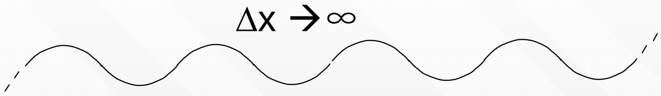
such a wavepulse will move with a velocity

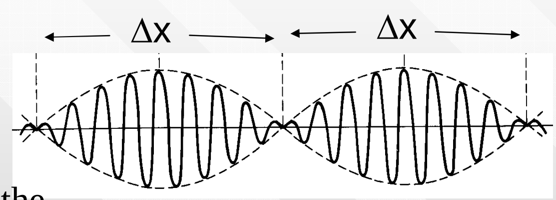
$$v_g = \left.\frac{d\omega}{dk}\right]_{k_0}$$

(c.f the group velocity considered earlier $v_g = \Delta\omega/\Delta k$)

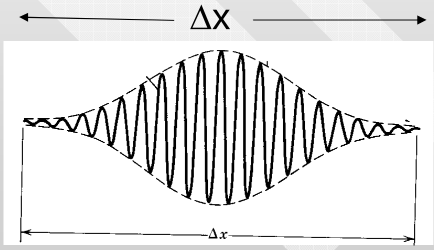
Comparing the three kinds of wave

$\Delta x \rightarrow \infty$





Which wave is the most localised?



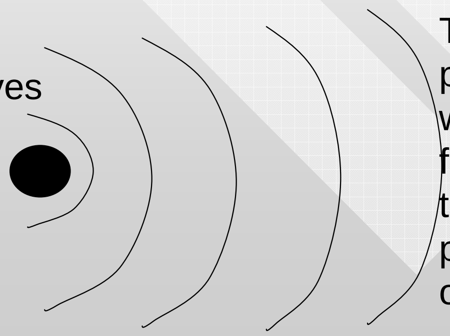
Why are waves and particles so important in physics?

- Waves and particles are important in physics because they represent the only modes of energy transport (interaction) between two points.
- E.g we signal another person with a thrown rock (a particle), a shout (sound waves), a gesture (light waves), a telephone call (electric waves in conductors), or a radio message (electromagnetic waves in space).

Interactions take place between

(i) particles and particles (e.g. in particle-particle collision, a girl bangs into a guy) or

(ii)waves and particle, in which a particle gives up all or part of its energy to generate a wave, or when all or part of the energy carried by a wave is absorbed/dissipated by a nearby particle (e.g. a wood chip dropped into water, or an electric charge under acceleration, generates EM wave)

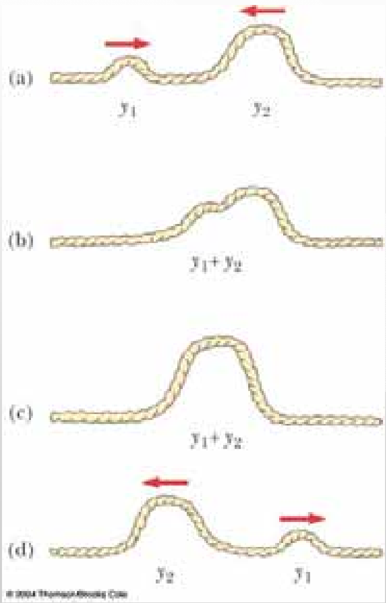


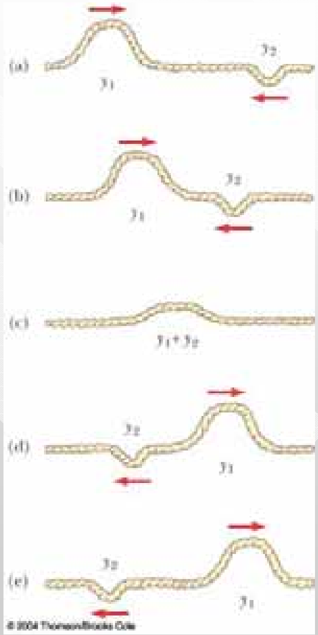
This is an example where particle is interacting with wave; energy transform from the electron’s K.E. to the energy propagating in the form of EM wave wave

Waves superimpose, not collide

- In contrast, two waves do not interact in the manner as particle-particle or particle-wave do
- Wave and wave simply “**superimpose**”: they pass through each other essentially unchanged, and their respective effects at every point in space simply add together according to the principle of superposition to form a resultant at that point -- a sharp contrast with that of two small, impenetrable particles

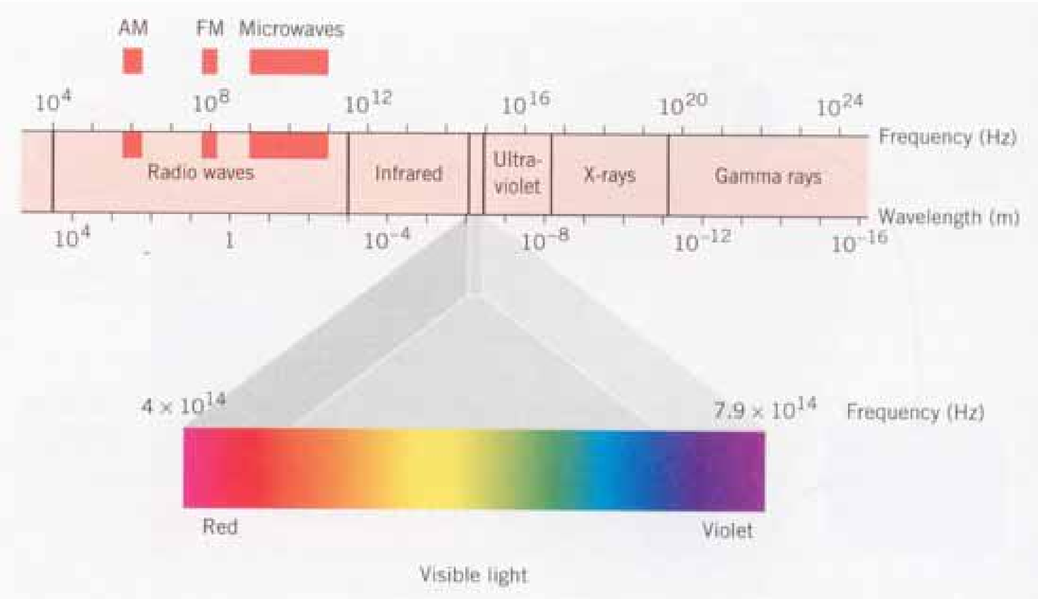
Superposition of waves



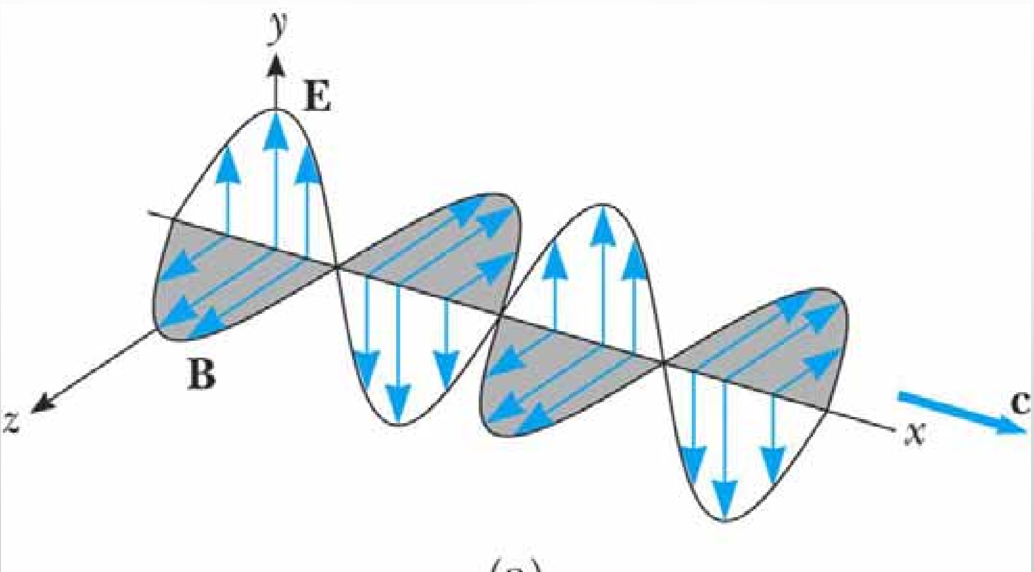


Electromagnetic (EM) wave

- According to Maxwell theory, light is a form of energy that propagates in the form of electromagnetic wave
- In Maxwell theory light is synonym to electromagnetic radiation is synonym to electromagnetic wave
- Other forms of EM radiation include heat in the form of infra red radiation, visible light, gamma rays, radio waves, microwaves, x-rays



A pure EM wave



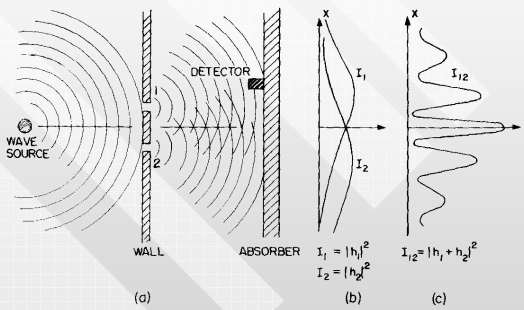
(a)

Heinrich Hertz (1857-1894), German, Established experimentally that light is EM wave

Interference experiment with waves

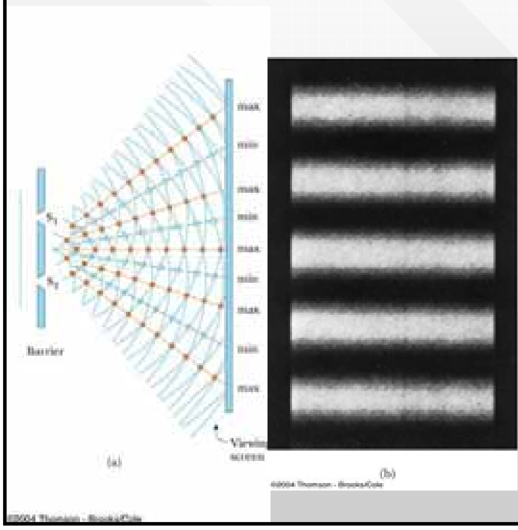
- If hole 1 (2) is block, intensity distribution of (I_1) I_2 is observed
- However, if both holes are opened, the intensity of I_{12} is such that $I_{12} \neq I_1 + I_2$
- Due to the wave nature, the intensities do not simply add
- In addition, and interference term exist,

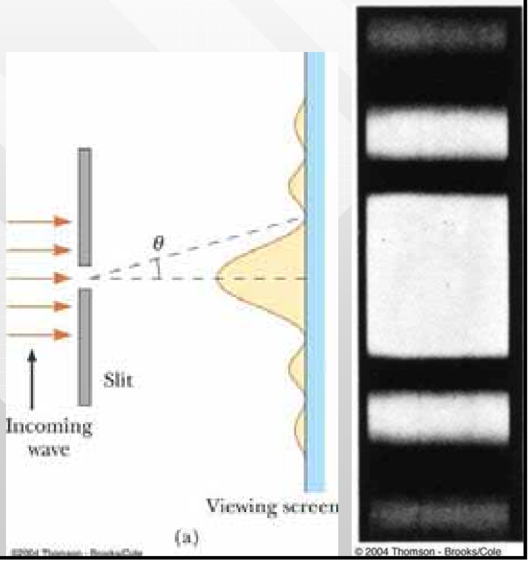
  $I_{12} = I_1 + I_2 + 2\cos\delta\,(I_1 + I_2)$
- "waves interfere"



Pictures of interference and diffraction pattern in waves

- Interference
- diffraction



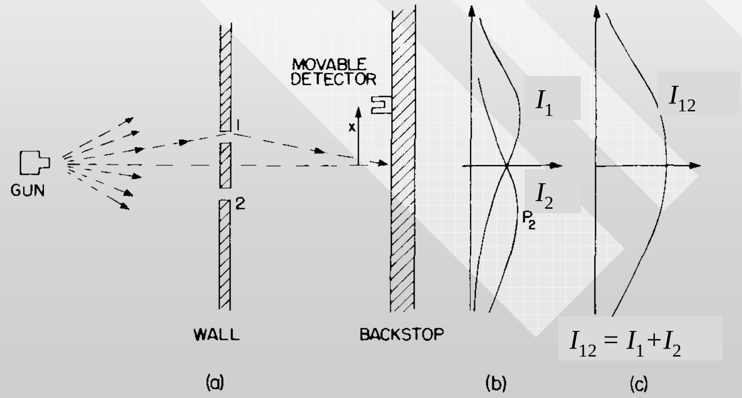


Since light display interference and diffraction pattern, it is wave

- Furthermore, Maxwell theory tell us what kind of wave light is
- It is electromagnetic wave
- (In other words it is not mechanical wave)

Interference experiment with bullets (particles)

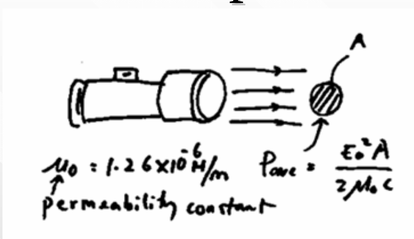
- I_2, I_1 are distribution of intensity of bullet detected with either one hole covered. I_{12} the distribution of bullets detected when both holes opened
- Experimentally, $I_{12} = I_1 + I_2$ (the individual intensity simply adds when both holes opened)
- Bullets always arrive in identical lump (corpuscular) and display no interference

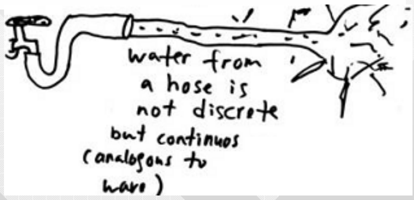


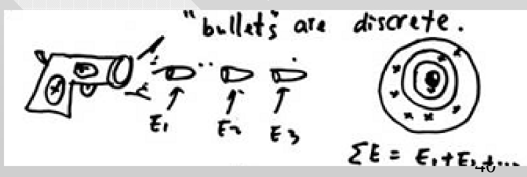
EM radiation transports energy in flux, not in bundles of particles

- The way how wave carries energy is described in terms of 'energy flux', in unit of energy per unit area per unit time
- Think of the continuous energy transported by a stream of water in a hose

This is in contrast to a stream of 'bullet' from a machine gun where the energy transported by such a steam is discrete in nature

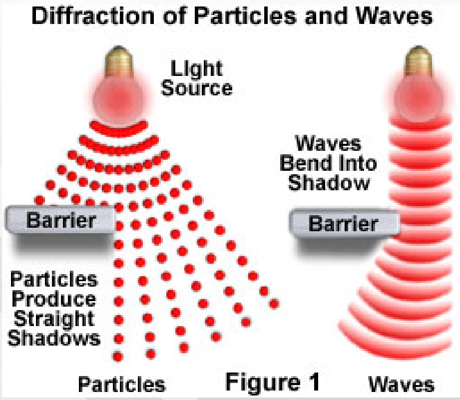






Essentially,

- Particles and wave are disparately distinct phenomena and are fundamentally different in their physical behaviour
- Free particles only travel in straight line and they don't bend when passing by a corner
- However, for light, it does
- Light, according to Maxwell's EM theory, is EM wave
- It displays wave phenomena such as diffraction and interference that is not possible for particles
- Energy of the EM wave is transported in terms of energy flux



BLACK BODY RADIATION

- Object that is HOT (anything > 0 K is considered "hot") emits EM radiation
- For example, an incandescent lamp is red HOT because it emits a lot of EM wave, especially in the IR region

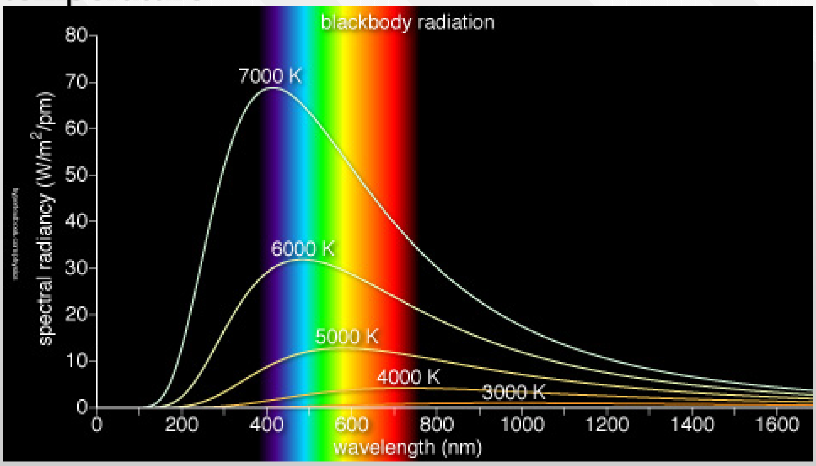
Attempt to understand the origin of radiation from hot bodies from classical theories

- In the early years, around 1888 – 1900, light is understood to be EM radiation
- Since hot body radiate EM radiation, hence physicists at that time naturally attempted to understand the origin of hot body in terms of classical EM theory and thermodynamics (which has been well established at that time)

- All hot object radiate EM wave of all wavelengths
- However, the energy intensities of the wavelengths differ continuously from wavelength to wavelength (or equivalently, frequency)
- Hence the term: the spectral distribution of energy as a function of wavelength

Spectral distribution of energy in radiation depends only on temperature

- The distribution of intensity of the emitted radiation from a hot body at a given wavelength depends on the temperature



Radiance

- In the measurement of the distribution of intensity of the emitted radiation from a hot body, one measures dI where dI is the intensity of EM radiation emitted between λ and λ +dλ about a particular wavelength λ.
- Intensity = power per unit area, in unit if Watt per m^2.
- **Radiance** $R(\lambda, T)$ is defined as per dI = $R(\lambda, T)$ dλ
- $R(\lambda, T)$ is the power radiated per unit area (*intensity*) per unit wavelength interval at a given wavelength λ and a given temperature T.
- It's unit could be in Watt per meter square per m or
- W per meter square per nm.

Total radiated power per unit area

- The total power radiated per unit area (intensity) of the BB is given by the integral

$$I(T) = \int_0^\infty R(\lambda, T)\,\mathrm{d}\lambda$$

- For a blackbody with a total area of *A*, its total power emitted at temperature *T* is

$$P(T) = AI(T)$$

- Note: The SI unit for *P* is Watt, SI unit for *I* is Watt per meter square; for *A*, the SI unit is meter square

Introducing idealised black body

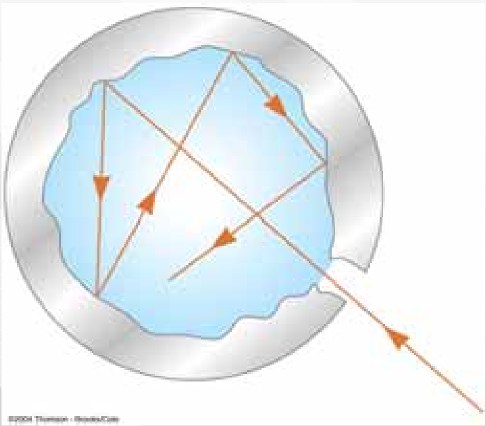
- In reality the spectral distribution of intensity of radiation of a given body could depend on the type of the surface which may differ in absorption and radiation efficiency (i.e. frequency-dependent)
- This renders the study of the origin of radiation by hot bodies case-dependent (which means no good because the conclusions made based on one body cannot be applicable to other bodies that have different surface absorption characteristics)
- E.g. At the same temperature, the spectral distribution by the exhaust pipe from a Proton GEN2 and a Toyota Altis is different

Emmissivity, *e*

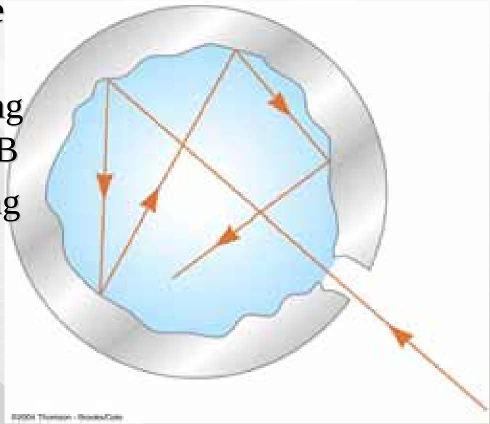
- As a strategy to overcome this non-generality, we introduce an idealised black body which, by definition, absorbs all radiation incident upon it, regardless of frequency
- Such idealised body is universal and allows one to disregard the precise nature of whatever is radiating, since all BB behave identically
- All real surfaces could be approximate to the behavior of a black body via a parameter EMMISSIVITY *e* (*e*=1 means ideally approximated, 0< *e* < 1 means poorly approximated)

Blackbody Approximation

- A good approximation of a black body is a small hole leading to the inside of a hollow object
- The HOLE acts as a perfect absorber
- The Black Body is the HOLE

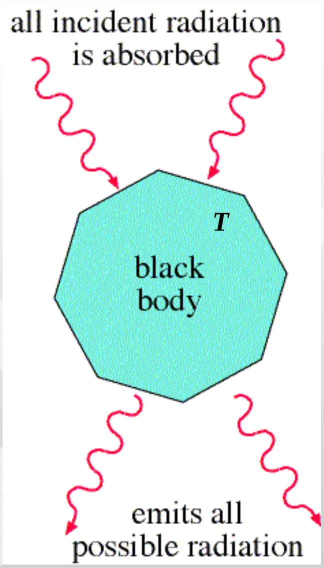


- Any radiation striking the HOLE enters the cavity, trapped by reflection until is absorbed by the inner walls
- The walls are constantly absorbing and emitting energy at thermal EB
- The nature of the radiation leaving the cavity through the hole depends only on the temperature of the cavity and not the detail of the surfaces nor frequency of the radiation



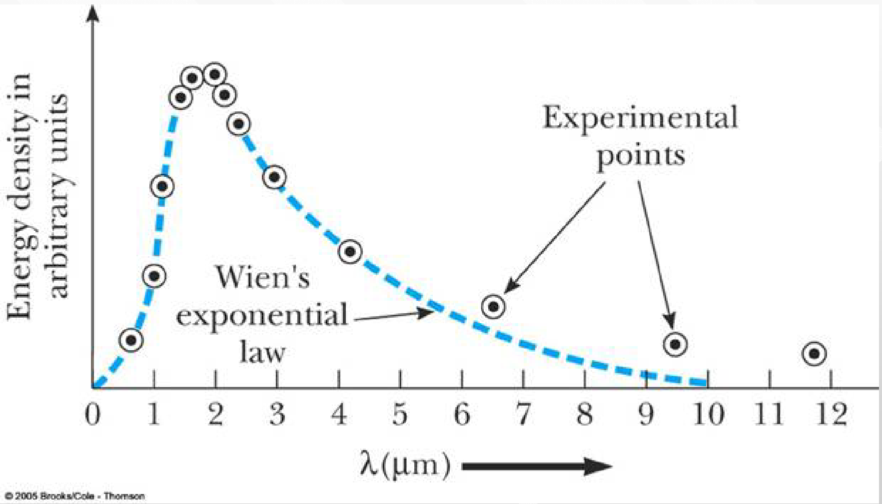
Essentially

- A black body in thermal EB absorbs and emits radiation at the same rate
- The HOLE effectively behave like a Black Body because it effectively absorbs all radiation fall upon it
- And at the same time, it also emits all the absorbed radiations at the same rate as the radiations are absorbed
- The measured spectral distribution of black bodies is universal and depends only on temperature.
- In other words: THE SPECTRAL DISTRIBUTION OF EMISSION DEPENDS SOLELY ON THE TEMPERATURE AND NOT OTHER DETAILS.



BB at thermodynamic equilibrium at a fixed temperature

Experimentally measured curve of a BB



Stefan's Law (an empirical law)

- $P = \sigma AeT^4$
  - P total power output of a BB
  - A total surface area of a BB
  - σ Stefan-Boltzmann constant (experimentally measured)

    $\sigma = 5.670 \times 10^{-8}\ \mathrm{W / m^2 \cdot K^4}$
- Stefan's law can be written in terms of intensity
  - $I = P/A = \sigma T^4$
    - For a blackbody, where $e = 1$

Wien's Displacement Law

- $\lambda_{max}T = 2.898 \times 10^{-3}$ m·K
  - λ_{max} is the wavelength at which the curve peaks
  - T is the absolute temperature
- The wavelength at which the intensity peaks, λ_{max}, is inversely proportional to the absolute temperature
  - As the temperature increases, the peak wavelength λ_{max} is "displaced" to shorter wavelengths.

Example

This figure shows two stars in the constellation Orion. Betelgeuse appears to glow red, while Rigel looks blue in color. Which star has a higher surface temperature?

(a) Betelgeuse

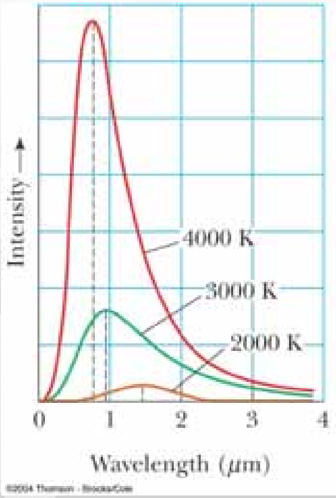
(b) Rigel

(c) They both have the same surface temperature.

(d) Impossible to determine.

Intensity of Blackbody Radiation, Summary

- The intensity increases with increasing temperature
- The amount of radiation emitted increases with increasing temperature
  - The area under the curve
- The peak wavelength decreases with increasing temperature



Example

- Find the peak wavelength of the blackbody radiation emitted by
- (A) the Sun (2000 K)
- (B) the tungsten of a light bulb at 3000 K

Solutions

- (A) the sun (2000 K)
- By Wein's displacement law,

$$\lambda_{\max} = 2.898\times10^{-3}\,\mathrm{m\cdot K} \big/ 2000\mathrm{K}$$

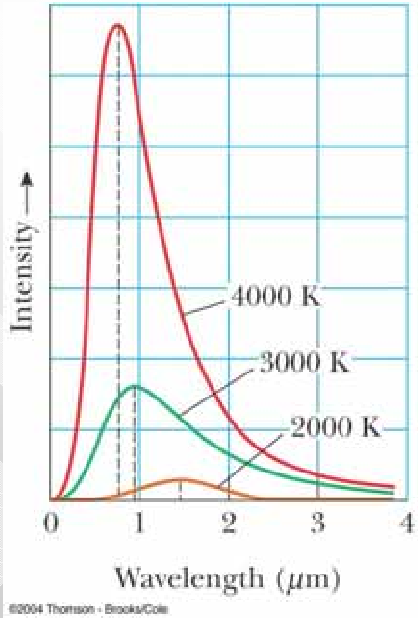
$$= 1.4\mu\mathrm{m}$$

- (infrared)
- (B) the tungsten of a lightbulb at 3000 K

$$\lambda_{\max} = 2.898\times10^{-3}\,\mathrm{m\cdot K} \big/ 5800\mathrm{K}$$

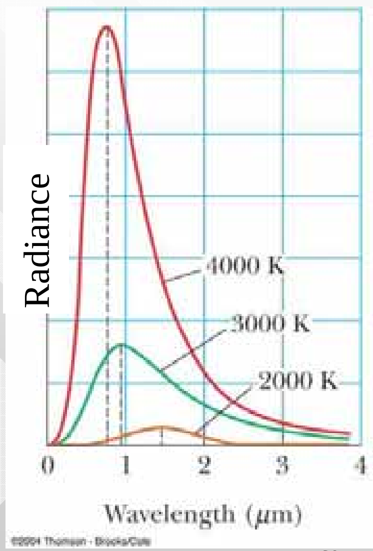
$$= 0.5\mu\mathrm{m}$$

- Yellow-green



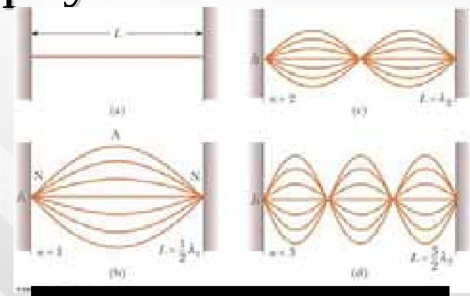
Why does the spectral distribution of black bodies have the shape as measured?

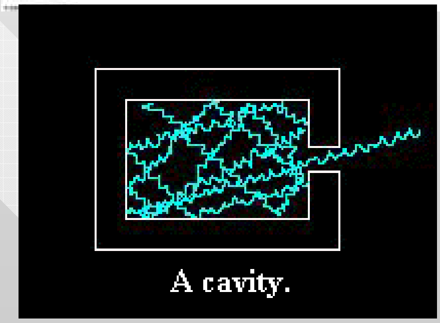
- Lord Rayleigh and James Jeans at 1890's try to theoretically derive the distribution based on statistical mechanics (some kind of generalised thermodynamics) and classical Maxwell theory
- (Details omitted, u will learn this when u study statistical mechanics later)



RJ's model of BB radiation with classical EM theory and statistical physics

- Consider a cavity at temperature *T* whose walls are considered as perfect reflectors
- The cavity supports many modes of oscillation of the EM field caused by accelerated charges in the cavity walls, resulting in the emission of EM waves at all wavelength
- These EM waves inside the cavity are the BB radiation
- They are considered to be a series of standing EM wave set up within the cavity





Number density of EM standing wave modes in the cavity

- The number of independent standing waves $G(\nu)d\nu$ in the frequency interval between ν and $\nu+d\nu$ per unit volume in the cavity is (by applying statistical mechanics)

$$G(\nu)d\nu = \frac{8\pi\nu^2 d\nu}{c^3}$$

- The next step is to find the average energy per standing wave

The average energy per standing wave, $\langle \varepsilon \rangle$

- Theorem of equipartition of energy (a mainstay theorem from statistical mechanics) says that the average energy per standing wave is
- $\langle \varepsilon \rangle = kT$

  $k = 1.38 \times 10^{-23}$ J/K, Boltzmann constant
- In classical physics, $\langle \varepsilon \rangle$ can take any value CONTINOUSLY and there is not reason to limit it to take only discrete values
- (this is because the temperature T is continuous and not discrete, hence ε must also be continuous)

Energy density in the BB cavity

- Energy density of the radiation inside the BB cavity in the frequency interval between ν and ν + dν,

  $u(\nu, T)\mathrm{d}\nu$ = the total energy per unit volume in the cavity in the frequency interval between ν and ν + dν

  = the number of independent standing waves in the frequency interval between ν and $\nu + \mathrm{d}\nu$ per unit volume, $G(\nu)\mathrm{d}\nu$, × the average energy per standing wave.

$$\Rightarrow \quad u(\nu, T)\mathrm{d}\nu = G(\nu)\mathrm{d}\nu \times \langle \varepsilon \rangle = \frac{8\pi \nu^2 kT d\nu}{c^3}$$

Energy density in terms of radiance

- The energy density in the cavity in the frequency interval between ν and $\nu + d\nu$ can be easily expressed in terms of wavelength, λ via $c = \nu\lambda$

$$u(\nu,T)\,d\nu = \frac{8\pi\nu^2 kT d\nu}{c^3} \rightarrow u(\lambda,T)\,d\lambda = \frac{8\pi kT}{\lambda^4}\,d\lambda$$

can you show this?

- In experiment we measure the BB in terms of radiance $R(\lambda,T)$ which is related to the energy density via a factor of $c/4$:
- $R(\lambda,T) = (c/4)u(\lambda,T) = \dfrac{2\pi ckT}{\lambda^4}$

Rayleigh-Jeans Law

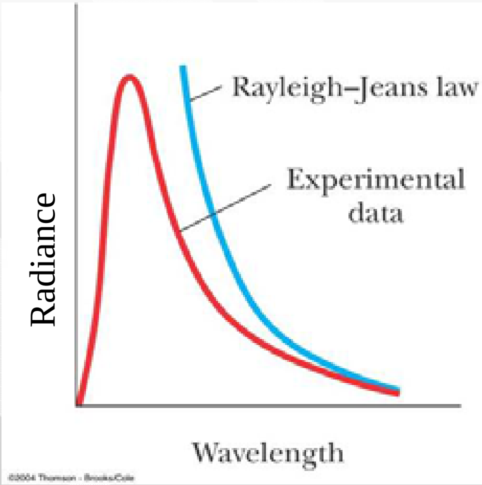
- **Rayleigh-Jeans law for the radiance (based on classical physics):**

$$R(\lambda,T) = \frac{2\pi ckT}{\lambda^4}$$

- At long wavelengths, the law matched experimental results fairly well

Rayleigh-Jeans Law, cont.

- At short wavelengths, there was a major disagreement between the Rayleigh-Jeans law and experiment
- This mismatch became known as the *ultraviolet catastrophe*
  - You would have infinite energy as the wavelength approaches zero



Max Planck

- Introduced the concept of "quantum of action"
- In 1918 he was awarded the Nobel Prize for the discovery of the quantized nature of energy

Planck’s Theory of Blackbody Radiation

- In 1900 Planck developed a theory of blackbody radiation that leads to an equation for the intensity of the radiation
- This equation is in complete agreement with experimental observations

Planck’s Wavelength Distribution Function

- Planck generated a theoretical expression for the wavelength distribution (radiance)

$$R(\lambda,T) = \frac{2\pi hc^2}{\lambda^5\left(e^{hc/\lambda kT} - 1\right)}$$

  - h = 6.626 x 10^{-34} J·s
  - h is a fundamental constant of nature

Planck’s Wavelength Distribution Function, cont.

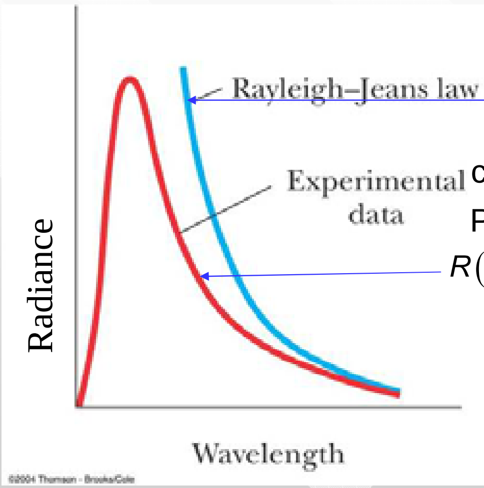
- At long wavelengths, Planck’s equation reduces to the Rayleigh-Jeans expression
- This can be shown by expanding the exponential term

$$e^{hc/\lambda kT} = 1 + \frac{hc}{\lambda kT} + \frac{1}{2!}\left(\frac{hc}{\lambda kT}\right)^2 + \ldots \approx 1 + \frac{hc}{\lambda kT}$$

in the long wavelength limit $hc \ll \lambda kT$

- At short wavelengths, it predicts an exponential decrease in intensity with decreasing wavelength
  - This is in agreement with experimental results

Comparison between Planck’s law of BB radiation and RJ’s law



$$R(\lambda,T) = \frac{2\pi ckT}{\lambda^4}$$

correctely fit by Planck's derivation

$$R(\lambda,T) = \frac{2\pi hc^2}{\lambda^5\left(e^{hc/\lambda kT} - 1\right)}$$

How Planck modeled the BB

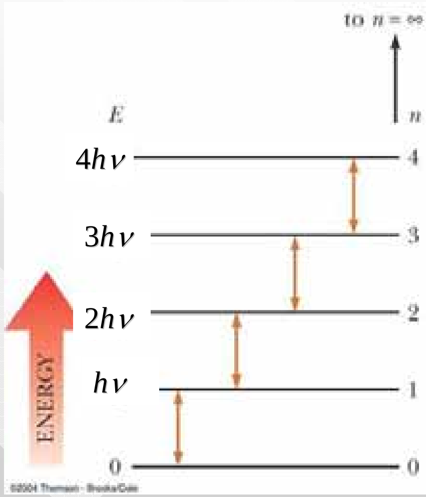
- He assumed the cavity radiation comes from the atomic oscillations in the cavity walls
- Planck made two assumptions about the nature of the oscillators in the cavity walls

Planck's Assumption, 1

- The energy of an oscillator can have only certain *discrete* values E_n
  - $E_n = nh\nu$
    - n =0,1,2,…; n is called the quantum number
    - h is Planck's constant = 6.63 x 10^{-34} Js
    - ν is the frequency of oscillation
  - the energy of the oscillator is **quantized**
  - Each discrete energy value corresponds to a different **quantum state**
  - This is in stark contrast to the case of RJ derivation according to classical theories, in which the energies of oscillators in the cavity must assume a continuous distribution

Energy-Level Diagram of the Planck Oscillator

- An **energy-level diagram** of the oscillators showing the quantized energy levels and allowed transitions
- Energy is on the vertical axis
- Horizontal lines represent the allowed energy levels of the oscillators
- The double-headed arrows indicate allowed transitions



Oscillator in Planck's theory is quantised in energies (taking only discrete values)

- The energy of an oscillator can have only certain *discrete* values $E_n = nh\nu$, n=0,1,2,3,…
- The average energy per standing wave in the Planck oscillator is

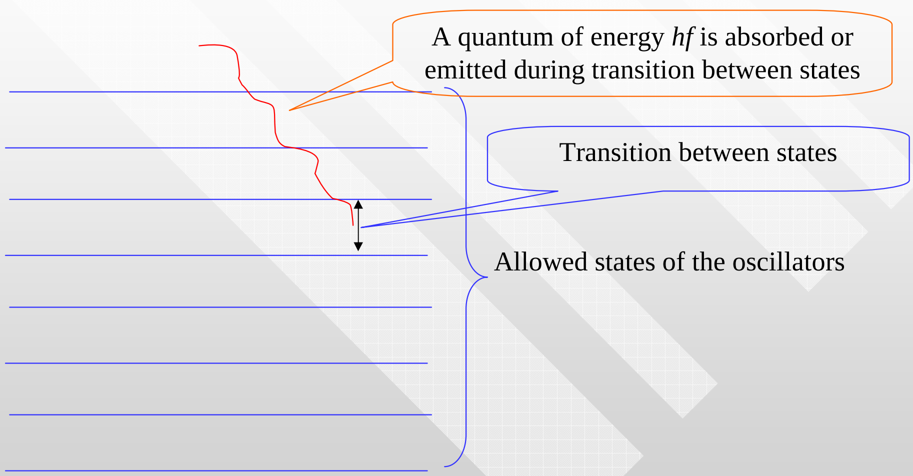
$$\langle \varepsilon \rangle = \frac{h\nu}{e^{h\nu/kT} - 1}$$ (instead of $\langle \varepsilon \rangle = kT$ in classical theories)

For the details of derivation, see slides in the appendix section

Planck's Assumption, 2

- The oscillators emit or absorb energy when making a transition from one quantum state to another
  - The entire energy difference between the initial and final states in the transition is emitted or absorbed as a single quantum of radiation
  - An oscillator emits or absorbs energy only when it changes quantum states

Pictorial representation of oscillator transition between states



Example: quantised oscillator vs classical oscillator

- A 2.0 kg block is attached to a massless spring that has a force constant k=25 N/m. The spring is stretched 0.40 m from its EB position and released.
- (A) Find the total energy of the system and the frequency of oscillation according to classical mechanics.

Solution

- In classical mechanics, $E = \frac{1}{2}kA^2 = \ldots 2.0$ J
- The frequency of oscillation is

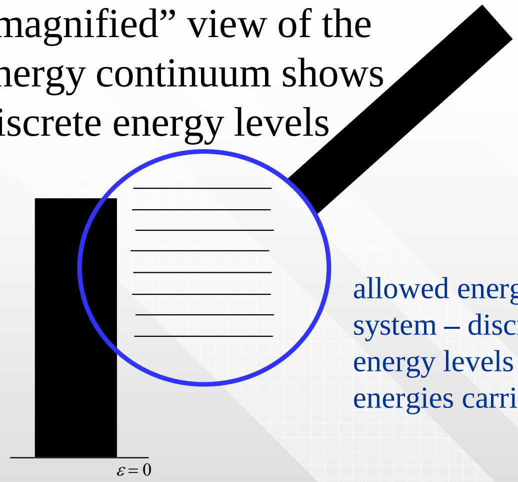
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \ldots = 0.56 \text{ Hz}$$

(B)

- (B) Assuming that the energy is quantised, find the quantum number *n* for the system oscillating with this amplitude
- Solution: This is a quantum analysis of the oscillator
- $E_n = nhf = n\,(6.63 \times 10^{-34}\ \mathrm{Js})(0.56\ \mathrm{Hz}) = 2.0\ \mathrm{J}$
- $\Rightarrow n = 5.4 \times 10^{33}$ *!!!* A very large quantum number, typical for macroscopin system

- The previous example illustrated the fact that the quantum of action, *h*, is so tiny that, from macroscopic point of view, the quantisation of the energy level is so tiny that it is almost undetectable.
- Effectively, the energy level of a macroscopic system such as the energy of a harmonic oscillator form a ‘continuum’ despite it is granular at the quantum scale

"magnified" view of the energy continuum shows discrete energy levels



allowed energies in quantised system – discrete (such as energy levels in an atom, energies carried by a photon)

allowed energies in classical system – continuous (such as an harmonic oscillator, energy carried by a wave; total mechanical energy of an orbiting planet, etc.)

To summarise

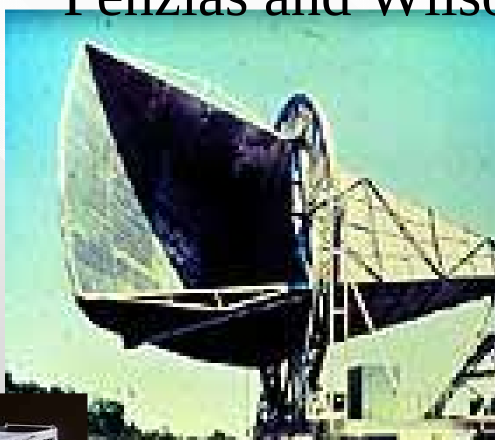
- Classical BB presents a "ultraviolet catastrophe"
- The spectral energy distribution of electromagnetic radiation in a black body CANNOT be explained in terms of classical Maxwell EM theory, in which the average energy in the cavity assumes continuous values of $<\varepsilon> = kT$ (this is the result of the wave nature of radiation)
- To solve the BB catastrophe one has to assume that the energy of individual radiation oscillator in the cavity of a BB is quantised as per $E_n = nh\nu$
- As a reslt the average energy of the radiation in the cavity
- This picture is in conflict with classical physics because in classical physics energy is in principle a continuous variable that can take any value between $0 \rightarrow \infty$
- One is then lead to the revolutionary concept that

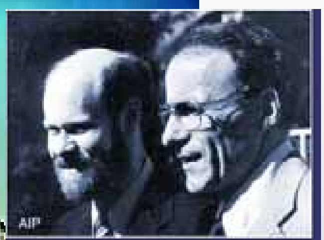
ENERGY OF AN OSCILLATOR IS QUANTISED

Cosmic microwave background (CMBR) as perfect black body radiation

1965, cosmic microwave background was first detected by Penzias and Wilson

Nobel Prize 1976





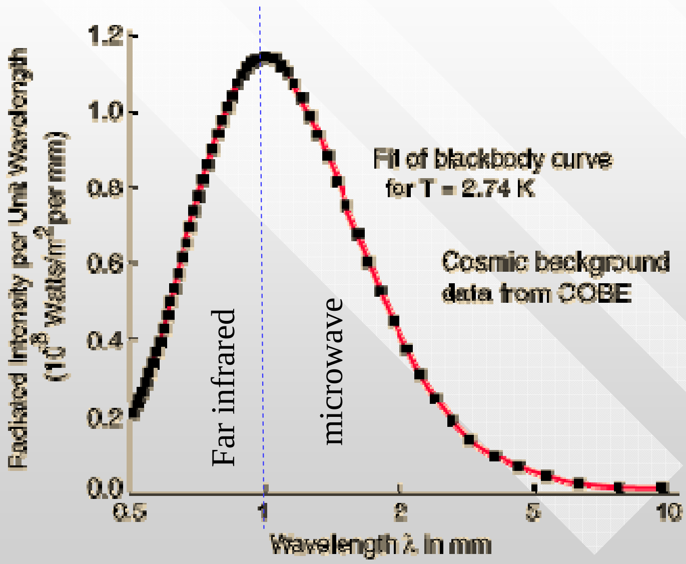
Pigeon Trap Used
Penzias and Wilson thought the static their radio antenna was picking up might be due to droppings from pigeons roosting in the antenna horn. They captured the pigeons with this trap and cleaned out the horn, but the static persisted.

CMBR – the most perfect Black Body

- Measurements of the cosmic microwave background radiation allow us to determine the temperature of the universe today.

- The brightness of the relic radiation is measured as a function of the radio frequency. To an excellent approximation it is described by a thermal of blackbody distribution with a temperature of *T*=2.735 degrees above absolute zero.
- This is a dramatic and direct confirmation of one of the predictions of the Hot Big Bang model.
- The COBE satellite measured the spectrum of the cosmic microwave background in 1990, showing remarkable agreement between theory and experiment.

The Temperature of the Universe Today, as implied from CMBR

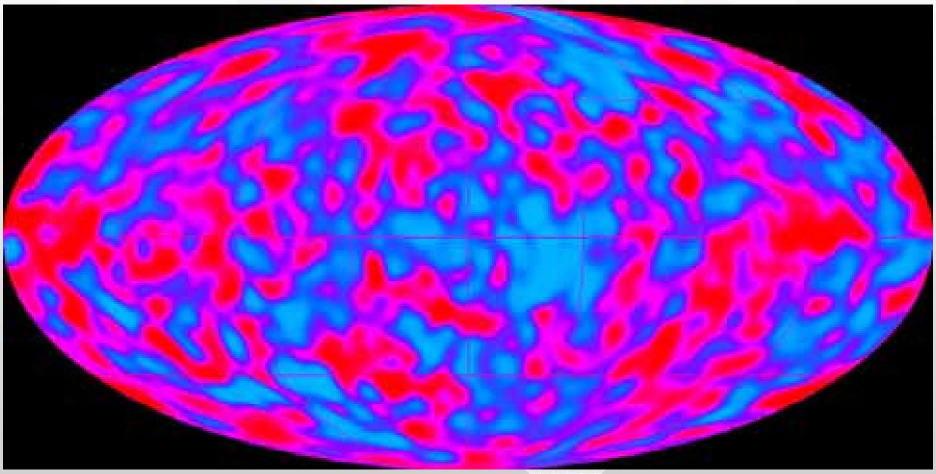


The diagram shows the results plotted in waves per centimeter versus intensity. The theoretical best fit curve (the solid line) is indistinguishable from the experimental data points (the point-size is greater than the experimental errors).

COBE

- The Cosmic Background Explorer satellite was launched twenty five years after the discovery of the microwave background radiation in 1964.
- In spectacular fashion in 1992, the COBE team announces that they had discovered `ripples at the edge of the universe', that is, the first sign of primordial fluctuations at 100,000 years after the Big Bang.
- These are the imprint of the seeds of galaxy formation.

“Faces of God”

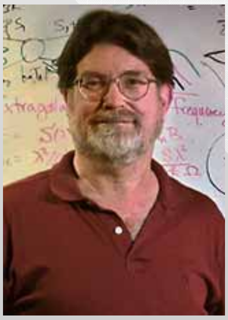


- The “faces of God”: a map of temperature variations on the full sky picture that COBE obtained.
- They are at the level of only one part in one hundred thousand.
- Viewed in reverse the Universe is highly uniform in every direction lending strong support for the cosmological principle.

The Nobel Prize in Physics 2006

"for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation"

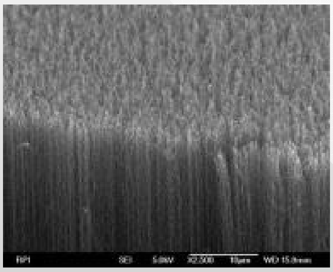
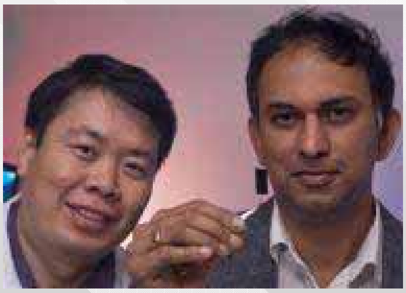
John C. Mather

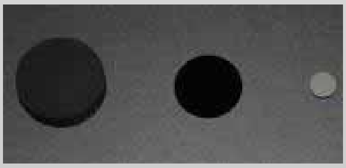


George F. Smoot

New material pushes the boundary of blackness – darkest material

- http://www.reuters.com/article/scienceNews/idUSN15550306200801 16?sp=true



Appendix

- Details of the derivation of Planck’s law

How does $E_n = nh\nu$, n=0,1,2,3,… leads to $\langle \varepsilon \rangle = \dfrac{h\nu}{e^{h\nu/kT} - 1}$?

- Statistical mechanics
- See, for example, Fizik Moden dan mekanik kuantum, by Elmer E. Anderson (English version is also available, with title Modern physics and quantum mechanics)
- Assume the modes of oscillator follow Maxwell-Boltzman distribution $N(n) = N_0 \exp\left(-\dfrac{E_n}{kT}\right)$
- so that the average energy for each oscillator is given by

(continue) how does $E_n = nhf$, n=0,1,2,3,… leads to $\langle \varepsilon \rangle = \dfrac{h\nu}{e^{h\nu/kT} - 1}$?

$$\langle \varepsilon \rangle = \frac{\sum_{n=0}^{\infty} N(n) E_n}{\sum_{n=0}^{\infty} N(n)} = \frac{\sum_{n=0}^{\infty} N_0 nh\nu \exp\left(-\dfrac{nh\nu}{kT}\right)}{\sum_{n=0}^{\infty} N_0 \exp\left(-\dfrac{nh\nu}{kT}\right)}$$

$$= \frac{0 + h\nu e^{-\frac{h\nu}{kT}} + 2h\nu e^{-\frac{2h\nu}{kT}} + 3h\nu e^{-\frac{3h\nu}{kT}} + \cdots}{1 + e^{-\frac{h\nu}{kT}} + e^{-\frac{2h\nu}{kT}} + e^{-\frac{3h\nu}{kT}} + \cdots}$$

$$= \cdots = \frac{h\nu}{e^{h\nu/kT} - 1}$$

Derivation of $R(\lambda,T)=\dfrac{2\pi hc^2}{\lambda^5\left(e^{hc/\lambda kT}-1\right)}$

- Energy density in the interval between ν dan ν+dν of the blackbody which has average energy $\langle\varepsilon\rangle=\dfrac{h\nu}{e^{h\nu/kT}-1}$ can be written down in a similar manner as for the case before,

$$G(\nu)d\nu=\frac{8\pi\nu^2 d\nu}{c^3}$$

$$u(\nu,T)d\nu=G(\nu)d\nu\cdot\langle\varepsilon\rangle=\frac{8\pi\nu^2\mathrm{d}\nu}{c^3}\cdot\frac{h\nu}{e^{h\nu/kT}-1}$$

$$h\nu\rightarrow\frac{hc}{\lambda},\frac{\nu^2 d\nu}{c^3}\rightarrow\frac{\mathrm{d}\lambda}{\lambda^4};u(\nu,T)d\nu\rightarrow u(\lambda,T)d\lambda$$

$$u(\lambda,T)\mathrm{d}\lambda=\frac{8\pi\mathrm{d}\lambda}{\lambda^4}\cdot\frac{hc/\lambda}{e^{hc/\lambda kT}-1}\rightarrow R(\lambda,T)=\frac{c}{4}\frac{u(\lambda,T)\mathrm{d}\lambda}{\mathrm{d}\lambda}=\frac{2\pi hc^2}{\lambda^5\left(e^{hc/\lambda kT}-1\right)}$$

CHAPTER 3

EXPERIMENTAL EVIDENCES
FOR PARTICLE-LIKE
PROPERTIES OF WAVES

1

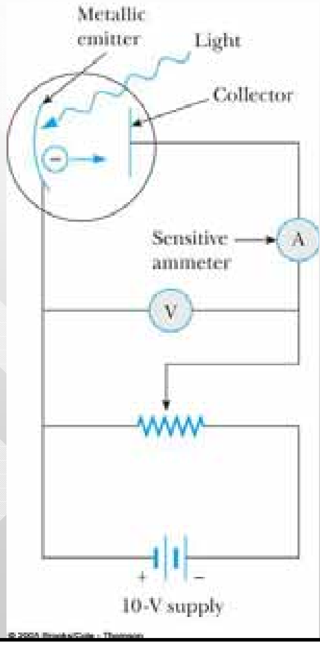
Photoelectricity

- Classically, light is treated as EM wave according to Maxwell equation
- However, in a few types of experiments, light behave in ways that is not consistent with the wave picture
- In these experiments, light behave like particle instead
- So, is light particle or wave? (recall that wave and particle are two mutually exclusive attributes of existence)
- This is a paradox that we will discuss in the rest of the course – wave particle duality

2

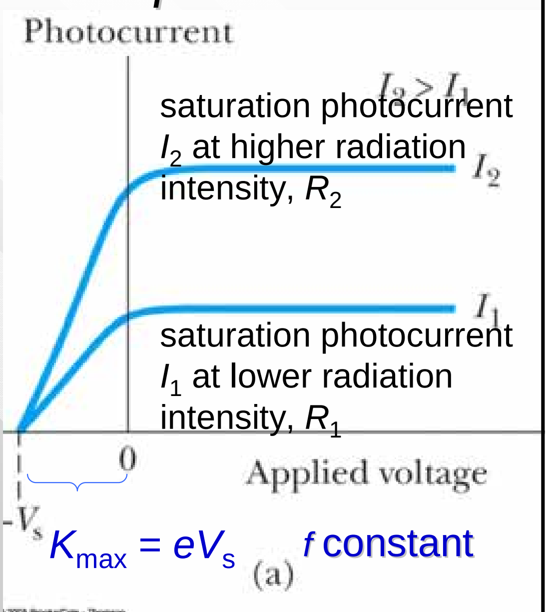
Photoelectric effect

- Photoelectrons are ejected from a metal surface when hit by radiation of sufficiently high frequency f (usually in the uv region)
- The photoelectrons are attracted to the collecting anode (positive) by potential difference applied on the anode and detected as electric current by the external circuits
- A negative voltage, relative to that of the emitter, can be applied to the collector.
- When this retarding voltage is sufficiently large the emitted electrons are repelled, and the current to the collector drops to zero (see later explanation).



Photocurrent I vs applied voltage V at constant f

- No current flows for a retarding potential more negative than $-V_s$
- The photocurrent I saturates for potentials near or above zero
- Why does the I-V curve rises gradually from $-V_s$ towards more positive V before it flat off ?



Features of the experimental result

- When the external potential difference $V = 0$, the current is not zero because the photoelectrons carry some kinetic energy, K
- K range from 0 to a maximal value, K_{max}
- As V becomes more and more positive, there are more electrons attracted towards the anode within a given time interval. Hence the pthotocurrent, I, increases with V
- Saturation of I will be achieved when all of the ejected electron are immediately attracted towards the anode once they are kicked out from the metal plates (from the curve this happens approximately when $V \approx 0$ or larger

5

- On the other direction, when V becomes more negative, the photocurrent detected decreases in magnitude because the electrons are now moving against the potential
- K_{max} can be measured. It is given by eV_s, where V_s, is the value of $|V|$ when the current flowing in the external circuit = 0
- V_s is called the '**stopping potential**'
- When $V = -V_s$, e of the highest KE will be sufficiently retarded by the external electric potential such that they wont be able to reach the collector

6

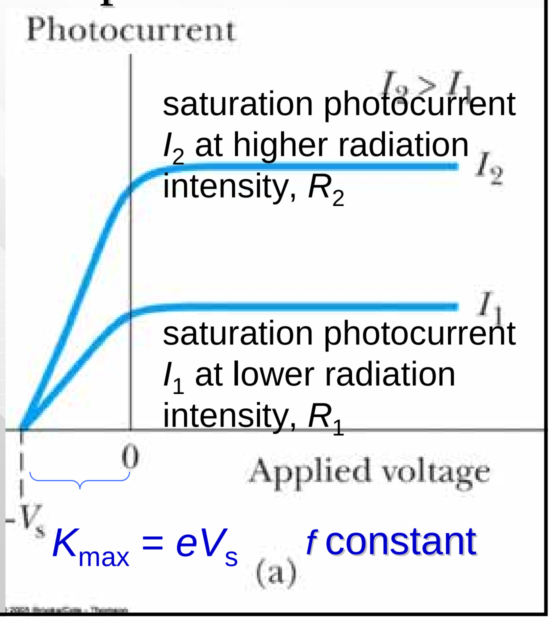
$I_2 > I_1$ because more electrons are kicked out per unit time by radiation of larger intensity, R

- The photocurrent saturates at a larger value of I_2 when it is irradiated by higher radiation intensity R_2
- This is expected as larger R means energy are imparted at a higher rate on the metal surface

7

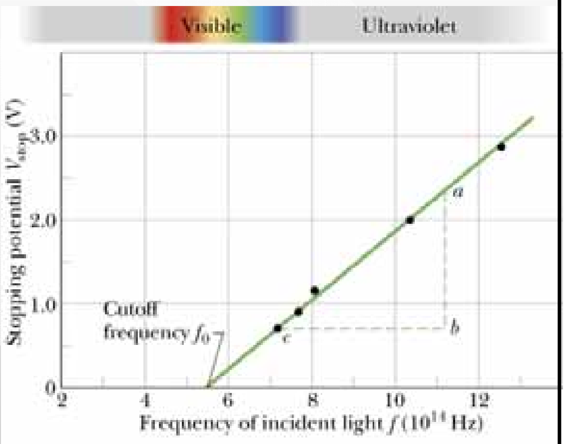
Stopping potential V_s is radiation intensity-independent

- Experimentalists observe that for a given type of surface:
- At constant frequency the maximal kinetic energy of the photoelectrons is measured to be a constant independent of the intensity of light.
- (this is a puzzle to those who thinks that light is wave)



K_{max} of photoelectrons is frequency-dependent at constant radiation intensity

- One can also detect the stopping potential V_s for a given material at different frequency (at constant radiation intensity)
- K_{max} $(=eV_s) = K_{max}$ is measured to increase linearly in the radiation frequency,
- i.e. if f increases, K_{max} too increases

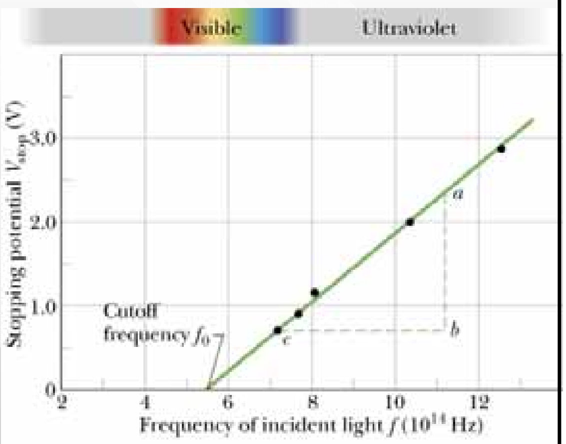


Sodium

9

Cutoff frequency, f_0

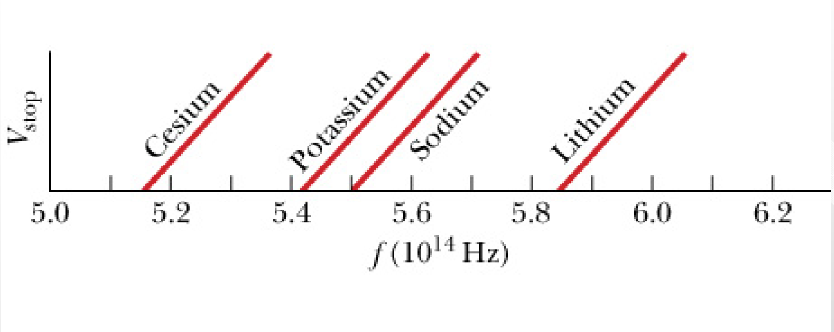
- From the same graph one also found that there exist a **cut-off frequency**, f_0, below which no PE effect occurs no matter how intense is the radiation shined on the metal surface



Sodium

10

Different material have different cut-off frequency f_0



- For different material, the cut-off frequency is different

Classical physics can't explain PE

- The experimental results of PE pose difficulty to classical physicists as they cannot explain PE effect in terms of classical physics (Maxwell EM theory, thermodynamics, classical mechanics etc.)

Puzzle one

- If light were wave, the energy carried by the radiation will increases as the intensity of the monochromatic light increases
- Hence we would also expect K_{max} of the electron to increase as the intensity of radiation increases (because K.E. of the photoelectron must come from the energy of the radiation)
- YET THE OBSERVATION IS OTHERWISE.

13

Puzzle two

- Existence of a characteristic cut-off frequency, ν_0. (previously I use f_0)
- Wave theory predicts that photoelectric effect should occur for any frequency as long as the light is intense enough to give the energy to eject the photoelectrons.
- No cut-off frequency is predicted in classical physics.

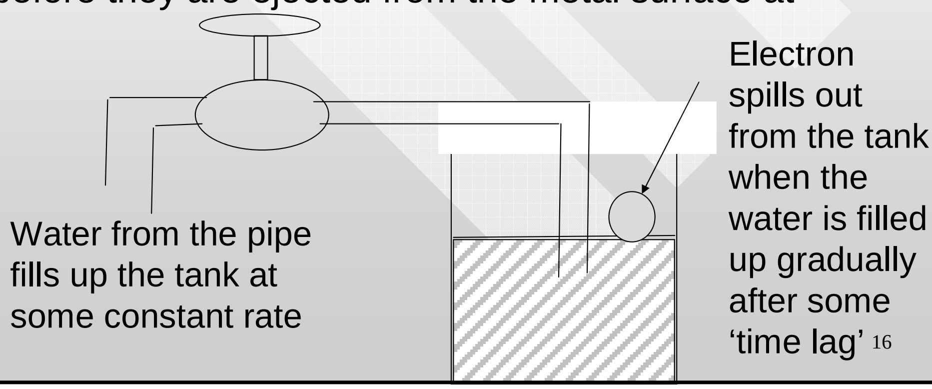
14

Puzzle three

- No detection time lag measured.
- Classical wave theory needs a time lag between the instance the light impinge on the surface with the instance the photoelectrons being ejected. Energy needs to be accumulated for the wave front, at a rate proportional to $S = \frac{E_0}{2\mu_0 c}$, before it has enough energy to eject photoelectrons. (*S* = energy flux of the EM radiation)
- But, in the PE experiments, PE is almost immediate

15

Cartoon analogy: in the wave picture, accumulating the energy required to eject an photoelectron from an atom is analogous to filling up a tank with water from a pipe until the tank is full. One must wait for certain length of time (time lag) before the tank can be filled up with water at a give rate. The total water filled is analogous to the total energy absorbed by electrons before they are ejected from the metal surface at



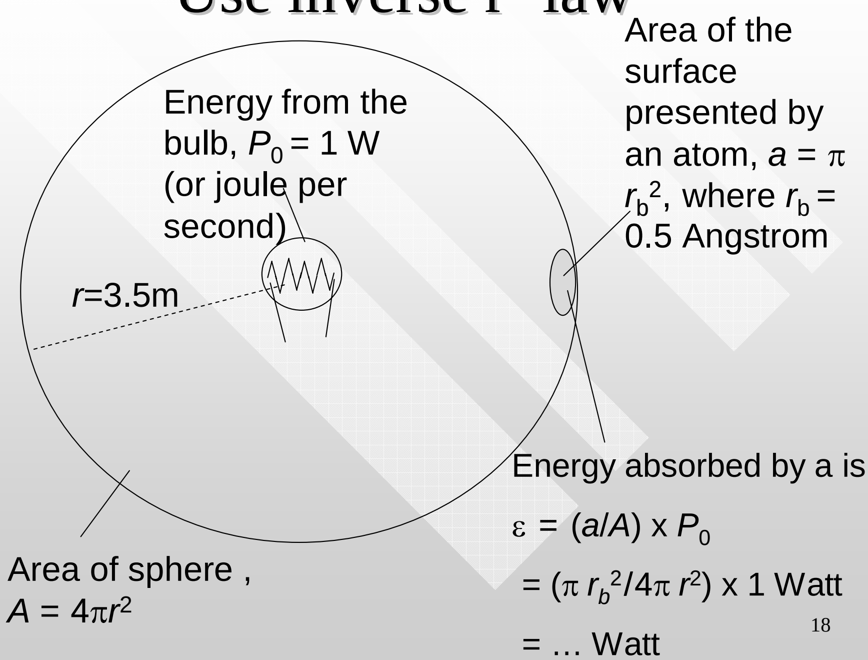
16

Wave theory and the time delay problem

- A potassium foil is placed at a distance r = 3.5 m from a light source whose output power P_0 is 1.0 W. How long would it take for the foil to soak up enough energy (=1.8 eV) from the beam to eject an electron? Assume that the ejected electron collected the energy from a circular area of the foil whose radius is 5.3 x 10^{-11} m

17

Use inverse r^2 law



18

- Time taken for a to absorb 1.8 eV is simply 1.8 x 1.6 x 10^{-19} J / ε = 5000 s = 1.4 h!!!
- In PE, the photoelectrons are ejected almost immediately but not 1.4 hour later
- This shows that the wave model used to calculate the time lag in this example fails to account for the almost instantaneous ejection of photoelectron in the PE experiment

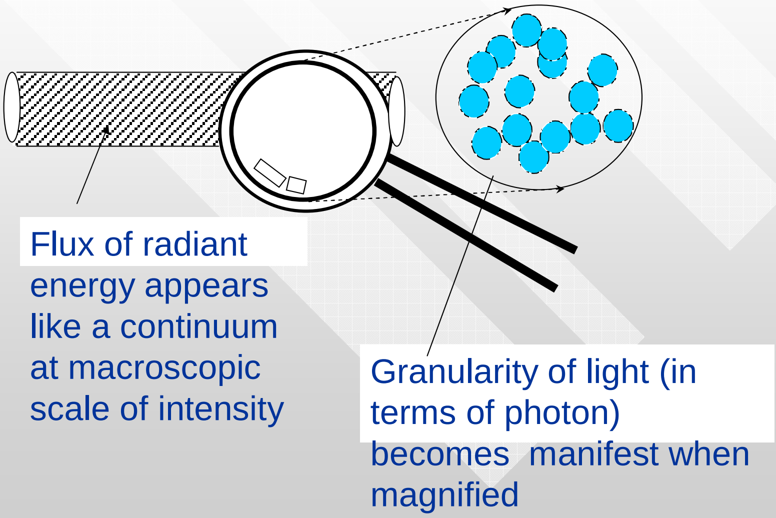
19

Einstein's quantum theory of the photoelectricity (1905)

- A Noble-prize winning theory
- To explain PE, Einstein postulates that the radiant energy of light is quantized into concentrated bundle. The discrete entity that carries the energy of the radiant energy is called photon
- Or, in quantum physics jargon, we say "photon is the quantum of light"
- Wave behaviour of light is a result of collective behaviour of very large numbers of photons

20

Photon is granular



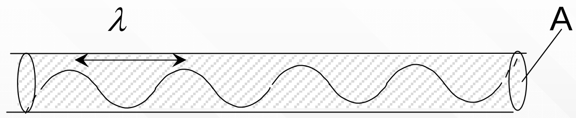
21

Wave and particle carries energy differently

- The way how photon carries energy is in in contrast to the way wave carries energy.
- For wave the radiant energy is continuously distributed over a region in space and not in separate bundles
- (always recall the analogy of water in a hose and a stream of ping pong ball to help visualisation)

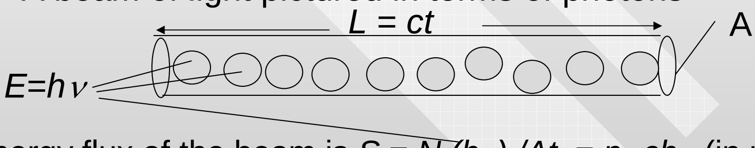
22

A beam of light if pictured as monochromatic wave (λ, ν)



Energy flux of the beam is $S = \frac{E_0}{2\mu_0 c}$ (in unit of joule per unit time per unit area), analogous to fluid in a host

A beam of light pictured in terms of photons



Energy flux of the beam is $S = N\,(h\nu)\,/At = n_0\,ch\nu$ (in unit of joule per unit time per unit area). N is obtained by 'counting' the total number of photons in the beam volume, $N = n_0 V = n_0$ x ($A\,ct$), where n_0 is the photon number density of the radiation (in unit of number per unit volume)

23

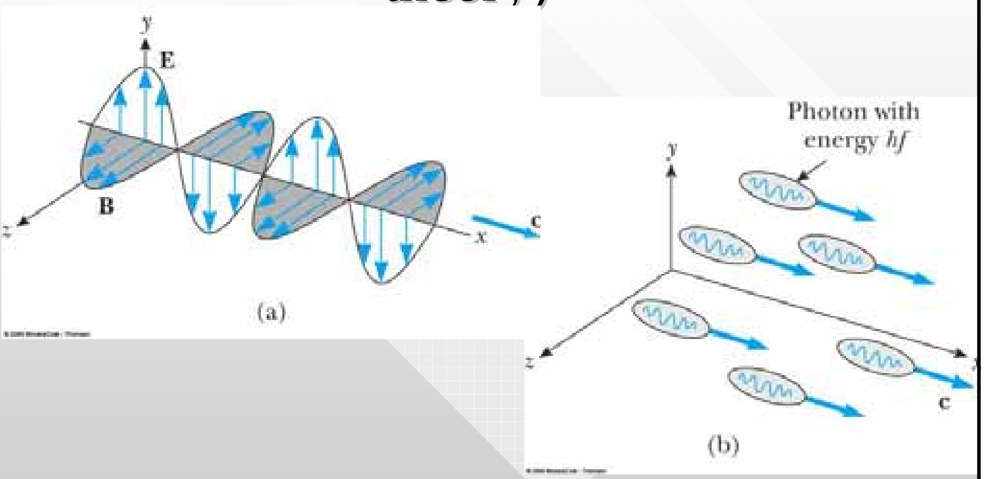
Einstein's 1st postulate

1. The energy of a single photon is $E = h\nu$. h is a proportional constant, called the Planck constant, that is to be determined experimentally.

- With this assumption, a photon will have a momentum given by $p = E/c = h/\lambda$.
- This relation is obtained from SR relationship
  $E^2 = p^2c^2 + (m_0c^2)^2$, for which the mass of a photon is zero.
- Note that in classical physics momentum is intrinsically a particle attribute not defined for wave.
  By picturing light as particle (photon), the definition of momentum for radiation now becomes feasible

24

Light as photon (in Einstein theory) instead of wave (in Classical EM theory)



$p=h/\lambda$, $E=h\nu=hc/\lambda$

$\{\nu,\lambda\}$

25

Example

- (a) What are the energy and momentum of a photon of red light of wavelength 650nm?
- (b) What is the wavelength of a photon of energy 2.40 eV?
- In atomic scale we usually express energy in eV, momentum in unit of eV/*c*, length in nm; the combination of constants, *hc*, is conveniently expressed in
- 1 eV = $1.6\text{x}10^{-19}$ J
- $hc = (6.62\text{x}10^{-34}\ \text{Js})\cdot(3\text{x}10^{8}\ \text{m/s})$
  $= [6.62\text{x}10^{-34} \cdot (1.6\text{x}10^{-19})^{-1}\text{eV}\cdot\text{s}]\cdot(3\text{x}10^{8}\ \text{m/s})$
  $= 1.24\text{eV}\cdot 10^{-6}\text{m} = 1240\text{eV}\cdot\text{nm}$
- 1 eV/*c* = $(1.6\text{x}10^{-19})\text{J}/\ (3\text{x}10^{8}\ \text{m/s}) = 5.3\text{x}10^{-28}$ Ns

26

solution

- (a) $E = hc/\lambda$

  $= 1240$ eV·nm /650 nm

  $= 1.91$ eV ($= 3.1\times10^{-19}$J)
- (b) $p = E/c = 1.91$ eV/c ($= 1\text{x}10^{-27}$ Ns)
- (c) $\lambda = hc/E$

  $= 1240$eV·nm /2.40 eV

  $= 517$ nm

27

Einstein's 2nd postulate

- In PE one photon is completely absorbed by one atom in the photocathode.
- Upon the absorption, one electron is 'kicked out' by the absorbent atom.
- The kinetic energy for the ejected electron is

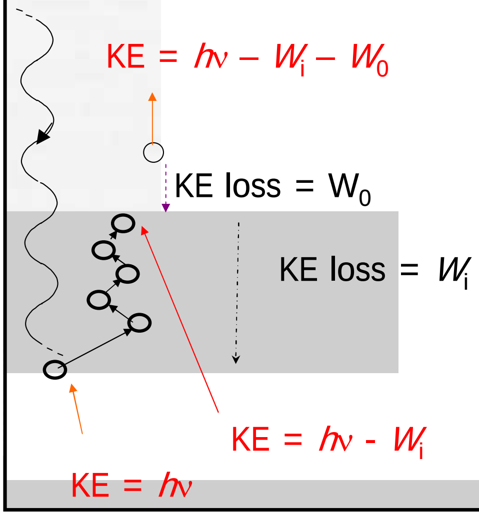
  $K = h\nu - W$
- W is the worked required to
- (i) cater for losses of kinetic energy due to internal collision of the electrons (W_i),
- (ii) overcome the attraction from the atoms in the surface (W_0)
- When no internal kinetic energy loss (happens to electrons just below the surface which suffers minimal loss in internal collisions), K is maximum:
- $K_{max} = h\nu - W_0$

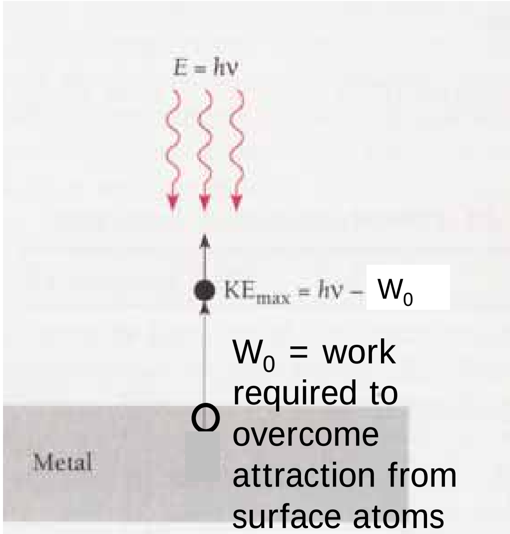
28

In general,

$K = h\nu - W$, where

$W = W_0 + W_i$





Einstein theory manage to solve the three unexplained features:

- First feature:
- In Einstein's theory of PE, $K_{max} = h\nu - W_0$
- Both $h\nu$ and W_0 do not depend on the radiation intensity
- Hence K_{max} is independent of irradiation intensity
- Doubling the intensity of light wont change K_{max} because only depend on the energy $h\nu$ of individual photons and W_0
- W_0 is the intrinsic property of a given metal surface

Second feature explained

- **The cut-off frequency is explained**

- Recall that in Einstein assumption, a photon is completely absorbed by one atom to kick out one electron.
- Hence each absorption of photon by the atom transfers a discrete amount of energy by $h\nu$ only.
- If $h\nu$ is not enough to provide sufficient energy to overcome the required work function, W_0, no photoelectrons would be ejected from the metal surface and be detected as photocurrent

31

Cut-off frequency is related to work function of metal surface $W_0 = h\nu_0$

- A photon having the cut-off frequency ν_0 has just enough energy to eject the photoelectron and none extra to appear as kinetic energy.
- Photon of energy less than $h\nu_0$ has not sufficient energy to kick out any electron
- Approximately, electrons that are eject at the cut-off frequency will not leave the surface.
- This amount to saying that the have got zero kinetic energy: $K_{max} = 0$
- Hence, from $K_{max} = h\nu - W_0$, we find that the cut-off frequency and the work function is simply related by
  - $W_0 = h\nu_0$
- Measurement of the cut-off frequency tell us what the work function is for a given metal

32

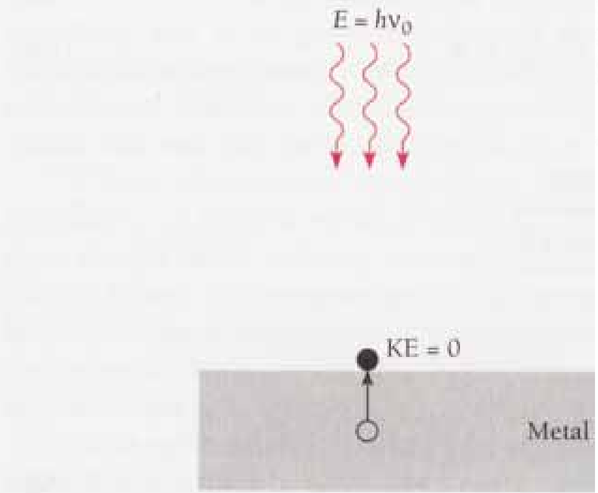


Table 3.1 Some Photoelectric Work Functions $W_0 = h\nu_0$

Material	W(eV
Na	2.28
Al	4.08
Co	3.90
Cu	4.70
Zn	4.31
Ag	4.73
Pt	6.35
Pb	4.14

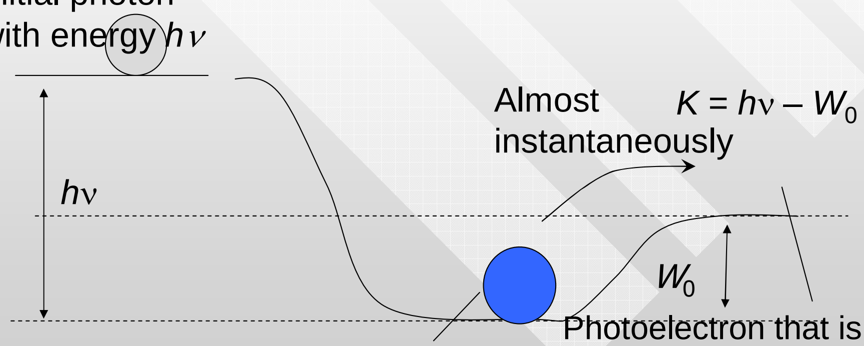
Third feature explained

- The required energy to eject photoelectrons is supplied in concentrated bundles of photons, not spread uniformly over a large area in the wave front.
- Any photon absorbed by the atoms in the target shall eject photoelectron immediately.
- Absorption of photon is a discrete process at quantum time scale (almost ‘instantaneously’): it either got absorbed by the atoms, or otherwise.
- Hence no time lag is expected in this picture

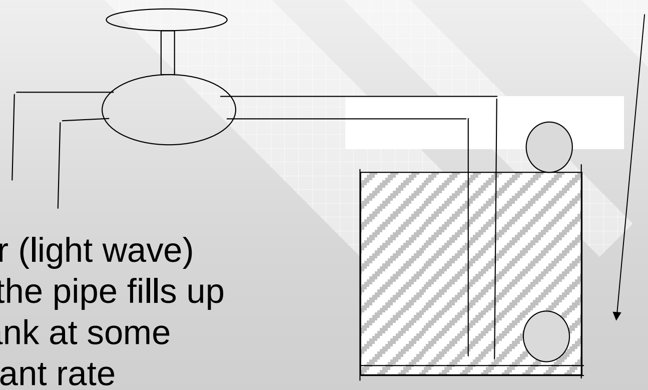
34

A simple way to picture photoelectricity in terms of particle-particle collision:

Energy of photon is transferred during the instantaneous collision with the electron. The electron will either get kicked up against the barrier threshold of W_0 almost instantaneously, or fall back to the bottom of the valley if $h\nu$ is less than W_0



Compare the particle-particle collision model with the water-filling-tank model:



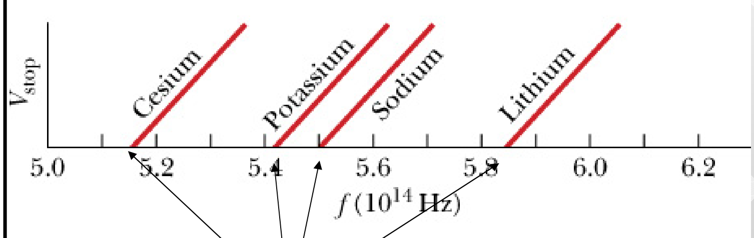
Experimental determination of Planck constant from PE

- Experiment can measure eV_s (= K_{max}) for a given metallic surface (e.g. sodium) at different frequency of impinging radiation
- We know that the work function and the stopping potential of a given metal is given by
- $eV_s = h\nu - W_0$

37

In experiment, we can measure the slope in the graph of V_s verses frequency ν for different metal surfaces. It gives a universal value of $h/e = 4.1\text{x}10^{-15}$ Vs. Hence, $h = 6.626 \text{ x } 10^{-34}$ Js

$$V_s = (h/e)\nu - \nu_0$$



Different metal surfaces have different ν_0

38

PYQ 2.16, Final Exam 2003/04

- Planck constant
- (i) is a universal constant
- (ii) is the same for all metals
- (iii) is different for different metals
- (iv) characterises the quantum scale
- **A.** I,IV **B.** I,II, IV **C.** I, III,IV
- **D.** I, III **E.** II,III
- **ANS: B, Machlup, Review question 8, pg. 496, modified**

39

PYQ 4(a,b) Final Exam 2003/04

- (a) Lithium, beryllium and mercury have work functions of 2.3 eV, 3.9 eV and 4.5 eV, respectively. If a 400-nm light is incident on each of these metals, determine
- (i) which metals exhibit the photoelectric effect, and
- (ii) the maximum kinetic energy for the photoelectron in each case (in eV)

40

Solution for Q3a

- The energy of a 400 nm photon is $E = hc/\lambda$ = **3.11 eV**
- The effect will occur only in **lithium***
- **Q3a(ii)**
- For lithium, $K_{max} = h\nu - W_0$

  $= 3.11 \text{ eV} - 2.30 \text{ eV}$

  $=$ **0.81 eV**

*marks are deducted for calculating "K_{max}" for beryllium and mercury which is meaningless

41

PYQ 4(a,b) Final Exam 2003/04

- (b) Molybdenum has a work function of 4.2 eV.
- (i) Find the cut-off wavelength (in nm) and threshold frequency for the photoelectric effect.
- (ii) Calculate the stopping potential if the incident radiation has a wavelength of 180 nm.

42

Solution for Q4b

- **Q3a(ii)**
- Known $h\nu_{cutoff} = W_0$
- Cut-off wavelength = $\lambda_{cutoff} = c/\nu_{cutoff}$

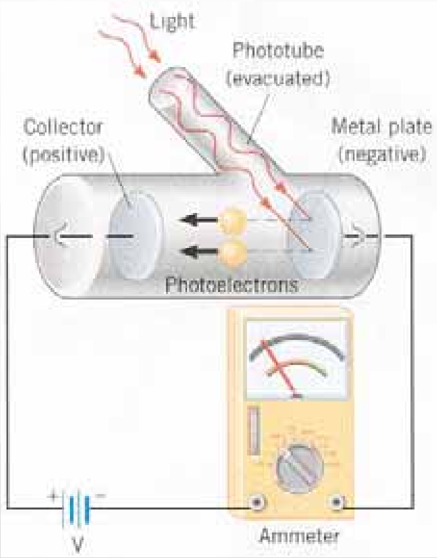
  $= hc/W_0$ = 1240 nm eV / 4.2 eV = **295 nm**
- Cut-off frequency (or threshold frequency), ν_{cutoff}

  $= c / \lambda_{cutoff}$ = 1.01 x 1015 Hz
- **Q3b(ii)**
- Stopping potential $V_{stop} = (hc/\lambda - W_0) / e$ = (1240 nm·eV/180 nm – 4.2 eV)/e = **2. 7 V**

43

Example (read it yourself)

- Light of wavelength 400 nm is incident upon lithium (W_0 = 2.9 eV). Calculate
- (a) the photon energy and
- (b) the stopping potential, V_s
- (c) What frequency of light is needed to produce electrons of kinetic energy 3 eV from illumination of lithium?



Solution:

- (a) $E = h\nu = hc/\lambda$ = 1240eV·nm/400 nm = 3.1 eV
- (b) The stopping potential x e = Max Kinetic energy of the photon
- => $eV_s = K_{max} = h\nu - W_0$ = (3.1 - 2.9) eV
- Hence, V_s = 0.2 V
- i.e. a retarding potential of 0.2 V will stop all photoelectrons
- (c) $h\nu = K_{max} + W_0$ = 3 eV + 2.9 eV = 5.9 eV. Hence the frequency of the photon is

  $\nu = 5.9 \times (1.6 \times 10^{-19}\text{ J}) / 6.63 \times 10^{-34}\text{ Js}$

  $= 1.42 \times 10^{15}$ Hz

45

PYQ, 1.12 KSCP 2003/04

Which of the following statement(s) is (are) true?

- **I** The energy of the quantum of light is proportional to the frequency of the wave model of light
- **II** In photoelectricity, the photoelectrons has as much energy as the quantum of light which causes it to be ejected
- **III** In photoelectricity, no time delay in the emission of photoelectrons would be expected in the quantum theory
- **A. II, III** **B. I, III** **C. I, II, III** **D. I ONLY**
- **E.** Non of the above
- **Ans: B**
- Murugeshan, S. Chand & Company, New Delhi, pg. 136, Q28 (for I), Q29, Q30 (for II,III)

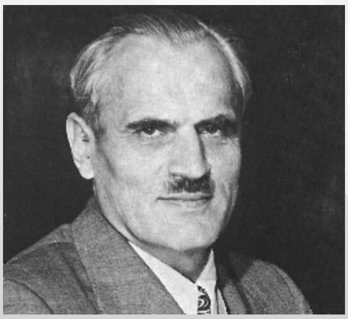
46

To summerise: In photoelectricity (PE), light behaves like particle rather than like wave.

47

Compton effect

- Another experiment revealing the particle nature of X-ray (radiation, with wavelength ~ 10^{-10} m)



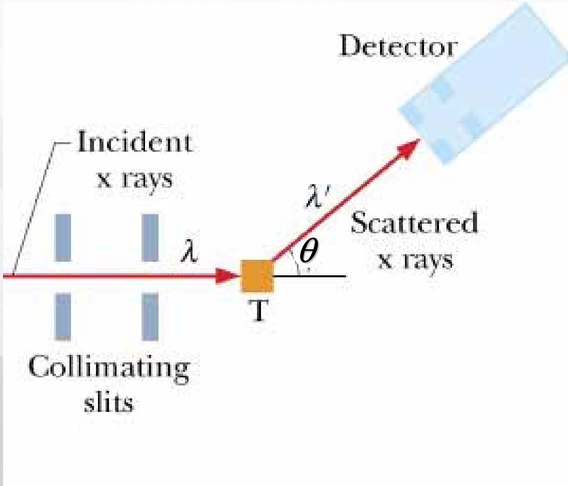
Compton, Arthur Holly (1892-1962), American physicist and Nobel laureate whose studies of X rays led to his discovery in 1922 of the so-called Compton effect.

The Compton effect is the change in wavelength of high energy electromagnetic radiation when it scatters off electrons. The discovery of the Compton effect confirmed that electromagnetic radiation has both wave and particle properties, a central principle of quantum theory.

48

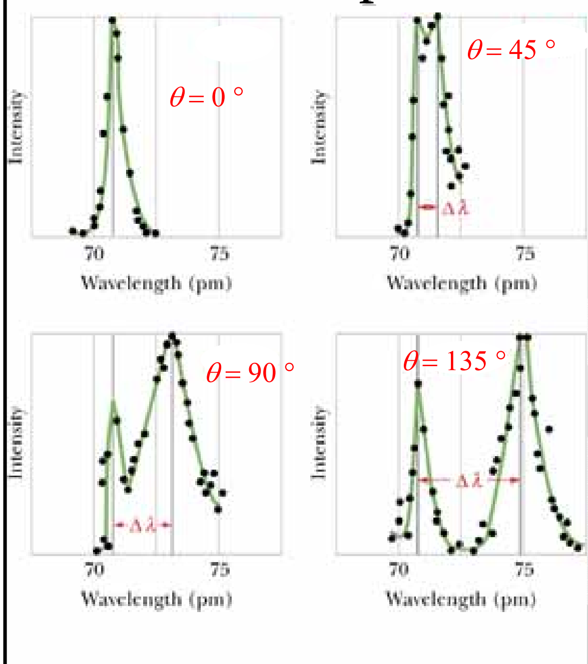
Compton's experimental setup

- A beam of x rays of wavelength 71.1 pm is directed onto a carbon target T. The x rays scattered from the target are observed at various angle θ to the direction of the incident beam. The detector measures both the intensity of the scattered x rays and their wavelength



49

Experimental data



Although initially the incident beam consists of only a single well-defined wavelength (λ) the scattered x-rays at a given angle θ have intensity peaks at two wavelength (λ' in addition), where $\lambda' > \lambda$

50

Compton shouldn't shift, according to classical wave theory of light

- **Unexplained by classical wave theory for radiation**
- **No shift of wavelength is predicted in wave theory of light**

51

Modelling Compton shift as "particle-particle" collision

- Compton (and independently by Debye) explain this in terms of collision between collections of (particle-like) photon, each with energy $E = h\nu = pc$, with the *free* electrons in the target graphite (imagine billard balls collision)
- $E^2=(mc^2)^2+c^2p^2$
- $E_\gamma^2=(m_\gamma c^2)^2+c^2p^2=c^2p^2$

52

Photographic picture of a Compton electron

- ***Part of a bubble chamber picture (Fermilab'15 foot Bubble Chamber', found at the University of Birmingham). An electron was knocked out of an atom by a high energy photon.***
- Photon is not shown as the photographic plate only captures the track of charged particle, not light.

Two particle collision in 2D

Scattered photon,
$E' = hc/\lambda'$,
$p' = h/\lambda'$

Initial photon,
$E = hc/\lambda$,

$p = h/\lambda$

Initial electron,
at rest,
$E_{ei} = m_e c^2$,
$p_{ei} = 0$

y

θ

x

ϕ

1: Conservation of E:

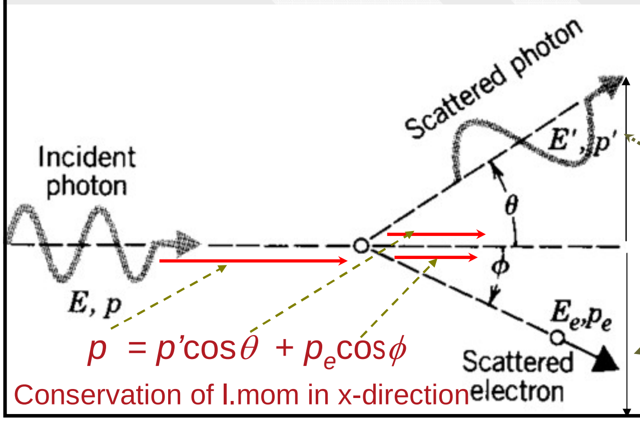
$cp + m_e c^2 = cp' + E_e$

Scattered
electron, E_e, p_e

2: Conservation of momentum: **p = p' + p$_e$ (vector sum)**

Conservation of momentum in 2-D

- $\mathbf{p} = \mathbf{p}' + \mathbf{p}_e$ (vector sum) actually comprised of two equation for both conservation of momentum in x- and y- directions



$p = p'\cos\theta + p_e\cos\phi$

Conservation of l.mom in x-direction

Conservation of l.mom in y-direction

$p'\sin\theta = p_e\sin\phi$

55

Some algebra…

Mom conservation in y : $p'\sin\theta = p_e\sin\phi$ (PY)

Mom conservation in x : $p - p'\cos\theta = p_e\cos\phi$ (PX)

Conservation of total relativistic energy:

$cp + m_e c^2 = cp' + E_e$ (RE)

$(PY)^2 + (PX)^2$, substitute into $(RE)^2$ to eliminate ϕ, p_e and E_e (and using $E_e^2 = c^2 p_e^2 + m_e^2 c^4$):

$$\Delta\lambda \equiv \lambda' - \lambda = (h/m_e c)(1 - \cos\theta)$$

56

Compton wavelength

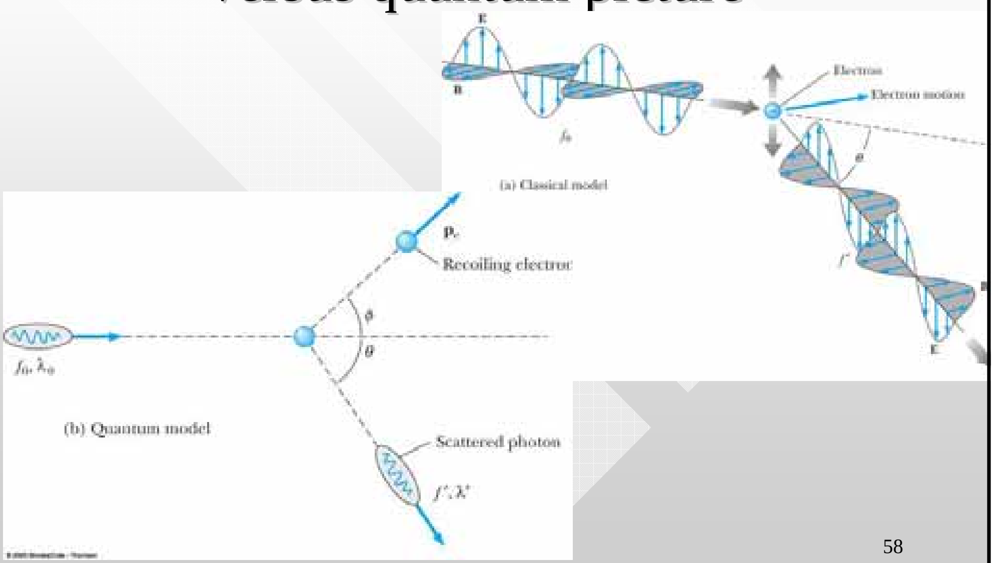
$\lambda_e = h/m_e c$ = 0.0243 Angstrom, is the Compton wavelength (for electron)

- Note that the wavelength of the x-ray used in the scattering is of the similar length scale to the Compton wavelength of electron

- The Compton scattering experiment can now be perfectly explained by the Compton shift relationship
  $$\Delta\lambda \equiv \lambda' - \lambda = \lambda_e(1 - \cos\theta)$$
  as a function of the photon scattered angle
- Be reminded that the relationship is derived by assuming light behave like particle (photon)

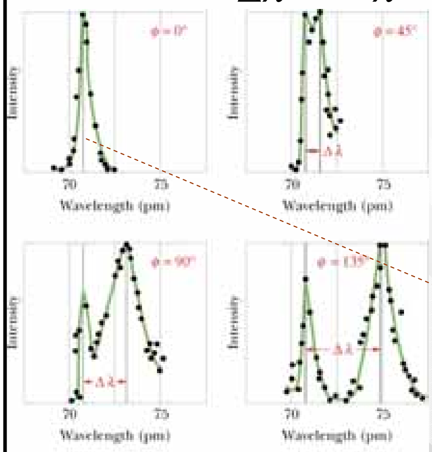
57

X-ray scattering from an electron (Compton scattering): classical versus quantum picture



58

$$\Delta\lambda \equiv \lambda' - \lambda = (h/m_e c)(1 - \cos\theta)$$

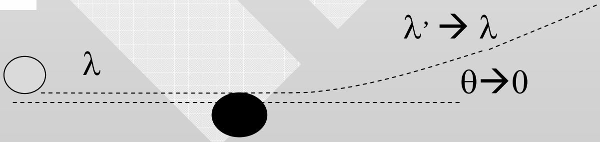


λ'=0.1795 nm

Notice that $\Delta\lambda$ depend on θ only, not on the incident wavelength, λ..

Consider some limiting behaviour of the Compton shift:

For $\theta = 0^0 \rightarrow$ “grazing” collision => $\Delta\lambda = 0$



For $\theta \rightarrow 180^0$ “head-on” collision
=> $\Delta\lambda = \Delta\lambda_{max}$

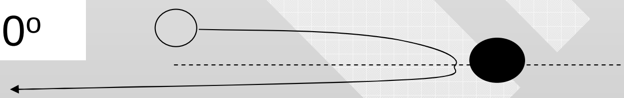
$\theta \rightarrow 180^0$ photon being reversed in direction

$\Delta\lambda_{max} = \lambda_{max}' - \lambda = (h/m_e c)(1 - \cos 180°)$

- $= 2\lambda_e = 2(\ 0.00243\text{nm})$

initially λ

θ=180°



After collision

$\lambda'_{max} = \lambda + \Delta\lambda_{max}$

PYQ 2.2 Final Exam 2003/04

Suppose that a beam of 0.2-MeV photon is scattered by the electrons in a carbon target. What is the wavelength of those photon scattered through an angle of 90^o?

A. 0.00620 nm

B. 0.00863 nm

C. 0.01106 nm

D. 0.00243 nm

E. Non of the above

61

Solution

First calculate the wavelength of a 0.2 MeV photon:

$E = hc/\lambda$= 1240 eV·nm/λ = 0.2 MeV

λ =1240 nm / 0.2 x 10^6 = 0.062 nm

From Compton scattering formula, the shift is

$\Delta\lambda = \lambda' - \lambda = \lambda_e (1 - \cos 90°) = \lambda_e$

Hence, the final wavelength is simply

$\lambda' = \Delta\lambda + \lambda = \lambda_e + \lambda =$ 0.00243nm + 0.062 nm = 0.00863 nm

ANS: B, Schaum's 3000 solved problems, Q38.31, pg. 712

62

Example

- X-rays of wavelength 0.2400 nm are Compton scattered and the scattered beam is observed at an angle of 60 degree relative to the incident beam.
- Find (a) the wave length of the scattered x-rays, (b) the energy of the scattered x-ray photons, (c) the kinetic energy of the scattered electrons, and (d) the direction of travel of the scattered electrons

63

solution

$\lambda' = \lambda + \lambda_e (1 - \cos\theta)$

$= 0.2400\text{nm} + 0.00243\text{nm}(1 - \cos 60^o)$

$= 0.2412 \text{ nm}$

$E' = hc/\lambda'$

$= 1240 \text{ eV}\cdot\text{nm} / 0.2412 \text{ nm}$

$= 5141 \text{ eV}$

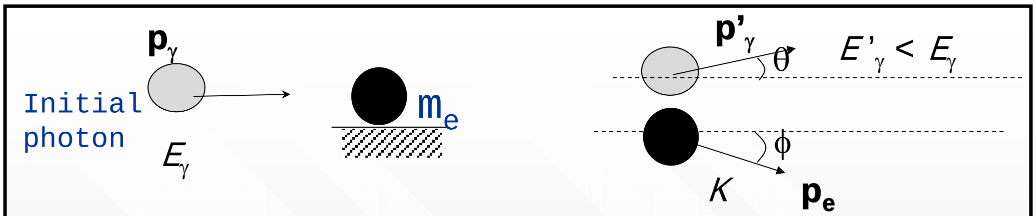
64

kinetic energy gained by the scattered electron
= energy transferred by the incident photon during the scattering:

$K = hc/\lambda - hc/\lambda' = (5167–5141)\text{eV} = 26\text{ eV}$

Note that we ignore SR effect here because K << rest mass of electron, $m_e = 0.5$ MeV

65



By conservation of momentum in the x- and y-direction:

$p_\gamma = p'_\gamma \cos\theta + p_e \cos\phi$; $p'_\gamma \sin\theta = p_e \sin\phi$;

$\tan\phi = p_e \sin\phi \,/\, p_e \cos\phi = (p'_\gamma \sin\theta) / (p_\gamma - p'_\gamma \cos\theta)$

$= (E'_\gamma \sin\theta) / (E_\gamma - E'_\gamma \cos\theta)$

$= (5141 \sin 60^0 / [5167–5141 (\cos 60^0] = 0.43 = 1.71$

Hence, $\phi = 59.7$ degree

66

PYQ 3(c), Final exam 2003/04

- (c) A 0.0016-nm photon scatters from a free electron. For what scattering angle of the photon do the recoiling electron and the scattered photon have the same kinetic energy?
- **Serway solution manual 2, Q35, pg. 358**

67

Solution

- The energy of the incoming photon is
  $E_i = hc/\lambda = 0.775$ MeV
- Since the outgoing photon and the electron each have half of this energy in kinetic form,
- $E_f = hc/\lambda' = 0.775$ MeV / 2 = 0.388 MeV and
  $\lambda' = hc/E_f = 1240$ eV· nm / 0.388 MeV = 0.0032 nm
- The Compton shift is
  $\Delta\lambda = \lambda' - \lambda = (0.0032 - 0.0016)$ nm $= 0.0016$ nm
- By $\Delta\lambda = \lambda_c (1 - \cos\theta)$
- $= (h/m_e c)\ (1 - \cos\theta)\ 0.0016$ nm
- $= 0.00243$ nm $(1 - \cos\theta)$

  $\theta = 70^o$

68

PYQ 1.10 KSCP 2003/04

Which of the following statements is (are) true?

- **I.** Photoelectric effect arises due to the absorption of electrons by photons
- **II.** Compton effect arises due to the scattering of photons by free electrons
- **III.** In the photoelectric effect, only part of the energy of the incident photon is lost in the process
- **IV**.*I*n the Compton effect, the photon completely disappears and all of its energy is given to the Compton electron
- **A. I,II** **B. II,III,IV** **C. I, II, III**
- **D. III,IV** **Ans: E**
- **[I = false; II = true; III = false; IV = false]**
- Murugeshan, S. Chand & Company, New Delhi, pg. 134, Q13,

69

X-ray:
The inverse of photoelectricity

- X-ray, discovered by Wilhelm Konrad Roentgen (1845-1923). He won the first Nobel prize in 1902. He refused to benefit financially from his work and died in poverty in the German inflation that followed the end of World War 1.

70

X-rays are simply EM radiation with very short wavelength, ~ 0.01 nm –10 nm

Some properties:

- energetic, according to $E = hc/\lambda \sim 0.1 - 100$ keV (c.f. $E \sim$ a few eV for visible light)
- travels in straight lines
- is unaffected by electric and magnetic fields
- passes readily through opaque materials – highly penetrative
- causes phosphorescent substances to glow
- exposes photographic plates

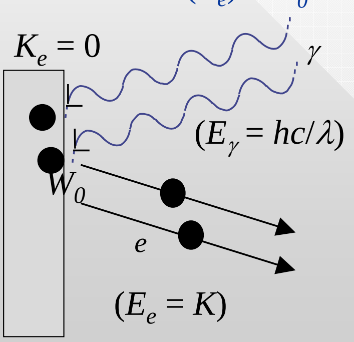
71

In photoelectricity, energy is transferred from photons to kinetic energy of electrons. The inverse of this process produces x-rays

P.E:

electron (K_e=0) + photon (hc/λ)

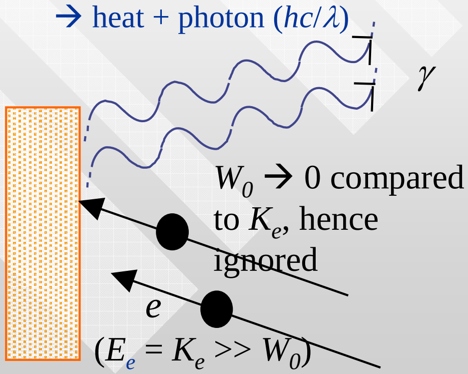
→ electron (K_e) + W_0



x-ray:

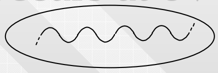
electron (K_e)

→ heat + photon (hc/λ)



72

PE and x-rays production happen at different energy scale

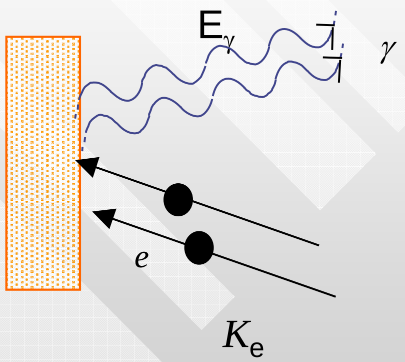
- However, both process occur at disparately different energy scale
- Roughly, for PE, it occurs at eV scale with ultraviolet radiation 
- For x-ray production, the energy scale involved is much higher - at the order of 100 eV - 100 keV 

73

X-ray production

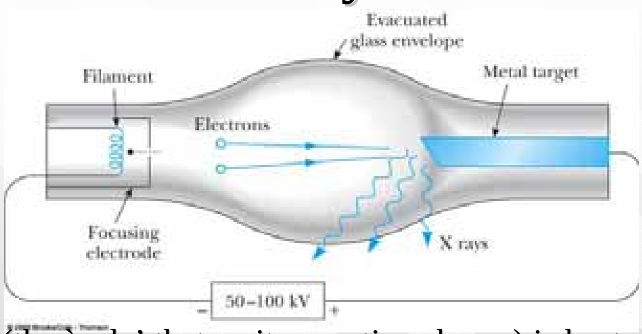
- X-rays is produced when electrons, accelerated by an electric field in a vacuum cathode-ray tube, are impacted on the glass end of the tube

- Part or all of the kinetic energy of a moving electron is converted into a x-ray photon



74

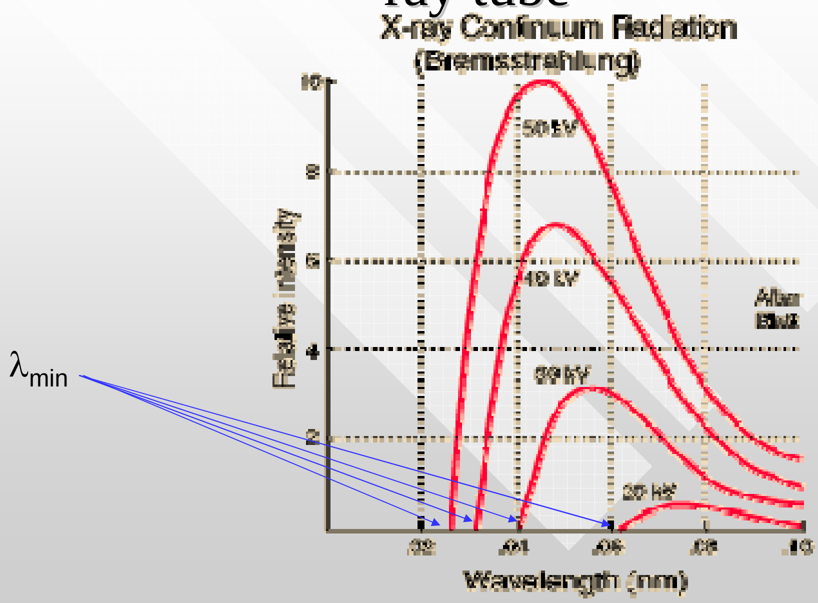
The x-ray tube



- A cathode (the `pole' that emits negative charge) is heated by means of electric current to produce thermionic emission of the electrons from the target
- A high potential difference *V* is maintained between the cathode and a metallic target
- The thermionic electrons will get accelerated toward the latter
- The higher the accelerating potential *V*, the faster the electron and the shorter the wavelengths of the x-rays

75

Typical x-ray spectrum from the x-ray tube



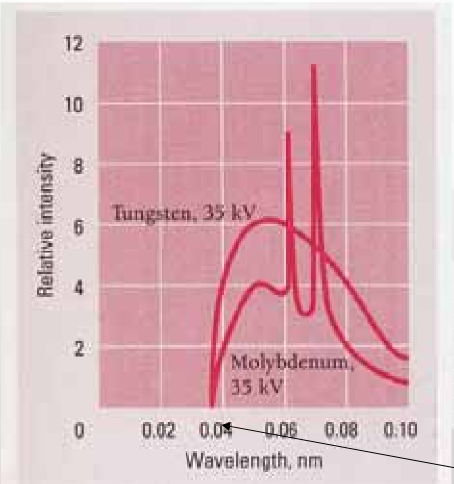
76

Important features of the x-ray spectrum

1. The spectrum is continuous
2. The existence of a minimum wavelength $\lambda_{\min}$ for a given V, below which no x-ray is observed
3. Increasing V decreases $\lambda_{\min}$.

77

$\lambda_{\min} \propto 1/V$, the same for all material surface



- At a particular V, $\lambda_{\min}$ is approximately the same for different target materials. Experimentally one finds that $\lambda_{\min}$ is inversely proportional to V,

$$\lambda_{\min} = \left(\frac{1.24 \times 10^{-6}}{V} \right) \mathrm{m} \cdot \mathrm{V}$$

The peaks in the spectrum are due to the electronic transition occurring between the adjacent shells (orbit) in the atom. We would not discuss them further here.

78

X-ray production heats up the target material

- Due to conversion of energy from the impacting electrons to x-ray photons is not efficient, the difference between input energy, K_e and the output x-ray energy E_γ becomes heat
- Hence the target materials have to be made from metal that can stand heat and must have high melting point (such as Tungsten and Molybdenum)

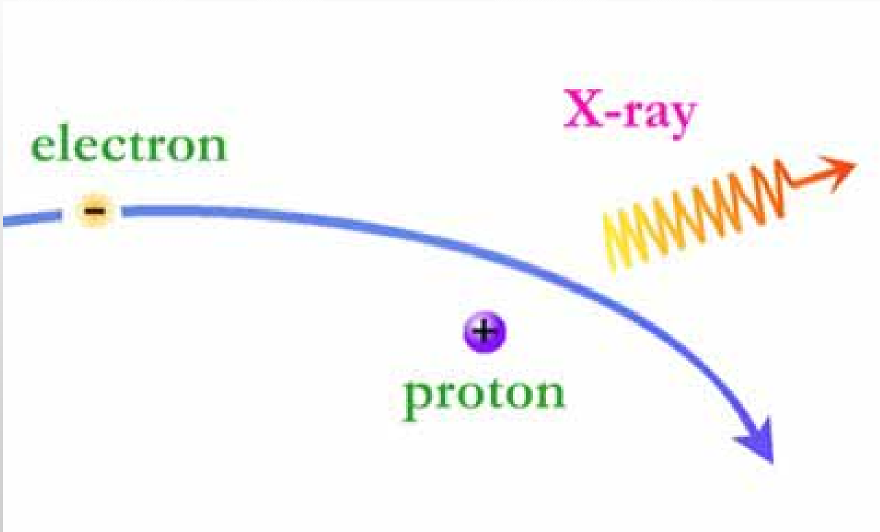
79

Classical explanation of continuous x-ray spectrum:

- The continuous X-ray spectrum is explained in terms of **Bremsstrahlung:** radiation emitted when a moving electron "tekan brake"
- According to classical EM theory, an accelerating or decelerating electric charge will radiate EM radiation
- Electrons striking the target get slowed down and brought to eventual rest because of collisions with the atoms of the target material
- Within the target, many electrons collides with many atoms for many times before they are brought to rest
- Each collision causes some non-unique losses to the kinetic energy of the Bremsstrahlung electron
- As a net effect of the collective behavior by many individual collisions, the radiation emitted (a result due to the lost of KE of the electron) forms a continuous spectrum

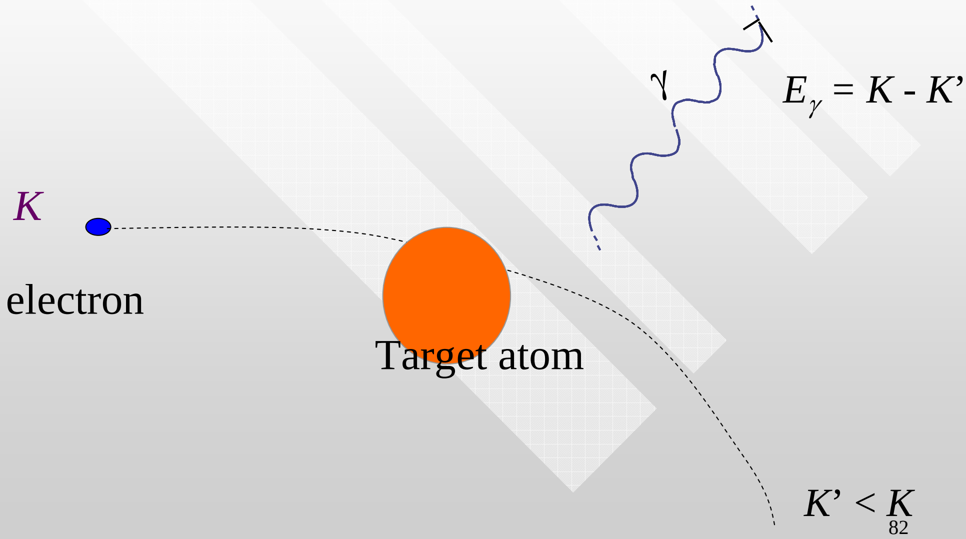
80

Bremsstrahlung



81

Bremsstrahlung, simulation



82

Bremsstrahlung cannot explain λ_{min}

- Notice that in the classical **Bremsstrahlung** process the x-ray radiated is continuous and there is no lower limit on the value of the wavelength emitted (because classical physics does not relate energy with wavelength). Hence, the existence of λ_{min} is not explained with the classical **Bremsstrahlung** mechanism. All range of λ from 0 to a maximum should be possible in this classical picture.

λ_{min} can only be explained by assuming light as photons but not as EM wave

83

Energy of the x-ray photon in the quantum picture

- According to Einstein assumption on the energy of a photon, the energy of the photon emitted in the Bremsstrahlung is simply the difference between the initial and final kinetic energy of the electron:

$$h\nu = K - K'$$

- The shortest wavelength of the emitted photon gains its energy, $E = h\nu_{max} = hc/\lambda_{min}$ corresponds to the maximal loss of the K.E. of an electron in a single collision (happen when $K' = 0$ in a single collision)
- This (i.e. the maximal lose on KE) only happens to a small sample of collisions. Most of the other collisions loss their KE gradually in smaller amount in an almost continuous manner.

84

Theoretical explanation of the experimental Value of λ_{min}

- K (of the Bremsstrahlung electron) is converted into the photon with $E = hc/\lambda_{min}$
- Experimentally K is caused by the external potential V that accelerates the electron before it bombards with the target, hence

$$K = eV$$

- Conservation of energy requires

$$K = eV = hc/\lambda_{min}$$

- or, $\lambda_{min} = hc/eV$ = (1240 nm·eV)$/eV$ = (1240V/V) nm

which is the value measured in x-ray experiments

85

Why is λ_{min} the same for different material?

- The production of the x-ray can be considered as an inverse process of PE
- Hence, to be more rigorous, the conservation of energy should take into account the effects due to the work potential of the target material during the emission of x-ray process, W_0
- However, so far we have ignored the effect of W_0 when we were calculating the relationship between λ_{min} and K
- This approximation is justified because of the following reason:
- The accelerating potentials that is used to produce x-ray in a x-ray vacuum tube, V, is in the range of 10,000 V
- Whereas the work function W_0 is only of a few eV
- Hence, in comparison, W_0 is ignored wrp to eV
- This explains why λ_{min} is the same for different target materials

86

Example

- Find the shortest wavelength present in the radiation from an x-ray machine whose accelerating potential is 50,000 V
- Solution:

$$\lambda_{\min} = \frac{hc}{eV} = \frac{1.24\times10^{-6}\,\mathrm{V\cdot m}}{5.00\times10^{4}\,\mathrm{V}} = 2.48\times10^{-11}\,\mathrm{m} = 0.0248\mathrm{nm}$$

This wavelength corresponds to the frequency

$$\nu_{\max} = \frac{c}{\lambda_{\min}} = \frac{3\times10^{8}\,\mathrm{m/s}}{2.48\times10^{-11}\,\mathrm{m}} = 1.21\times10^{19}\,\mathrm{Hz}$$

87

PYQ 1. 9 Final Exam 2003/04

- To produce an x-ray quantum energy of 10^{-15} J electrons must be accelerated through a potential difference of about
- A. 4 kV
- B. 6 kV
- C. 8 kV
- D. 9 kV
- E. 10 kV
- **ANS: B, OCR ADVANCED SUBSIDIARY GCE PHYSICS B (PDF), Q10, pg. 36**

Solution:

The energy of the x-rays photon comes from the external accelerating potential, V

$$E_\lambda = eV$$

$$V = E_\lambda / e = 1\times10^{-15}\,\mathrm{J}/e = \left(\frac{1\times10^{-15}}{1.6\times10^{-19}}\right)\mathrm{eV}/e = 6250\mathrm{V}$$

88

PYQ 1.9 KSCP 2003/04

Which of the following statement(s) is (are) true?

- **I.** γ-rays have much shorter wavelength than x-rays
- **II.** The wavelength of x-rays in a x-ray tube can be controlled by varying the accelerating potential
- **III**. x-rays are electromagnetic waves
- **IV.** x-rays show diffraction pattern when passing through crystals
- **A. I,II** **B. I,II,III,IV** **C. I, II, III**
- **D. III.IV** **E.** Non of the above
- **Ans: B** Murugeshan, S. Chand & Company, New Delhi, pg. 132, Q1.(for I), pg. 132, Q3 (for II), pg. 132, Q4 (for III,IV)

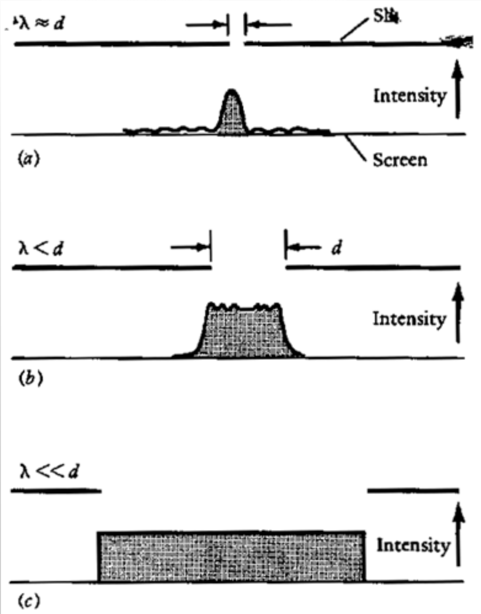
89

X-ray diffraction

- X-ray wavelengths can be determined through diffraction in which the x-ray is diffracted by the crystal planes that are of the order of the wavelength of the x-ray, ~ 0.1 nm
- The diffraction of x-ray by crystal lattice is called 'Bragg's diffraction'
- It is also used to study crystal lattice structure (by analysing the diffraction pattern)

90

Condition for diffraction



- Note that as a general rule in wave optics, diffraction effect is prominent only when the wavelength and the hole/obstacle are comparable in their length scale

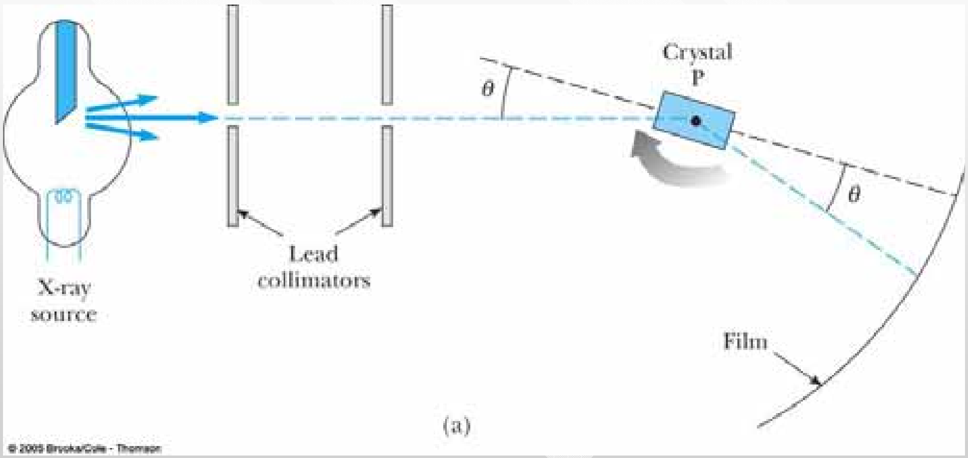
91

Use atoms in a crystal lattice to diffract X-rays

- Since wavelength of x-rays is very small, what kind of “scatterer” has sufficiently tiny separation to produce diffraction for x-rays?
- ANS: Atoms in a crystal lattice. Only the atomic separation in a crystal lattice is small enough (~ nm) to diffract X-rays which are of the similar order of length scale.

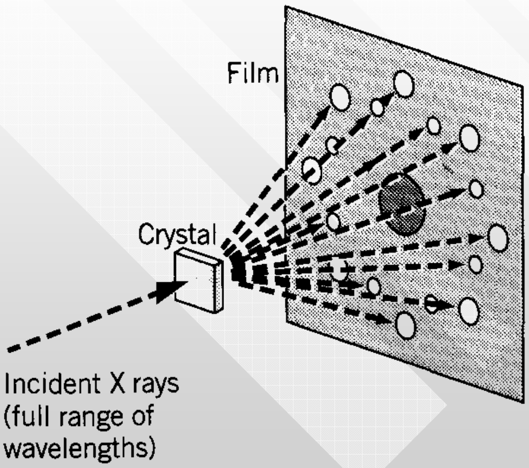
92

Experimental setup of Bragg's diffraction



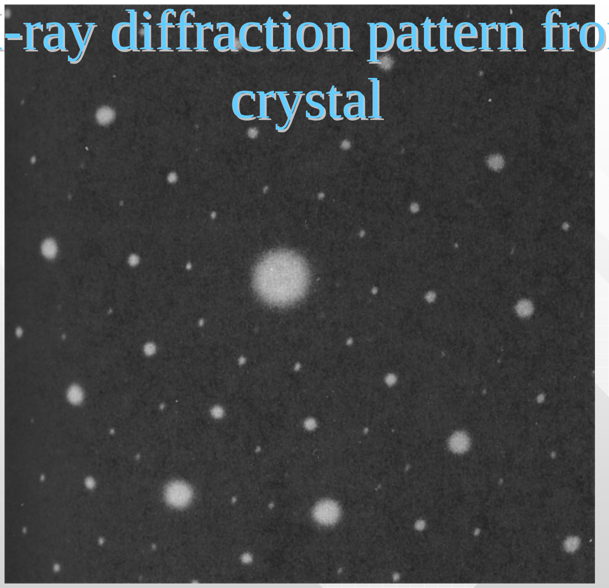
93

Experimental setup of Bragg's diffraction



94

X-ray diffraction pattern from crystal



The bright spots correspond to the directions where x-rays (full ranges of wavelengths) scattered from various layers (different Braggs planes) in the crystal interfere constructively.

95

Bragg's law for x-rays diffraction

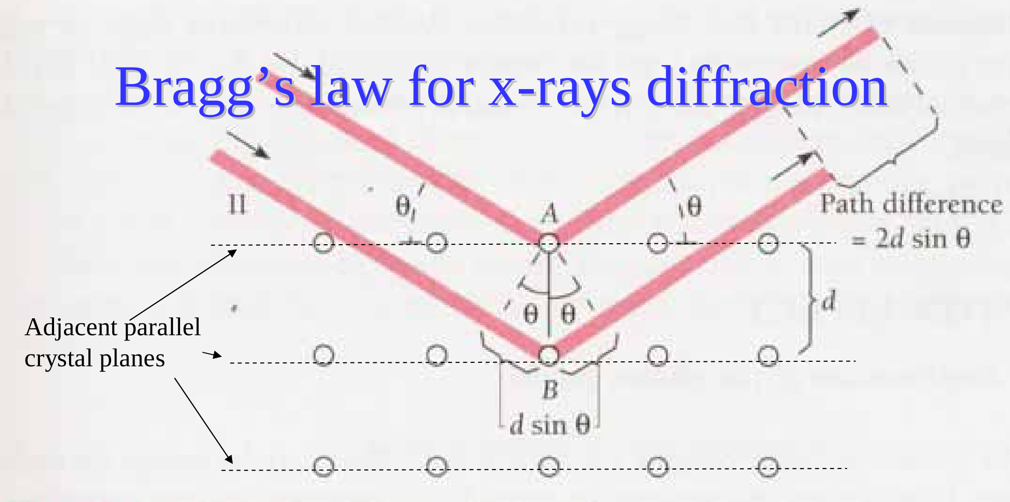


Figure 2.20 X-ray scattering from a cubic crystal.

Constructive interference takes place only between those scattered rays that are parallel and whose paths differ by exactly λ, 2λ, 3λ and so on (beam I, II):

$2d \sin\theta = n\lambda$, $n = 1, 2, 3$ …**Bragg's law for x-ray diffraction** 96

An X-rays can be reflected from many different crystal planes

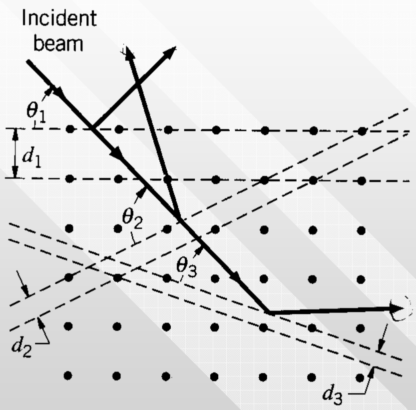


FIGURE 3.6 An incident beam of X rays can be reflected from many different crystal planes.

97

Example

- **A single crystal of table salt (NaCl) is irradiated with a beam of x-rays of unknown wavelength. The first Bragg's reflection is observed at an angle of 26.3 degree. Given that the spacing between the interatomic planes in the NaCl crystal to be 0.282 nm, what is the wavelength of the x-ray?**

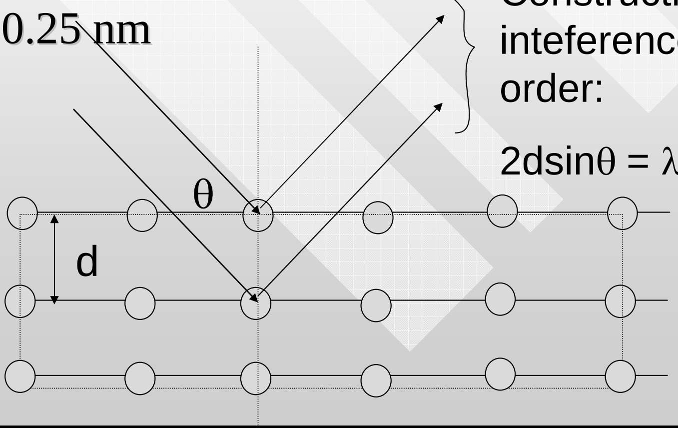
98

Solution

- Solving Bragg's law for the n = 1 order,

  $\lambda = 2d \sin \theta = 2 \times 0.282\ \text{nm} \times \sin(26.3^o)$

  $= 0.25$ nm



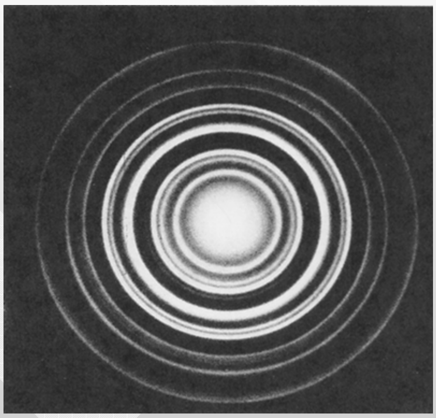
Constructive inteference of n=1 order:

$2d\sin\theta = \lambda$

99

If powder specimen is used (instead of single crystal)

- We get diffraction ring due to the large randomness in the orientation of the planes of scattering in the power specimen



100

Why ring for powdered sample?

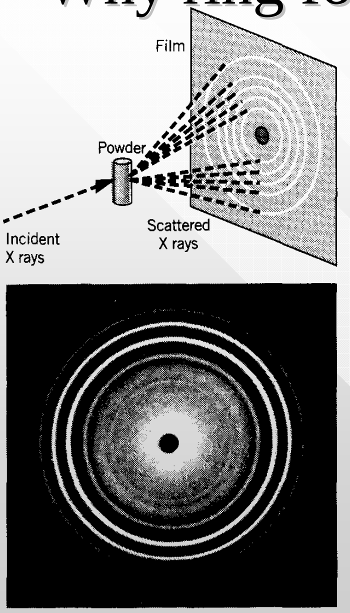


FIGURE 3.9 (Top) Apparatus for observing X-ray scattering from a powdered sample. Because the many crystals in a powder have all possible different orientations, each scattered ray of Figure 3.7 becomes a cone which forms a circle on the film. (Bottom) Diffraction pattern (known as *Debye-Scherrer* pattern) of a powder sample.

101

X-rays "finger print" of crystals

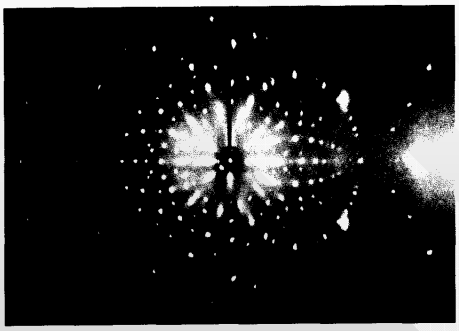


FIGURE 3.7 (Top) Apparatus for observing X-ray scattering by a crystal. An interference maximum (dot) appears on the film whenever a set of crystal planes happens to satisfy the Bragg condition for a particular wavelength. (Bottom) Laue pattern of NaCl crystal.

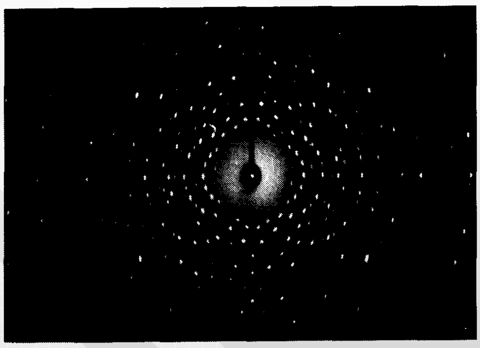


FIGURE 3.8 Laue pattern of a quartz crystal. The difference in crystal structure and spacing between quartz and NaCl makes this pattern look different from Figure 3.7.

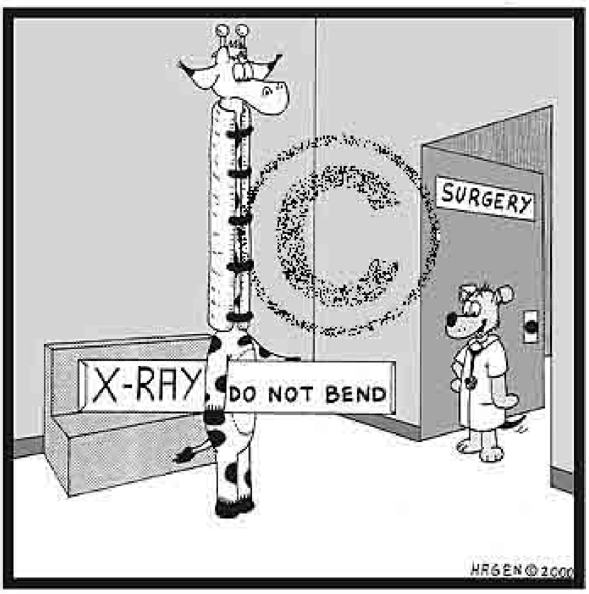
102

PYQ 6 Test I, 2003/04

- X-ray of wavelength 1.2 Angstrom strikes a crystal of *d*-spacing 4.4 Angstrom. Where does the diffraction angle of the second order occur?
- **A.**16° **B.** 33° **C.**55 °
- **D.** 90° **E.** Non of the above
- Solution: $n\lambda = 2d \sin\theta$
- $\sin\theta = n\lambda/2d$ = 2 x 2.2 / (2 x 4.4) = 0. 5

  $\theta = 30$ °
- ANS: B, Schaum's 3000 solved problems, Q38.46, pg. 715

103



I hope you didn't come by bus!

104

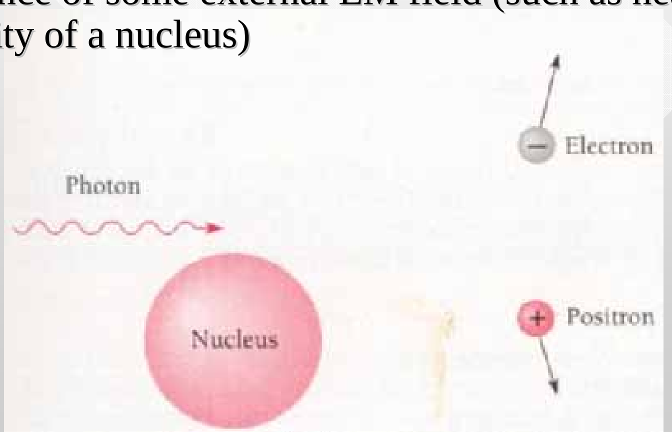
Pair Production: Energy into matter

- In photoelectric effect, a photon gives an electron all of its energy. In Compton effect, a photon give parts of its energy to an electron
- A photon can also materialize into an electron and a positron
- Positron = anti-electron, positively charged electron with the exactly same physical characteristics as electron except opposite in charge and spin
- In this process, called pair production, electromagnetic energy is converted into matter
- Creation of something (electron-positron pair) out of nothing (pure EM energy) triggered by strong external EM field

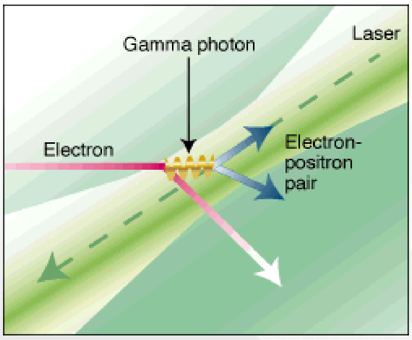
105

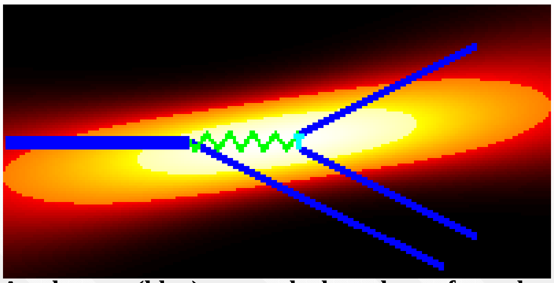
Pictorial visualisation of pair production

- In the process of pair production, a photon of sufficient energy is converted into electron-positron pair. The conversion process must occur only in the presence of some external EM field (such as near the vicinity of a nucleus)

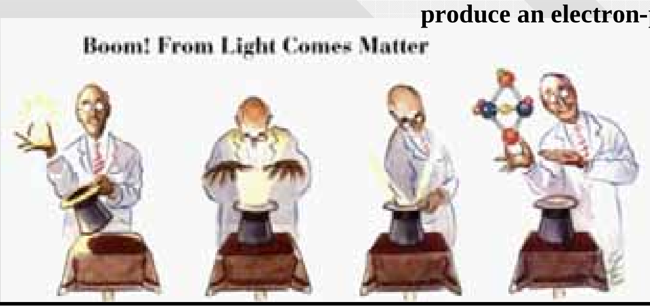


106





An electron (blue) enters the laser beam from the left, and collides with a laser photon to produce a high-energy gamma ray (wiggly yellow line). The electron is deflected downwards. The gamma ray then collides with four or more laser photons to produce an electron-positron pair



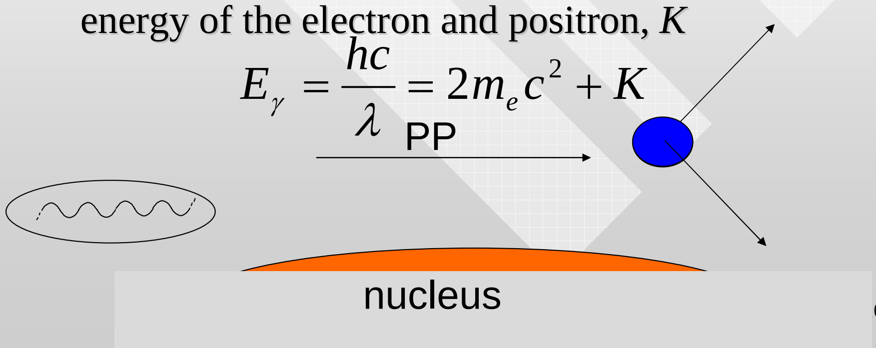
Conservational laws in pair-production

- The pair-production must not violate some very fundamental laws in physics:
- Charge conservation, total linear momentum, total relativistic energy are to be obeyed in the process
- Due to kinematical consideration (energy and linear momentum conservations) pair production cannot occur in empty space
- Must occur in the proximity of a nucleus
- Will see this in an example

Energy threshold

- Due to conservation of relativistic energy, pair production can only occur if E_γ is larger than 2 m_e = 2 × 0.51 MeV = 1.02 MeV
- Any additional photon energy becomes kinetic energy of the electron and positron, K

$$E_\gamma = \frac{hc}{\lambda} = 2m_e c^2 + K$$



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Example

- What is the minimal wavelength of a EM radiation to pair-produce an electron-positron pair?
- Solutions: minimal photon energy occurs if the pair have no kinetic energy after being created, K = 0. Hence,

$$\lambda_{\min} = \frac{hc}{2m_e c^2} = \frac{1240\,\mathrm{nm}\cdot\mathrm{eV}}{2\cdot 0.51\mathrm{MeV}} = 1.21\times 10^{-12}\,\mathrm{m}$$

These are very energetic EM radiation called gamma rays and are found in nature as one of the emissions from radioactive nuclei and in cosmic rays.

110

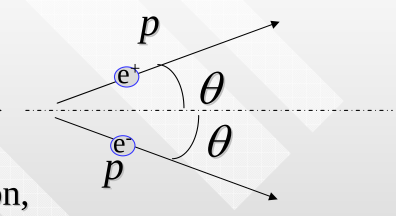
Electron-positron creation

- ***Part of a bubble chamber picture (Fermilab'15 foot Bubble Chamber', found at the University of Birmingham). The curly line which turns to the left is an electron. Positron looks similar but turn to the right The magnetic field is perpendicular to the picture plan***

Pair Production cannot occur in empty space

- Conservation of energy must me fulfilled, $hf = 2mc^2$
- Conservation of linear momentum must be fulfilled:

$E_\gamma = hf$



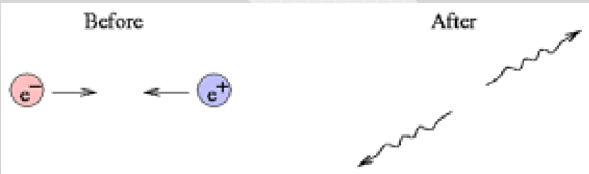
- $\Rightarrow p_\gamma = hf/c = 2p \cos\theta$
- Since $p = mv$ for electron and positron,
- $\Rightarrow hf = 2c(mv) \cos\theta = 2mc^2 (v/c) \cos\theta$
- Because $v/c < 1$ and $\cos\theta \leqslant 1$, $hf < 2mc^2$
- But conservation of energy requires $hf = 2mc^2$. Hence it is impossible for pair production to conserve both energy and momentum unless some other object (such as a nucleus) in involved in the process to carry away part of the initial of the photon momentum

Pair-annihilation

- The inverse of pair production occurs when a positron is near an electron and the two come together under the influence of their opposite electric charges

$$e^{+} + e^{-} \rightarrow \gamma + \gamma$$

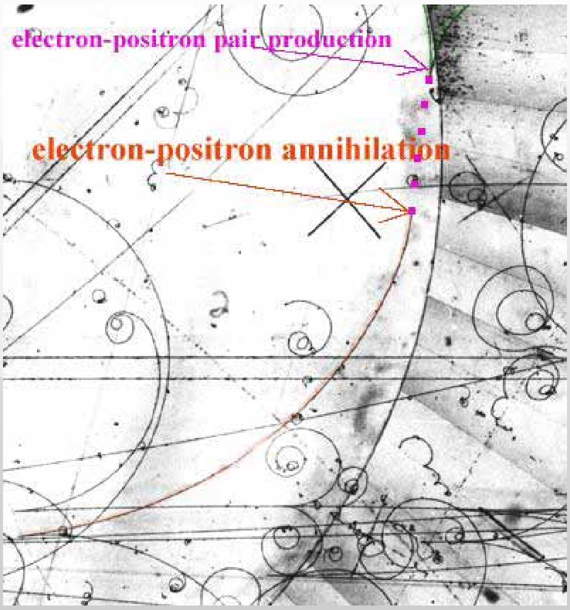
- Both particles vanish simultaneously, with the lost masses becoming energies in the form of two gamma-ray photons
- Positron and electron annihilate because they are anti particles to each other



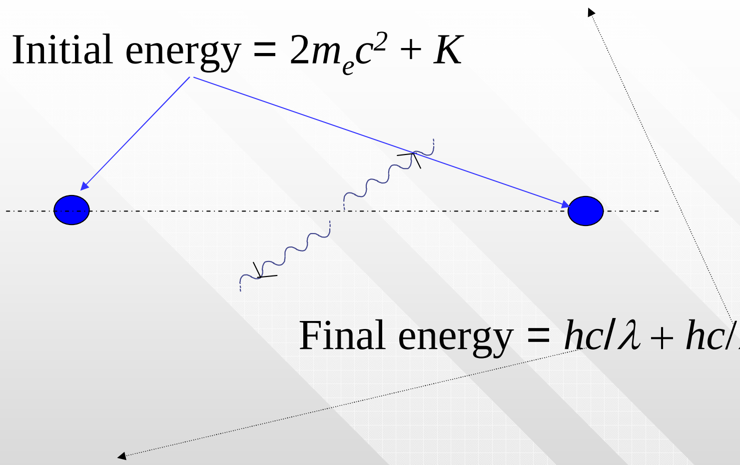
113

Pair annihilation

- *Part of a bubble-chamber picture from a neutrino experiment performed at the Fermilab (found at the University of Birmingham). A positron in flight annihilate with an electron. The photon that is produced materializes at a certain distance, along the line of flight, resulting a new electron-positron pair (marked with green)*



Initial energy = $2m_ec^2 + K$



Final energy = $hc/\lambda + hc/\lambda$

Conservation of relativistic energy:

$2m_ec^2 + K = 2\ hc/\lambda$

115

Energy and linear momentum are always conserved in pair annihilation

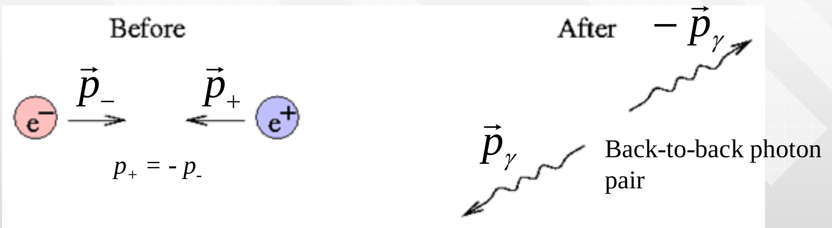
- The total relativistic energy of the e^--e^+ pair is
  - $E = 2m_ec^2 + K = 1.02$ MeV $+ K$
- where K the total kinetic energy of the electron-positron pair before annihilation
- Each resultant gamma ray photon has an energy

$hf = 0.51$ MeV $+ K/2$

- Both energy and linear momentum are automatically conserved in pair annihilation (else it wont occur at all)
- For e^--e^+ pair annihilation in which each particle collide in a head-on manner with same magnitude of momentum, i.e., $p_+ = - p_-$, the gamma photons are always emitted in a back-to-back manner due to kinematical reasons (conservation of linear momentum). (see explanation below and figure next page)
- In such a momentum-symmetric collision, the sum of momentum of the system is zero. Hence, after the photon pair is created, the sum their momentum must also be zero. Such kinematical reason demands that the photon pair be emitted back-to-back.
- No nucleus or other particle is needed for pair annihilation to take place
- Pair annihilation always occurs whenever a matter comes into contact with its antimatter

116

Collision of e^+-e^- pair in a center of momentum (CM) frame



Sum of momentum before annihilation = $\vec{p}_+ + \vec{p}_-$

= Sum of momentum after annihilation = $\vec{p}_\gamma - \vec{p}_\gamma$

= 0

117

As a tool to observe anti-world

- What is the characteristic energy of a gamma-ray that is produced in a pair-annihilation production process? What is its wavelength?
- Answer: 0.51 MeV, λ_{annih} = hc / 0.51 MeV = 0.0243 nm
- The detection of such characteristic gamma ray in astrophysics indicates the annihilation of matter-antimatter in deep space

118

PYQ 4, Test I, 2003/04

- An electron and a positron collide and undergo pair-annihilation. If each particle is moving at a speed of 0.8*c* relative to the laboratory before the collision, determine the energy of each of the resultant photon.
- **A.** 0.85MeV **B.** 1.67 MeV
- **C.** 0.51 MeV **D.** 0.72MeV
- **E.** Non of the above

119

Solution

Total energy before and after anniliation must remain the same: *i.e.* the energy of each electron is converted into the energy of each photon. Hence the energy of each photon is simple equal to the total relativistic energy of each electron travelling at $0.8c$:

$$E_\gamma = E_e = \gamma m_e c^2$$

where $\gamma = 1/\sqrt{1-(0.8)^2} = 1.678$

Hence $E_\gamma = 1.678 \times 0.51$ MeV = 0.85 MeV

- ANS: A, Cutnell, Q17, pg. 878, modified

120

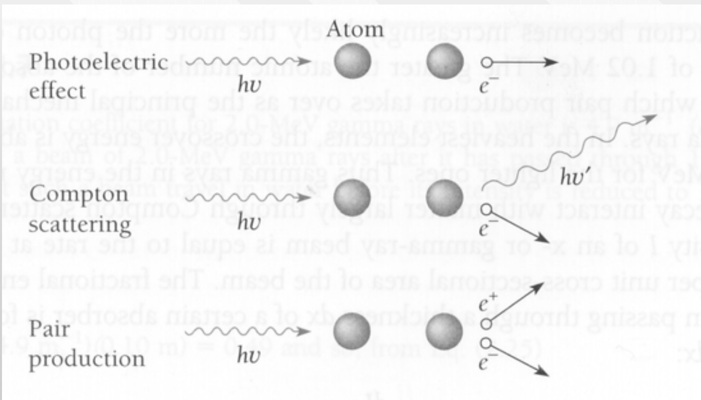
What is the wavelength of the gamma ray in a proton-antiproton annihilation?

$\lambda_{annih} = hc/M_p c^2 = 1240\ \text{nm eV}/937\ \text{MeV} = 1.3\times10^{-15}\ \text{m}$

121

Photon absorption

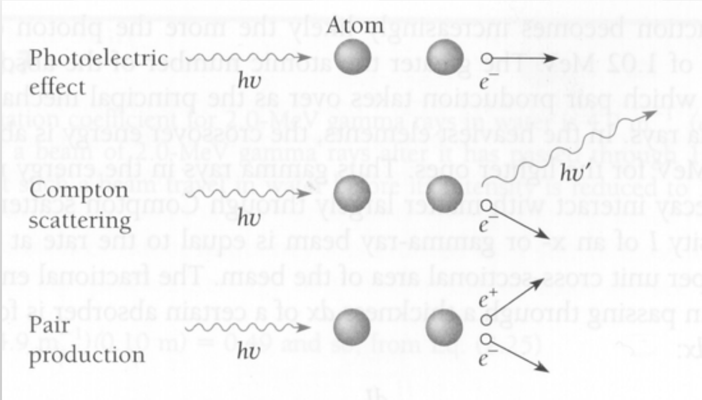
- Three chief “channels” photons interact with matter are:
- Photoelectric effect, Compton scattering effect and Pair-production
- In all of these process, photon energy is transferred to electrons which in turn lose energy to atoms in the absorbing material



122

Photon absorption

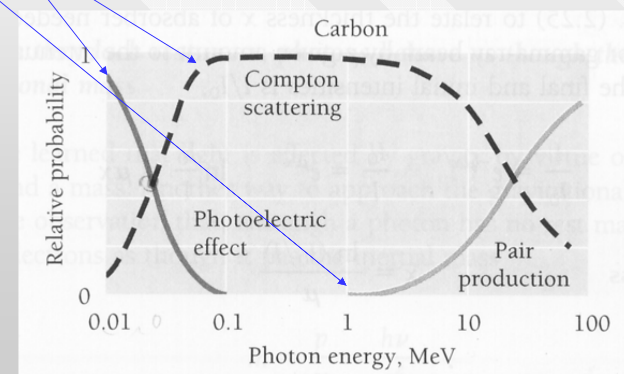
- The probability (cross section) of a photon undergoes a given channel of interaction with matter depends on
- (1) Photon energy, and
- (2) Atomic number of the absorbing material



123

Relative probabilities of photon absorption channels

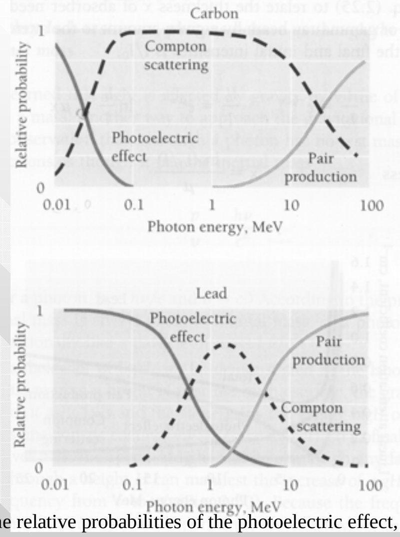
- For a fixed atomic number (say Carbon, A = 12)
- At low energy photoelectric effect dominates. It diminishes fast when E_γ approaches tens of keV
- At E_γ = a few tens of keV, Compton scattering start to take over
- Once E_γ exceeds the threshold of $2m_ec^2$ = 1.02 MeV, pair production becomes more likely. Compton scattering diminishes as energy increases from 1 MeV.



124

Relative probabilities between different absorbers different

- Compare with Lead absorber (much higher A):
- Photoelectric effect remains dominant up to a higher energy of a few hundreds of keV (c.f. Carbon of a few tens of keV)
- This is because the heavier the nucleus the better it is in absorbing the momentum transfer that occurs when the energetic photon imparts its momentum to the atom
- Compton scattering starts to appears after a much higher energy of 1 MeV (c.f. a few tens of keV for Carbon).
- This is because a larger atomic number binds an electron stronger, rendering the electron less ‘free’> In this case, to Compton scatter off an “free” electron the photon has to be more energetic
- (recall that in Compton scattering, only free electrons are scattered by photon).

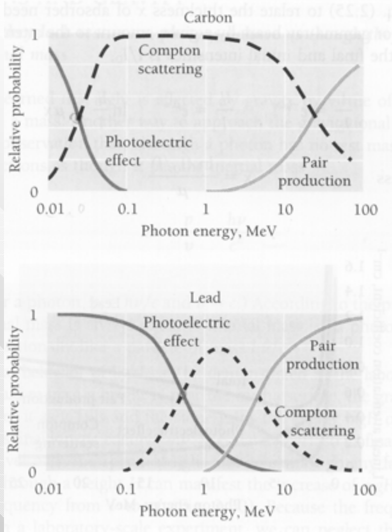


The relative probabilities of the photoelectric effect, Compton scattering, and pair production as functions of energy in carbon (a light element) and lead (a heavy element).

125

Relative probabilities between different absorbers different

- The energy at which pair production takes over as the principle mechanism of energy loss is called the crossover energy
- The crossover energy is 10 MeV for Carbon, 4 for Lead
- The greater atomic number, the lower the crossover energy
- This is because nuclear with larger atomic number has stronger electric field that is necessary to trigger pair-creation



126

What is a photon?

- Like an EM wave, photons move with speed of light *c*
- They have zero mass and rest energy
- The carry energy and momentum, which are related to the frequency and wavelength of the EM wave by $E=hf$ and $p = h/\lambda$
- They can be created or destroyed when radiation is emitted or absorbed
- They can have particle-like collisions with other particles such as electrons

127

Contradictory nature of light

- In Photoelectric effect, Compton scatterings, inverse photoelectric effect, pair creation/annihilation, light behaves as particle. The energy of the EM radiation is confined to localised bundles
- In Young's Double slit interference, diffraction, Bragg's diffraction of X-ray, light behave as waves. In the wave picture of EM radiation, the energy of wave is spread smoothly and continuously over the wavefronts.

128

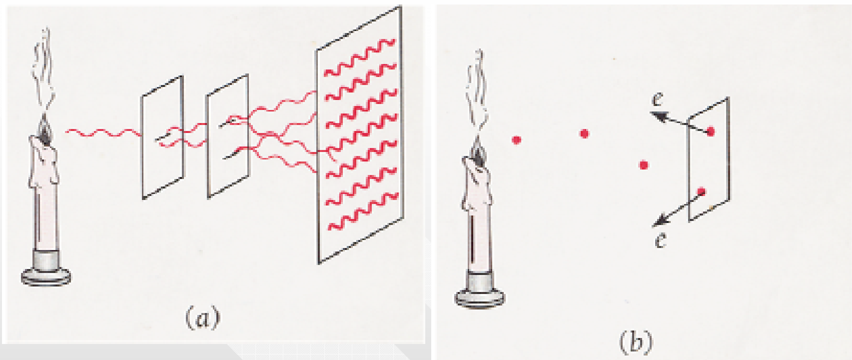


Figure 2.14 (a) The wave theory of light explains diffraction and interference, which the quantum theory cannot account for. (b) The quantum theory explains the photoelectric effect, which the wave theory cannot account

129

Is light particle? Or is it wave?

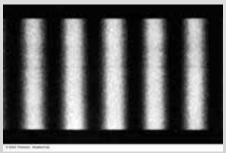
- Both the wave and particle explanations of EM radiation are obviously mutually exclusive
- So how could we reconcile these seemingly contradictory characteristics of light?
- The way out to the conundrum:
    - WAVE-PARTICLE DUALITY

130

Gedanken experiment with remote light source

- The same remote light source is used to simultaneously go through two experimental set up separated at a huge distance of say 100 M light years away.
- In the left experiment, the EM radiation behaves as wave; the right one behave like particle
- This is weird: the “light source” from 100 M light years away seems to “know” in which direction to aim the waves and in which direction to aim the particles

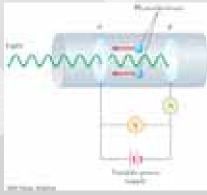
Light source is 100 M light years away
from the detection sites



Interference pattern observed

Double slit experiment

Photoelectric experiment



Photoelectron observed

131

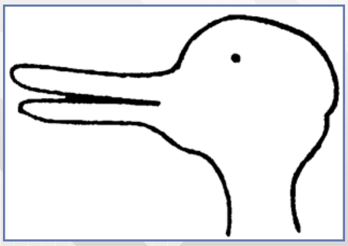
So, (asking for the second time) is light wave of particle?

- So, it is not *either* particle or wave but *both* particles and waves
- However, both typed of nature cannot be simultaneously measured in a single experiment
- The light only shows one or the other aspect, depending on the kind of experiment we are doing
- Particle experiments show the particle nature, while a wave-type experiment shows the wave nature

132

The identity of photon depends on how the experimenter decide to look at it

The face of a young or an old woman?

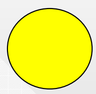


Is this a rabbit or a duck?

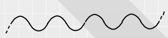
133

Coin a simile of wave-particle duality

- It's like a coin with two faces. One can only sees one side of the coin but not the other at any instance
- This is the so-called wave-particle duality
- Neither the wave nor the particle picture is wholly correct all of the time, that both are needed for a complete description pf physical phenomena
- The two are *complementary* to another



photon as particle

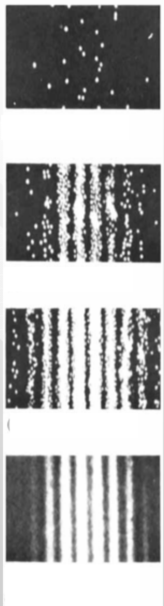


Photon as wave

134

Interference experiment with a single photon

- Consider an double slit experiment using an extremely weak source (say, a black body filament) that emits only one photon a time through the double slit and then detected on a photographic plate by darkening individual grains.
- When one follows the time evolution of the pattern created by these individual photons, interference pattern is observed
- At the source the light is being emitted as photon (radiated from a dark body) and is experimentally detected as a photon which is absorbed by an individual atom on the photographic plate to form a grain
- In between (e.g. between emission and detection), we must interpret the light as electromagnetic energy that propagates smoothly and continuously as a wave
- However, the wave nature between the emission and detection **is not directly detected**. Only the particle nature are detected in this procedure.
- The correct explanation of the origin and appearance of the interference pattern comes from the wave picture, and the correct interpretation of the evolution of the pattern on the screen comes from the particle picture;
- Hence to completely explain the experiment, the two pictures must somehow be taken together – this is an example for which *both pictures are complimentary to each other*



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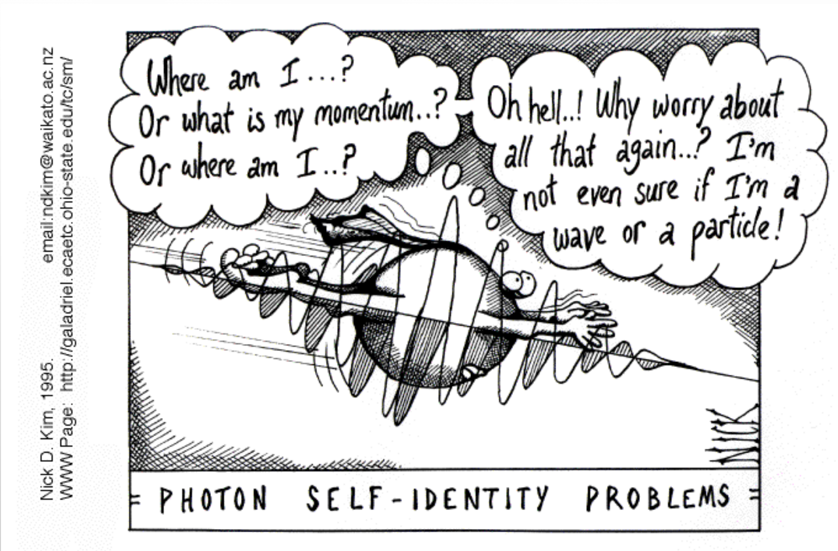


136

Both light and material particle display wave-particle duality

- Not only light manifest such wave-particle duality, but other microscopic material particles (e.g. electrons, atoms, muons, pions well).
- In other words:
- Light, as initially thought to be wave, turns out to have particle nature;
- Material particles, which are initially thought to be corpuscular, also turns out to have wave nature (next topic)

CHAPTER 4

The wavelike properties of particles

Schroedinger's Cat: "Am I a particle or wave?"

Wave particle duality

- "Quantum nature of light" refers to the particle attribute of light
- "Quantum nature of particle" refers to the wave attribute of a particle
- Light (classically EM waves) is said to display "wave-particle duality" – it behave like wave in one experiment but as particle in others (c.f. a person with schizophrenia)

- Not only light does have "schizophrenia", so are other microscopic "particle'' such as electron, i.e. particles also manifest wave characteristics in some experiments
- Wave-particle duality is essentially the manifestation of the quantum nature of things
- This is an very weird picture quite contradicts to our conventional assumption with is deeply rooted on classical physics or intuitive notion on things

Planck constant as a measure of quantum effect

- When investigating physical systems involving its quantum nature, the theory usually involves the appearance of the constant *h*
- e.g. in Compton scattering, the Compton shift is proportional to *h*; So is photoelectricity involves *h* in its formula
- In general, when *h* appears, it means quantum effects arise
- In contrary, in classical mechanics or classical EM theory, *h* never appear as both theories do not take into account of quantum effects
- Roughly quantum effects arise in microscopic system (e.g. on the scale approximately of the order 10^{-10} m or smaller)

Wavelike properties of particle

- In 1923, while still a graduate student at the University of Paris, Louis de Broglie published a brief note in the journal *Comptes rendus* containing an idea that was to revolutionize our understanding of the physical world at the most fundamental level:
- *That particle has intrinsic wave properties*
- *For more interesting details:*
- *http://www.davis-inc.com/physics/index.shtml*

Prince de Broglie, 1892-1987

de Broglie's postulate (1924)

- The postulate: there should be a symmetry between matter and wave. The wave aspect of matter is related to its particle aspect in exactly the same quantitative manner that is in the case for radiation. The total (i.e. relativistic) energy E and momentum p of an entity, for both matter and wave alike, is related to the frequency f of the wave associated with its motion via Planck constant

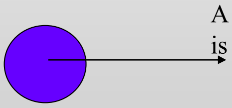
$$p = h/\lambda,$$
$$E = hf$$

A particle has wavelength!!!

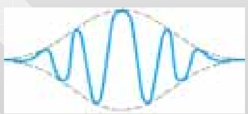
$$\lambda = h/p$$

- is the de Broglie relation predicting the wave length of the matter wave λ associated with the motion of a material particle with momentum p
- Note that classically the property of wavelength is only reserved for wave and particle was never associate with any wavelength
- But, following de Broglie's postulate, such distinction is removed

A particle with momentum p is pictured as a wave (wavepulse)



Particle with linear momentum p



Matter wave with de Broglie wavelength

$$\lambda = p/h$$

A physical entity possess both aspects of particle and wave in a complimentary manner

BUT <u>why</u> is the wave nature of material particle

not observed?

Because …

- Because…we are too large and quantum effects are too small
- Consider two extreme cases:
- (i) an electron with kinetic energy K = 54 eV, de Broglie wavelength, $\lambda = h/p = h / (2m_eK)^{1/2}$ = 1.65 Angstrom.
- Such a wavelength is comparable to the size of atomic lattice, and is experimentally detectable
- (ii) As a comparison, consider an macroscopic object, a billard ball of mass m = 100 g moving with momentum p
- $p = mv \approx 0.1$ kg × 10 m/s = 1 Ns (relativistic correction is negligible)
- It has de Broglie wavelength $\lambda = h/p \approx 10^{-34}$ m, too tiny to be observed in any experiments
- The total energy of the billard ball is
  - $E = K + m_0c^2 \approx m_0c^2 = 0.1\times(3\times10^8)^2$ J $= 9\times10^{15}$J
- (K is ignored since $K << m_0c^2$)
- The frequency of the de Broglie wave associated with the billard ball is $f = E/h = m_0c^2/h = (9\times10^{15}/6.63\times10^{34})$ Hz $=10^{78}$ Hz, impossibly high for any experiment to detect

Matter wave is a quantum phenomena

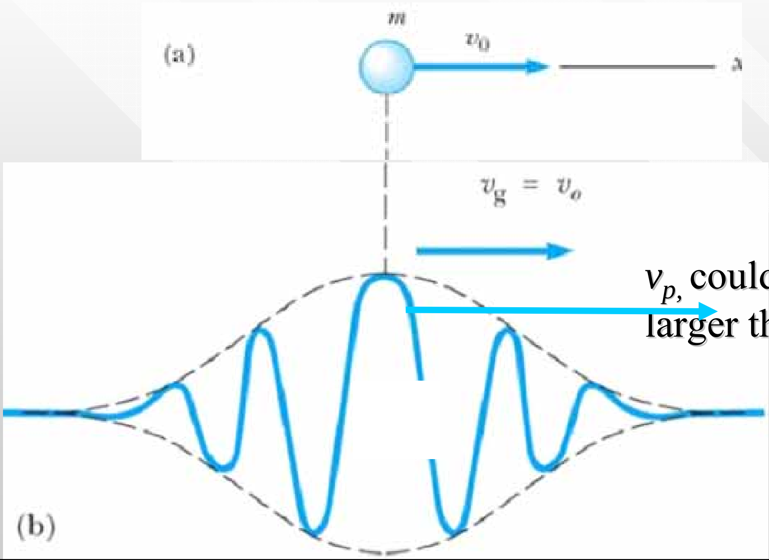
- This also means that the wave properties of matter is difficult to observe for macroscopic system (unless with the aid of some specially designed apparatus)
- The smallness of h in the relation $\lambda = h/p$ makes wave characteristic of particles hard to be observed
- The statement that when $h \rightarrow 0$, λ becomes vanishingly small means that:
- the wave nature will become effectively "shut-off" and appear to loss its wave nature whenever the relevant p of the particle is too large in comparison with the quantum scale characterised by h

How small is small?

- More quantitatively, we could not detect the quantum effect if $h/p \sim 10^{-34}$ Js/p (dimension: length, L) becomes too tiny in comparison to the length scale discernable by an experimental setup (e.g. slit spacing in a diffraction experiment)
- For a numerical example: For a slit spacing of $l \sim$ nm (inter-atomic layer in a crystal), and a momentum of p=10 Ns (100 g billard ball moving with 10 m/s),

  $h/p = 10^{-34}$ Js/p = 10^{-34} Js/10 Ns $\sim 10^{-35}$ m << $l \sim$ nm
- LHS, i.e. h/p ($\sim 10^{-35}$ m) , is the length scale of the de Broglie (quantum) wavelength;
- RHS, i.e. l ($\sim$ nm), is the length scale charactering the experiment
- Such an experimental set up could not detect the wave length of the moving billard ball.

The particle's velocity v_0 is identified with the de Broglie' group wave, v_g but not its phase wave v_p



$v_{p,}$ could be larger than c

Example

- An electron has a de Broglie wavelength of 2.00 pm. Find its kinetic energy and the group velocity of its de Broglie waves.
- *Hint:*
- The group velocity of the dB wave of electron v_g is equal to the velocity of the electron, v.
- Must treat the problem relativistically.
- If the electron's de Broglie wavelength λ is known, so is the momentum, p. Once p is known, so is the total energy, E and velocity v. Once E is known, so will the kinetic energy, K.

Solution

- Total energy $E^2 = c^2p^2 + m_0{}^2c^4$
- $K = E - m_0c^2$
  $= (c^2p^2 + m_0{}^2c^4)^{1/2} - m_0c^2$
  $= ((hc/\lambda)^2 + m_0{}^2c^4)^{1/2} - m_0c^2 = 297$ keV
- $v_g = v$; $1/\gamma^2 = 1 - (v/c)^2$;
- $(pc)^2 = (\gamma m_0vc)^2 = (hc/\lambda)^2$ (from Relativity and de Broglie's postulate)

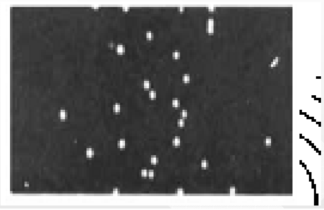
$\Rightarrow (\gamma v/c)^2 = (hc/\lambda)^2/(m_0c^2)^2 = (620 \text{ keV}/510 \text{ keV})^2 = 1.4884$;
$(\gamma v/c)^2 = (v/c)^2 / 1 - (v/c)^2$
$\Rightarrow v_g/c = \surd(1.4884/(1+1.4884)) = 0.77$

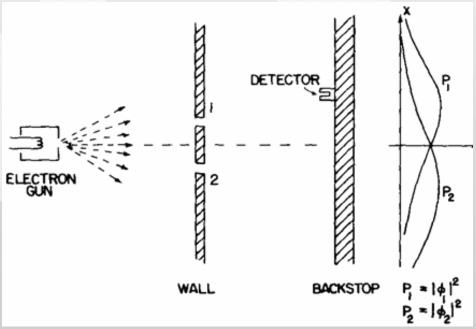
Alternatively

- The previous calculation can also proceed via:
- $K = (\gamma - 1)m_ec^2$
- $\Rightarrow \gamma = K/(m_ec^2) + 1 = 297 \text{ keV}/(510 \text{ keV}) + 1$
  $= 1.582$;
- $p = h/\lambda = \gamma m_e v \Rightarrow v = hc/(\lambda\gamma m_e c)$
- $\Rightarrow v/c = hc/(\lambda\, \gamma\, m_e c^2)$
- $= (1240 \text{ nm}\cdot\text{eV}) / (2\text{pm}\cdot 1.582\cdot 0.51\text{MeV})$
- $= 0.77$

Interference experiment with a single electron, firing one in a time

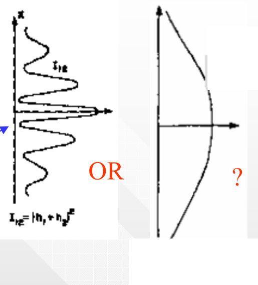
- Consider an double slit experiment using an extremely small electron source that emits only one electron a time through the double slit and then detected on a fluorescent plate
- When hole 1 (hole 2) is blocked, distribution P1 (P2) is observed.
- P1 are P2 are the distribution pattern as expected from the behaviour of particles.
- Hence, electron behaves like particle when one of the holes is blocked
- What about if both holes are not blocked? Shall we see the distribution simply be P1 + P2? (This would be our expectation for particle: Their distribution simply adds)

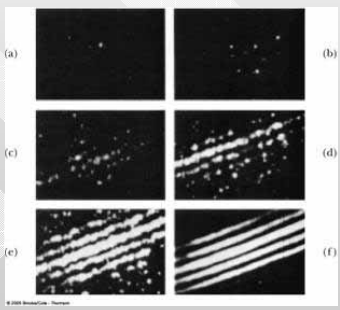




Electrons display interference pattern

- When one follows the time evolution of the pattern created by these individual electron with both hole opened, what sort of pattern do you think you will observed?
- It's the interference pattern that are in fact observed in experiments
- At the source the electron is being emitted as particle and is experimentally detected as a electron which is absorbed by an individual atom in the fluorescent plate
- In between, we must interpret the electron in the form of a wave. The double slits change the propagation of the electron wave so that it is 'processed' to forms diffraction pattern on the screen.
- Such process would be impossible if electrons are particle (because no one particle can go through both slits at the same time. Such a simultaneous penetration is only possible for wave.)
- Be reminded that the wave nature in the intermediate states is not measured. Only the particle nature are detected in this procedure.

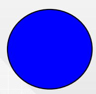




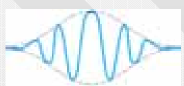
- The correct explanation of the origin and appearance of the interference pattern comes from the wave picture
- Hence to completely explain the experiment, the two pictures must somehow be taken together – this is an example for which *both pictures are complimentary to each other*
- Try to compare the last few slides with the slides from previous chapter for photon, which also displays wave-particle duality

So, is electron wave or particle?

- They are both…but not simultaneously
- In any experiment (or empirical observation) only one aspect of either wave or particle, but not both can be observed simultaneously.
- It's like a coin with two faces. But one can only see one side of the coin but not the other at any instance
- This is the so-called wave-particle duality



Electron as particle



Electron as wave

Detection of electron as particle destroy the interference pattern

- If in the electron interference experiment one tries to place a detector on each hole to determine through which an electron passes, the wave nature of electron in the intermediate states are destroyed
- i.e. the interference pattern on the screen shall be destroyed
- Why? It is the consistency of the wave-particle duality that demands such destruction must happen (think of the logics yourself or read up from the text)

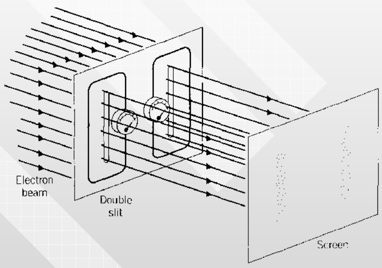


FIGURE 4.15 Apparatus to record passage of electrons through slits. Each slit is surrounded by a loop with a meter that signals the passage of an electron through the slit. No interference fringes are seen on the screen.

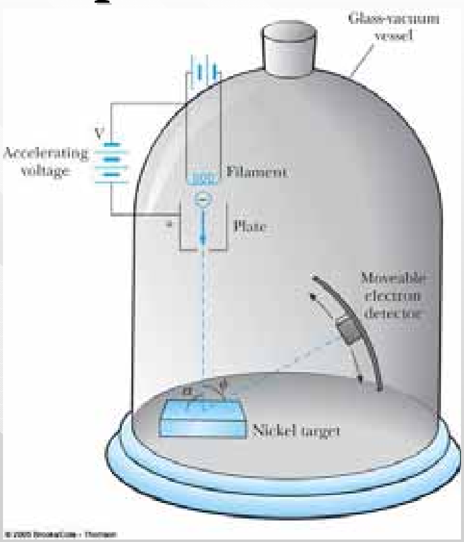
"Once and for all I want to know what I'm paying for. When the electric company tells me whether electron is a wave or a particle I'll write my check"

Extra readings

- Those quantum enthusiasts may like to read more about wave-particle duality in Section 5.7, page 179-185, Serway, Moses and Mayer.
- An even more recommended reading on wave-particle duality: the Feynman lectures on physics, vol. III, chapter 1 (Addison-Wesley Publishing)
- It's a very interesting and highly intellectual topic to investigate

Davisson and Gremer experiment, 1937 Nobel prize

- DG confirms the wave nature of electron in which it undergoes Bragg's diffraction
- Thermionic electrons are produced by hot filament, accelerated and focused onto the target (all apparatus is in vacuum condition)
- Electrons are scattered at an angle ϕ into a movable detector

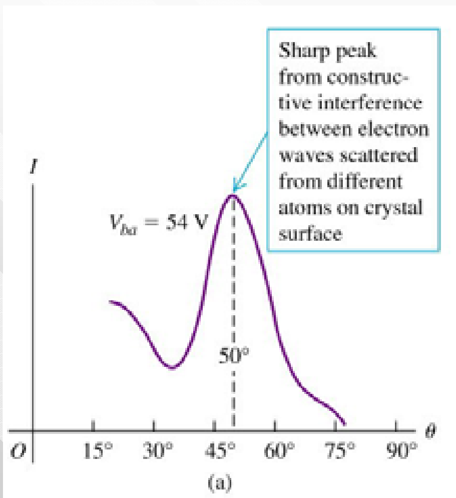


Note: the discussion of DG experiment in Beiser, 6th edition, is incorrect.

Pix of Davisson and Gremer

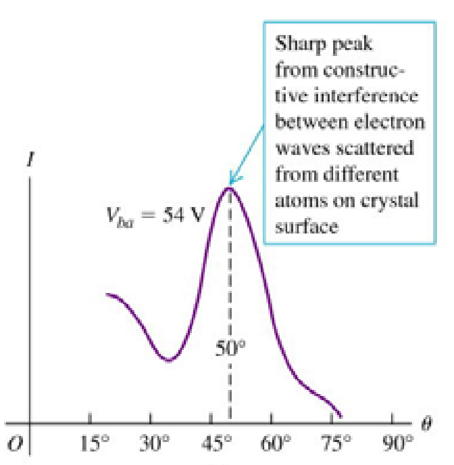
Result of the DG experiment

- Distribution of electrons is measured as a function of ϕ
- Strong scattered e-beam is detected at ϕ = 50 degree for V = 54 V



How to interpret the result of DG?

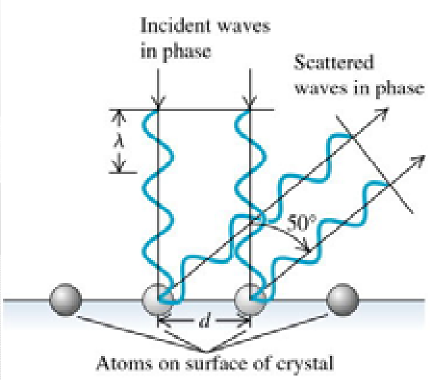
- Electrons get diffracted by the atoms on the surface (which acted as diffraction grating) of the metal as though the electron acting like they are WAVES
- Electrons do behave like waves as postulated by de Broglie



(a)

Diffraction of electron by atoms on the crystal surface

- Electron energy is low, hence electron did not penetrate far into the crystal
- Sufficient to consider only diffraction to take place in the plane of atoms on the surface (refer figure)
- The situation is similar to using a reflection grating for light; the spacing *d* between the rows of atoms on the crystal surface plays the role of the spacing between the slit in the optical grating.
- $d \sin \phi = n\lambda$, $n = 1,2,3\ldots$(for constructive diffraction)



(b)

Diffraction of electron by atoms on the crystal surface

- The peak of the diffraction pattern is the $n = 1^{st}$ order
- From x-ray diffraction experiment done independently on nickle, we know $d = 2.15$ Amstrong
- Hence the wavelength corresponds to the diffraction pattern observed in the DG experiment is

$$\lambda = d \sin\phi = 1.65\text{Angstrom}$$

- Here, 1.65 Angstrom is the experimentally inferred value, which is to be checked against the theoretical value predicted by de Broglie

Theoretical value of λ of the electron

- An external potential V accelerates the electron via $eV=K$
- In the DG experiment the kinetic energy of the electron is accelerated to $K = 54$ eV (non-relativistic treatment is suffice because $K << m_e c^2 = 0.51$ MeV)
- According to de Broglie, the wavelength of an electron accelerated to kinetic energy of $K = p^2/2m_e = 54$ eV has a equivalent matter wave wavelength

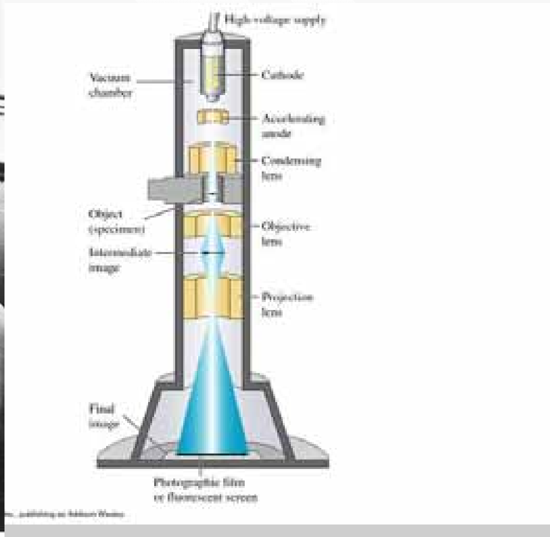
  $\lambda = h/p = h/(2Km_e)^{1/2} = 1.67$ Amstrong
- In terms of the external potential,

  $\lambda = h/(2eVm_e)^{1/2}$

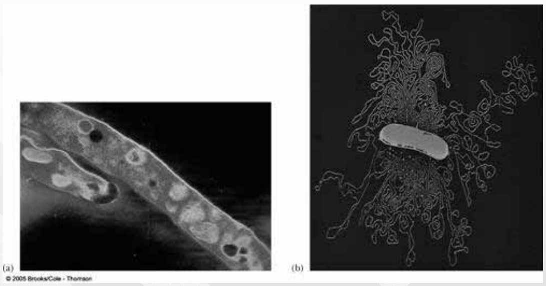
Theory's prediction matches measured value

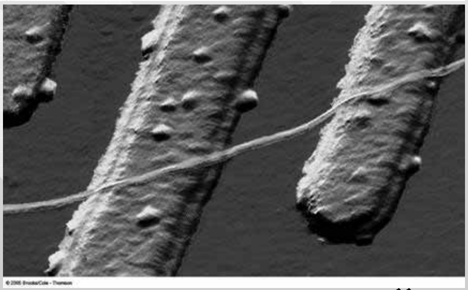
- The result of DG measurement agrees almost perfectly with the de Broglie's prediction: 1.65 Angstrom measured by DG experiment against 1.67 Angstrom according to theoretical prediction
- Wave nature of electron is hence experimentally confirmed
- In fact, wave nature of microscopic particles are observed not only in e- but also in other particles (e.g. neutron, proton, molecules etc. – most strikingly Bose-Einstein condensate)

Application of electrons wave: electron microscope, 1986 Nobel Prize in medicine (Ernst Ruska)



- Electron's de Broglie wavelength can be tuned via $\lambda = h/(2eVm_e)^{1/2}$
- Hence electron microscope can magnify specimen (x4000 times) for biological specimen or 120,000 times of wire of about 10 atoms in width

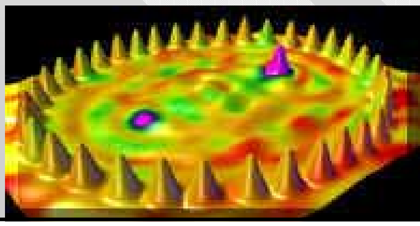




Not only electron, other microscopic particles also behave like wave at the quantum scale

- The following atomic structural images provide insight into the threshold between prime radiant flow and the interference structures called matter.
- In the right foci of the ellipse a real cobalt atom has been inserted. In the left foci of the ellipse a phantom of the real atom has appeared. The appearance of the phantom atom was not expected.
- The ellipsoid coral was constructed by placing 36 cobalt atom on a copper surface. This image is provided here to provide a visual demonstration of the attributes of material matter arising from the harmonious interference of background radiation.

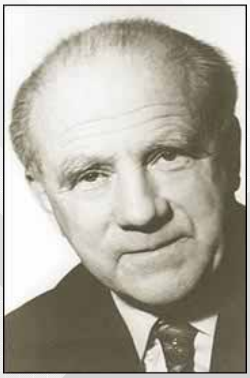
QUANTUM CORAL



http://home.netcom.com/~sbyers11/grav11E.htm

Heisenberg's uncertainty principle (Nobel Prize,1932)

- WERNER HEISENBERG (1901 - 1976)
- was one of the greatest physicists of the twentieth century. He is best known as a founder of quantum mechanics, the new physics of the atomic world, and especially for the uncertainty principle in quantum theory. He is also known for his controversial role as a leader of Germany's nuclear fission research during World War II. After the war he was active in elementary particle physics and West German science policy.
- *http://www.aip.org/history/heisenberg/p01.htm*



A particle is represented by a wave packet/pulse

- Since we experimentally confirmed that particles are wave in nature at the quantum scale *h* (matter wave) we now have to describe particles in term of waves (relevant only at the quantum scale)
- Since a real particle is localised in space (not extending over an infinite extent in space), the wave representation of a particle has to be in the form of wave packet/wave pulse

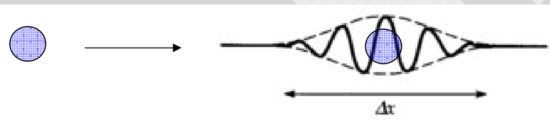
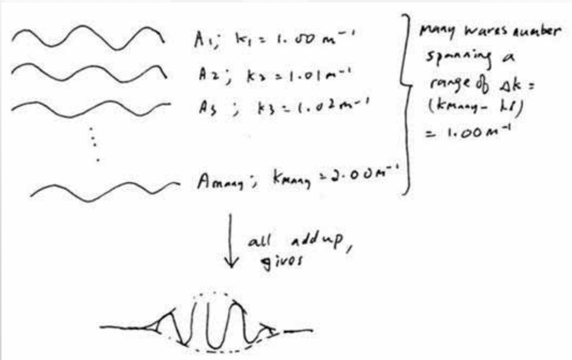


FIGURE 6.14 An idealized wave packet localized in space over a region Δx is the perposition of many waves of different amplitudes and frequencies.

- As mentioned before, wavepulse/wave packet is formed by adding many waves of different amplitudes and with the wave numbers spanning a range of Δk (or equivalently, $\Delta\lambda$)



Δx

Recall that k = $2\pi/\lambda$, hence

$\Delta k/k = \Delta\lambda/\lambda$

Still remember the uncertainty relationships for classical waves?

- As discussed earlier, due to its nature, a wave packet must obey the uncertainty relationships for classical waves (which are derived mathematically with some approximations)

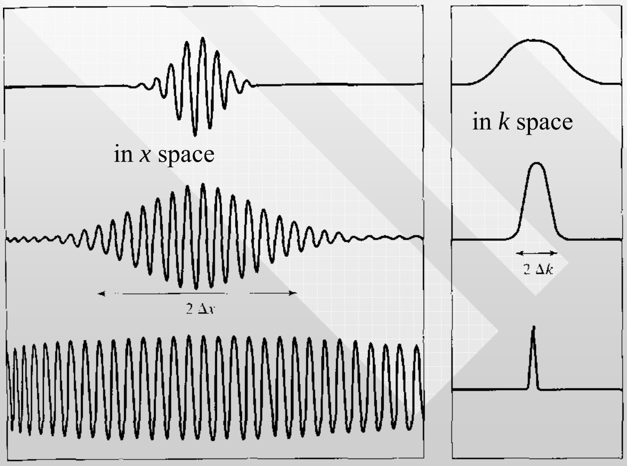
$$\Delta\lambda\Delta x \underset{\sim}{>} \lambda^2 \equiv \Delta k\Delta x \underset{\sim}{>} 2\pi \qquad \Delta t\Delta\nu \geq 1$$

- However a more rigorous mathematical treatment (without the approximation) gives the exact relations

$$\Delta\lambda\Delta x \geq \frac{\lambda^2}{4\pi} \equiv \Delta k\Delta x \geq 1/2 \qquad \Delta\nu\Delta t \geq \frac{1}{4\pi}$$

- To describe a particle with wave packet that is localised over a small region Δx requires a large range of wave number; that is, Δk is large. Conversely, a small range of wave number cannot produce a wave packet localised within a small distance.

- A narrow wave packet (small Δx) corresponds to a large spread of wavelengths (large Δk).
- A wide wave packet (large Δx) corresponds to a small spread of wavelengths (small Δk).



Matter wave representing a particle must also obey similar wave uncertainty relation

- For matter waves, for which their momentum and wavelength are related by $p = h/\lambda$, the uncertainty relationship of the classical wave
- $\Delta\lambda\Delta x \geq \dfrac{\lambda^2}{4\pi} \equiv \Delta k\Delta x \geq 1/2$ is translated into

$$\Delta p_x \Delta x \geq \frac{\hbar}{2}$$

- where $\hbar = h/2\pi$
- Prove this relation yourselves (hint: from $p = h/\lambda$, $\Delta p/p = \Delta\lambda/\lambda$)

Time-energy uncertainty

- Just as $\Delta p_x \Delta x \geq \frac{\hbar}{2}$ implies position-momentum uncertainty relation, the classical wave uncertainty relation $\Delta \nu \Delta t \geq \frac{1}{4\pi}$ also implies a corresponding relation between time and energy

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

- This uncertainty relation can be easily obtained:

$$h\Delta \nu \Delta t \geq \frac{h}{4\pi} = \frac{\hbar}{2};$$

$$\because E = h\nu, \Delta E = h\Delta \nu \Rightarrow \Delta E \Delta t = h\Delta \nu \Delta t = \frac{\hbar}{2}$$

Heisenberg uncertainty relations

$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

- The product of the uncertainty in momentum (energy) and in position (time) is at least as large as Planck's constant

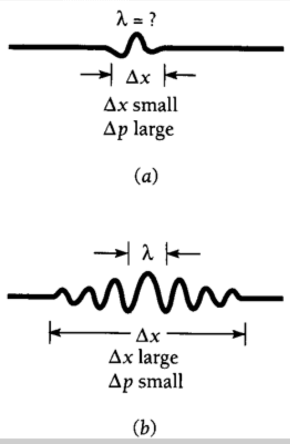
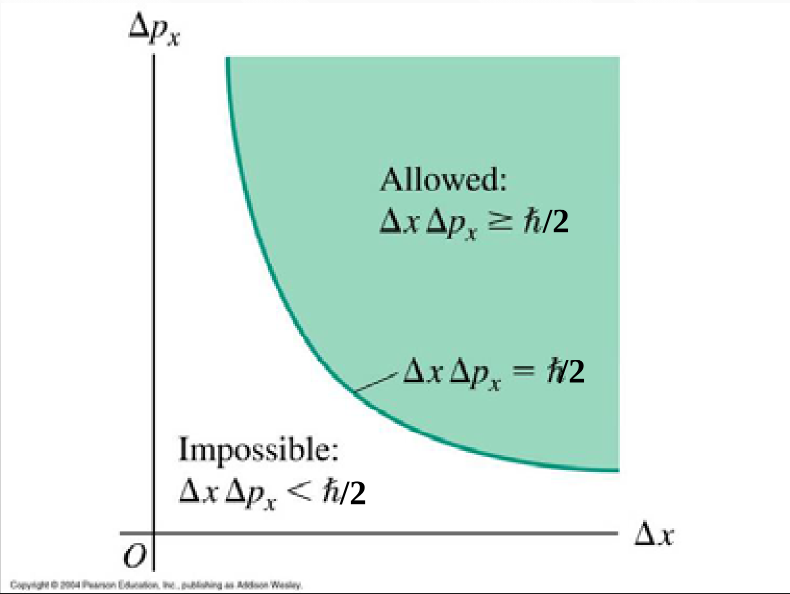


Figure 3.12 (*a*) A narrow de Broglie wave group. The position of the particle can be precisely determined, but the wavelength (and hence the particle's momentum) cannot be established because there are not enough waves to measure accurately. (*b*) A wide wave group. Now the wavelength can be precisely determined but not the position of the particle.

What $\Delta p_x \Delta x \geq \frac{\hbar}{2}$ means

- It sets the intrinsic lowest possible limits on the uncertainties in knowing the values of p_x and x, no matter how good an experiments is made
- It is impossible to specify simultaneously and with infinite precision the linear momentum and the corresponding position of a particle

It is impossible for the product $\Delta x \Delta p_x$ to be less than $h/4\pi$

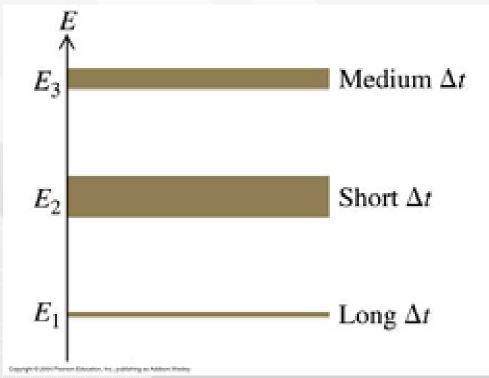


What $\Delta E \Delta t \geq \frac{\hbar}{2}$ means

- Uncertainty principle for energy.
- The energy of a system also has inherent uncertainty, ΔE
- ΔE *is* dependent on the *time interval* Δt during which the system remains in the given states.
- If a system is known to exist in a state of energy E over a limited period Δt, then this energy is uncertain by at least an amount $h/(4\pi\Delta t)$. This corresponds to the 'spread' in energy of that state (see next page)
- Therefore, the energy of an object or system can be measured with infinite precision (ΔE=0) only if the object of system exists for an infinite time ($\Delta t \rightarrow \infty$)

What $\Delta E \Delta t \geq \frac{\hbar}{2}$ means

- A system that remains in a metastable state for a very long time (large Δt) can have a very well-defined energy (small ΔE), but if remain in a state for only a short time (small Δt), the uncertainty in energy must be correspondingly greater (large ΔE).



Conjugate variables (Conjugate observables)

- $\{p_x, x\}$, $\{E, t\}$ are called *conjugate variables*
- The conjugate variables cannot in principle be measured (or known) to infinite precision simultaneously

Heisenberg's Gedanken experiment

- The U.P. can also be understood from the following gedanken experiment that tries to measure the position and momentum of an object, say, an electron at a certain moment
- In order to measure the momentum and position of an electron it is necessary to "interfere" it with some "probe" that will then carries the required information back to us – such as shining it with a photon of say a wavelength of λ

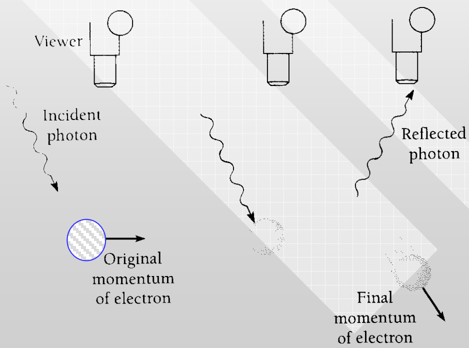
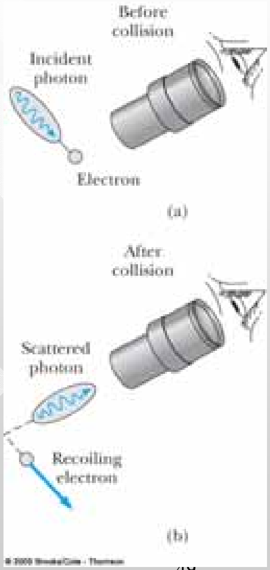


Figure 3.17 An electron cannot be observed without changing its momentum.

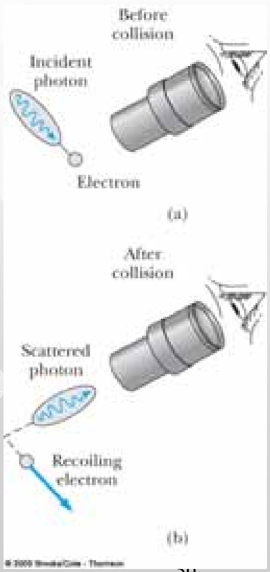
Heisenberg's Gedanken experiment

- Let's say the "unperturbed" electron was initially located at a "definite" location *x* and with a "definite" momentum *p*
- When the photon 'probes' the electron it will be bounced off, associated with a changed in its momentum by some uncertain amount, Δp.
- Δp cannot be predicted but must be of the similar order of magnitude as the photon's momentum h/λ
- Hence $\Delta p \approx h/\lambda$
- The longer λ (i.e. less energetic) the smaller the uncertainty in the measurement of the electron's momentum
- In other words, electron cannot be observed without changing its momentum



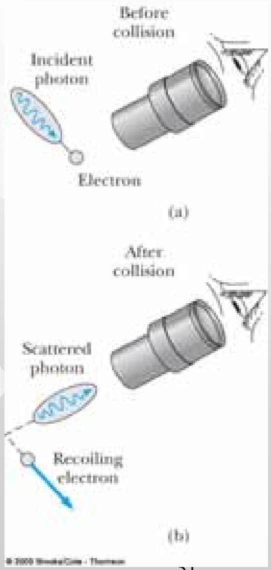
Heisenberg's Gedanken experiment

- How much is the uncertainty in the position of the electron?
- By using a photon of wavelength λ we cannot determine the location of the electron better than an accuracy of $\Delta x = \lambda$
- Hence $\Delta x \geq \lambda$
- Such is a fundamental constraint coming from optics (Rayleigh's criteria).
- The shorter the wavelength λ (i.e. more energetic) the smaller the uncertainty in the electron's position



Heisenberg's Gedanken experiment

- However, if shorter wavelength is employed (so that the accuracy in position is increased), there will be a corresponding decrease in the accuracy of the momentum, measurement (recall $\Delta p \approx h/\lambda$)
- A higher photon momentum will disturb the electron's motion to a greater extent
- Hence there is a 'zero sum game' here
- Combining the expression for Δx and Δp, we then have $\Delta p \, \Delta\lambda \geq h$, a result consistent with $\Delta p \, \Delta\lambda \geq h/2$



Heisenberg's kiosk

Example

- A typical atomic nucleus is about 5.0×10^{-15} m in radius. Use the uncertainty principle to place a lower limit on the energy an electron must have if it is to be part of a nucleus

Solution

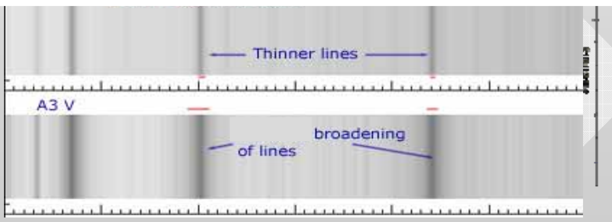
- Letting $\Delta x = 5.0\times10^{-15}$ m, we have
- $\Delta p \geq h/(4\pi\Delta x)=\ldots=1.1\times10^{-20}$ kg·m/s

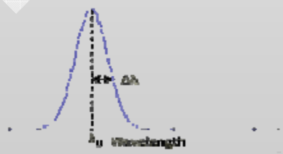
  If this is the uncertainty in a nuclear electron's momentum, the momentum p must be at lest comparable in magnitude. An electron of such a momentum has a
- KE $\approx pc \geq 3.3\times10^{-12}$ J

  $= 20.6$ MeV >> $m_e c^2 = 0.5$ MeV
- i.e., if electrons were contained within the nucleus, they must have an energy of at least 20.6 MeV
- However such an high energy electron from radioactive nuclei never observed
- Hence, by virtue of the uncertainty principle, we conclude that electrons emitted from an unstable nucleus cannot comes from within the nucleus

Broadening of spectral lines due to uncertainty principle

- An excited atom gives up it excess energy by emitting a photon of characteristic frequency. The average period that elapses between the excitation of an atom and the time it radiates is 1.0×10^{-8} s. Find the inherent uncertainty in the frequency of the photon.





Solution

- The photon energy is uncertain by the amount
- $\Delta E \geq hc/(4c\pi\Delta t)= 5.3\times10^{-27}$ J = 3.3×10^{-8} eV
- The corresponding uncertainty in the frequency of light is $\Delta\nu = \Delta E/h \geq 8\times10^{6}$ Hz
- This is the irreducible limit to the accuracy with which we can determine the frequency of the radiation emitted by an atom.
- As a result, the radiation from a group of excited atoms does not appear with the precise frequency ν.
- For a photon whose frequency is, say, 5.0×10^{14} Hz,
- $\Delta\nu/\nu = 1.6\times10^{-8}$

PYQ 2.11 Final Exam 2003/04

- Assume that the uncertainty in the position of a particle is equal to its de Broglie wavelength. What is the minimal uncertainty in its velocity, v_x?
- A. $v_x/4\pi$ B. $v_x/2\pi$ C. $v_x/8\pi$
- D. v_x E. v_x/π
- ANS: A, Schaum's 3000 solved problems, Q38.66, pg. 718

Solution

$$\Delta x \Delta p_x \geq \hbar/2; \Delta p_x = m\Delta v_x.$$

Given $\Delta x = \lambda$,

$$\Rightarrow m\Delta x \Delta v_x = m\lambda \Delta v_x \geq \hbar/2;$$

$$\Rightarrow \Delta v_x \geq \hbar/2m\lambda = h/4\pi m\lambda$$

But $p_x = h/\lambda$

$$\Rightarrow \Delta v_x \geq h/4\pi m\lambda = p_x/4\pi m$$

$$= mv_x/4\pi m = v_x/4\pi$$

Example

- A measurement established the position of a proton with an accuracy of $\pm 1.00 \times 10^{-11}$ m. Find the uncertainty in the proton's position 1.00 s later. Assume $v \ll c$.

59

Solution

- Let us call the uncertainty in the proton's position Δx_0 at the time $t = 0$.
- The uncertainty in its momentum at $t = 0$ is

$$\Delta p \geq h/(4\pi\, \Delta x_0)$$

- Since $v \ll c$, the momentum uncertainty is

$$\Delta p = m\Delta v$$

- The uncertainty in the proton's velocity is

$$\Delta v = \Delta p / m \geq h/(4\pi m\, \Delta x_0)$$

- The distance x of the proton covers in the time t cannot be known more accurately than

$$\Delta x = t\Delta v \geq ht/(4\pi\, m\Delta x_0)$$

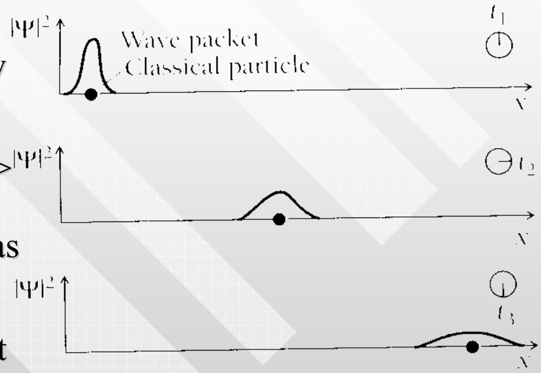
- m=970 MeV/c^2
- The value of Δx at t = 1.00 s is 3.15 km.

60

A moving wave packet spreads out in space

$\Delta x = t\Delta v \geq ht/(4\pi\, \Delta x_0)$

- Note that Δx is inversely proportional to Δx_0
- It means the more we know about the proton's position at $t = 0$ the less we know about its later position at $t > 0$.
- The original wave group has spread out to a much wider one because the phase velocities of the component wave vary with wave number and a large range of wave numbers must have been present to produce the narrow original wave group



Example

Estimating quantum effect of a macroscopic particle

- Estimate the minimum uncertainty velocity of a billard ball ($m \sim$ 100 g) confined to a billard table of dimension 1 m

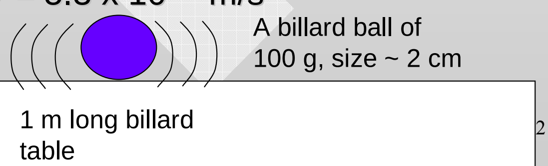
Solution

For $\Delta x \sim 1$ m, we have

$\Delta p \geq h/4\pi\Delta x = 5.3\text{x}10^{-35}$ Ns,

- So $\Delta v = (\Delta p)/m \geq 5.3\text{x}10^{-34}$ m/s
- One can consider $\Delta v = 5.3\text{x}10^{-34}$ m/s (extremely tiny) is the speed of the billard ball at anytime caused by quantum effects
- In quantum theory, no particle is absolutely at rest due to the Uncertainty Principle

$\Delta v = 5.3 \text{ x } 10^{-34}$ m/s

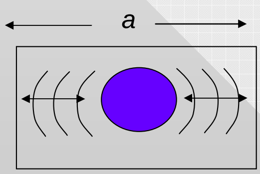


A particle contained within a finite region must has some minimal KE

- One of the most dramatic consequence of the uncertainty principle is that a particle confined in a small region of finite width cannot be exactly at rest (as already seen in the previous example)
- Why? Because…
- ...if it were, its momentum would be precisely zero, (meaning $\Delta p = 0$) which would in turn violate the uncertainty principle

What is the K_{ave} of a particle in a box due to Uncertainty Principle?

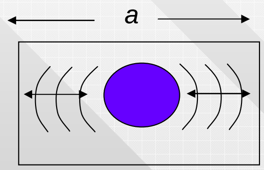
- We can estimate the minimal KE of a particle confined in a box of size a by making use of the U.P.
- If a particle is confined to a box, its location is uncertain by $\Delta x = a$
- Uncertainty principle requires that $\Delta p \geq (h/2\pi)a$
- (don't worry about the factor 2 in the uncertainty relation since we only perform an estimation)



Zero-point energy

$$K_{\text{ave}} = \left(\frac{p^2}{2m}\right)_{av} \gtrsim \frac{(\Delta p)^2}{2m} \gtrsim \frac{\hbar^2}{2ma^2}$$

This is the zero-point energy, the minimal possible kinetic energy for a quantum particle confined in a region of width *a*



Particle in a box of size *a* can never be at rest (e.g. has zero K.E) but has a minimal KE K_{ave} (its zero-point energy)

We will formally re-derived this result again when solving for the Schrodinger equation of this system (see later).

Recap

- Measurement necessarily involves interactions between observer and the observed system
- Matter and radiation are the entities available to us for such measurements
- The relations $p = h/\lambda$ and $E = h\nu$ are applicable to both matter and to radiation because of the intrinsic nature of wave-particle duality
- When combining these relations with the universal waves properties, we obtain the Heisenberg uncertainty relations
- In other words, the uncertainty principle is a necessary consequence of particle-wave duality


OH ALICE.. YOU'RE THE ONE FOR ME
BUT BOB.. IN A QUANTUM WORLD HOW CAN WE BE SURE?
Physics World

CHAPTER 5

Atomic Models

- Much of the luminous matter in the Universe is hydrogen. In fact hydrogen is the most abundance atom in the Universe. The colours of this Orion Nebula come from the transition between the quantized states in hydrogen atoms.

INTRODUCTION

- The purpose of this chapter is to build a simplest atomic model that will help us to understand the structure of atoms
- This is attained by referring to some basic experimental facts that have been gathered since 1900's (e.g. Rutherford scattering experiment, atomic spectral lines etc.)
- In order to build a model that well describes the atoms which are consistent with the experimental facts, we need to take into account the wave nature of electron
- This is one of the purpose we explore the wave nature of particles in previous chapters

Basic properties of atoms

- 1) Atoms are of microscopic size, ~ 10^{-10} m. Visible light is not enough to resolve (see) the detail structure of an atom as its size is only of the order of 100 nm.
- 2) Most atoms are stable (i.e. atoms that are non radioactive)
- 3) Atoms contain negatively charges, electrons, but are electrically neutral. An atom with Z electrons must also contain a net positive charge of $+Ze$.
- 4) Atoms emit and absorb EM radiation (in other words, atoms interact with light quite readily)

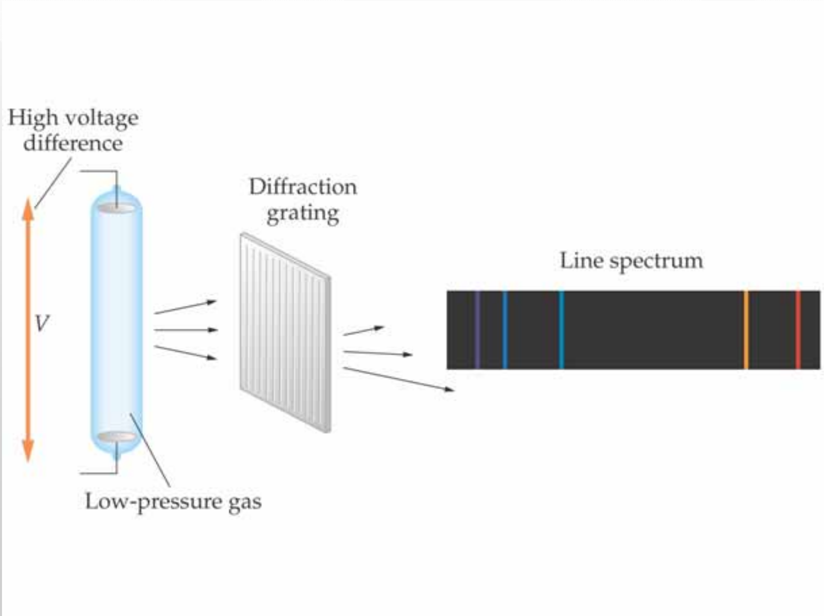
Because atoms interacts with EM radiation quite strongly, it is usually used to probe the structure of an atom. The typical of such EM probe can be found in the atomic spectrum as we will see now

Emission spectral lines

- Experimental fact: A single atom or molecule in a very diluted sample of gas emits radiation characteristic of the particular atom/molecule species
- The emission is due to the de-excitation of the atoms from their excited states
- e.g. if heating or passing electric current through the gas sample, the atoms get excited into higher energy states
- When a excited electron in the atom falls back to the lower energy states (de-excites), EM wave is emitted
- The spectral lines are analysed with ***spectrometer***, which give important physical information of the atom/molecules by analysing the wavelengths composition and pattern of these lines.

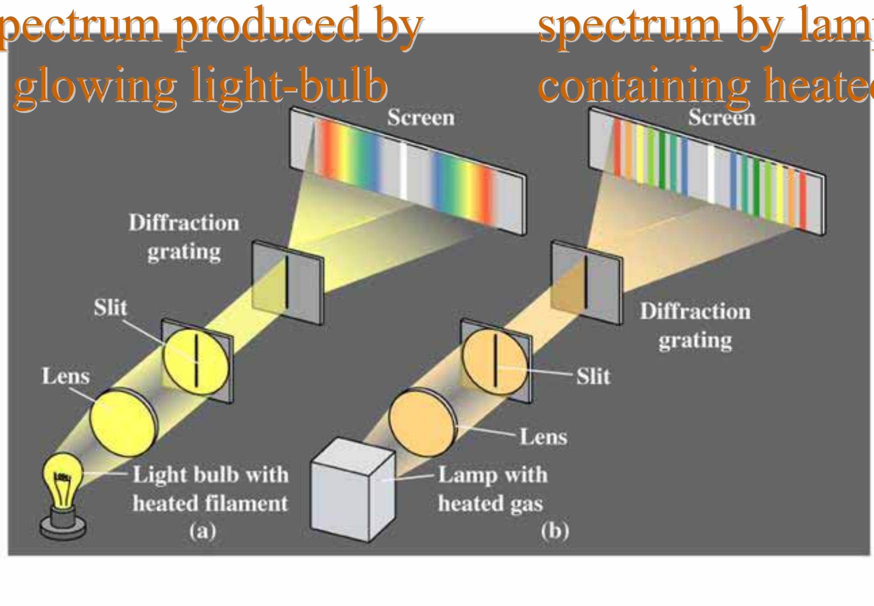
Line spectrum of an atom

- The light given off by individual atoms, as in a low-pressure gas, consist of a series of discrete wavelengths corresponding to different colour.



Comparing continuous and line spectrum

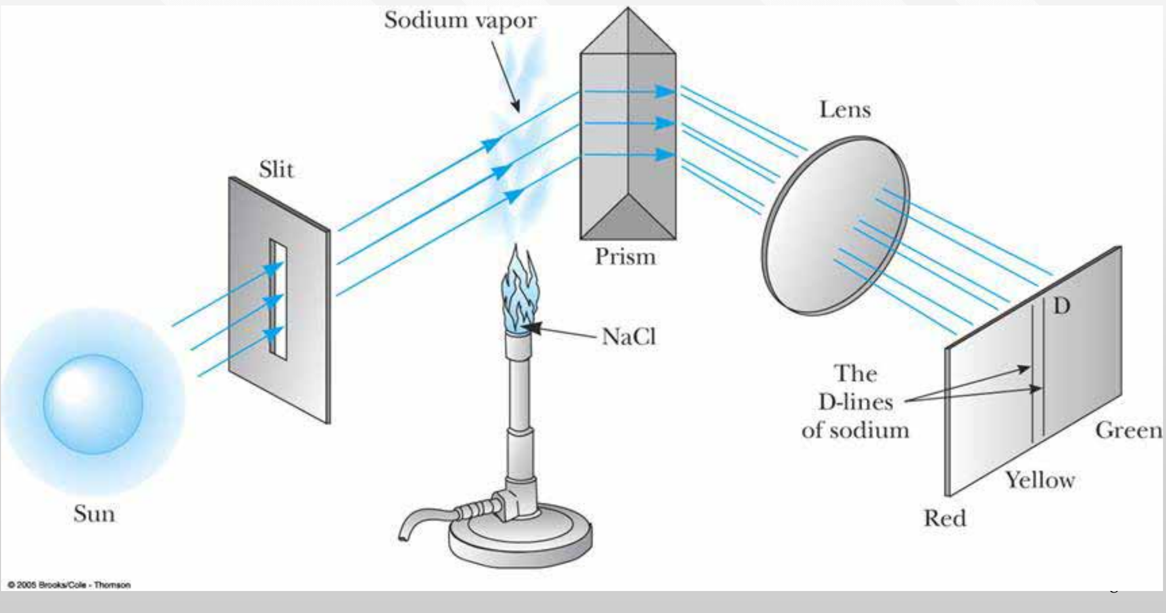
- (a) continuous spectrum produced by a glowing light-bulb
- (b) Emission line spectrum by lamp containing heated gas



Absorption line spectrum

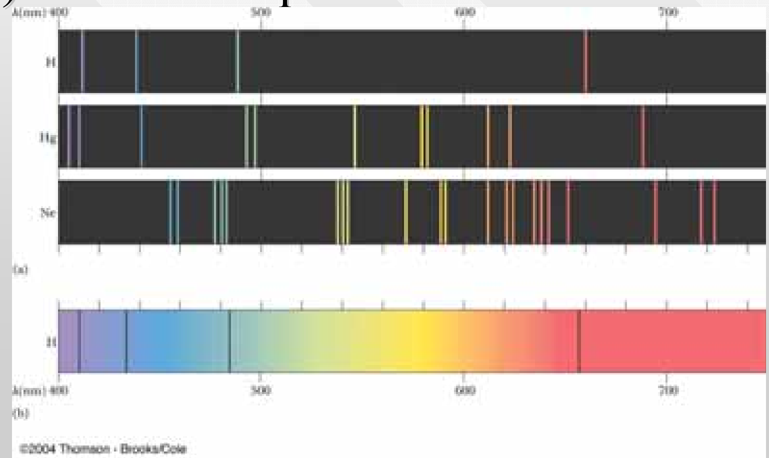
- We also have absorption spectral line, in which white light is passed through a gas. The absorption line spectrum consists of a bright background crossed by dark lines that correspond to the absorbed wavelengths by the gas atom/molecules.

Experimental arrangement for the observation of the absorptions lines of a sodium vapour



Comparing emission and absorption spectrum

The emitted and absorption radiation displays characteristic discrete sets of spectrum which contains certain discrete wavelengths only
(a) shows ‘finger print’ emission spectral lines of H, Hg and Ne. (b) shows absorption lines for H



A successful atomic model must be able to explain the observed discrete atomic spectrum

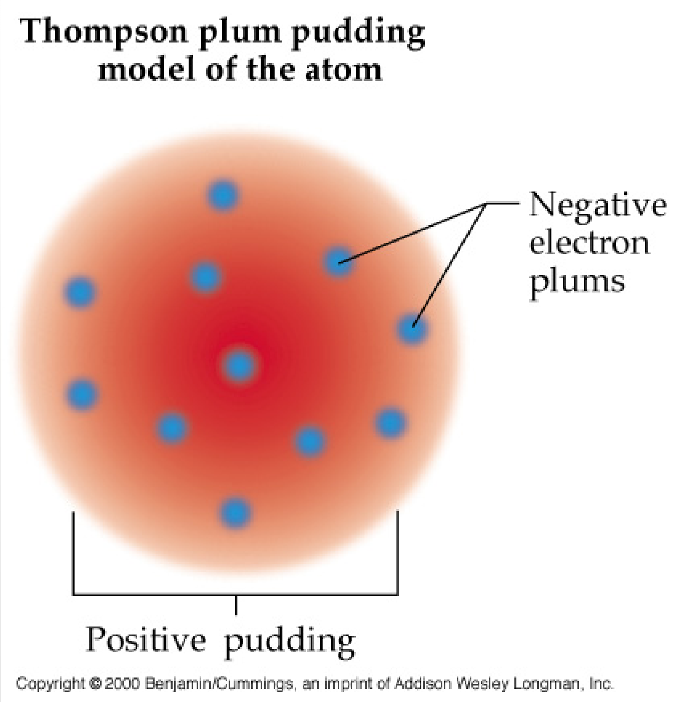
We are going to study two attempts to built model that describes the atoms: the Thompson Plum-pudding model (which fails) and the Rutherford-Bohr model (which succeeds)

The Thompson model – Plum-pudding model

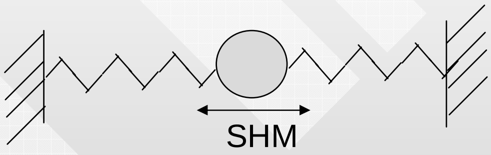
Sir J. J. Thompson (1856-1940) is the Cavandish professor in Cambridge who discovered electron in cathode rays. He was awarded Nobel prize in 1906 for his research on the conduction of electricity by bases at low pressure.
He is the first person to establish the particle nature of electron. Ironically his son, another renown physicist proves experimentally electron behaves like wave…

Plum-pudding model

- An atom consists of *Z* electrons is embedded in a cloud of positive charges that exactly neutralise that of the electrons'
- The positive cloud is heavy and comprising most of the atom's mass
- Inside a stable atom, the electrons sit at their respective equilibrium position where the attraction of the positive cloud on the electrons balances the electron's mutual repulsion



One can treat the electron in the pudding like a point mass stressed by two springs



The "electron plum" stuck on the pudding vibrates and executes SHM

- The electron at the EQ position shall vibrate like a simple harmonic oscillator with a frequency

$$\nu = \left(\frac{1}{2\pi}\right)\sqrt{\frac{k}{m}}$$

- Where $k = \frac{Ze^2}{4\pi\varepsilon_o R^3}$, *R* radius of the atom, *m* mass of the electron
- From classical EM theory, we know that an oscillating charge will emit radiation with frequency identical to the oscillation frequency ν as given above

The plum-pudding model predicts unique oscillation frequency

- Radiation with frequency identical to the oscillation frequency.
- Hence light emitted from the atom in the plum-pudding model is predicted to have exactly one unique frequency as given in the previous slide.
- This prediction has been falsified because observationally, light spectra from all atoms (such as the simplest atom, hydrogen,) have sets of discrete spectral lines correspond to many different frequencies (already discussed earlier).

Experimental verdict on the plum pudding model

- Theoretically one expect the deviation angle of a scattered particle by the plum-pudding atom to be small: $\Theta = \sqrt{N}\theta_{ave} \sim 1^{\circ}$
- This is a prediction of the model that can be checked experimentally
- Rutherford was the first one to carry out such experiment

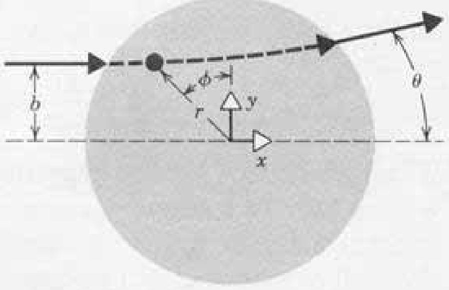


FIGURE 6.2 A positively charged alpha particle is deflected by an angle θ as it passes through a Thomson-model atom. The coordinates r and ϕ locate the alpha particle while it is inside the atom.

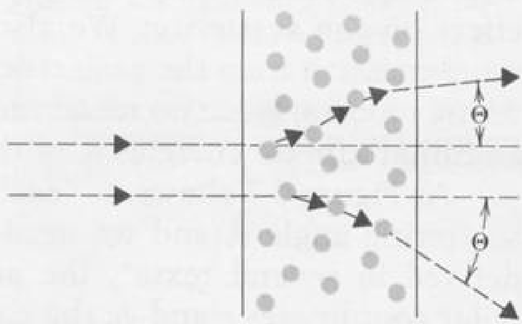
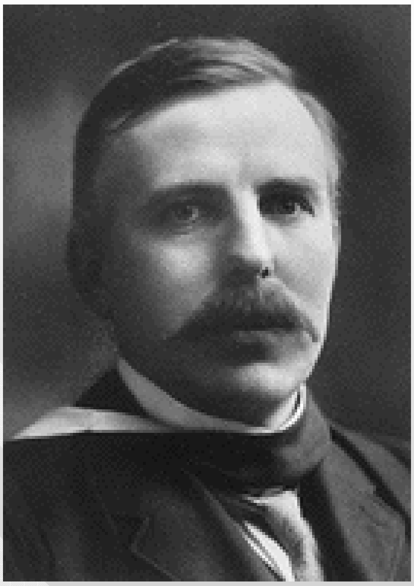


FIGURE 6.6 A microscopic representation of the scattering. Some individual scatterings tend to increase Θ, while others tend to decrease Θ.

Ernest Rutherford

British physicist Ernest Rutherford, winner of the 1908 Nobel Prize in chemistry, pioneered the field of nuclear physics with his research and development of the nuclear theory of atomic structure



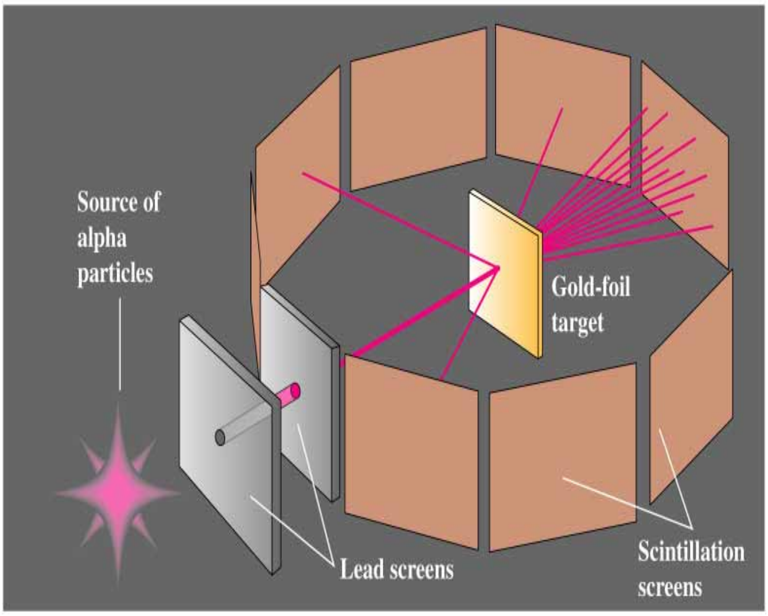
Born in New Zealand, teachers to many physicists who later become Nobel prize laureates

Rutherford stated that an atom consists largely of empty space, with an electrically positive nucleus in the center and electrically negative electrons orbiting the nucleus. By bombarding nitrogen gas with *alpha particles* (nuclear particles emitted through radioactivity), Rutherford engineered the transformation of an atom of nitrogen into both an atom of oxygen and an atom of hydrogen.

This experiment was an early stimulus to the development of nuclear energy, a form of energy in which nuclear transformation and disintegration release extraordinary power.

Rutherford's experimental setup

- Alpha particles from source is used to be scattered by atoms from the thin foil made of gold
- The scattered alpha particles are detected by the background screen



“…fire a 15 inch artillery shell at a tissue paper and it came back and hit you”

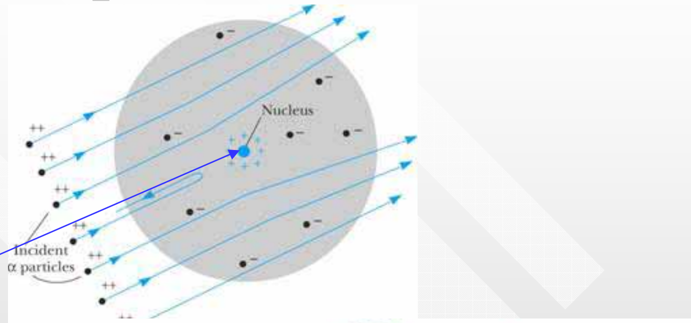
- In the scattering experiment Rutherford saw some electrons being bounced back at 180 degree.
- He said this is like firing “a 15-inch shell at a piece of a tissue paper and it came back and hit you”
- Hence Thompson plum-pudding model fails in the light of these experimental result

So, is the plum pudding model utterly useless?

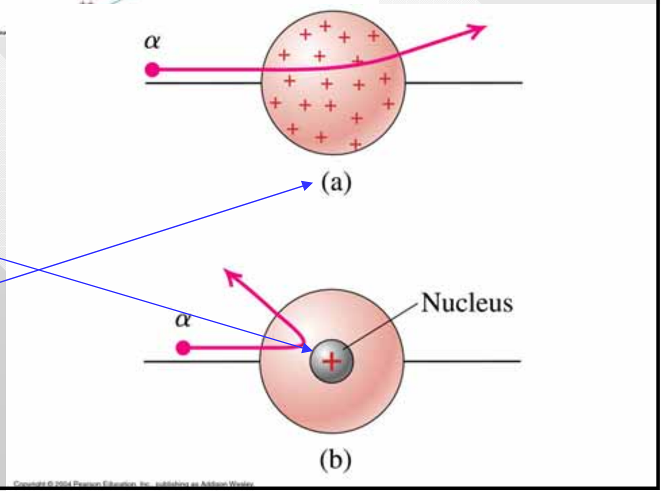
- So the plum pudding model does not work as its predictions fail to fit the experimental data as well as other observations
- Nevertheless it’s a perfectly sensible scientific theory because:
- It is a mathematical model built on sound and rigorous physical arguments
- It predicts some physical phenomenon with definiteness
- It can be verified or falsified with experiments
- It also serves as a prototype to the next model which is built on the experience gained from the failure of this model

How to interpret the Rutherford scattering experiment?

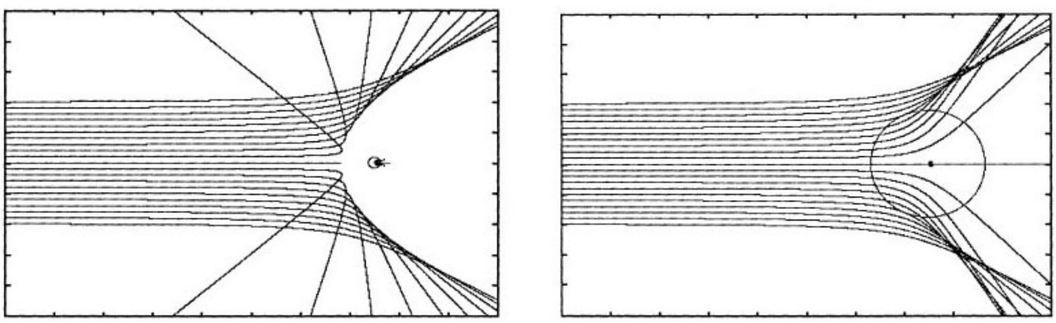
- The large deflection of alpha particle as seen in the scattering experiment with a thin gold foil must be produced by a close encounter between the alpha particle and a very small but massive kernel inside the atom



- In contrast, a diffused distribution of the positive charge as assumed in plum-pudding model cannot do the job

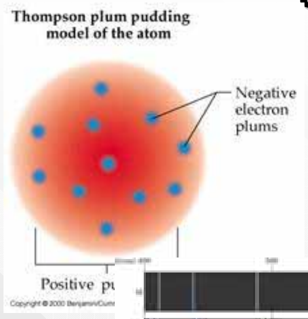


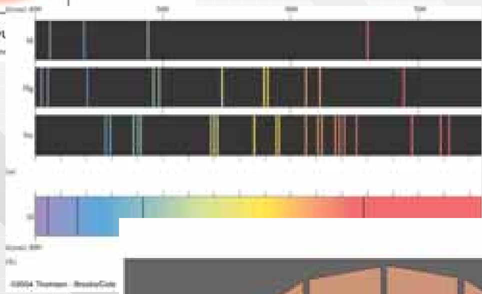
Comparing model with nucleus concentrated at a point-like nucleus and model with nucleus that has large size

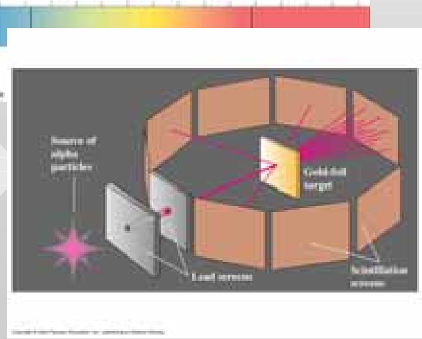


(a) (b)

Recap
the atomic model building story







- Plum-pudding model by Thompson
- It fails to explain the emission and absorption line spectrum from atoms because it predicts only a single emission frequency

$$\nu = \left(\frac{1}{2\pi}\right)\sqrt{\frac{k}{m}}$$

- Most importantly it fails to explain the back-scattering of alpha particle seen in Rutherford's scattering experiment because the model predicts only $\Theta = \sqrt{N}\theta_{ave} \sim 1^{\circ}$

The Rutherford model (planetary model)

- Rutherford put forward an model to explain the result of the scattering experiment: the Rutherford model
- An atom consists of a very small nucleus of charge $+Ze$ containing almost all of the mass of the atom; this nucleus is surrounded by a swarm of Z electrons
- The atom is largely comprised of empty space
- $R_{atom} \sim 10^{-10}$m
- $R_{nucleus} \sim 10^{-13} - 10^{-15}$ m

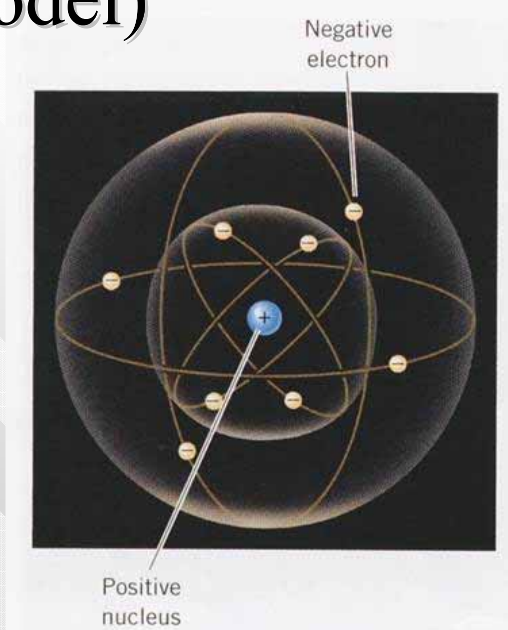
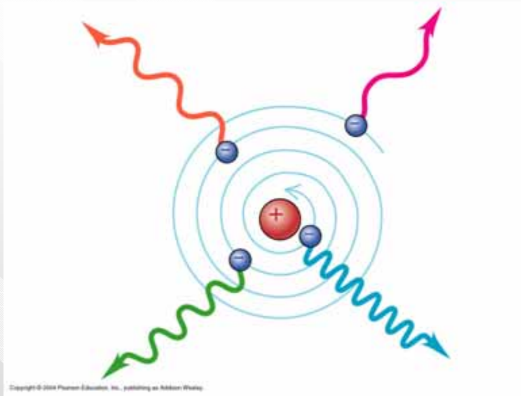
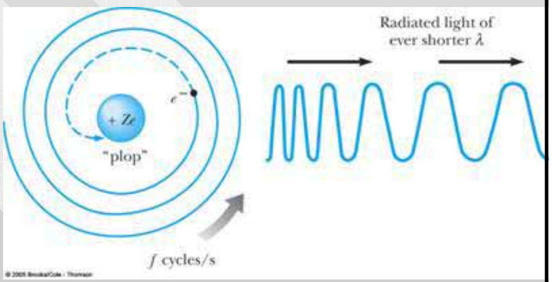


Figure 30.1 In the nuclear atom a small positively charged nucleus is surrounded at relatively large distances by a number of electrons.

Infrared catastrophe: insufficiency of the Rutherford model

- According to classical EM, the Rutherford model for atom (a classical model) has a fatal flaw: it predicts the collapse of the atom within 10^{-10} s
- A accelerated electron will radiate EM radiation, hence causing the orbiting electron to loss energy and consequently spiral inward and impact on the nucleus





Rutherford model also can't explain the discrete spectrum

- The Rutherford model also cannot explain the pattern of discrete spectral lines as the radiation predicted by Rutherford model is a continuous burst.

So how to fix up the problem?

NEILS BOHR COMES TO THE RESCUE

- **Niels Bohr** (1885 to 1962) is best known for the investigations of atomic structure and also for work on radiation, which won him the 1922 Nobel Prize for physics
- He was sometimes dubbed "the God Father" in the physicist community

- http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Bohr_Niels.html

To fix up the infrared catastrophe …

Neils Bohr put forward a model which is a hybrid of the Rutherford model with the wave nature of electron taken into account

Bohr’s model of hydrogen-like atom

- We shall consider a simple atom consists of a nucleus with charge Ze and mass of $M_{nucleus} >> m_e$, such that
- $(m_e / M_{nucleus})$ can be ignored.
- The nucleus is surrounded by only a single electron
- We will assume the centre of the circular motion of the electron coincides with the centre of the nucleus
- We term such type of simple system: hydrogen-like atoms
- For example, hydrogen atom corresponds to $Z = 1$; a singly ionised Helium atom He^+ corresponds to $Z = 2$ etc

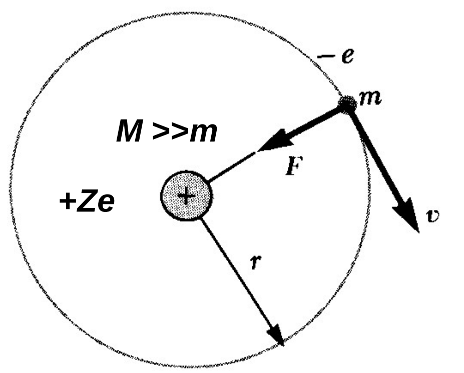
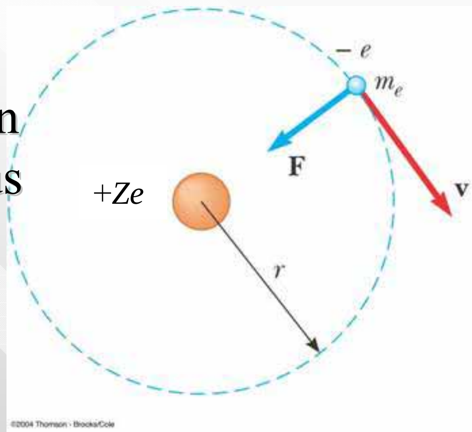


Diagram representing the model of a hydrogen-like atom

Bohr’s postulate, 1913

- Postulate No.1: Mechanical stability (classical mechanics)
- An electron in an atom moves in a circular orbit about the nucleus under Coulomb attraction obeying the law of classical mechanics



Coulomb’s attraction = centripetal force

$$\frac{1}{4\pi\varepsilon_0}\frac{(Ze)e}{r^2} = \frac{m_e v^2}{r}$$

Assumption: the mass of the nucleus is infinitely heavy compared to the electron’s

Postulate 2: condition for orbit stability

- Instead of the infinite orbit which could be possible in classical mechanics (c.f the orbits of satellites), it is only possible for an electron to move in an orbit that contains an integral number of de Broglie wavelengths,
- $n\lambda_n = 2\pi r_n$, $n = 1,2,3...$

Bohr's 2nd postulate means that *n* de Broglie wavelengths must fit into the circumference of an orbit

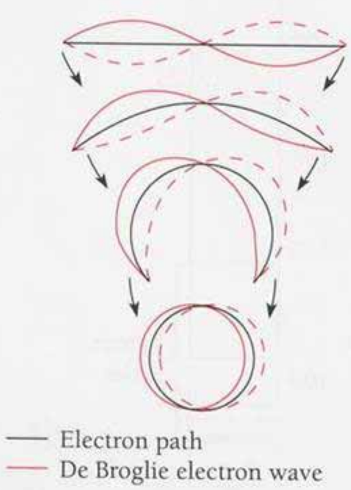
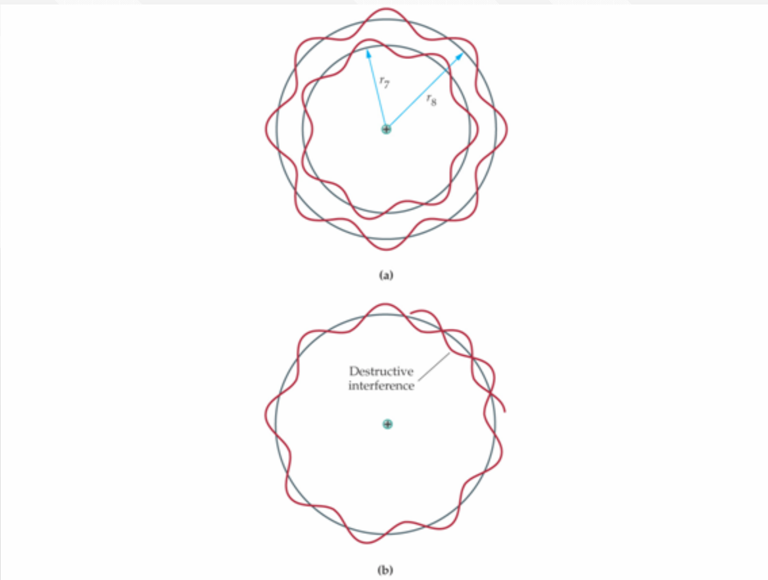


Figure 4.12 The orbit of the electron in a hydrogen atom corresponds to a complete electron de Broglie wave joined on itself.



Electron that don't form standing wave

- Since the electron must form standing waves in the orbits, the the orbits of the electron for each n is quantised
- Orbits with the perimeter that do not conform to the quantisation condition cannot persist
- All this simply means: all orbits of the electron in the atom must be quantised, and orbit that is not quantised is not allowed (hence can't exist)

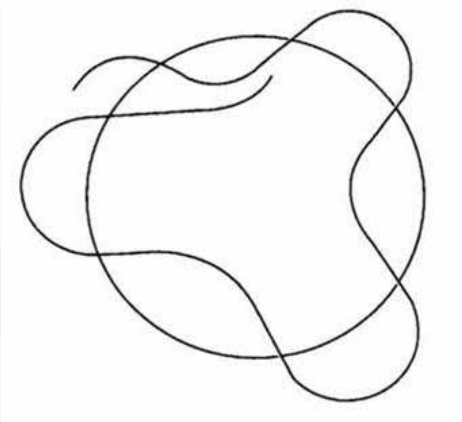
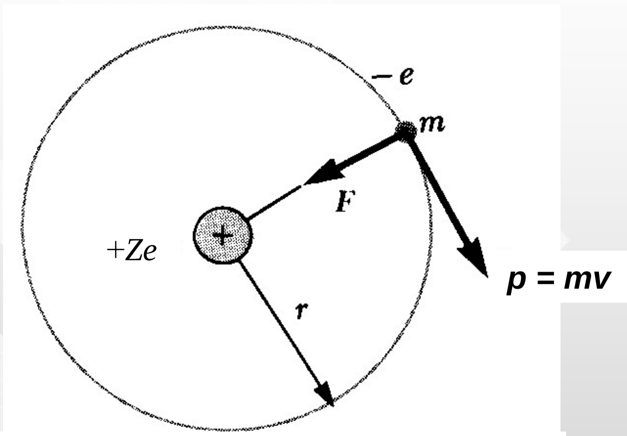


Figure 4.14 A fractional number of wavelengths cannot persist because destructive interference will occur.

Quantisation of angular momentum

- As a result of the orbit quantisation, the angular momentum of the orbiting electron is also quantised:
- $L = (m_e v)\, r = pr$ (definition)
- $n\lambda = 2\pi r$ (orbit quantisation)
- Combining both:
- $p = h/\lambda = nh/2\pi r$
- $L = m_e vr = p\, r = nh/2\pi$



Angular momentum of the electron,

$L = p$ x r. It is a vector quantity with its direction pointing to the direction perpendicular to the plane defined by p and r

Third postulate

- Despite the fact that it is constantly accelerating, an electron moving in such an allowed orbit does not radiate EM energy (hence total energy remains constant)
- The atom remains stable at the ground state because there is no other states of lower energies below the ground state (hence the atom cannot make any transition to a lower energy state)
- As far as the stability of atoms is concerned, classical physics is invalid here
- My Comment: At the quantum scale (inside the atoms) some of the classical EM predictions fail (e.g. an accelerating charge radiates EM wave)

Quantisation of velocity and radius

- Combining the quantisation of angular momentum and the equation of mechanical stability we arrive at the result that:
- the allowed radius and velocity at a given orbit are also quantised:

$$r_n = 4\pi\varepsilon_0 \frac{n^2\hbar^2}{m_e Z e^2} \qquad v_n = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{n\hbar}$$

Some mathematical steps leading to quantisation of orbits, $r_n = 4\pi\varepsilon_0 \frac{n^2\hbar^2}{m_e Z e^2}$

$$m_e v r = \frac{nh}{2\pi} \qquad \text{(Eq.1)}$$

$$\frac{1}{4\pi\varepsilon_0}\frac{(Ze)e}{r^2} = \frac{m_e v^2}{r} \Rightarrow v^2 = \frac{Ze^2}{4\pi\varepsilon_0 m_e}\frac{1}{r} \qquad \text{(Eq. 2)}$$

- (Eq.2) → (Eq.1)^2,
- $(m_e v r)^2 = (nh/2\pi)^2$
- LHS: $m_e^2 r^2 v^2 = m_e^2 r^2 (Ze^2 / 4\pi\varepsilon_0 m_e r)$

$= m_e r Ze^2 / 4\pi\varepsilon_0 =$ RHS $= (nh/2\pi)^2$

$r = n^2 (h/2\pi)^2 \, 4\pi\varepsilon_0 / Ze^2 m_e \equiv r_n$,

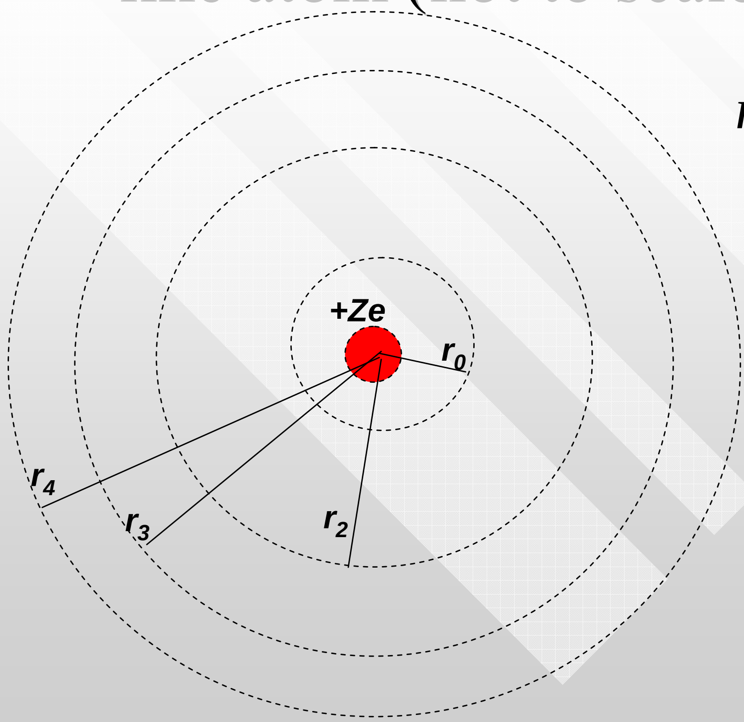
$n = 1,2,3\ldots$

Prove it yourself the quantisation of the electron velocity

$$v_n = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{n\hbar}$$

using Eq.(1) and Eq.(2)

The quantised orbits of hydrogen-like atom (not to scale)



$$r_n = n^2 \frac{r_0}{Z}$$

Important comments

- The smallest orbit charaterised by
- $Z = 1$, n=1 is the ground state orbit of the hydrogen

$$r_0 = \frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2} = 0.5 \overset{0}{A}$$

- It's called the Bohr's radius = the typical size of an atom
- In general, the radius of an hydrogen-like ion/atom with charge *Ze* in the nucleus is expressed in terms of the Bohr's radius as

$$r_n = n^2 \frac{r_0}{Z}$$

- Note also that the ground state velocity of the electron in the hydrogen atom is $v_0 = 2.2 \times 10^6$ m/s << c
- non-relativistic

PYQ 7 Test II 2003/04

- In Bohr's model for hydrogen-like atoms, an electron (mass *m*) revolves in a circle around a nucleus with positive charges *Ze*. How is the electron's velocity related to the radius *r* of its orbit?
- **A.** $v = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{mr}$ **B.** $v = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{mr^2}$ **C.** $v = \frac{1}{4\pi\varepsilon_0}\frac{Ze}{mr^2}$
- **D.** $v^2 = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{mr}$ **E.** Non of the above

- Solution: I expect you to be able to derive it from scratch without memorisation
- ANS: D, Schaum's series 3000 solved problems, Q39.13, pg 722 modified

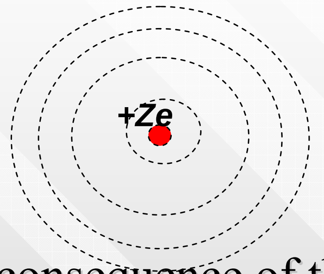
Strongly recommending the Physics 2000 interactive physics webpage by the University of Colorado

For example the page
http://www.colorado.edu/physics/2000/quantumzone/bohr.html
provides a very interesting explanation and simulation on atom and Bohr model in particular.
Please visit this page if you go online

Recap

- The hydrogen-like atom's radii are quantised according to:

$$r_n = n^2 \frac{r_0}{Z}$$



- The quantisation is a direct consequence of the postulate that electron wave forms stationary states (standing waves) at the allowed orbits
- The smallest orbit or hydrogen, the Bohr's radius

$$r_0 = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} = 0.5\overset{0}{A}$$

Postulate 4

- Similar to Einstein's postulate of the energy of a photon

EM radiation is emitted if an electron initially moving in an orbit of total energy E_i, discontinuously changes it motion so that it moves in an orbit of total energy E_f,(E_i > E_f). The frequency of the emitted radiation,

$$\nu = (E_i - E_f)/h;$$

$$E_i > E_f$$

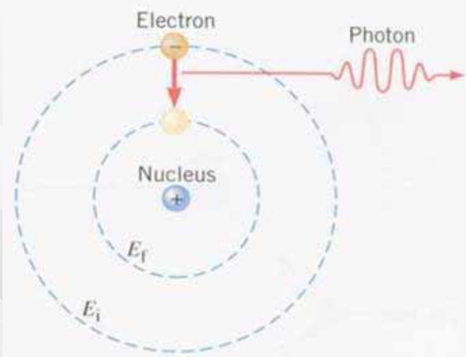


Figure 30.5 In the Bohr model, a photon is emitted when the electron drops from a larger, higher-energy orbit (energy = E_i) to a smaller, lower-energy orbit (energy = E_f).

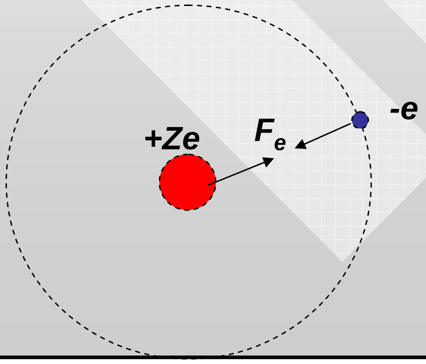
Energies in the hydrogen-like atom

- Potential energy of the electron at a distance *r* from the nucleus is, as we learned from standard electrostatics, ZCT 102, form 6, matriculation etc. is simply

$$V = -\int_r^{\infty} \frac{Ze^2}{4\pi\varepsilon_0 r^2} dr = -\frac{Ze^2}{4\pi\varepsilon_0 r}$$

- -ve means that the EM force is attractive

Check this sign to see if it's correct



Kinetic energy in the hydrogen-like atom

- According to definition, the KE of the electron is

$$K = \frac{m_e v^2}{2} = \frac{Ze^2}{8\pi\varepsilon_0 r}$$

The last step follows from the equation $\frac{m_e v^2}{r} = \frac{Ze^2}{4\pi\varepsilon_0 r^2}$

- Adding up KE + V, we obtain the total mechanical energy of the atom:

$$E = K + V = \frac{Ze^2}{8\pi\varepsilon_0 r} + \left(-\frac{Ze^2}{4\pi\varepsilon_0 r}\right) = -\frac{Ze^2}{8\pi\varepsilon_0}\left(\frac{1}{r}\right) = -\frac{Ze^2}{8\pi\varepsilon_0}\left[\frac{m_e Ze^2}{4\pi\varepsilon_0 n^2 \hbar^2}\right]$$

$$= -\frac{m_e Z^2 e^4}{(4\pi\varepsilon_0)^2 2\hbar^2}\frac{1}{n^2} \equiv E_n$$

The ground state energy

- For the hydrogen atom (Z = 1), the ground state energy (which is characterised by *n* =1)

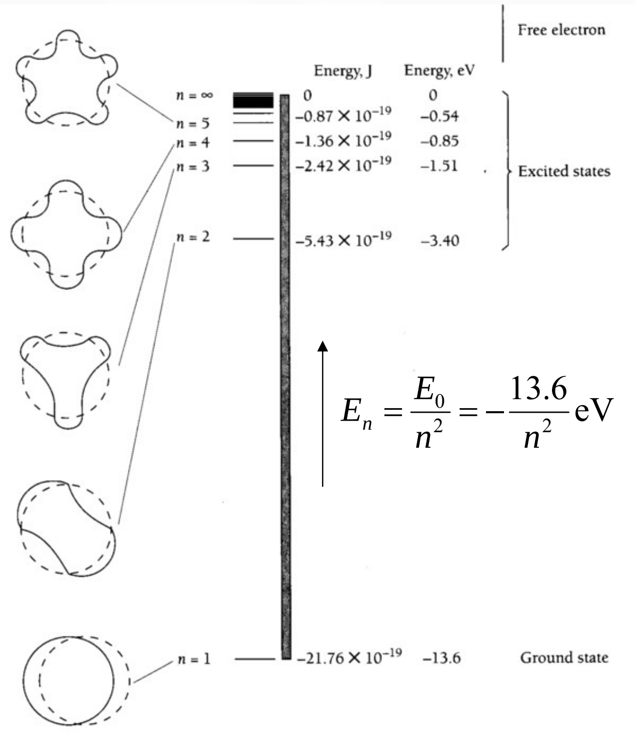
$$E_0 \equiv E_n(n=1) = -\frac{m_e e^4}{(4\pi\varepsilon_0)^2 2\hbar^2} = -13.6\text{eV}$$

In general the energy level of a hydrogen like atom with *Ze* nucleus charges can be expressed in terms of

$$E_n = \frac{Z^2 E_0}{n^2} = -\frac{13.6Z^2}{n^2}\text{eV}$$

Quantisation of energy levels

- The energy level of the electrons in the atomic orbit is quantised
- The quantum number, *n*, that characterises the electronic states is called principle quantum number
- Note that the energy state is –ve (because it's a bounded system)

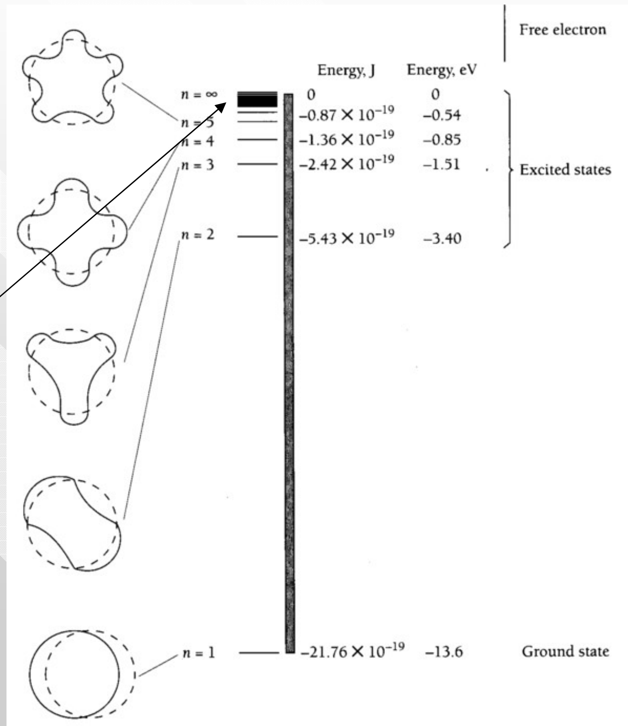


Energy of the electron at very large n

- An electron occupying an orbit with very large n is "almost free" because its energy approaches zero:

$$E_n(n \to \infty) = 0$$

- $E = 0$ means the electron is free from the bondage of the nucleus' potential field
- Electron at high n is not tightly bounded to the nucleus by the EM force
- Energy levels at high n approaches to that of a continuum, as the energy gap between adjacent energy levels become infinitesimal in the large n limit

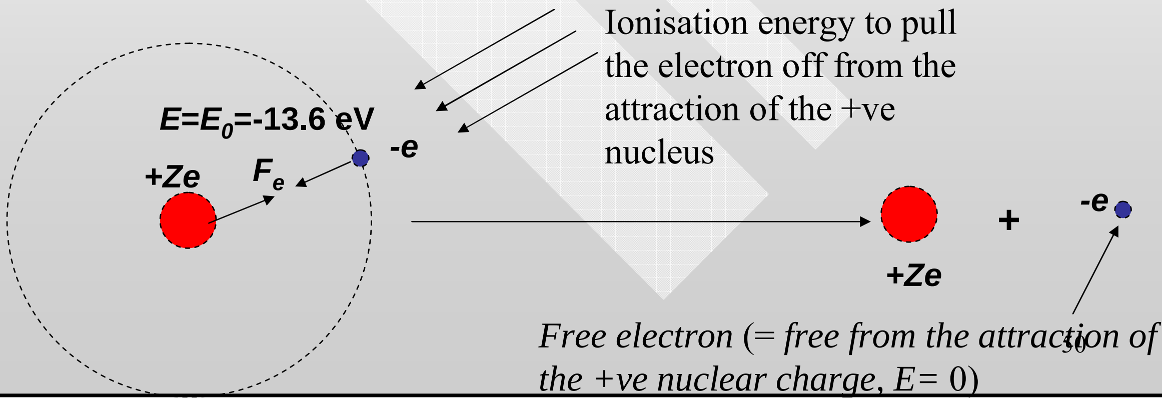


Ionisation energy of the hydrogen atom

- The energy input required to remove the electron from its ground state to infinity (ie. to totally remove the electron from the bound of the nucleus) is simply

$$E_{ionisation} = E_\infty - E_0 = -E_0 = 13.6\text{eV}$$

- this is the ionisation energy of hydrogen

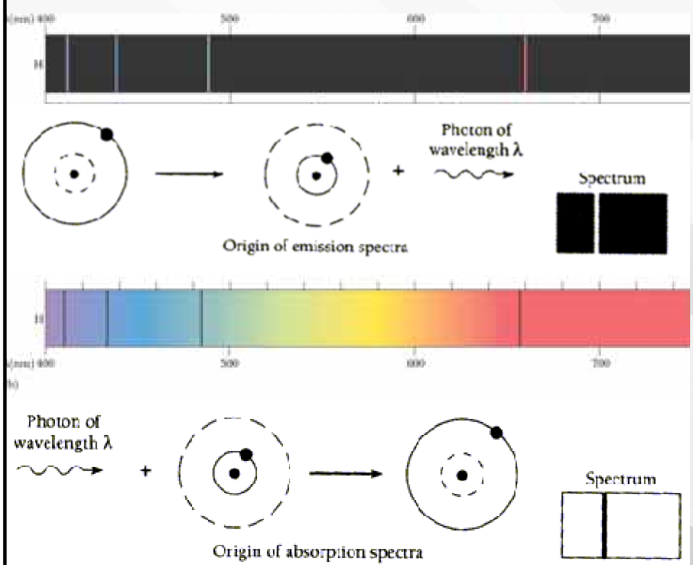


Two important quantities to remember

- As a practical rule, it is strongly advisable to remember the two very important values

- (i) the Bohr radius, $r_0 = 0.53A$ and
- (ii) the ground state energy of the hydrogen atom, $E_0 = -13.6$ eV

Bohr's 4th postulate explains the line spectrum



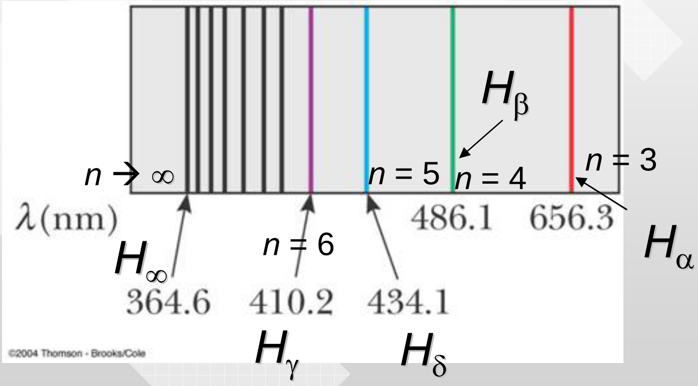
- When atoms are excited to an energy state above its ground state, they shall radiate out energy (in forms of photon) within at the time scale of ~10^{-8} s upon their de-excitations to lower energy states –emission spectrum explained

- When a beam of light with a range of wavelength sees an atom, the few particular wavelengths that matches the allowed energy gaps of the atom will be absorbed, leaving behind other unabsorbed wavelengthsto become the bright background in the absorption spectrum. Hence absorption spectrum explained

Balmer series and the empirical emission spectrum equation

- Since 1860 – 1898 Balmer have found an empirical formula that correctly predicted the wavelength of four visible lines of hydrogen:

$$\frac{1}{\lambda} = R_H\left(\frac{1}{2^2} - \frac{1}{n^2}\right)$$



where n = 3,4,5,….R_H is called the Rydberg constant, experimentally measured to be R_H = 1.0973732 x 10^7 m^{-1}

Example

- For example, for the H_β (486.1 nm) line, n = 4 in the empirical formula

$$\frac{1}{\lambda} = R_H\left(\frac{1}{2^2} - \frac{1}{n^2}\right)$$

- According to the empirical formula the wavelength of the hydrogen beta line is

$$\frac{1}{\lambda_\beta} = R_H\left(\frac{1}{2^2} - \frac{1}{4^2}\right) = R_H\left(\frac{3}{16}\right) = \frac{3(1.0973732 \times 10^7\,\mathrm{m}^{-1})}{16}$$

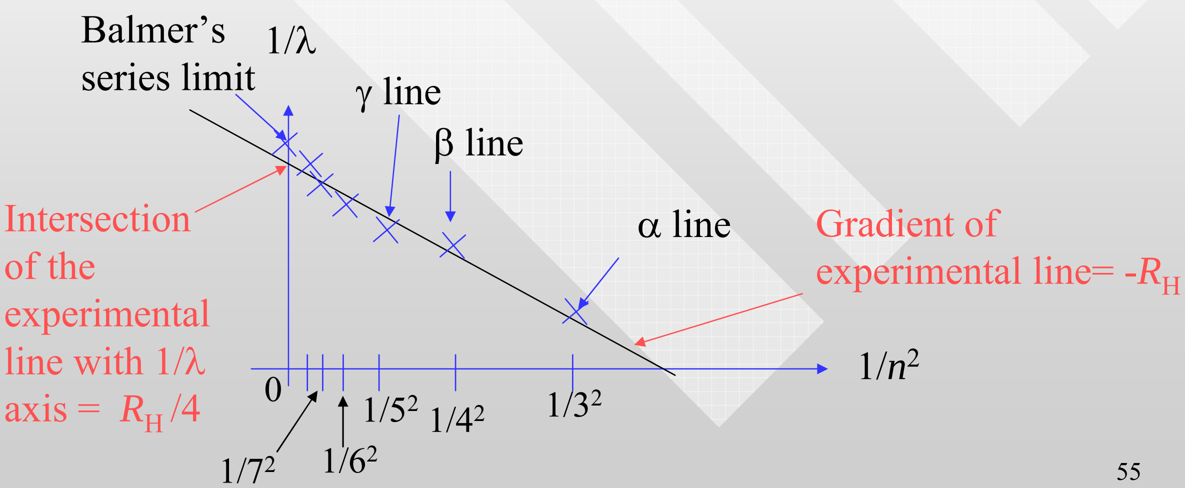
$$\Rightarrow \lambda_\beta = 486\mathrm{nm}$$

- which is consistent with the observed value

Experimental measurement of the Rydberg constant, R_H

One measures the wavelengths of the α, β, γ, …lines (corresponding to $n = 3, 4, 5, \ldots\infty$) in Balmer's empirical formula $\frac{1}{\lambda} = R_H\left(\frac{1}{2^2} - \frac{1}{n^2}\right)$

Then plot $1/\lambda$ as a function of $1/n^2$. Note that here $n \geq 3$.



Other spectra series

- Apart from the Balmer series others spectral series are also discovered: Lyman, Paschen and Brackett series
- The wavelengths of these spectral lines are also given by the similar empirical equation as

$$\frac{1}{\lambda} = R_H\left(\frac{1}{1} - \frac{1}{n^2}\right), \quad n = 2,3,4,\ldots$$

Lyman series, ultraviolet region

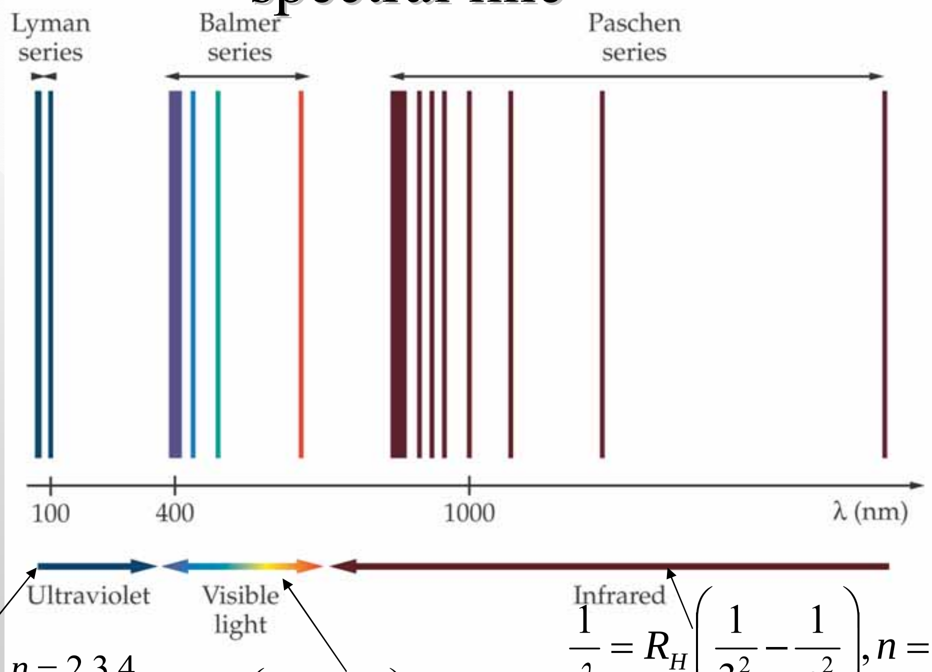
$$\frac{1}{\lambda} = R_H\left(\frac{1}{3^2} - \frac{1}{n^2}\right), \quad n = 4,5,6,\ldots$$

Paschen series, infrared region

$$\frac{1}{\lambda} = R_H\left(\frac{1}{4^2} - \frac{1}{n^2}\right), \quad n = 5,6,7,\ldots$$

Brackett series, infrared region

These are experimentally measured spectral line



$$\frac{1}{\lambda} = R_H\left(\frac{1}{1} - \frac{1}{n^2}\right), n = 2,3,4...$$

$$\frac{1}{\lambda} = R_H\left(\frac{1}{2^2} - \frac{1}{n^2}\right), \quad n = 3,4,5,6,$$

$$\frac{1}{\lambda} = R_H\left(\frac{1}{3^2} - \frac{1}{n^2}\right), n = 4,5,6...$$

The empirical formula needs a theoretical explanation

$$\frac{1}{\lambda} = R_H\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

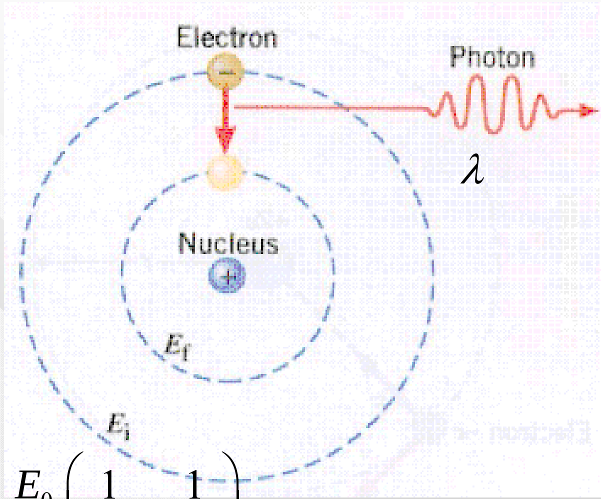
is an empirical formula with R_H measured to be $R_H = 1.0973732 \times 10^7 \text{ m}^{-1}$.

Can the Bohr model provide a sound theoretical explanation to the form of this formula and the numerical value of R_H in terms of known physical constants?

The answer is: YES

Theoretical derivation of the empirical formula from Bohr's model

- According to the 4th postulate:
  $\Delta E = E_i - E_f = h\nu = hc/\lambda$, and
- $E_k = E_0 / n_k^2$
- $\quad = -13.6 \text{ eV} / n_k^2$
- where $k = i$ or j
- Hence we can easily obtain the theoretical expression for the emission line spectrum of hydrogen-like atom



$$\frac{1}{\lambda} = \frac{\nu}{c} = \frac{E_i - E_f}{ch} = \frac{E_0}{ch}\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right)$$

$$= \frac{m_e e^4}{4c\pi\hbar^3 (4\pi\varepsilon_0)^2}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \equiv R_\infty \left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

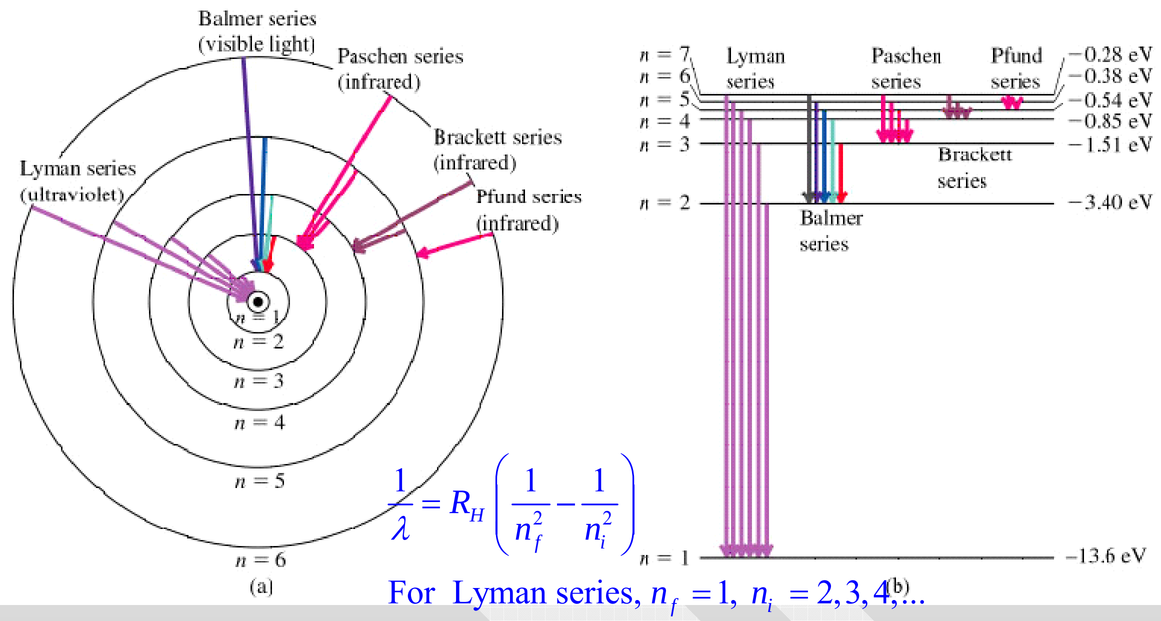
The theoretical Rydberg constant

$$R_\infty \equiv \frac{m_e e^4}{4c\pi\hbar^3 (4\pi\varepsilon_0)^2} = 1.0973732 \times 10^7 \text{ m}^{-1}$$

- The theoretical Rydberg constant, R_∞, agrees with the experimental one up a precision of less than 1%

$$R_H = 1.0xxxx \times 10^7 \text{ m}^{-1}$$

This is a remarkable experimental verification of the correctness of the Bohr model



$$\frac{1}{\lambda} = R_H\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

For Lyman series, $n_f = 1$, $n_i = 2, 3, 4, ...$

For Balmer series, $n_f = 2$, $n_i = 3, 4, 5...$

For Paschen series, $n_f = 3$, $n_i = 4, 5, 6...$

For Brackett series, $n_f = 4$, $n_i = 5, 6, 7...$

For Pfund series, $n_f = 5$, $n_i = 6, 7, 8...$

Real life example of atomic emission

- AURORA are caused by streams of fast photons and electrons from the sun that excite atoms in the upper atmosphere. The green hues of an auroral display come from oxygen

Example

- Suppose that, as a result of a collision, the electron in a hydrogen atom is raised to the second excited state (n = 3).
- What is (i) the energy and (ii) wavelength of the photon emitted if the electron makes a direct transition to the ground state?
- What are the energies and the wavelengths of the two photons emitted if, instead, the electron makes a transition to the first excited state (n=2) and from there a subsequent transition to the ground state?

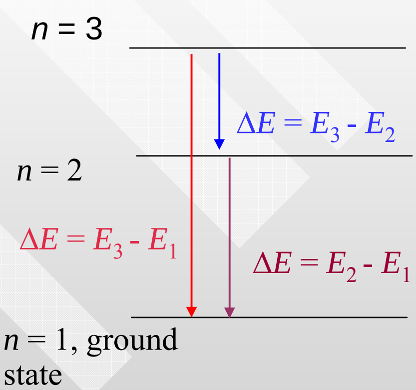
Make use of $E_k = E_0 / n_k^2 = -13.6 \text{ eV} / n_k^2$

The energy of the proton emitted in the transition from the n = 3 to the n = 1 state is

$$\Delta E = E_3 - E_1 = -13.6\left(\frac{1}{3^2} - \frac{1}{1^2}\right)\text{eV} = 12.1\text{eV}$$

the wavelength of this photon is

$$\lambda = \frac{c}{\nu} = \frac{ch}{\Delta E} = \frac{1242 eV \cdot nm}{12.1 eV} = 102\text{nm}$$



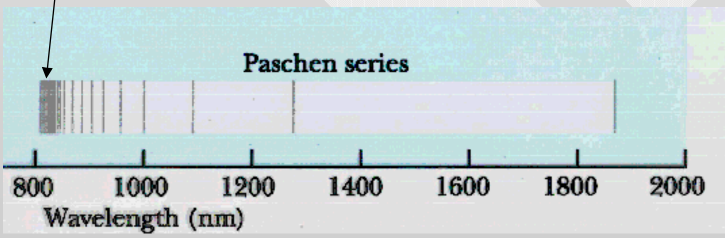
Likewise the energies of the two photons emitted in the transitions from n = 3 → n = 2 and n = 2 → n = 1 are, respectively,

$$\Delta E = E_3 - E_2 = -13.6\left(\frac{1}{3^2} - \frac{1}{2^2}\right) = 1.89\text{eV}$$ with wavelength $$\lambda = \frac{ch}{\Delta E} = \frac{1242 eV \cdot nm}{1.89 eV} = 657\text{nm}$$

$$\Delta E = E_2 - E_1 = -13.6\left(\frac{1}{2^2} - \frac{1}{1^2}\right) = 10.2\text{eV}$$ with wavelength $$\lambda = \frac{ch}{\Delta E} = \frac{1242 eV \cdot nm}{10.2 eV} = 121\text{nm}$$

Example

- The series limit of the Paschen ($n_f = 3$)is 820.1 nm
- The series limit of a given spectral series is the shortest photon wavelength for that series
- The series limit of a spectral series is the wavelength corresponds to $n_i \rightarrow \infty$
- What are two longest wavelengths of the Paschen series?



Solution

- Note that the Rydberg constant is not provided
- But by definition the series limit and the Rydberg constant is closely related
- We got to make use of the series limit to solve that problem
- By referring to the definition of the series limit,

$$\frac{1}{\lambda} = R_H\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \xrightarrow{n_i \rightarrow \infty} \frac{1}{\lambda_\infty} = \frac{R_H}{n_f^2}$$

- Hence we can substitute $R_H = n_f^2 / \lambda_\infty$ into

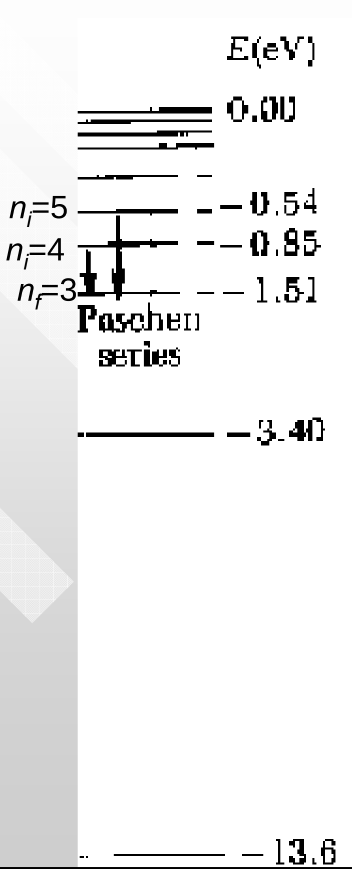
$$\frac{1}{\lambda} = R_H\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

- and express it in terms of the series limit as $\frac{1}{\lambda} = \frac{1}{\lambda_\infty}\left(1 - \frac{n_f^2}{n_i^2}\right)$
- $n_i = 4,5,6\ldots$; $n_f = 3$

- For Paschen series, $n_f = 3, \lambda_\infty = 820.1$ nm

$$\frac{1}{\lambda} = \frac{1}{820.1\,\text{nm}}\left(1 - \frac{3^2}{n_i^2}\right)$$

- The two longest wavelengths correspond to transitions of the two smallest energy gaps from the energy levels closest to $n = 3$ state (i.e the $n = 4$, $n = 5$ states) to the $n = 3$ state



$$n_i = 4: \lambda = 820.1\,\text{nm}\left(\frac{n_i^2}{n_i^2 - 9}\right) = 820.1\,\text{nm}\left(\frac{4^2}{4^2 - 9}\right) = 1875\,\text{nm}$$

$$n_i = 5: \lambda = 820.1\,\text{nm}\left(\frac{n_i^2}{n_i^2 - 9}\right) = 820.1\,\text{nm}\left(\frac{5^2}{5^2 - 9}\right) = 1281\,\text{nm}$$

Example

- Given the ground state energy of hydrogen atom -13.6 eV, what is the longest wavelength in the hydrogen's Balmer series?
- Solution:

  $\Delta E = E_i - E_f = -13.6 \text{ eV } (1/n_i^2 - 1/n_f^2) = hc/\lambda$
- Balmer series: $n_f = 2$. Hence, in terms of 13.6 eV the wavelengths in Balmer series is given by

$$\lambda_{Balmer} = \frac{hc}{13.6\text{eV}\left(\frac{1}{4} - \frac{1}{n_i^2}\right)} = \frac{1240\text{eV}\cdot\text{nm}}{13.6\text{eV}\left(\frac{1}{4} - \frac{1}{n_i^2}\right)} = \frac{91\text{nm}}{\left(\frac{1}{4} - \frac{1}{n_i^2}\right)}, \quad n_i = 3,4,5...$$

$$\lambda_{Balmer} = \frac{91\text{nm}}{\left(\frac{1}{4} - \frac{1}{n_i^2}\right)}, \quad n_i = 3,4,5...$$

- longest wavelength corresponds to the transition from the $n_i = 3$ states to the $n_f = 2$ states
- Hence

$$\lambda_{Balmer,\max} = \frac{91\text{nm}}{\left(\frac{1}{4} - \frac{1}{3^2}\right)} = 655.2\text{nm}$$

- This is the red H_α line in the hydrogen's Balmer series
- Can you calculate the shortest wavelength (the series limit) for the Balmer series? Ans = 364 nm

PYQ 2.18 Final Exam 2003/04

- Which of the following statements are true?
- **I.**. the ground states are states with lowest energy
- **II.** ionisation energy is the energy required to raise an electron from ground state to free state
- **III**. Balmer series is the lines in the spectrum of atomic hydrogen that corresponds to the transitions to the $n = 1$ state from higher energy states

- **A.** I,IV **B.** I,II, IV **C.** I, III,IV
- **D.** I, II **E.** II,III

- **ANS: D, My own question**
- **(note: this is an obvious typo error with the statement IV missing. In any case, only statement I, II are true.)**

PYQ 1.5 KSCP 2003/04

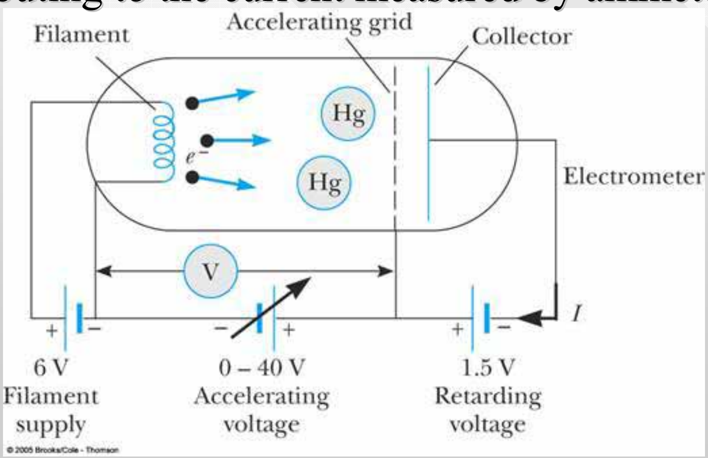
- An electron collides with a hydrogen atom in its ground state and excites it to a state of *n* =3. How much energy was given to the hydrogen atom in this collision?
- **A.** -12.1 eV **B.** 12.1 eV **C.** -13.6 eV
- **D.** 13.6 eV **E.** Non of the above
- **Solution:**

$$\Delta E = E_3 - E_0 = \frac{E_0}{3^2} - E_0 = \frac{(-13.6\text{eV})}{3^2} - (-13.6\text{eV}) = 12.1\text{eV}$$

- **ANS: B**, Modern Technical Physics, Beiser, Example 25.6, pg. 786

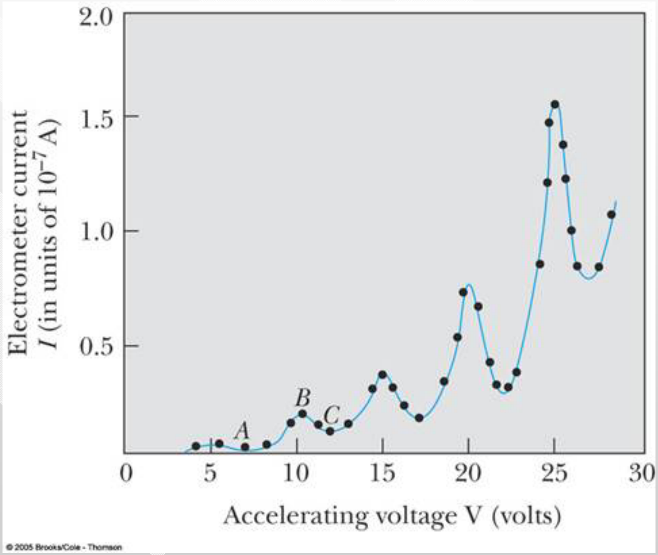
Frank-Hertz experiment

- The famous experiment that shows the excitation of atoms to discrete energy levels and is consistent with the results suggested by line spectra
- Mercury vapour is bombarded with electron accelerated under the potential *V* (between the grid and the filament)
- A small potential V_0 between the grid and collecting plate prevents electrons having energies less than a certain minimum from contributing to the current measured by ammeter



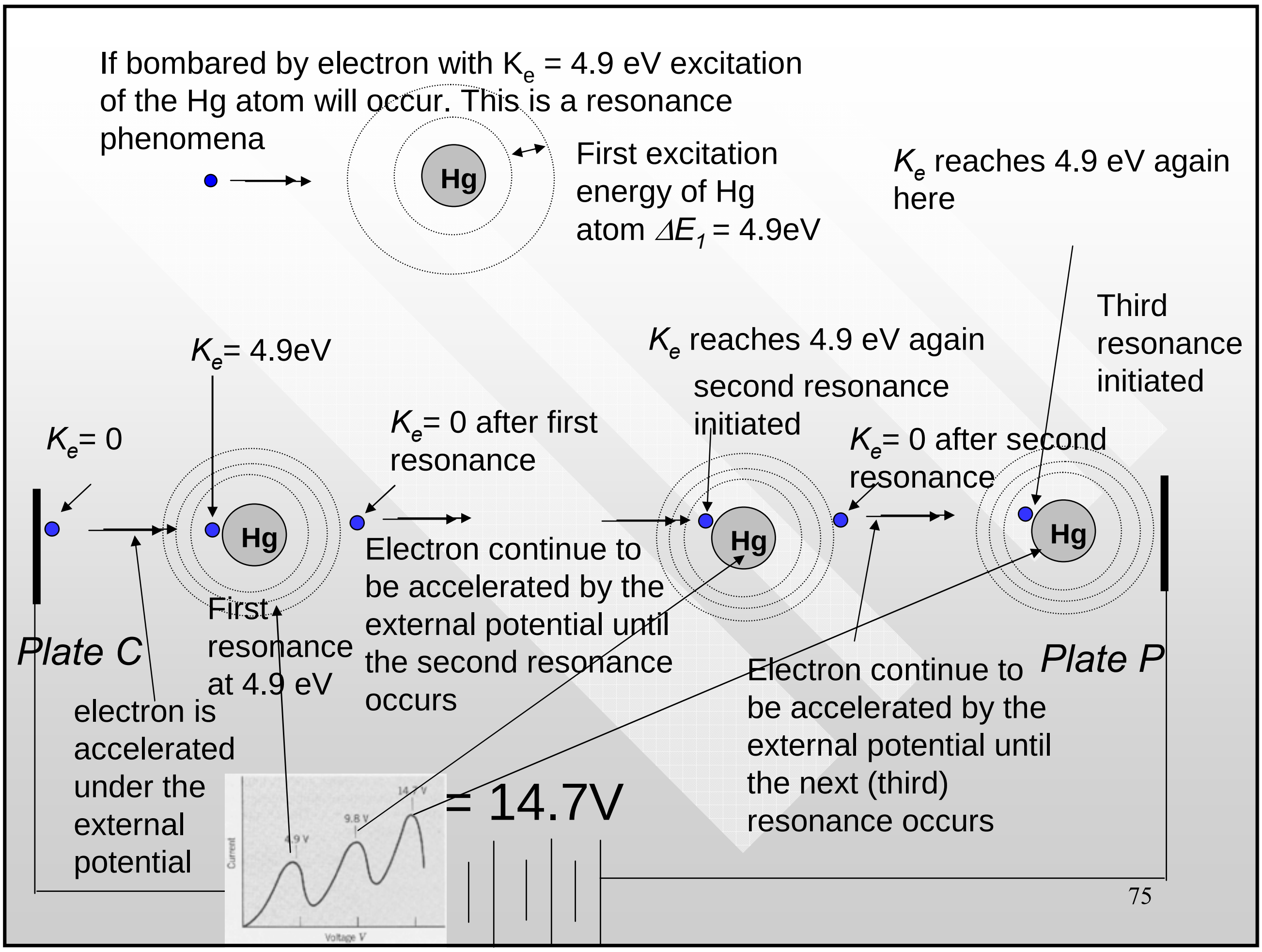
The electrons that arrive at the anode peaks at equal voltage intervals of 4.9 V

- As V increases, the current measured also increases
- The measured current drops at multiples of a critical potential
- V = 4.9 V, 9.8V, 14.7V



Interpretation

- As a result of inelastic collisions between the accelerated electrons of KE 4.9 eV with the the Hg atom, the Hg atoms are excited to an energy level above its ground state
- At this critical point, the energy of the accelerating electron equals to that of the energy gap between the ground state and the excited state
- This is a resonance phenomena, hence current increases abruptly
- After inelastically exciting the atom, the original (the bombarding) electron move off with too little energy to overcome the small retarding potential and reach the plate
- As the accelerating potential is raised further, the plate current again increases, since the electrons now have enough energy to reach the plate
- Eventually another sharp drop (at 9.8 V) in the current occurs because, again, the electron has collected just the same energy to excite the same energy level in the other atoms



- The higher critical potentials result from two or more inelastic collisions and are multiple of the lowest (4.9 V)
- The excited mercury atom will then de-excite by radiating out a photon of exactly the energy (4.9 eV) which is also detected in the Frank-Hertz experiment
- The critical potential verifies the existence of atomic levels

Bohr's correspondence principle

- The predictions of the quantum theory for the behaviour of any physical system must correspond to the prediction of classical physics in the limit in which the quantum number specifying the state of the system becomes very large:
- $\lim_{n \to \infty}$ quantum theory = classical theory
- At large *n* limit, the Bohr model must reduce to a "classical atom" which obeys classical theory

In other words…

- The laws of quantum physics are valid in the atomic domain; while the laws of classical physics is valid in the classical domain; where the two domains overlaps, both sets of laws must give the same result.

PYQ 20 Test II 2003/04

- Which of the following statements are correct?

- **I** Frank-Hertz experiment shows that atoms are excited to discrete energy levels
- **II** Frank-Hertz experimental result is consistent with the results suggested by the line spectra
- **III** The predictions of the quantum theory for the behaviour of any physical system must correspond to the prediction of classical physics in the limit in which the quantum number specifying the state of the system becomes very large
- **IV**The structure of atoms can be probed by using electromagnetic radiation
- **A. II,III** **B. I, II,IV** **C. II, III, IV**
- **D. I,II, III, IV** **E. Non of the above**
- **ANS:D, My own questions**

Example
(Read it yourself)

- Classical EM predicts that an electron in a circular motion will radiate EM wave at the same frequency
- According to the correspondence principle,

  the Bohr model must also reproduce this result in the large *n* limit

  More quantitatively
- In the limit, $n = 10^3 - 10^4$, the Bohr atom would have a size of 10^{-3} m
- This is a large quantum atom which is in classical domain
- The prediction for the photon emitted during transition around the $n = 10^3 - 10^4$ states should equals to that predicted by classical EM theory.

$n \rightarrow$ large

V_n **(Bohr)** **=** V **(classical theory)**

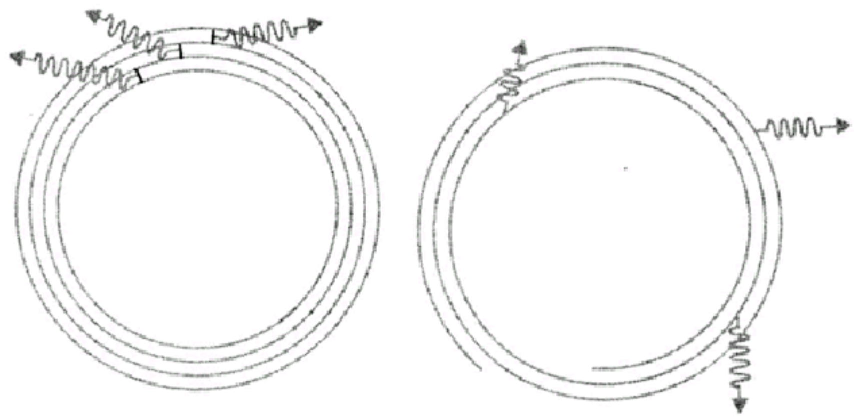


FIGURE 6.26 (Top) A large quantum atom. Photons are emitted in discrete transitions as the electron jumps to lower states. (Bottom) A classical atom. Photons are emitted continuously by the accelerated electron.

Classical physics calculation

- The period of a circulating electron is

$$T = 2\pi r/(2K/m)^{1/2}$$
$$= \pi r(2m)^{1/2}(8\pi\varepsilon_0 r)^{1/2}/e$$

- This result can be easily derived from the mechanical stability of the atom as per

$$\frac{1}{4\pi\varepsilon_0}\frac{(Ze)e}{r^2} = \frac{m_e v^2}{r}$$

- Substitute the quantised atomic radius $r_n = n^2 r_0$ into T, we obtain the frequency as per

$$\nu_n = 1/T = me^4/32\pi^3\varepsilon_0^2\hbar^3 n^3$$

Based on Bohr's theory

- Now, for an electron in the Bohr atom at energy level n = 10^3 - 10^4, the frequency of an radiated photon when electron makes a transition from the n state to n–1 state is given by

$$\nu_n = (E_n - E_{n-1}) / h$$
$$= (me^4/64\pi^3\varepsilon_0^2\hbar^3)[(n-1)^{-2} - n^{-2}]$$
$$= (me^4/64\pi^3\varepsilon_0^2\hbar^3)[(2n-1)/n^2(n-1)^2]$$

Where we have made use of

$$E_n / h = E_0 / n^2h = (-me^4/64\pi^3\varepsilon_0^2\hbar^3 n^2).$$

- In the limit of large n,

$$\nu \approx (me^4/64\pi^3\varepsilon_0^2\hbar^3)[2n/n^4]$$
$$= (me^4/32\pi^3\varepsilon_0^2\hbar^3)[1/n^3]$$

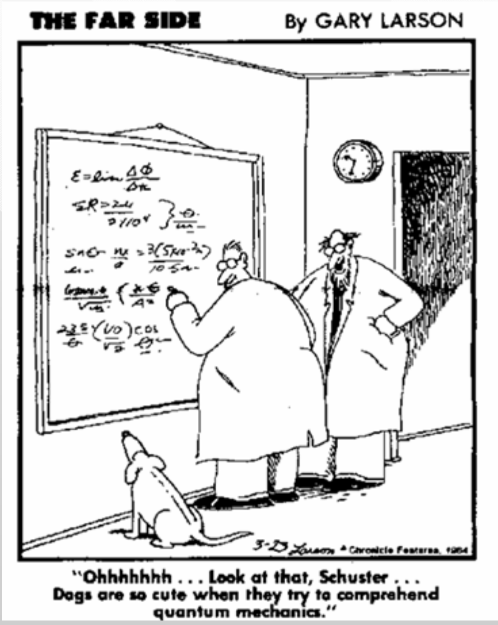
Classical result and Quantum calculation meets at $n \to \infty$

- Hence, in the region of large n, where classical and quantum physics overlap, the classical prediction and that of the quantum one is identical

$$\nu_{classical} = \nu_{Bohr} = (me^4/32\pi^3\varepsilon_0^2\hbar^3)[1/n^3]$$

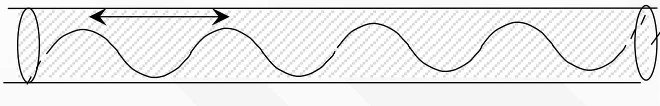
CHAPTER 6

Very Brief introduction to Quantum mechanics



Probabilistic interpretation of matter wave

λ



A = 1 unit area

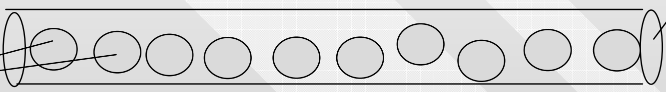
A beam of light if pictured as monochromatic wave (λ, ν)

Intensity of the light beam is $I = \varepsilon_0 c E^2$

A beam of light pictured in terms of photons

A = 1 unit area

$E=h\nu$



Intensity of the light beam is $I = Nh\nu$

N = average number of photons per unit time crossing unit area perpendicular to the direction of propagation

Intensity = energy crossing one unit area per unit time. I is in unit of joule per m^2 per second

Probability of observing a photon

- Consider a beam of light
- In wave picture, $E = E_0 \sin(kx-\omega t)$, electric field in radiation
- Intensity of radiation in wave picture is

$$I = \varepsilon_0 c \overline{E^2}$$

- On the other hand, in the photon picture, $I = Nh\nu$
- Correspondence principle: what is explained in the wave picture has to be consistent with what is explained in the photon picture in the limit $N \rightarrow$ infinity:

$$I = \varepsilon_0 c \overline{E^2} = Nh\nu$$

Statistical interpretation of radiation

- The probability of observing a photon at a point in unit time is proportional to N
- However, since $Nh\nu = \varepsilon_0 c \overline{E^2} \propto \overline{E^2}$
- the probability of observing a photon must also
- This means that the probability of observing a photon at any point in space is proportional to the square of the averaged electric field strength at that point

$$Prob\,(x) \propto \overline{E^2}$$

Square of the mean of the square of the **wave field amplitude**

What is the physical interpretation of matter wave?

- we will call the mathematical representation of the de Broglie's wave / matter wave associated with a given particle (or an physical entity) as

The wave function, Ψ

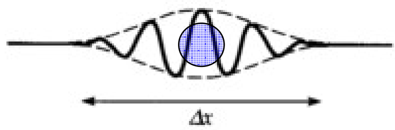


FIGURE 6.14 An idealized wave packet localized in space over a region Δx is the perposition of many waves of different amplitudes and frequencies.

- We wish to answer the following questions:
- Where is exactly the particle located within Δx? the locality of a particle becomes fuzzy when it's represented by its matter wave. We can no more tell for sure where it is exactly located.
- Recall that in the case of conventional wave physics, |field amplitude|^2 is proportional to the intensity of the wave). Now, what does $|\Psi|^2$ physically mean?

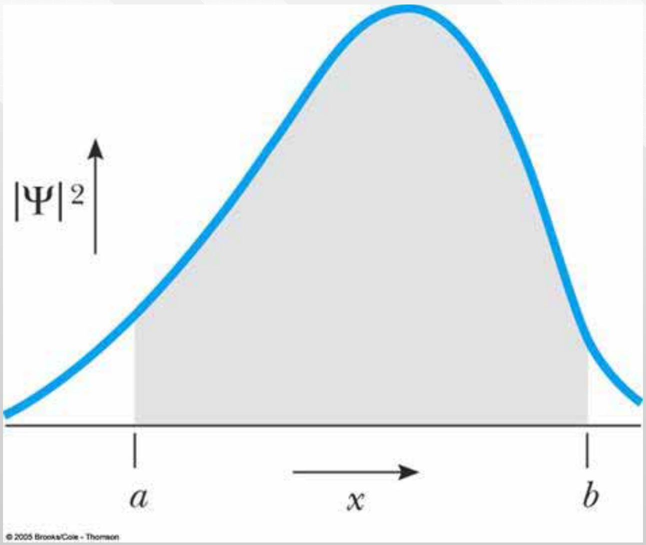
Probabilistic interpretation of (the square of) matter wave

- As seen in the case of radiation field, |electric field's amplitude|^2 is proportional to the probability of finding a photon
- In exact analogy to the statistical interpretation of the radiation field,
- $P(x) = |\Psi|^2$ is interpreted as the probability density of observing a material particle
- More quantitatively,
- Probability for a particle to be found between point a and b is

$$p(a \le x \le b) = \int_a^b P(x)dx = \int_a^b |\Psi(x,t)|^2 \, dx$$

$$p_{ab} = \int_a^b |\Psi(x,t)|^2 \, dx$$ is the probability to find the particle between a and b

- It value is given by the area under the curve of probability density between *a* and *b*



Expectation value

- Any physical observable in quantum mechanics, *O* (which is a function of position, *x*), when measured repeatedly, will yield an expectation value of given by

$$\langle O \rangle = \frac{\int_{-\infty}^{\infty} \Psi O \Psi^* dx}{\int_{-\infty}^{\infty} \Psi \Psi^* dx} = \frac{\int_{-\infty}^{\infty} O |\Psi|^2 \, dx}{\int_{-\infty}^{\infty} \Psi \Psi^* dx}$$

- Example, *O* can be the potential energy, position, etc.
- (Note: the above statement is not applicable to energy and linear momentum because they cannot be express explicitly as a function of *x* due to uncertainty principle)…

Example of expectation value: average position measured for a quantum particle

- If the position of a quantum particle is measured repeatedly with the same initial conditions, the averaged value of the position measured is given by

$$\langle x \rangle = \frac{\int_{-\infty}^{\infty} x \left|\Psi\right|^2 dx}{1} = \int_{-\infty}^{\infty} x \left|\Psi\right|^2 dx$$

Example

- A particle limited to the *x* axis has the wave function $\Psi = ax$ between x=0 and x=1; Ψ = 0 else where.
- (a) Find the probability that the particle can be found between x=0.45 and x=0.55.
- (b) Find the expectation value $<x>$ of the particle's position

Solution

- (a) the probability is

$$\int_{-\infty}^{\infty} |\Psi|^2 \, dx = \int_{0.45}^{0.55\infty} x^2 dx = a^2 \left[\frac{x^3}{3} \right]_{0.45}^{0.55} = 0.0251a^2$$

- (b) The expectation value is

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi|^2 \, dx = \int_0^1 x^3 dx = a^2 \left[\frac{x^3}{4} \right]_0^1 = \frac{a^2}{4}$$

Max Born and probabilistic interpretation

- Hence, a particle's wave function gives rise to a *probabilistic interpretation* of the position of a particle
- Max Born in 1926

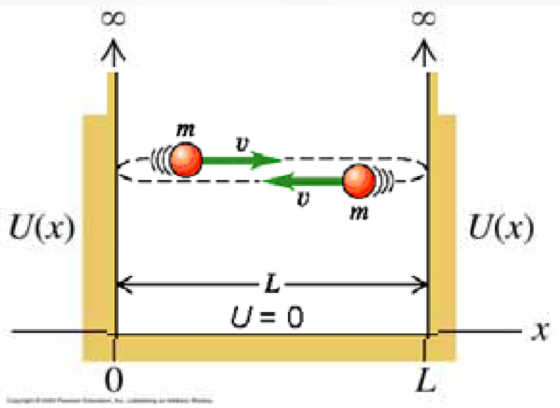
German-British physicist who worked on the mathematical basis for quantum mechanics. Born's most important contribution was his suggestion that the absolute square of the wavefunction in the Schrödinger equation was a measure of the probability of finding the particle at a given location. Born shared the 1954 Nobel Prize in physics with Bethe

PYQ 2.7, Final Exam 2003/04

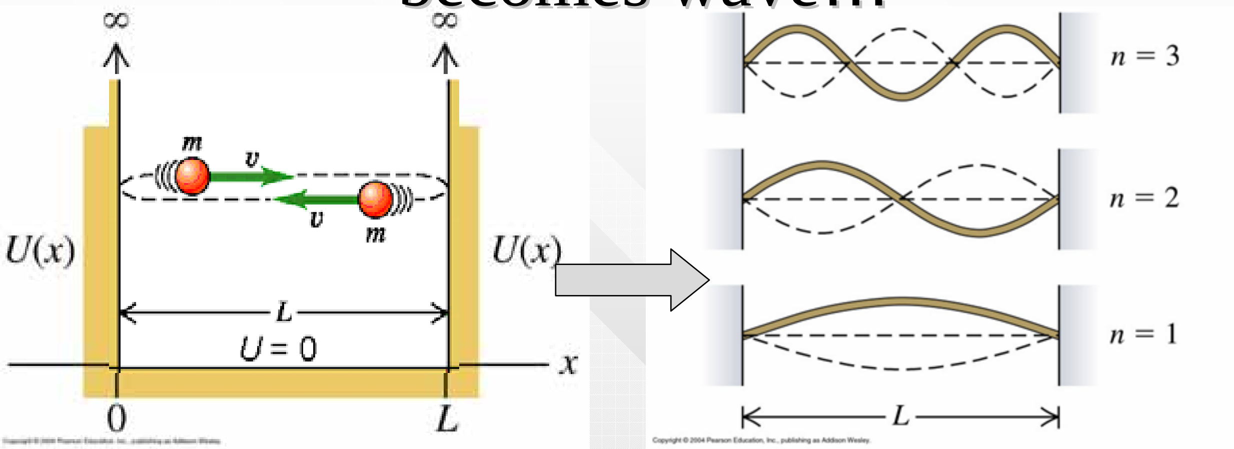
- A large value of the probability density of an atomic electron at a certain place and time signifies that the electron
- **A.** is likely to be found there
- **B.** is certain to be found there
- **C.** has a great deal of energy there
- **D.** has a great deal of charge
- **E.** is unlikely to be found there
- ANS:A, Modern physical technique, Beiser, MCP 25, pg. 802

Particle in in an infinite well (sometimes called particle in a box)

- Imagine that we put particle (e.g. an electron) into an "infinite well" with width L (e.g. a potential trap with sufficiently high barrier)
- In other words, the particle is confined within $0 < x < L$
- In Newtonian view the particle is traveling along a straight line bouncing between two rigid walls



However, in quantum view, particle becomes wave…

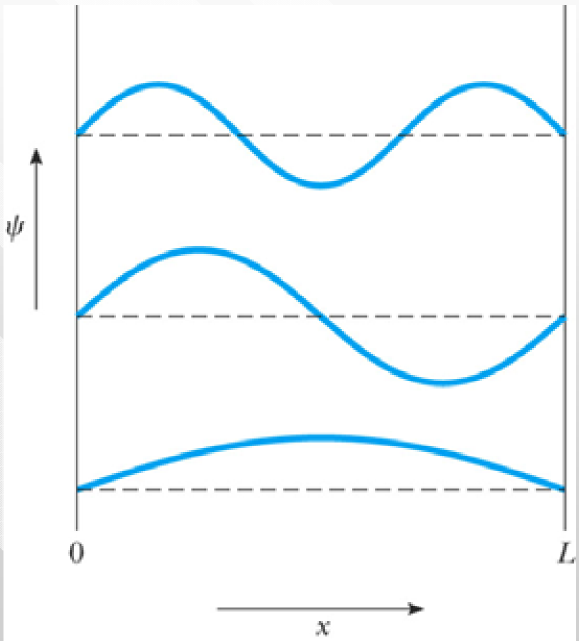


- The ‘particle’ is no more pictured as a particle bouncing between the walls but a de Broglie wave that is trapped inside the infinite quantum well, in which they form standing waves

Particle forms standing wave within the infinite well

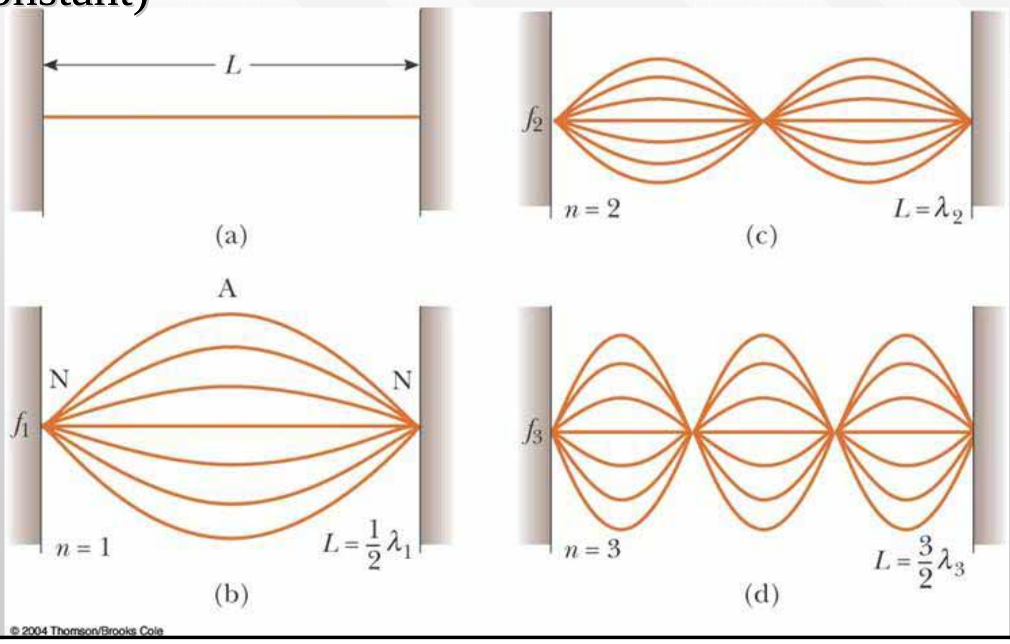
- How would the wave function of the particle behave inside the well?

- They form standing waves which are confined within $0 \leqslant x \leqslant L$



Standing wave in general

- Shown below are standing waves which ends are fixed at $x = 0$ and $x = L$
- For standing wave, the speed is constant, ($v = \lambda f$ = constant)



Mathematical description of standing waves

- In general, the equation that describes a standing wave (with a constant width L) is simply:

$$L = n\lambda_n/2$$

  n = 1, 2, 3, … (positive, discrete integer)
- n characterises the "mode" of the standing wave
- n = 1 mode is called the 'fundamental' or the first harmonic
- n = 2 is called the second harmonics, etc.
- λ_n are the wavelengths associated with the n-th mode standing waves
- The lengths of λ_n is "quantised" as it can take only discrete values according to $\lambda_n = 2L/n$

Energy of the particle in the box

- Recall that

$$V(x)=\begin{cases}\infty, & x\le 0, x\ge L\\ 0, & 0<x<L\end{cases}$$

- For such a free particle that forms standing waves in the box, it has no potential energy
- It has all of its mechanical energy in the form of kinetic energy only
- Hence, for the region $0 < x < L$, we write the total energy of the particle as

$$E = K + V = p^2/2m + 0 = p^2/2m$$

Energies of the particle are quantised

- Due to the quantisation of the standing wave (which comes in the form of $\lambda_n = 2L/n$), the momentum of the particle must also be quantised due to de Broglie's postulate:

$$p \to p_n = \frac{h}{\lambda_n} = \frac{nh}{2L}$$

It follows that the total energy of the particle is also quantised:

$$E \to E_n = \frac{p_n^2}{2m} = n^2\frac{\pi^2\hbar^2}{2mL^2}$$

$$E_n = \frac{p_n^2}{2m} = n^2 \frac{h^2}{8mL^2} = n^2 \frac{\pi^2 \hbar^2}{2mL^2}$$

The n = 1 state is a characteristic state called the ground state = the state with lowest possible energy (also called zero-point energy)

$$E_n(n=1) \equiv E_0 = \frac{\pi^2 \hbar^2}{2mL^2}$$

Ground state is usually used as the reference state when we refer to ``excited states'' (n = 2, 3 or higher)

The total energy of the n-th state can be expressed in term of the ground state energy as

$$E_n = n^2 E_0 \quad (n = 1,2,3,4\ldots)$$

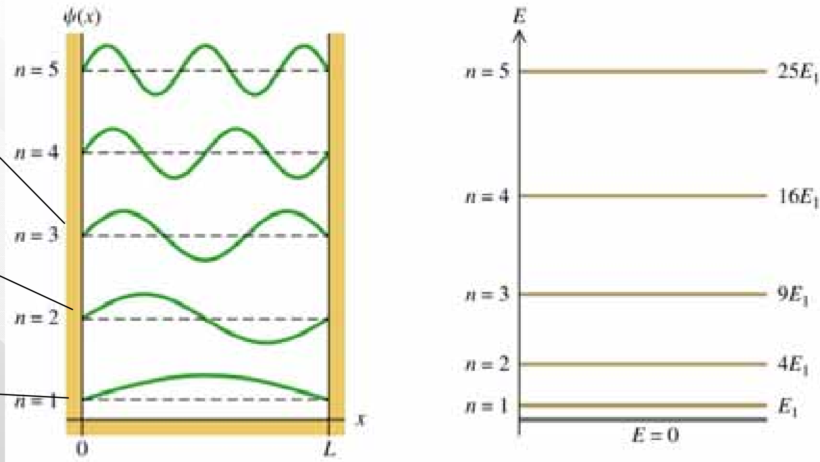
The higher n the larger is the energy level

- Some terminology
- n = 1 corresponds to the ground state
- n = 2 corresponds to the first excited state, etc

n = 3 is the second excited state, 4 nodes, 3 antinodes

n = 2 is the first excited state, 3 nodes, 2 antinodes

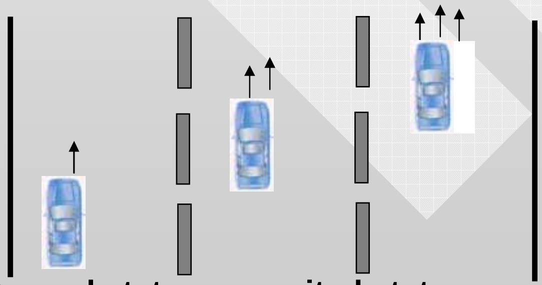
n = 1 is the ground state (fundamental mode): 2 nodes, 1 antinode



- Note that lowest possible energy for a particle in the box is not zero but E_0 (= E_1), the zero-point energy.
- This a result consistent with the Heisenberg uncertainty principle

Simple analogy

- Cars moving in the right lane on the highway are in 'excited states' as they must travel faster (at least according to the traffic rules). Cars travelling in the left lane are in the ``ground state'' as they can move with a relaxingly lower speed. Cars in the excited states must finally resume to the ground state (i.e. back to the left lane) when they slow down



Ground state excited states

Example on energy levels

- Consider an electron confined by electrical force to an infinitely deep potential well whose length L is 100 pm, which is roughly one atomic diameter. What are the energies of its three lowest allowed states and of the state with n = 15?
- **SOLUTION**
- For n = 1, the ground state, we have

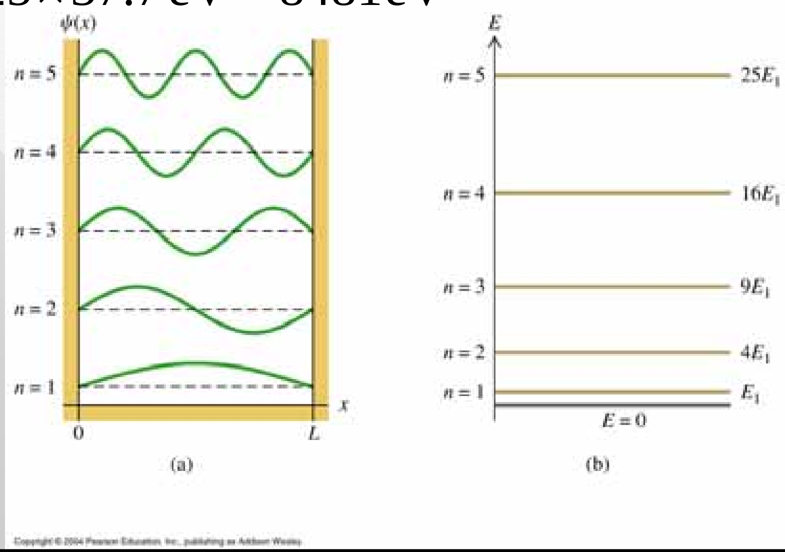
$$E_1 = (1)^2 \frac{h^2}{8m_e L^2} = \frac{\left(6.63\times10^{-34}\,\mathrm{Js}\right)^2}{\left(9.1\times10^{-31}\,\mathrm{kg}\right)\left(100\times10^{-12}\,\mathrm{m}\right)^2} = 6.3\times10^{-18}\,\mathrm{J} = 37.7\mathrm{eV}$$

- The energy of the remaining states (n=2,3,15) are

$$E_2 = (2)^2 E_1 = 4 \times 37.7\,\text{eV} = 150\,\text{eV}$$

$$E_3 = (3)^2 E_1 = 9 \times 37.7\,\text{eV} = 339\,\text{eV}$$

$$E_{15} = (15)^2 E_1 = 225 \times 37.7\,\text{eV} = 8481\,\text{eV}$$



Question continued

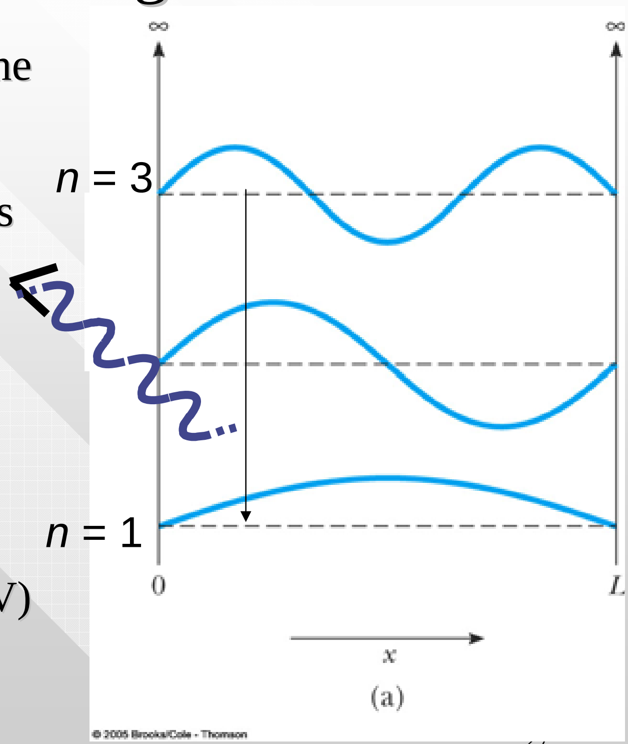
- When electron makes a transition from the n = 3 excited state back to the ground state, does the energy of the system increase or decrease?
- **Solution:**
- The energy of the system decreases as energy drops from 339 eV to 150 eV
- The lost amount $|\Delta E| = E_3 - E_1$ = 339 eV – 150 eV is radiated away in the form of electromagnetic wave with wavelength λ obeying $\Delta E = hc/\lambda$

Photon with

λ = *xx* nm

Example
Radiation emitted during de-excitation

- Calculate the wavelength of the electromagnetic radiation emitted when the excited system at $n = 3$ in the previous example de-excites to its ground state



- **Solution**

$\lambda = hc/|\Delta E|$

$= 1240$ nm. eV / $(|E_3 - E_1|)$

$= 1240$ nm. eV/(339 eV–150 eV)

$=$ *xx* nm

Example
A macroscopic particle’s quantum state

- Consider a 1 microgram speck of dust moving back and forth between two rigid walls separated by 0.1 mm. It moves so slowly that it takes 100 s for the particle to cross this gap. What quantum number describes this motion?

Solution

- The energy of the particle is

$$E(=K)=\frac{1}{2}mv^2=\frac{1}{2}\left(1\times10^{-9}\,\text{kg}\right)\times\left(1\times10^{-6}\,\text{m/s}\right)^2=5\times10^{-22}\,\text{J}$$

- Solving for n in $E_n = n^2\dfrac{\pi^2\hbar^2}{2mL^2}$
- yields $n=\dfrac{L}{h}\sqrt{8mE}\approx3\times10^{14}$
- This is a very large number
- It is experimentally impossible to distinguish between the n = 3 x 10^{14} and n = 1 + (3 x 10^{14}) states, so that the quantized nature of this motion would never reveal itself

- The quantum states of a macroscopic particle cannot be experimentally discerned (as seen in previous example)
- Effectively its quantum states appear as a continuum

allowed energies in classical system – appear continuous (such as energy carried by a wave; total mechanical energy of an orbiting planet, etc.)

descret energies in quantised system – discrete (such as energy levels in an atom, energies carried by a photon)

PYQ 4(a) Final Exam 2003/04

- An electron is contained in a one-dimensional box of width 0.100 nm. Using the particle-in-a-box model,
- (i) Calculate the n = 1 energy level and n = 4 energy level for the electron in eV.
- (ii) Find the wavelength of the photon (in nm) in making transitions that will eventually get it from the the n = 4 to n = 1 state
- Serway solution manual 2, Q33, pg. 380, modified

Solution

- 4a(i) In the particle-in-a-box model, standing wave is formed in the box of dimension L: $\lambda_n = \frac{2L}{n}$
- The energy of the particle in the box is given by

$$K_n = E_n = \frac{p_n^2}{2m_e} = \frac{(h/\lambda_n)^2}{2m_e} = \frac{n^2 h^2}{8m_e L^2} = \frac{n^2 \pi^2 \hbar^2}{2m_e L^2}$$

$$E_1 = \frac{\pi^2 \hbar^2}{2m_e L^2} = 37.7\text{ eV} \qquad E_4 = 4^2 E_1 = 603\text{ eV}$$

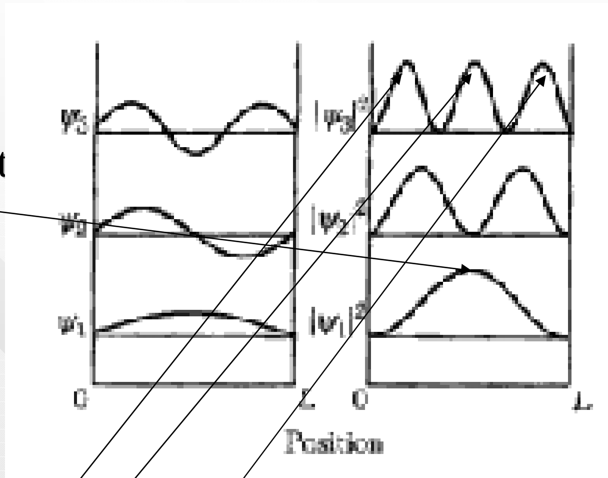
- 4a(ii)
- The wavelength of the photon going from n = 4 to n = 1 is $\lambda = hc/(E_6 - E_1)$
- = 1240 eV nm/ (603 – 37.7) eV = **2.2 nm**

Example on the probabilistic interpretation: Where in the well the particle spend most of its time?

- The particle spend most of its time in places where its probability to be found is largest
- Find, for the *n* = 1 and for *n* =3 quantum states respectively, the points where the electron is most likely to be found

Solution

- For electron in the n = 1 state, the probability to find the particle is highest at $x = L/2$
- Hence electron in the n =1 stat spend most of its time there compared to other places

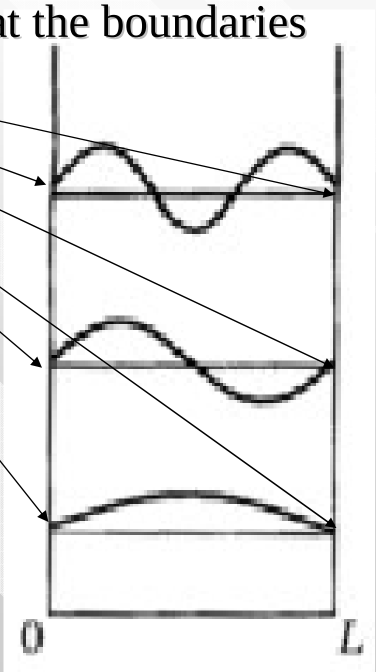


- For electron in the n = 3 state, the probability to find the particle is highest at $x = L/6, L/2, 5L/6$
- Hence electron in the n =3 state spend most of its time at this three places

Boundary conditions and normalisation of the wave function in the infinite well

- Due to the probabilistic interpretation of the wave function, the probability density $P(x) = |\Psi|^2$ must be such that
- $P(x) = |\Psi|^2 > 0$ for $0 < x < L$
- The particle has no where to be found at the boundary as well as outside the well, i.e $P(x) = |\Psi|^2 = 0$ for $x \leqslant 0$ and $x \geqslant L$

- The probability density is zero at the boundaries



- Inside the well, the particle is bouncing back and forth between the walls
- It is obvious that it must exist within somewhere within the well
- This means:

$$\int_{-\infty}^{\infty} P(x)dx = \int_{0}^{L} |\Psi|^2 \, dx = 1$$

$$\int_{-\infty}^{\infty} P(x)dx = \int_{0}^{L} |\Psi|^2 \, dx = 1$$

- is called the normalisation condition of the wave function
- It represents the physical fact that the particle is contained inside the well and the integrated possibility to find it inside the well must be 1
- The normalisation condition will be used to determine the normalisaton constant when we solve for the wave function in the Schrodinder equation

See if you could answer this question

- Can you list down the main differences between the particle-in-a-box system (infinite square well) and the Bohr's hydrogen like atom? E.g. their energies level, their quantum number, their energy gap as a function of *n*, the sign of the energies, the potential etc.

Schrodinger Equation

Schrödinger, Erwin (1887-1961), Austrian physicist and Nobel laureate. Schrödinger formulated the theory of wave mechanics, which describes the behavior of the tiny particles that make up matter in terms of waves. Schrödinger formulated the Schrödinger wave equation to describe the behavior of electrons (tiny, negatively charged particles) in atoms. For this achievement, he was awarded the 1933 Nobel Prize in physics with British physicist Paul Dirac

What is the general equation that governs the evolution and behaviour of the wave function?

- Consider a particle subjected to some time-independent but space-dependent potential $V(x)$ within some boundaries

- The behaviour of a particle subjected to a time-independent potential is governed by the famous (1-D, time independent, non relativistic) Schrodinger equation:

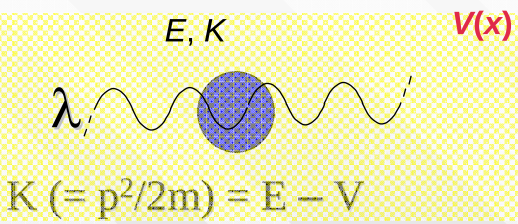
$$\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2}+(E-V)\psi(x)=0$$

How to derive the T.I.S.E

- 1) Energy must be conserved: $E = K + U$
- 2) Must be consistent with de Brolie hypothesis that
  $p = h/\lambda$
- 3) Mathematically well-behaved and sensible (e.g. finite, single valued, linear so that superposition prevails, conserved in probability etc.)
- Read the msword notes or text books for more technical details (which we will skip here)

Energy of the particle

- The kinetic energy of a particle subjected to potential $V(x)$ is



- E is conserved if there is no net change in the total mechanical energy between the particle and the surrounding
  (Recall that this is just the definition of total mechanical energy)
- It is essential to relate the de Broglie wavelength to the energies of the particle:

$$\lambda = h / p = h / \sqrt{[2m(E-V)]}$$

- Note that, as $V \rightarrow 0$, the above equation reduces to the no-potential case (as we have discussed earlier)

$\lambda = h / p \rightarrow h / \sqrt{[2mE]}$, where $E = K$ only

Infinite potential revisited

- Armed with the T.I.S.E we now revisit the particle in the infinite well
- By using appropriate boundary condition to the T.I.S.E, the solution of T.I.S.E for the wave function Ψ should reproduces the quantisation of energy level as have been deduced earlier, i.e. $E_n = \dfrac{n^2\pi^2\hbar^2}{2mL^2}$

In the next slide we will need to do some mathematics to solve for Ψ(x) in the second order differential equation of TISE to recover this result. This is a more formal way compared to the previous standing waves argument which is more qualitative

Why do we need to solve the Shrodinger equation?

- The potential $V(x)$ represents the environmental influence on the particle
- Knowledge of the solution to the T.I.S.E, i.e. $\psi(x)$ allows us to obtain essential physical information of the particle (which is subjected to the influence of the external potential $V(x)$), e.g the probability of its existence in certain space interval, its momentum, energies etc.

Take a classical example: A particle that are subjected to a gravity field $U(x) = GMm/r^2$ is governed by the Newton equations of motion,

$$-\frac{GMm}{r^2} = m\frac{d^2r}{dt^2}$$

- Solution of this equation of motion allows us to predict, e.g. the position of the object m as a function of time, $r = r(t)$, its instantaneous momentum, energies, etc.

S.E. is the quantum equivalent of the Newton's law of motion

- The equivalent of "Newton laws of motion" for quantum particles = Shroedinger equation
- Solving for the wave function in the S.E. allows us to extract all possible physical information about the particle (energy, expectation values for position, momentum, etc.)

The infinite well in the light of TISE

$$V(x) = \begin{cases} \infty, & x \le 0, x \ge L \\ 0, & 0 < x < L \end{cases}$$

Plug the potential function *V*(x) into the T.I.S.E

$$\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + (E - V)\psi(x) = 0$$

Within 0 < x < *L*, *V* (*x*) = 0, hence the TISE becomes

$$\frac{\partial^2\psi(x)}{\partial x^2} = -\frac{2m}{\hbar^2}E\psi(x) \equiv -B^2\psi(x)$$

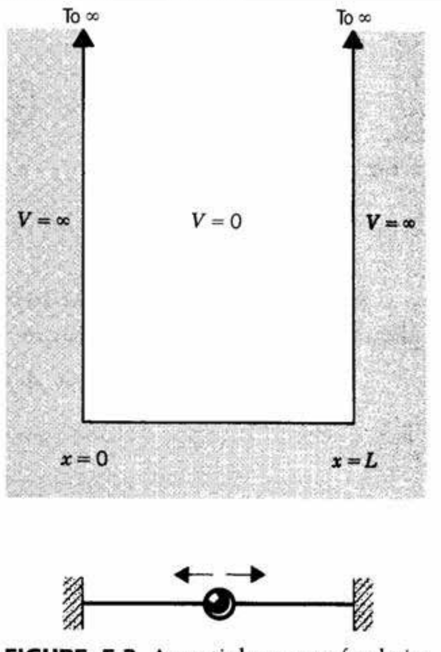


FIGURE 5.3 A particle moves freely in the one-dimensional region $0 \le x \le L$, but is excluded completely from $x < 0$ and $x > L$.

The behavior of the particle inside the box is governed by the equation

$$\frac{\partial^2 \psi(x)}{\partial x^2} = -B^2 \psi(x)$$

$$B^2 = \frac{2mE}{\hbar^2}$$

This term contain the information of the energies of the particle, which in terns governs the behaviour (manifested in terms of its mathematical solution) of ψ(x) inside the well. Note that in a fixed quantum state *n*, *B* is a constant because *E* is conserved.

However, if the particle jumps to a state *n'* $\neq$ *n*, *E* takes on other values. In this case, *E* is not conserved because there is an net change in the total energy of the system due to interactions with external environment (e.g. the particle is excited by external photon)

If you still recall the elementary mathematics of second order differential equations, you will recognise that the solution to the above TISE is simply

$$\psi(x) = A\sin Bx + C\cos Bx$$

Where *A*, *C* are constants to be determined by ultilising the boundary conditions pertaining to the infinite well system

You can prove that indeed

$$\psi(x) = A\sin Bx + C\cos Bx \quad \textbf{(EQ 1)}$$

is the solution to the TISE $$\frac{\partial^2 \psi(x)}{\partial x^2} = -B^2 \psi(x) \quad \textbf{(EQ 2)}$$

- I will show the steps in the following:
- Mathematically, to show that EQ 1 is a solution to EQ 2, we just need to show that when EQ1 is plugged into the LHS of EQ. 2, the resultant expression is the same as the expression to the RHS of EQ. 2.

Plug

$\psi(x) = A\sin Bx + C\cos Bx$ into the LHS of EQ 2:

$$\frac{\partial^2 \psi(x)}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left[A\sin Bx + C\cos Bx\right]$$

$$= \frac{\partial}{\partial x}\left[BA\cos Bx - BC\sin Bx\right]$$

$$= -B^2 A\sin Bx - B^2 C\cos Bx$$

$$= -B^2\left[A\sin Bx + C\cos Bx\right]$$

$$= -B^2\psi(x) = \text{RHS of EQ2}$$

Proven that EQ1 is indeed the solution to EQ2

Boundary conditions

- Next, we would like to solve for the constants A, C in the solution $\psi(x)$, as well as the constraint that is imposed on the constant B
- We know that the wave function forms nodes at the boundaries. Translate this boundary conditions into mathematical terms, this simply means

$$\psi(x = 0) = \psi(x = L) = 0$$

- First,
- Plug $\psi(x = 0) = 0$ into
  $\psi = A\sin Bx + C\cos Bx$, we obtain
  $\psi(x=0)) = 0 = A\sin 0 + C \cos 0 = C$
- i.e, $C = 0$
- Hence the solution is reduced to
  $\psi(x) = A\sin Bx$

- Next we apply the second boundary condition

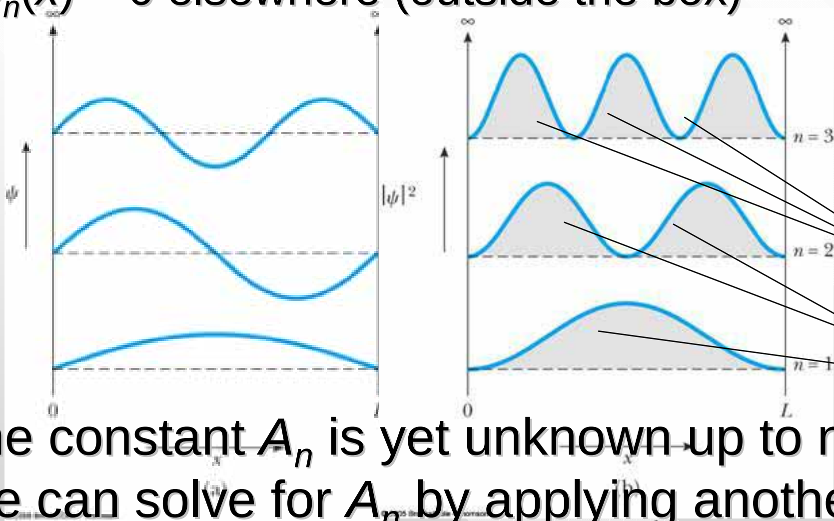
  $\psi(x = L) = 0 = A\sin(BL)$

- Only either A or $\sin(BL)$ must be zero but not both

- A cannot be zero else this would mean $\psi(x)$ is zero everywhere inside the box, conflicting the fact that the particle must exist inside the box
- The upshot is: A cannot be zero

- This means it must be $\sin BL = 0$, or in other words
- $B = n\pi/L \equiv B_n$, $n = 1,2,3,\ldots$
- n is used to characterise the quantum states of $\psi_n(x)$
- B is characterised by the positive integer n, hence we use B_n instead of B
- The relationship $B_n = n\pi/L$ translates into the familiar quantisation of energy condition:
- $(B_n = n\pi/L)^2 \rightarrow B_n^2 = \dfrac{2mE_n}{\hbar^2} = \dfrac{n^2\pi^2}{L^2} \Rightarrow E_n = n^2 \dfrac{\pi^2\hbar^2}{2mL^2}$

- Hence, up to this stage, the solution is
- $\psi_n(x) = A_n \sin(n\pi x/L)$, $n = 1, 2, 3,\ldots$ for $0 < x < L$
- $\psi_n(x) = 0$ elsewhere (outside the box)



The area under the curves of $|\Psi_n|^2 = 1$ for every n

- The constant A_n is yet unknown up to now
- We can solve for A_n by applying another "boundary condition" – the normalisation condition that:

$$\int_{-\infty}^{\infty} \psi_n^2(x)\,dx = \int_0^L \psi_n^2(x)\,dx = 1$$

Solve for A_n with normalisation

$$\int_{-\infty}^{\infty} \psi_n^2(x)dx = \int_0^L \psi_n^2(x)dx = A_n^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx = \frac{A_n^2 L}{2} = 1$$

- thus

$$A_n = \sqrt{\frac{2}{L}}$$

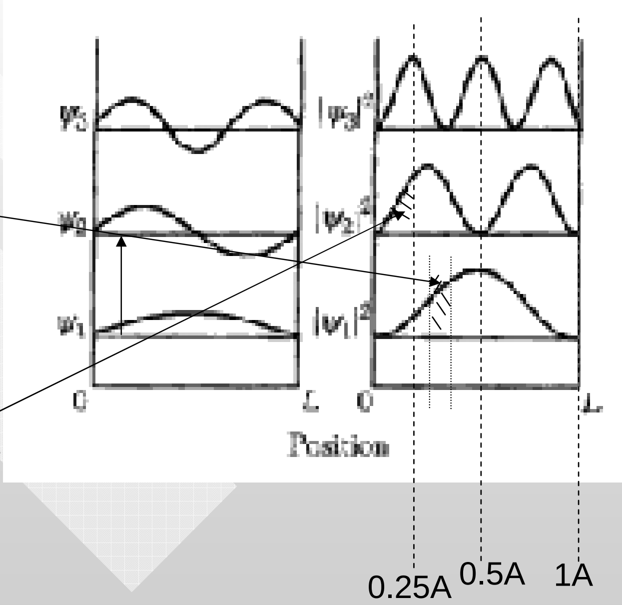
- We hence arrive at the final solution that

➢ $\psi_n(x) = (2/L)^{1/2} \sin(n\pi x/L)$, $n = 1, 2, 3, \ldots$ for $0 < x < L$

➢ $\psi_n(x) = 0$ elsewhere (i.e. outside the box)

Example

- An electron is trapped in a one-dimensional region of length $L = 1.0 \times 10^{-10}$ m.
- (a) How much energy must be supplied to excite the electron from the ground state to the first state?
- (b) In the ground state, what is the probability of finding the electron in the region from $x = 0.090 \times 10^{-10}$ m to 0.110×10^{-10} m?
- (c) In the first excited state, what is the probability of finding the electron between $x = 0$ and $x = 0.250 \times 10^{-10}$ m?



Solutions

(a) $E_1 \equiv E_0 = \dfrac{\hbar^2\pi^2}{2mL^2} = 37\text{eV}$ $\quad E_2 = n^2 E_0 = (2)^2 E_0 = 148\text{eV}$

$$\Rightarrow \Delta E = |E_2 - E_0| = 111\text{eV}$$

(b) $P_{n=1}(x_1 \le x \le x_2) = \int_{x_1}^{x_2} \psi_0^2 dx = \dfrac{2}{L}\int_{x_1}^{x_2} \sin^2 \dfrac{\pi x}{L} dx$

For ground state

$$= \left(\frac{x}{L} - \frac{1}{2\pi}\sin\frac{2\pi x}{L} \right)\Bigg|_{x_1 = 0.09\mathring{A}}^{x_2 = 0.11\mathring{A}} = 0.0038$$

On average the particle in the ground state spend only 0.04 % of its time in the region between x=0.11A and x=0.09 A

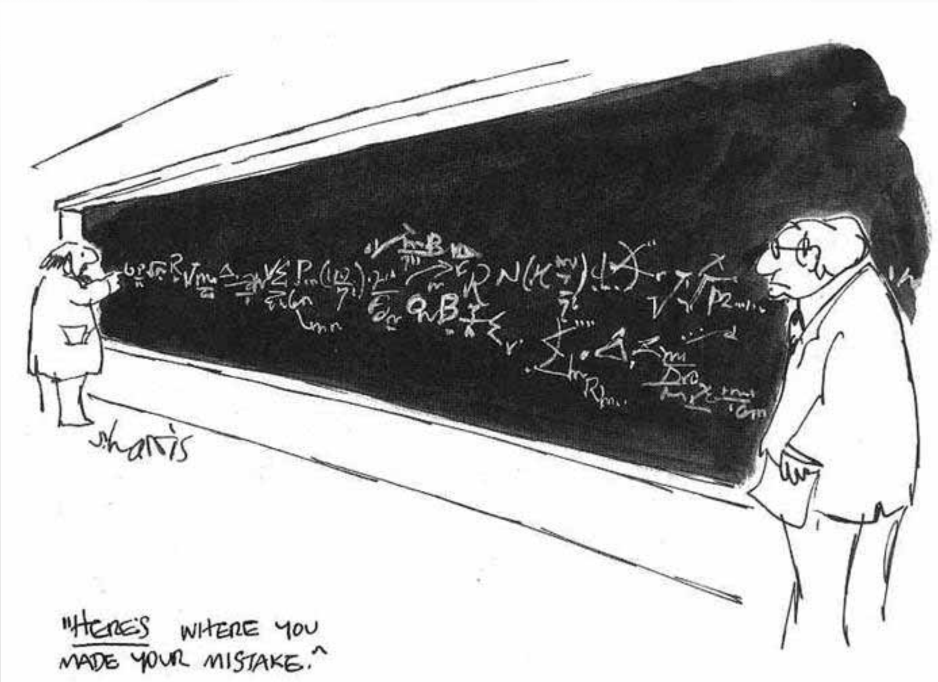
(c) **For *n* = 2,** $\psi_2 = \sqrt{\dfrac{2}{L}} \sin\dfrac{2\pi x}{L};$

$$P_{n=2}(x_1 \le x \le x_2) = \int_{x_1}^{x_2} \psi_2^2 dx = \frac{2}{L}\int_{x_1}^{x_2} \sin^2 \frac{2\pi x}{L} dx$$

$$= \left(\frac{x}{L} - \frac{1}{4\pi}\sin\frac{4\pi x}{L} \right)\Bigg|_{x_1 = 0}^{x_2 = 0.25\mathring{A}} = 0.25$$

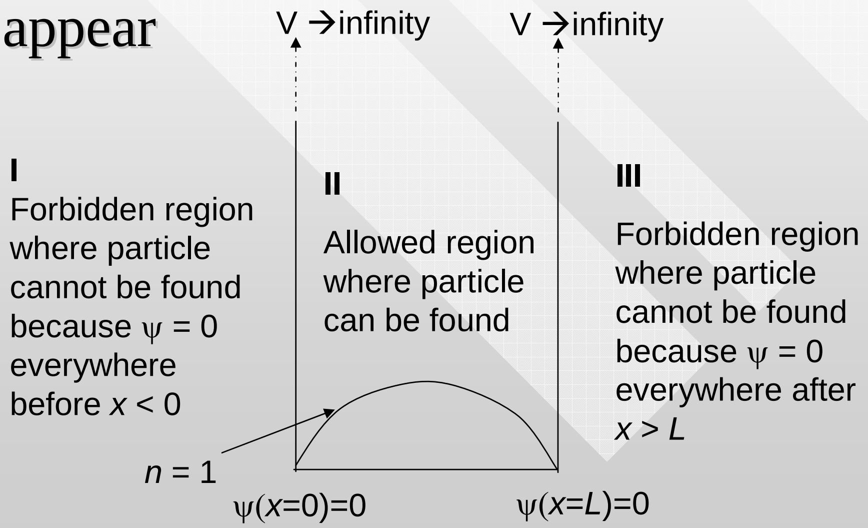
On average the particle in the n = 2 state spend 25% of its time in the region between x=0 and x=0.25 A

The nightmare of a lengthy calculation



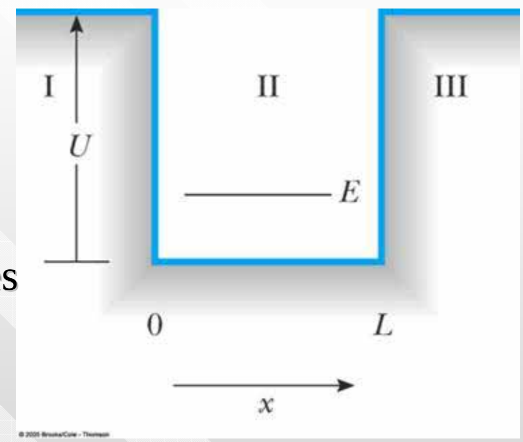
Quantum tunneling

- In the infinite quantum well, there are regions where the particle is "forbidden" to appear



Finite quantum well

- The fact that *y* is 0 everywhere x ≤0, x ≥ *L* is because of the infiniteness of the potential, *V* → ∞
- If *V* has only finite height, the solution to the TISE will be modified such that a non-zero value of *y* can exist beyond the boundaries at $x = 0$ and $x = L$
- In this case, the pertaining boundaries conditions are

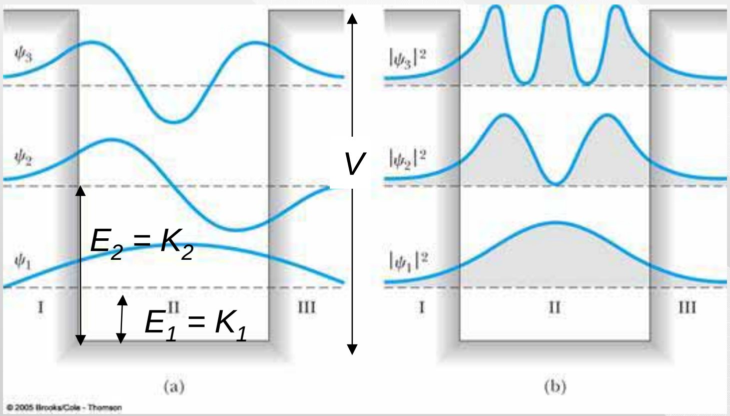


$$\psi_I(x=0) = \psi_{II}(x=0), \psi_{II}(x=L) = \psi_{III}(x=L)$$

$$\left.\frac{d\psi_I}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0}, \left.\frac{d\psi_{II}}{dx}\right|_{x=L} = \left.\frac{d\psi_{III}}{dx}\right|_{x=L}$$

- For such finite well, the wave function is not vanished at the boundaries, and may extent into the region I, III which is not allowed in the infinite potential limit
- Such ψ that penetrates beyond the classically forbidden regions diminishes very fast (exponentially) once x extents beyond x = 0 and $x = L$
- The mathematical solution for the wave function in the "classically forbidden" regions are
-

$$\psi(x) = \begin{cases} A_+ \exp(Cx) \neq 0, & x \leq 0 \\ A_- \exp(-Cx) \neq 0, & x \geq L \end{cases}$$



The total energy of the particle

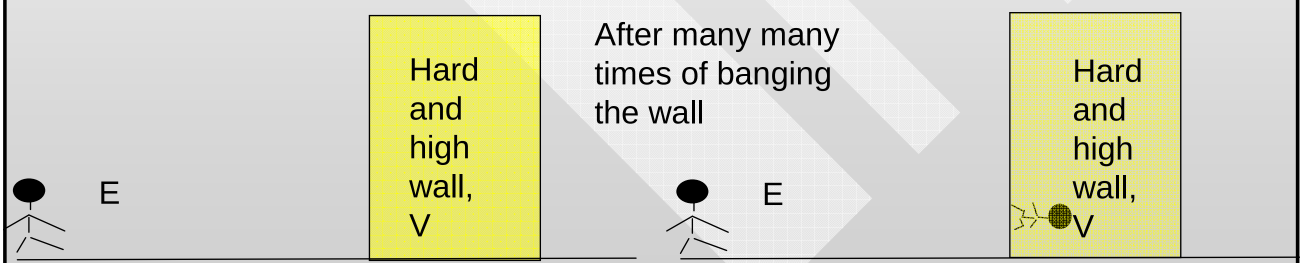
E = K inside the well.

The height of the potential well V is larger than E for a particle trapped inside the well

Hence, classically, the particle inside the well would not have enough kinetic energy to overcome the potential barrier and escape into the forbidden regions I, III

However, in QM, there is a slight chance to find the particle outside the well due to the quantum tunneling effect

- The quantum tunnelling effect allows a confined particle within a finite potential well to penetrate through the classically impenetrable potential wall



Quantum tunneling effect

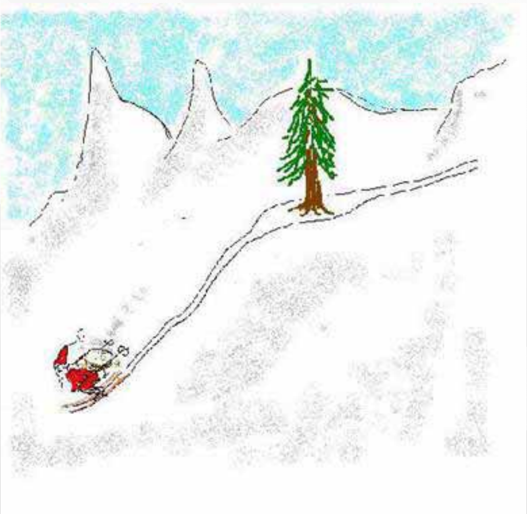
Why tunneling phenomena can happen

- It's due to the continuity requirement of the wave function at the boundaries when solving the T.I.S.E
- The wave function cannot just "die off" suddenly at the boundaries of a finite potential well
- The wave function can only diminishes in an exponential manner which then allow the wave function to extent slightly beyond the boundaries

$$\psi(x) = \begin{cases} A_+ \exp(Cx) \neq 0, & x \leq 0 \\ A_- \exp(-Cx) \neq 0, & x \geq L \end{cases}$$

- The quantum tunneling effect is a manifestation of the wave nature of particle, which is in turns governed by the T.I.S.E.
- In classical physics, particles are just particles, hence never display such tunneling effect

Quantum tunneling effect



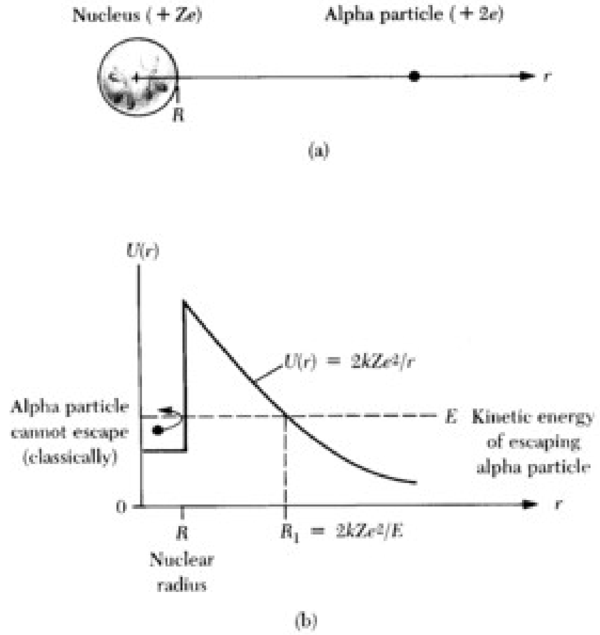


Figure 6.7 (a) Alpha decay of a radioactive nucleus. (b) The potential energy seen by an alpha particle emitted with energy E. R is the nuclear radius, about 10^{-14} m or 10 fm. Alpha particles tunneling through the potential barrier between R and R_1 escape the nucleus to be detected as radioactive decay products.

Real example of tunneling phenomena: alpha decay

Real example of tunneling phenomena: Atomic force microscope

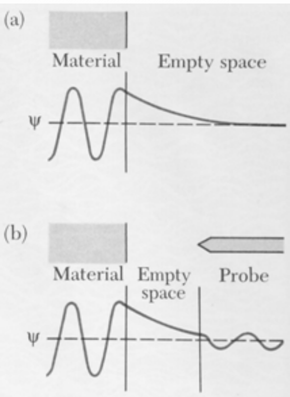


Figure 3 (a) The wavefunction of an electron in the surface of the material to be studied. The wavefunction extends beyond the surface into the empty region. (b) The sharp tip of a conducting probe is brought close to the surface. The wavefunction of a surface electron penetrates into the tip, so that the electron can "tunnel" from surface to tip.

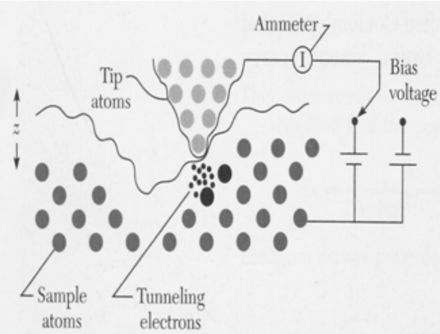


FIGURE A Highly schematic diagram of the scanning tunneling microscope process. Electrons, represented in the figure as small dots, tunnel across the gap between the atoms of the tip and sample. A feedback system that keeps the tunneling current constant causes the tip to move up and down tracing out the contours of the sample atoms.

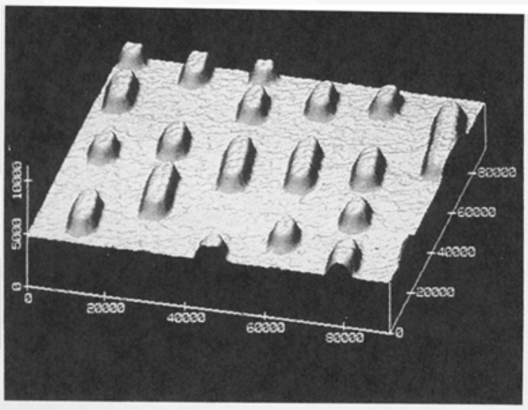


FIGURE D An atomic force microscope scan of a stamper used to mold compact disks. The numbers given are in nm. The bumps on this metallic mold stamp out 60 nm-deep holes in tracks that are 1.6 μm apart in the optical disks. *Photo courtesy of Digital Instruments.*