V
Ψ
$r =$
(x, y, z)
t
$V =$
$V(, t)$
$\Psi =$
$\Psi(, t)$
$V() =$
$V(r)$
$V()$
r
$V(r)$
$V(r) \sim$
$-\frac{1}{r^2}$
$V(r)$
$V =$
$V()$
$\Psi_n(r, t) = \psi_n() e^{-iE_n t/\hbar}$

$\psi_n()$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi.$$

$$\Psi(, t) = \sum c_n \psi_n() e^{-iE_n t/\hbar}$$

c_n
$\Psi(x, 0)$
(r, θ, ϕ)
∇

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right).$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial\psi}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2\psi}{\partial\phi^2}\right)\right] + V()\psi = E\psi.$$

(1)

$\psi(r, \theta, \phi) = R(r) Y(\theta, \phi)$

$$\frac{\partial\psi}{\partial r} = Y\frac{dR}{dr}; \quad \frac{\partial\psi}{\partial\theta} = R\frac{\partial Y}{\partial\theta}, \quad \frac{\partial^2\psi}{\partial\phi^2} = R\frac{\partial^2 Y}{\partial\phi^2}.$$

$$-\frac{\hbar^2}{2m}\left[\frac{Y}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\left(\frac{\partial^2 Y}{\partial\phi^2}\right)\right] + V()RY = ERY.$$

(2)

RY
$-2mr^2/\hbar^2$

$$\left[\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}(V() - E)\right] + \frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\left(\frac{\partial^2 Y}{\partial\phi^2}\right)\right] = -\ell(\ell+1).$$

r
θ
ϕ
$\ell(\ell+$
$1)$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}(V() - E) = \ell(\ell+1)$$

(3)

$$\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\left(\frac{\partial^2 Y}{\partial\phi^2}\right)\right] = -\ell(\ell+1).$$

ℓ
ψ
θ
ϕ
$Y\sin^2\theta$

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2} = -\ell(\ell+1)\sin^2\theta Y.$$

$Y(\theta, \phi) =$

m
$m = 0, \pm 1, \pm 2, \cdots.$

θ
$\Theta(\theta) = A P_\ell^m(\cos\theta),$

$$P_\ell^m(x) = (1-x^2)^{|m|/2} \left(\frac{d}{dx}\right)^{|m|} P_\ell(x),$$

$P_\ell(x)$
ℓ
$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell.$$

$P_\ell^m(x) =$
$P_\ell^{-m}(x)$
ℓ
$(\frac{d}{dx})^\ell$
ℓ
$P_\ell(x)$
ℓ
$(\frac{d}{dx})^{|m|} P_\ell(x)$
$|m| >$
ℓ
$P_\ell^{|m|}(x)$
$|m| \leq \ell \Rightarrow m = 0, \pm 1, \pm 2, \cdots, \pm \ell$

ℓ
ℓ
$2\ell +$
1
m
θ
$\theta =$
0π

$n_0(x)$
$n_1(x)$
$j_0(x)$
$j_1(x)$

$$R(r) = Aj_\ell(x).$$

ℓ
$\ell = 0, 1, 2, \cdots$
$R(r = a) = 0$

$$R(r = a) = Aj_\ell(ka) = 0 \Rightarrow j_\ell(ka) = 0,$$

ka
ℓ
ℓ
$j_\ell(x)$
x
$j_\ell(x)$
$j_\ell(x) = 0$
ℓ
$\beta_{n\ell}$
$n = 1, n = 3, \cdots$
$\beta_{n\ell}$
$\{n, \ell\}$
n
ℓ
$n = 0$

$$k = k_{n\ell} = \beta_{n\ell}/a \Leftrightarrow E = E_{n\ell} = \frac{\hbar^2}{2ma}\beta_{nl}.$$

$$\psi() = \psi_{nlm}(r, \theta, \phi) = A_{nl}j_\ell(\frac{r\beta_{n\ell}}{a})Y_\ell^m(\theta, \phi).$$

$\phi_{n,\ell,m}(r, \theta, \phi)$
$\{n, \ell, m\}$
$A_{n\ell}$
$\{n, \ell\}$
$\int_0^\infty A_{n\ell}^2 j_\ell(\beta_{n\ell})^2 r^2 dr = 1$

$\{n, \ell\}$
n
ℓ
$\{n, \ell\}$
$E_{n\ell}$
m
$E_{n\ell}$
$2\ell + 1$
$2\ell + 1$
m
ℓ
$E_{n=0,\ell=1}$
$\psi_{n=0,\ell=1,m=-1}, \psi_{n=0,\ell=1,m=0}, \psi_{n=0,\ell=1,m=+1}$
$\ell = 0$
$\ell = 0$

$$\psi_{n\ell m} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}} e^{-r/na} \left(\frac{2r}{na}\right)^\ell \left[L_{n-\ell-1}^{2\ell+1}(2r/na)\right] Y_\ell^m(\theta,\phi).$$

n
n, ℓ

$$\int \psi_{n'\ell' m'}^* \psi_{n\ell m} r^2 \sin\theta dr d\phi = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}.$$

n, ℓ, m
$|\psi_{n,\ell,m}|^2$

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