Test 2 (29 May 2014)
ZCT 205 Quantum Mechanics

Name:

Matrix Number:

1. The ground state of a particle bounded to a Dirac potential well

$$V(x) = -\alpha\delta(x),$$

with α a positive constant and energy $E < 0$, is

$$\psi_0(x) = \sqrt{\kappa}e^{-\kappa|x|}, \ \ \kappa = \frac{m\alpha}{\hbar^2}$$

(a) State the ground state energy, E_0 (you do not need to perform the calculation).

(b) Show that $\psi_0(x)$ is normalised. Show your steps clearly.

(c) Write down the wavefuction of a particle bounded to the Dirac potential in the ground state, $\Psi_0(x,t)$.

(d) What is the ground state wavefunction of the particle in momentum space, $\Phi_0(p,t)$? Show your steps clearly.

(e) Consider the particle above in momentum space representation. The position operator in momentum space is $\hat{x} = -i\hbar\frac{\partial}{\partial p}$. Any operator $\hat{Q}$ in momentum space is given in terms of $\hat{x} = -i\hbar\frac{\partial}{\partial p}$ and $\hat{p}$, i.e., $\hat{Q} = \hat{Q}(-i\hbar\frac{\partial}{\partial p}, \hat{p})$. The expectation value of any operator $\hat{Q}$ in momentum space is hence expressed in terms of

$$\langle\hat{Q}\rangle = \int \Phi(p,t)^* Q(-i\hbar\frac{\partial}{\partial p}, \hat{p})\Phi(p,t)dp$$

i. Calculate the expectation of the momentum for the particle in the ground state of the Dirac potential, $\langle\hat{p}\rangle$, in momentum space representation.

ii. Calculate the expectation of (momentum)^2 for the particle in the ground state of the Dirac potential, $\langle\hat{p}^2\rangle$ in momentum space representation.

iii. What is the uncertainty of the momentum, σ_p, of the particle in the ground state?

Hint:

$$\int_{-\infty}^{\infty} \frac{p^n}{(p^2+p_0^2)^2}dp = 0 \text{ if } n = 1$$

$$\int_{-\infty}^{\infty} \frac{p^n}{(p^2+p_0^2)^2}dp = \frac{\pi}{2p_0} \text{ if } n = 2$$

Solution Test 2

Q1(a) $E_0 = -\dfrac{m\alpha^2}{2\hbar^2}$ eq(2.127) from Griffiths.

Q1(b) $\displaystyle\int_{-\infty}^{\infty} \Psi_0^*(x,t)\Psi_0(x,t)\,dx = \int_{-\infty}^{\infty}\left(\sqrt{K}e^{-K|x|}\right)^2 dx$

$\displaystyle = K\int_{-\infty}^{\infty} e^{-2K|x|}\,dx = K\left\{\int_{-\infty}^{0} e^{+2Kx}\,dx + \int_{0}^{\infty} e^{-2Kx}\,dx\right\}$

$\displaystyle = K\left\{\left[\frac{e^{2Kx}}{2K}\right]_{-\infty}^{0} + \left[\frac{e^{-2Kx}}{(-2K)}\right]_{0}^{\infty}\right\}$

$\displaystyle = \frac{K}{2K}\left[(1-0) - (0-1)\right] = 1$ #

Q1(c) $\Psi_0(x,t) = \psi_0(x)e^{-iE_0t/\hbar}$; $\psi_0(x) = \sqrt{K}e^{-K|x|}$

Q1(d)

$$\Psi_0(x,t) = e^{-iEt/\hbar}\sqrt{K}\,e^{-K|x|} \,;$$

$$K = \frac{m\alpha}{\hbar^2} \;;\; p_0 = \frac{m\alpha}{\hbar} \;;\; p_0 = K\hbar$$

$$\Phi_0(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\,\Psi(x,t)\,dx$$

$$\vdots$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{p_0^{3/2}\,e^{-iEt/\hbar}}{p^2+p_0^2}$$

show these steps clearly.

(e)(i)

$$\langle p \rangle = \int_{-\infty}^{\infty} \Phi_0^*(p,t)\, p\, \Phi_0(p,t)\, dp$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{p_0^3\, p\, dp}{(p^2+p_0^2)^2}$$

$$= \frac{2p_0^3}{\pi}\int_{-\infty}^{\infty} \frac{p\, dp}{(p^2+p_0^2)^2} = 0$$

(ii)

$$\langle p^2 \rangle = \frac{2p_0^3}{\pi}\int_{-\infty}^{\infty} \frac{p^2\, dp}{(p^2+p_0^2)^2} = \frac{2p_0^3}{\pi}\cdot\frac{\pi}{2p_0} = p_0^2$$

(iii) Hence

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = p_0 = \frac{m\alpha}{\hbar}$$