Solution for Q1

Total Solution $\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar}$

Need to know: 1 $\psi_n(x)$; 2 E_n ; 3 c_n.

First, solve for $\psi_n(x)$, using TISE:

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

For $\frac{-a}{2} \le x \le \frac{a}{2}$, $V(x) = 0 \longrightarrow \frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$

or $$\frac{d^2\psi(x)}{dx^2} = -k^2\psi(x) \quad \text{— Eq(1)}$$

where $k^2 = \frac{2mE}{\hbar^2}$, real & positive.

Solution to Eq(1) is $\psi(x) = G\cos kx + D\sin kx$

Boundary condition at $x = -a/2$: $\psi(\frac{-a}{2}) = 0$

$$G\cos k(-a/2) + D\sin k(\tfrac{-a}{2}) = 0$$
$$G\cos\frac{ka}{2} - D\sin\frac{ka}{2} = 0 \quad \text{Eq(2)}$$

note: $\cos(-x) = \cos x$
$\sin(-x) = -\sin x$

Boundary condition at $x = a/2$, $\psi(\frac{a}{2}) = 0$

$$G\cos\frac{ka}{2} + D\sin\frac{ka}{2} = 0 \quad \text{— Eq(3)}.$$

Eq(3) - Eq(2): $2D\sin\frac{ka}{2} = 0$

$D \neq 0$, $\frac{ka}{2} = \pi, 2\pi, 3\pi, 4\pi \ldots$

$k = \frac{2\pi}{a}, \frac{4\pi}{a}, \frac{6\pi}{a}, \ldots$

or $k_n = n\pi/a$; $n = 2, 4, 6 \ldots$ (even)

Eq(2) + Eq(3):

$$2G\cos\frac{ka}{2} = 0$$

$G \neq 0$, $\frac{ka}{2} = \frac{\pi}{2}, 3(\frac{\pi}{2}), 5(\frac{\pi}{2}), 7(\frac{\pi}{2}), \ldots$

or $k_n = n\pi/a$; $n = 1, 3, 5, 7, \ldots$ (odd)

$\therefore$ Solution to T.I.S.E is

$$\psi_n(x) = \begin{cases} \cancel{D \sin} \\ D \sin k_n x & ; k_n = \frac{n\pi}{a} ; n = 2,4,6\ldots \text{(even)} \\ C \cos k_n x & ; k_n = \frac{n\pi}{a} ; n = 1,3,5,7,\ldots \text{(odd)}. \end{cases}$$

Normalisation : $\displaystyle\int_{-a/2}^{a/2} \psi_n^2(x)\,dx = 1$

$$D^2 \int_{-a/2}^{a/2} \sin^2 k_n x = C^2 \int_{-a/2}^{a/2} \cos^2 k_n x = 1$$

$$(\sin^2 k_n x + \cos^2 k_n x) = 1$$

$$\int_{-a/2}^{a/2} \sin^2 k_n x\,dx + \int_{-a/2}^{a/2} \cos^2 k_n a\,dx = \int_{-a/2}^{a/2} 1 . dx = a$$

$$\therefore \int_{-a/2}^{a/2} \sin^2 k_n x\,dx = \int_{-a/2}^{a/2} \cos^2 k_n x\,dx = \frac{a}{2}$$

$$\Rightarrow D^2\left(\frac{a}{2}\right) = C^2\left(\frac{a}{2}\right) = 1$$

$$\Rightarrow D = C = \sqrt{\frac{2}{a}}$$

$$\therefore \psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin \frac{k_n \pi}{2} & ; k_n = \frac{n\pi}{a} ; n = 2,4,6\ldots \\ \sqrt{\frac{2}{a}} \cos \frac{k_n \pi}{2} & ; k_n = \frac{n\pi}{a} ; n = 1,3,5,7\ldots \end{cases}$$

$$\because k^2 = \frac{2mE}{\hbar^2} \Rightarrow E_n = \frac{k_n^2 \hbar^2}{2m} ; k_n = \frac{n\pi}{a} ; n = 1,2,3,4\ldots$$

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2} ; n = 1,2,3,4\ldots$$