Solution for Q1 (cont. 4)

$$\int_{-a/2}^{a/2} \varphi_m^*(x)\, \psi(x,0)\, dx = \int_{-\frac{a}{2}}^{a/2} \varphi_m^*(x) \cdot \sum_n c_n \varphi_n(x)\, dx$$

$$= \sum_n c_n \overbrace{\int_{-a/2}^{a/2} \varphi_m^*(x)\varphi_n(x)\,dx = \sum_n c_n \delta_{m,n}}^{\text{orthonormality}} = c_m$$

$$\therefore\ c_m = \int_{-\frac{a}{2}}^{a/2} \varphi_m^*(x)\, \psi(x,0)\, dx$$

$$= \int_{-\frac{a}{2}}^{a/2} \varphi_m^*(x) \cdot A\left(x^2 - \frac{a^2}{4}\right) dx = A \int_{-\frac{a}{2}}^{a/2} \varphi_m^*(x)\left(x^2 - \frac{a^2}{4}\right) dx$$

m odd, $c_m = A \int_{-\frac{a}{2}}^{a/2} \sqrt{\frac{2}{a}} \cos \frac{k_n x}{2} \cdot \left(x^2 - \frac{a^2}{4}\right)$; $k_n = 1, 3, 5 \ldots$

m even, $c_m = A \int_{-\frac{a}{2}}^{a/2} \sqrt{\frac{2}{a}} \sin \frac{k_n x}{2} \left(x^2 - \frac{a^2}{4}\right)$; $k_n = 2, 4, 6 \ldots$

Look at m odd:

$$c_m = A \cdot \left[\sqrt{\frac{2}{a}} \left(\int_{-\frac{a}{2}}^{a/2} x^2 \cos \frac{k_n x}{a} dx - \frac{a^2}{4} \int_{-\frac{a}{2}}^{a/2} \cos \frac{k_n x}{a} dx \right) \right]$$; $\cancel{k_n = 1, 3, 5, 7, \ldots}$; $k_n = \frac{n\pi}{a}$; $n = 1, 3, 5, \ldots$

$\int_{-\frac{a}{2}}^{a/2} \cos \frac{n\pi x}{a} dx = \frac{2a}{\pi}\left(\frac{1}{n}\right)$; $n = 1, 3, 5 \ldots$ (even function)

$\int_{-\frac{a}{2}}^{a/2} \sin \frac{n\pi x}{a} dx = 0$ (odd function)

$\int_{-\frac{a}{2}}^{a/2} x^2 \cos \frac{n\pi x}{a} dx = \frac{a^3(-8 + n\pi^2)}{2n^3\pi^3}$ (even function). | $\int_{-\frac{a}{2}}^{a/2} x^2 \sin \frac{n\pi x}{a} dx = 0$ (odd function)

$$\therefore\ c_{m,\,\text{odd}} = A \cdot \sqrt{\frac{2}{a}} \cdot \left[\frac{a^3(-8+n\pi^2)}{2n^3\pi^3} - \frac{a^2}{4}\left(\frac{2a}{\pi n}\right) \right]$$; $n = 1, 3, 5, 7 \ldots$

$c_{m,\,\text{even}} = A \sqrt{\frac{2}{a}} \left[0 \right] = 0$; $A = \sqrt{\frac{30}{a^5}}$.