ZCT 205 Quantum Mechanics

Tutorial 1 (25%)

2 (i)

Prove that for a wave function that is the solution to the Schroedinger equation, the normalization of the wave function is time-independent,

$$\frac{d}{dt}\left(\int_{-\infty}^{\infty}|\Psi(x,t)|^2 dx\right) = 0$$

(7%)

Solution

$$\frac{d}{dt}\left(\int_{-\infty}^{\infty}|\Psi(x,t)|^2 dx\right) = \int_{-\infty}^{\infty}\left[\frac{\partial}{\partial t}(\Psi^*\Psi)\right]dx$$

$$= \int_{-\infty}^{\infty}\left[\Psi^*\left(\frac{\partial\Psi}{\partial t}\right) + \left(\frac{\partial\Psi^*}{\partial t}\right)\Psi\right]dx \qquad (1) \quad ✓$$

From Schroedinger equation : $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$ (2)

$$-i\hbar\frac{\partial\Psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + V\Psi^* \qquad (3) \quad ✓$$

Substitute (2) and (3) into (1)

$$\frac{d}{dt}\left(\int_{-\infty}^{\infty}|\Psi(x,t)|^2 dx\right) = \int_{-\infty}^{\infty}\left\{\frac{i\hbar}{2m}\left[\Psi^*\left(\frac{\partial^2\Psi}{\partial x^2}\right) - \Psi\left(\frac{\partial^2\Psi^*}{\partial x^2}\right)\right]\right\}dx \quad ✓$$

$$= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left[\Psi^*\left(\frac{\partial\Psi}{\partial x}\right) - \Psi\left(\frac{\partial\Psi^*}{\partial x}\right)\right]dx \quad ✓$$

$$= \frac{i\hbar}{2m}\left[\Psi^*\left(\frac{\partial\Psi}{\partial x}\right) - \Psi\left(\frac{\partial\Psi^*}{\partial x}\right)\right]_{-\infty}^{\infty} \qquad (4) \quad ✓$$

As $x \to \pm\infty$, $\Psi \to 0$, (4) becomes : ✓

$$\frac{d}{dt}\left(\int_{-\infty}^{\infty}|\Psi(x,t)|^2 dx\right) = 0 \quad \text{(Shown)} \quad ✓$$

2 (ii)

Prove this : $\langle p \rangle = \int \Psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) \Psi \, dx$ (7%)

<u>*Solution*</u>

$$\langle p \rangle = m \frac{d}{dt} \langle x \rangle \quad (1) \quad ✓$$

$$\frac{d}{dt}\langle x \rangle = \int_{-\infty}^{\infty} x \frac{\partial}{\partial t} |\Psi(x,t)|^2 dx$$

$$= \int_{-\infty}^{\infty} x \left[\Psi^* \left(\frac{\partial \Psi}{\partial t}\right) + \left(\frac{\partial \Psi^*}{\partial t}\right) \Psi\right] dx \quad (2) \quad ✓$$

From *<u>2(i)</u>* :

$$\frac{d}{dt}\langle x \rangle = \frac{\hbar}{2mi} \int_{-\infty}^{\infty} \left\{ x \frac{\partial}{\partial x} \left[\Psi \left(\frac{\partial \Psi^*}{\partial x}\right) - \Psi^* \left(\frac{\partial \Psi}{\partial x}\right)\right]\right\} dx \quad (3)$$

By using integration by parts :

$$\frac{d}{dt}\langle x \rangle = \frac{\hbar}{2mi} \left\{ x \left[\Psi \left(\frac{\partial \Psi^*}{\partial x}\right) - \Psi^* \left(\frac{\partial \Psi}{\partial x}\right)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left[\Psi \left(\frac{\partial \Psi^*}{\partial x}\right) - \Psi^* \left(\frac{\partial \Psi}{\partial x}\right)\right] dx \right\} \quad (4) \quad ✓$$

As $x \to \pm\infty$, $\Psi \to 0$, (4) becomes :

$$\frac{d}{dt}\langle x \rangle = -\frac{\hbar}{2mi} \int_{-\infty}^{\infty} \left[\Psi \left(\frac{\partial \Psi^*}{\partial x}\right) - \Psi^* \left(\frac{\partial \Psi}{\partial x}\right)\right] dx \quad (5) \quad ✓$$

$$= \frac{\hbar}{2mi} \left\{ \int_{-\infty}^{\infty} \left[\Psi^* \left(\frac{\partial \Psi}{\partial x}\right)\right] dx - \int_{-\infty}^{\infty} \left[\Psi \left(\frac{\partial \Psi^*}{\partial x}\right)\right] dx \right\}$$

By using integration by parts on second term :

$$\frac{d}{dt}\langle x \rangle = \frac{\hbar}{2mi} \left\{ 2 \int_{-\infty}^{\infty} \left[\Psi^* \left(\frac{\partial \Psi}{\partial x}\right)\right] dx - [\Psi \Psi^*]_{-\infty}^{\infty} \right\} \quad (6) \quad ✓$$

As $x \to \pm\infty$, $\Psi \to 0$, (6) becomes :

$$\frac{d}{dt}\langle x \rangle = \frac{\hbar}{mi} \int_{-\infty}^{\infty} \left[\Psi^* \left(\frac{\partial \Psi}{\partial x}\right)\right] dx \quad (7) \quad ✓$$

Substitute into (1)

$$\langle p \rangle = \int_{-\infty}^{\infty} \left[\Psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) \Psi\right] dx \qquad \text{(Shown)} \quad (8) \quad ✓$$

Problem 1.17

A particle is represented (at time $t = 0$) by the wave function

$$\Psi(x,0) = \begin{cases} A(a^2 - x^2), & \text{if } -a \le x \le +a \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the normalization constant A.

(b) What is the expectation value of x (at time $t = 0$)?

(c) What is the expectation value of p (at time $t = 0$)? (Note that you cannot get it from $p = m\, d\langle x\rangle/dt$. Why not?)

(d) Find the expectation value of x^2.

(e) Find the expectation value of p^2.

(f) Find the uncertainty in x (σ_x).

(6%)

Solution

(a) Normalization condition : $\int_{-\infty}^{\infty} |\Psi(x,t)|^2\, dx = 1$

$$|A|^2 \int_{-a}^{a} (a^2 - x^2)^2 dx = 1$$

$$A = \sqrt{\frac{15}{16a^5}}$$ ✓

(b) $\langle x\rangle = \int_{-a}^{a} x|\Psi|^2 dx = |A|^2 \int_{-a}^{a} x(a^2 - x^2)^2 dx = 0$ ✓

(c) $\langle p\rangle = \int_{-a}^{a} \left[\Psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) \Psi\right] dx = \frac{\hbar}{i} |A|^2 \int_{-a}^{a} \left\{(a^2 - x^2)\left[\frac{d}{dx}(a^2 - x^2)\right]\right\} dx = 0$ ✓

Since we only know $\langle x\rangle$ at $t = 0$, we cannot calculate $d\langle x\rangle/dt$.

(d) $\langle x^2\rangle = \int_{-a}^{a} x^2|\Psi|^2 dx = A^2 \int_{-a}^{a} x^2(a^2 - x^2)^2 dx = \frac{a^2}{7}$ ✓

(e) $\langle p\rangle = \int_{-a}^{a} \left[\Psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \Psi\right] dx = -A^2\hbar^2 \int_{-a}^{a} \left\{(a^2 - x^2)\left[\frac{d^2}{dx^2}(a^2 - x^2)\right]\right\} dx = \frac{5}{2}\frac{\hbar^2}{a^2}$ ✓

(f) $\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{a}{\sqrt{7}}$ ✓

Problem 1.16

Show that

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1{}^*\Psi_2\,dx = 0$$

for any two (normalizable) solutions to the Schroedinger equation, Ψ_1 and Ψ_2.

(5%)

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1{}^*\Psi_2\,dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}(\Psi_1{}^*\Psi_2)\,dx$$

$$= \int_{-\infty}^{\infty}\left[\frac{\partial \Psi_1{}^*}{\partial t}\Psi_2 + \frac{\partial \Psi_2}{\partial t}\Psi_1{}^*\right]dx \qquad (1) \quad ✓$$

From Schroedinger equation : $-i\hbar\frac{\partial \Psi_1{}^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi_1{}^*}{\partial x^2} + V\Psi_1{}^*$ (2)

$$i\hbar\frac{\partial \Psi_2}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi_2}{\partial x^2} + V\Psi_2 \qquad (3)$$

Substitute into (1)

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1{}^*\Psi_2\,dx = \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left[\Psi_1{}^*\left(\frac{\partial^2 \Psi_2}{\partial x^2}\right) - \Psi_2\left(\frac{\partial^2 \Psi_1{}^*}{\partial x^2}\right)\right]dx \quad ✓$$

$$= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left[\Psi_1{}^*\left(\frac{\partial \Psi_2}{\partial x}\right) - \Psi_2\left(\frac{\partial \Psi_1{}^*}{\partial x}\right)\right]dx \quad ✓$$

$$= \frac{i\hbar}{2m}\left[\Psi_1{}^*\left(\frac{\partial \Psi_2}{\partial x}\right) - \Psi_2\left(\frac{\partial \Psi_1{}^*}{\partial x}\right)\right]_{-\infty}^{\infty} \qquad (4)$$

As $x \to \pm\infty$, $\Psi \to 0$, (4) becomes : ✓

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1{}^*\Psi_2\,dx = 0$$ (Shown) ✓