Instruction [Arahan]:
Answer any three (3) questions only from 5 questions given. Each question carries equal weight. Total 3×100 marks = 300 marks.
[*Jawab mana-mana TIGA (3) soalan sahaja daripada 5 soalan. Setiap soalan membawa keberatan yang sama. Jumlah markah =* 3×100 *= 300*]

1. Consider specifically a spin-1/2 paramagnetic system as a microcanonical ensemble. The system is subjected to an external magnetic field *H*. The total number of spins in the system is *N*.
[Pertimbangkan secara spesifik sistem paramagnet berspin ½ yang merupakan suatu ensembel berkanun mikro. Sistem ini tertakluk kepada suatu medan magnet luar H. Jumlah bilangan spin dalam sustem tersebut ialah N.]

(a) Define *microcanonical ensemble*.
[*Takrifkan ensembel berkanun mikro*]

(b) Let $\Gamma(E)$ be the number of mircostates with energy *E*. What is the probability to find the system in any particular microstate $\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$?
[*Biar $\Gamma(E)$ bilangan keadaan mikro yang tenaganya E. Apakah kebarangkalian sistem ini berada dalam sesuatu keadaan yang tertentu, $\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$?*]

(c) Let $q = N_+ - N_-$, where N_+ (N_-) is the number of spin with projection +1 (-1). What is the energy of the paramagnetic system in terms of *q* and *H* ? Note that the total number of spin in the system is $N = N_+ + N_-$.
[*Biar $q = N_+ - N_-$, di mana N_+ (N_-) menandakan bilangan spin yang projeksinya +1 (-1). Apakah tenaga sistem paramagnet tersebut dalam sebutan q dan H? Dimaklumkan bahawa jumlah bilangan spin dalam sistem ialah $N = N_+ + N_-$.*]

(d) State $\Gamma(E)$ in terms of N, N_+ and N_-.
[*Nyatakan $\Gamma(E)$ dalam sebutan N, N_+ dan N_-.*]

(e) Based on your answers in (c and d), express $\Gamma(E)$ in terms of *N*, *E* and *H*.
[*Berdasarkan jawapan anda dalam (c dan d), nyatakan $\Gamma(E)$ dalam sebutan N, E dan H.*]

(f) State the relationship between entropy *S* and $\Gamma(E)$.
[*Nyatakan hubungan di antara entropi S dan $\Gamma(E)$.*]

(g) Based on your answer to (f), express the entropy *S* in terms of *N*, *E*, *H* with the aid of the Stirling approximation $n! \simeq n^n e^{-n} \sqrt{2\pi n}$.
[*Berdasarkan jawapan anda dalam (f), nyatakan entropi S dalam sebutan N, E, H dengan bantuan penghampiran Stirling* $n! \simeq n^n e^{-n} \sqrt{2\pi n}$.]

(100/100)

...3/-

2. Consider a spin-1/2 paramagnetic system at temperature *T* subjected to an external

magnetic field H. The total number of spins in the system is N (a constant). Also, $\beta = 1/kT$.
[*Pertimbangkan sistem paramagnet berspin ½ pada suhu T, dan tertakluk kepada medan magnet luar H. Jumlah bilangan spin dalam sistem ini ialah N (suatu pemalar). Diberi juga* $\beta = 1/kT$.]

(a) What is the energy, ε, of any single spin in the system in terms of the magnetic field H and the spin projection, σ?
[*Apakah tenaga* ε *bagi mana-mana suatu spin tunggal dalam sistem, dalam sebutan medan magnet H dan projeksi spin,* σ?]

(b) Based on your answer in (a), what is the partition function for a single spin, $z(\beta, H)$?
[*Berdasarkan jawapan anda di (a), apakah fungsi partisi untuk spin tunggal* $z(\beta, H)$?]

(c) What is the partition function for the paramagnetic system, $Z(\beta, H)$, in terms of $z(\beta, H)$?
[*Apakah fungsi partisi untuk sistem paramagnet* $Z(\beta, H)$, *dalam sebutan* $z(\beta, H)$?]

(d) Write down the expectation value (i.e., average) of the energy of the system, $\langle E \rangle$, in terms of the partition function $Z(\beta, H)$.
[*Tuliskan nilan jangkaan (iaitu purata) tenaga sistem tersebut* $\langle E \rangle$, *dalam sebutan fungsi partisi* $Z(\beta, H)$.]

(e) What is $\langle E \rangle$ in terms of N, β and H?
[*Apakah* $\langle E \rangle$ *dalam sebutan N,* β *dan H?*]

(f) Is $\langle E \rangle$ an extensive or intensive quantity? Explain your answer.
[*Adakah* $\langle E \rangle$ *suatu kuantiti ekstensif atau intensif? Terangkan jawapan anda.*]

(g) Sketch $\langle E \rangle$ versus $x = \beta H \mu_B$. Label your graph appropriately.
[*Lakarkan* $\langle E \rangle$ *lawan* $x = \beta H \mu_B$. *Labelkan graf anda dengan sepadannya.*]

(100/100)

3. Consider an Einstein solid which is modeled as a collection of N atoms pegged at their respective equilibrium lattice sites in a three dimensional space. The system is in thermal equilibrium at inverse temperature $\beta = 1/kT$.
[*Pertimbangkan pepejal Einstein yang dimodel sebagai suatu koleksi N atom yang terikat pada tapak kekisi dalam kesimbangan tiap-tiap satunya dalam ruang tiga dimensi. Sistem tersebut berada pada keseimbangan terma pada suhu songsang* $\beta = 1/kT$.]

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(a) (i) What is the energy, ε, of a 1-D quantum harmonic oscillator with constant frequency ω?
[*Akapah tenaga* ε *satu pengayun harmonik kuantum 1-D dengan frekuensi*

malar, ω?]

(ii) How many quantum harmonic oscillator is required to model the Einstein solid?
[*Berapakah bilangan pengayun harmonik kuantum yang diperlukan untuk memodelkan pepejal Einstein tersebut?*]

(iii) What is the partition function of a single oscillator, z? (Simplify the summation in z)
[*Apakah fungsi partisi suatu pengayun tunggal z? (Permudahkan penjumlahan dalam z)*]

Hint: You may use the following result for the sum of an infinite geometric series:
[*Anda boleh menggunakan keputusan berikut daripada penjumlahan siri geometri infinit:*]

$$\sum_{n=0}^{n=\infty} x^n = \frac{1}{1-x},\ x<1$$

(iv) What is the partition function Z for the Einstein solid in terms of z?
[*Apakah fungsi partisi Z bagi pepejal Einstein dalam sebutan z?*]

(v) Derive the molar specific heat of the Einstein solid (as a function of inverse temperature).
[*Terbitkan kapasiti haba spesifik molar pepejal Einstein (sebagai fungsi suhu songsang).*]

(60/100)

(b) List the essential differences between Einstein and Debye models for the heat capacities of solid.
[*Senaraikan perbezaan di antara model Einstein dan model Debye bagi kapasiti haba pepejal.*]

(40/100)

4. (a) Write a concise, qualitative essay on blackbody radiation in the context of statistical physics.
[*Karangkan secara kualitatif, ringkas dan padat berkenaan dengan pancaran jasad hitam dalam konteks fizik statistik*]

(40/100)

(b) Statistical physics analysis as applied to a blackbody cavity at thermal equilibrium leads to the expression for the energy density (i.e. energy per unit volume and unit frequency) stored in the electromagnetic radiation:

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[*Analisis dengan menggunakan fizik statistik ke atas lohong jasad hitam pada keseimbangan terma membawa kepada ekspresi ketumpatan tenaga (iaitu tenaga per unit isipadu per unit frekuensi) terkandung dalam pancaran elektromagnet:*]

$$\rho(\omega)=\frac{1}{\pi^2 c^3}\cdot\frac{\hbar\omega^3}{e^{\beta\hbar\omega}-1}$$

ω is the angular frequency of the electromagnetic radiation, β inverse temperature, c speed of light.
[*ω frekuensi sudut pancaran elektromagnet, β suhu songsang dan c laju cahaya*.]

(i) State the definition for photon number density. Obtain the explicit expression for it from $\rho(\omega)$.
[*Nyatakan definasi bagi ketumpatan bilangan foton. Peroleh ungkapan eksplisitnya daripada $\rho(\omega)$.*]

(20/100)

(ii) State the relation between emissivity $I(\omega)$ and energy density.
[*Nyatakan hubungan di antara emisiviti $I(\omega)$ dan ketumpatan tenaga.*]

(10/100)

(iii) State the Stefan-Boltzmann law.
[*Nyatakan hukum Stefan-Boltzmann.*]

(10/100)

(iv) Derive the expression for Stefan-Boltzmann constant, σ.
[*Terbitkan ekspresi bagi pemalar Stefan-Boltzmann, σ.*]

(20/100)

Hint: $\int \frac{x^3}{e^x-1}dx=\frac{\pi^4}{15}$

5. Consider a grand canonical ensemble, where each particle in the ensemble is identical and with a common chemical potential, μ.
[*Pertimbangkan suatu ensembel berkanun raya, di mana setiap zarahnya adalah seiras dengan keupayaan kimia sama μ.*]

(a) (i) Define occupation number in the context of a grand canonical ensemble.
[*Takrifkan nombor penghunian dalam konteks ensembel berkanun raya.*]

(ii) List down the fundamental differences between a fermion and a boson.
[*Senaraikan perbezaan asas di antara fermion dan boson.*]

(iii) Discuss concisely in what way a grand canonical ensemble is different from a canonical ensemble.
[*Bincangkan secara padat dan ringkas dari segi apakah ensembel berkanun raya berbeza daripada ensembel berkanun.*]

(40/100)

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(b) (i) Derive the single particle state (SPS) grand partition function for bosons, $z_k^{(B)}$?
State clearly the specific mathematical assumption you have made in deriving $z_k^{(B)}$ (but which does not apply for the fermion case).

[*Terbitkan fungsi partisi ray*   *an zarah tunggal bagi boson-boson,* $z_k^{(B)}$ *?* *Nyatakan dengan jelas angg*   *atematik yang anda buat dalam menerbitkan* $z_k^{(B)}$ *(yang tidak boleh digu*   *atas kes fermion).*]

(20/100)

(ii) What is the grand thermodynamic potential for bosons, $\Omega^{(B)}$?
[*Apakah keupayaan termodinamik raya bagi boson-boson,* $\Omega^{(B)}$ *?*]

(20/100)

(iii) From (ii), derive the Bose-Einstein distribution.
[*Daripada (ii), terbitkan taburan Bose-Einstein.*]

(20/100)

Hint: You may use the following result for the sum of an infinite geometric series
[*Anda boleh menggunakan keputusan berikut daripada penjumlahan siri geometri infinit.*]

$$\sum_{n=0}^{n=\infty} x^n = \frac{1}{1-x}, \; x < 1$$

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ZCT 207 Statistical Mechanics
Final Exam Solution

Solution:

1(a) Page 148, Amit.

Microcanonical ensemble: The collection of states (an ensemble) of a system in which the microscopic states appear at the same occurrence rate, $p = 1/G(E)$, where the energy of the system E is fixed.

(Or, alternatively: Microcanonical ensemble is the collection of all accessible microstates with constant energy, and the probability to find the system in any particular microstate is equally probable.)

10 marks

(b) $P\{\sigma_1, \sigma_2, \ldots, \sigma_N\} = 1/\Gamma(E)$

10 marks

(c) Energy of the paramagnetic system is:

$$E = -\mu_B Hq,$$

where μ_B *the Bohr magneton*, $q = N_+ + N\_$.

10 marks

(d) $\Gamma = N!/(N_+!\ N_-!)$

10 marks

(e) Eq. (2.3.5), page 148, Amit.

$$q = N_+ - N_- = -E/(\mu_B H); \quad N = N_+ + N_-;$$

$$\Rightarrow \quad N_{+¿} = \frac{1}{2}(N+q) = \frac{1}{2}\left(N - \frac{E}{\mu_B}H\right)$$

¿

$$N_{-¿} = \frac{1}{2}(N-q) = \frac{1}{2}\left(N + \frac{E}{\mu_B}H\right)$$

¿

Expressing N_+, $N\_$ in terms of N, E, H: 15 marks

$$\Gamma(E) = \frac{N!}{N_+!\,N_-!} = \frac{N!}{\left(\frac{N}{2} - \frac{E}{2\mu_B H}\right)!\left(\frac{N}{2} + \frac{E}{2\mu_B H}\right)!}$$

Arriving at right, final expression: 5 marks

(f) $S/k = \ln \Gamma$

10 marks

(g) $S = k \ln \Gamma$, Eq. (2.3.13), page 153, Amit.

Stirling formula:
$n! = n^n e^{-n} (2\pi n)^{1/2} \Rightarrow \lim_{n\to\infty} \ln n! = \lim_{n\to\infty} n \ln n - n + ½ \ln(2\pi n) = n \ln n - n$

Displaying the logarithmic version of Stirling formula: 10 marks

$$S/k = \ln\left[\frac{N!}{\left(\frac{N}{2}-\frac{E}{2\mu_B H}\right)!\left(\frac{N}{2}+\frac{E}{2\mu_B H}\right)!}\right] = \ln(N!) - \ln\left[\left(\frac{N}{2}-\frac{E}{2\mu_B H}\right)!\right] - \ln\left[\left(\frac{N}{2}+\frac{E}{2\mu_B H}\right)!\right]$$

Displaying mathematical steps: 5 + 10 = 15 marks

$$\frac{1}{k}S \equiv \ln\Gamma \simeq -\left(\frac{N}{2}-\frac{E}{2\mu_B H}\right)\ln\left(\frac{1}{2}-\frac{E/N}{2\mu_B H}\right) - \left(\frac{N}{2}+\frac{E}{2\mu_B H}\right)\ln\left(\frac{1}{2}+\frac{E/N}{2\mu_B H}\right)$$

Arriving at the right final expression: 5 marks

Solution:

2(a) Page 141, Amit, Eq. (2.2.8)

The energy of any single spin, of which projection is σ, in the magnetic field is given by $\varepsilon(\sigma) = -\mu_B H\sigma$.

The negative sign must be included, else no mark be given. 10 marks

(b) Page 169, Amit, Eq, (2.5.11)

$$z(\beta, H) = \Sigma_{|\sigma|} \exp[-\beta\epsilon(\sigma)] = \exp[-\beta\epsilon(\sigma=-1)] + \exp[-\beta\epsilon(\sigma=+1)] = \exp(\beta\mu_B H) + \exp(-\beta\mu_B H)$$
$$= 2\cosh(\beta\mu_B H)$$

Must show (i) the summation of all spin symbol, $\Sigma_{|\sigma|}$, (ii) the procedure of summing up all spin projections. These gives 10 + 10 = 20 marks.
5 marks be given for final correct answer, $2\cosh(\beta\mu_B H)$. (total 25 marks)

(c) Page 169, Amit, Eq, (2.5.12)
$Z = z^N$ 10 marks

(d) Page 167, Amit; Eq. (2.5.7)

$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta}$$ 10 marks

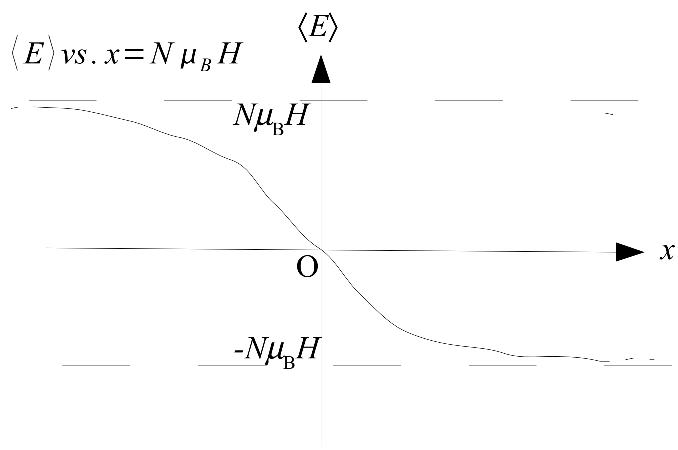
(e) Page 169, Amit; Eq. (2.5.13)

$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta} = -N\frac{\partial \ln z}{\partial \beta} = -N\frac{\partial}{\partial \beta}\ln[2\cosh(\beta \mu_B H)] = -N\mu_B H \tanh(\mu \beta H)$$

Must show procedure of calculation. 15 marks.
Final and correct answer: 5 marks

(f) $\langle E \rangle$ is an extensive quantity. 5 marks
Because it scales with the size of the system, i.e., as N increases, so is $\langle E \rangle$. 10 marks

(g) The graph of $\langle E \rangle$ $vs.$ $x = N\mu_B H$



Right trend of the graph 7 marks
Labeling of the asymptotic values $\pm N\mu_B H$ 3 marks

Solution:
3(a)(i) page 244, Amit., eq. (3.2.1)

$$\epsilon_n = \left(n + \frac{1}{2}\right)\hbar\omega, \quad n = 0, 1, 2, 3 \ldots$$

ε_n correctly stated, 5 marks. n correctly stated, 5 marks 5 + 5 = 10 marks

(a)(ii) $3N$ (N is not acceptable as answer)

5 marks

(a)(iii) page 246, Amit., eq. (3.2.7)

$$z = e^{-\beta\hbar\omega/2}\sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}.$$

Full marks be given only if: (i) Both definitions of the partition function and the correct expression of z are given, (ii) the convergence of the infinite series $\Sigma_{n=0}^{n=\infty} e^{-n\beta\hbar\omega} = \frac{1}{1-e^{-\beta\hbar\omega}}$ is used.

5 + 5 marks = 10 marks

(a)(iv) $Z = z^{3N}$

5 marks

(a)(v) page 250, Amit., eq. (3.2.16)

$$\ln Z = 3\text{N} \ln z = 3\text{N}\left[-\beta\hbar\frac{\omega}{2} - \ln\left(1-e^{-\beta\hbar\omega}\right)\right]$$

$$\langle E\rangle = -\frac{\partial \ln Z}{\partial\beta} = 3\text{N}\frac{\partial}{\partial\beta}\left[\beta\hbar\frac{\omega}{2} + \ln\left(1-e^{-\beta\hbar\omega}\right)\right] = 3\text{N}\hbar\omega\left[\frac{1}{2}+\frac{1}{e^{\beta\hbar\omega}-1}\right]$$

$$C = \frac{\partial\langle E\rangle}{\partial\beta} = \frac{\partial\beta}{\partial T}\frac{\partial\langle E\rangle}{\partial T} = -\frac{1}{kT^2}\cdot 3\text{N}\hbar\omega\cdot\frac{\partial}{\partial T}\left(e^{\beta\hbar\omega}-1\right)^{-2} = \frac{3\text{N}(\hbar\omega)^2}{kT^2}\frac{e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega}-1\right)^2} = \frac{3\text{R}(\hbar\omega)^2}{(kT)^2}\frac{e^{\frac{\hbar\omega}{kT}}}{\left(e^{\frac{\hbar\omega}{kT}}-1\right)^2}$$

Expression of ln Z: 5 marks;
Expression of average E: 5 marks.
Definition of C: 5 marks.
Working: 10
Correct final expression: 5 marks.
Total: 30 marks.

(b) The major difference between Einstein model and Debye model is that the former assume the oscillators are non-interacting whereas the latter assume the oscillators are interacting.

In Debye model, the basic material-dependent property parameter is the speed of sound wave, v.
In Einstein model, the basic material-dependent property parameter is the stiffness of the material, ω.

In Debye model the oscillations mode of the harmonic oscillators are associated to the sound wave, whereas in Einstein model the oscillations are not associated to the sound wave.

In Debye model a cut-off frequency (the Debye frequency) has to be imposed in the summation of the oscillation modes; in Einstein model the oscillation frequency is a constant, hence there is no cut-off frequency.

At least four essentials differences are required to deserve a full 40 marks. Points that are not essential difference between the two model will not be allocated marks.

Solution:

4(a) Refer notes. Marks allocated according to whether candidate can display understanding of what blackbody radiation is by stating sensible qualitative statements about the essential properties, features of it.

40 marks

(b)(i) page 390, Amit, Eq. (4.4.13)

photon number density = number of photon per unit volume per unit frequency.

Inexact statement only gets 3 marks. 5 marks

$$n(\omega)=\frac{\rho(\omega)}{\hbar\omega}=\frac{\omega^2}{c^3\pi^2}(e^{\beta\hbar\omega}-1)^{-1}$$

15 marks

(b)(ii) page 387, Amit, Eq. (4.4.2)

$$I(\omega)\omega=\frac{c}{4}\rho(\omega)d\omega$$

10 marks

(b)(iii) Stafan-Boltzmann law: The total power emitter per unit area of a black body is proportional to the fourth power of the temperature.

10 marks

(b)(iv) Page 392, Amit, Eq. (4.4.16) .

Let $x=\beta\hbar\omega$

$$I(T)=\int_{\omega=0}^{\omega=\infty} I(\omega)d\omega=\frac{1}{4\pi^2c^2}\int_0^\infty \frac{\hbar\omega^3}{e^{\beta\hbar\omega}-1}d\omega=\frac{1}{(2\pi\hbar c)^2}\int_0^\infty \frac{\left(\frac{x}{\beta}\right)^3\cdot\frac{dx}{\hbar\beta}}{e^x-1}=\frac{1}{(2\pi\hbar c)^2}\cdot\frac{1}{\hbar\beta^4}\int_0^\infty\frac{x^4dx}{e^x-1}$$

$$=\frac{1}{(hc)^2}\cdot\frac{2\pi(kT)^4}{h}\cdot\frac{\pi^4}{15}=\sigma T^4$$

$\sigma=\frac{2\pi^5k^4}{15h^3c^2}$ Numerical value of σ not required.

Definition of $I(T)=\int_{\omega=0}^{\omega=\infty} I(\omega)d\omega$ worth 5 marks.
Working of the integration: 10 marks.
Correct expression for σ: 5 marks.
Total: 20 marks

Solution:
5(a)(i) n_k, the occupation number (or occupancy number) of a grand canonical ensemble is the number of states that can be occupied in a given single-particle-state *k*.

10 marks

(a)(ii)
Fermion: spin restricted to only *n*/2, where *n* = 1,3,5,...
Boson: spin restricted to only *n*/2, where *n* = 2,4, 6,...

Fermion: wavefunction is antisymmetric
Boson: wavefunction is symmetric.

Fermion: Occupation number in a given single-particle state is limited only to $n_k=\{0,1\}$. (Or equivalently, fermions obey Pauli's exclusion principle).
Boson: Occupation number in a given single-particle state is not restricted as in the case of fermion. n_k for boson can take {0,1, 2,3,...}. (Or equivalently, bosons do not obey Pauli's exclusion principle)

3 points x 5 marks = 15 marks

5(a)(iii)

GCE: A generalisation of GE.
GCE: chemical potential is introduced; CE: no such potential.
GCE: The occupation number n_k for a fixed *k* is to become a variable. No such variable in GE.
GCE: Total number of particle is not conserved; GE: the total particle number is conserved.

Any 3 points x 5 marks = 15 marks

5b(i) Amit, page 463, Eq. (5.2.9a)

$$z_k^{(B)}=\Sigma_{n_k=0}^{n_k=\infty} e^{\beta n_k(\mu-\epsilon_k)}$$

Definition: 5 marks

For bosons $n_k \in \{0,1,2,...\}$.

5 marks

To obtain $z_k^{(B)}$, we substitute $x=e^{\beta(\mu-\epsilon)}$, so that $z_k^{(B)}=\Sigma_{n_k=0}^{n_k=\infty} x^{n_k}$.

The sum $\Sigma_{n_k=0}^{n_k=\infty} x^{n_k}$ converges to $\frac{1}{1-x}$ if $x=e^{\beta(\mu-\epsilon)}<1$ (or or equivalently $\mu<\epsilon_k$)

Assumption: 5 marks

The SPS grand partition function is then $z_k^{(B)}=(1-e^{\beta(\mu-\epsilon)})^{-1}$

correct expression: 5 marks
(total 20 marks)

5b(ii) Amit, page 463, Eq. (5.2.10b)

$$\Omega^{(B)}(T,V,\mu) = -kT\sum_{k=1}^{k=\infty}\ln z_k^{(B)} = kT\sum_{k=1}^{k=\infty}\ln\left(1-e^{\beta(\mu-\epsilon_k)}\right)$$

20 marks

5b(iii) Amit, page 464, Eq. (5.2.12b)

$$\langle N^B\rangle = -\left(\frac{\partial\Omega^{(B)}}{\partial\mu}\right)_{V,T} = \sum_k \frac{1}{e^{\beta(\epsilon_k-\mu)}-1} = \sum_k\langle n_k^{(B)}\rangle; \qquad \mu<\epsilon_k\,\forall k$$

BE distribution:

$$\langle n_k^{(B)}\rangle = \frac{1}{e^{\beta(\epsilon_k-\mu)}-1}$$

Definition of $\langle N^B\rangle$: 5 marks
Taking derivative wrp to μ: 5 marks
Identification of $\langle n^B\rangle$: 5 marks
Correct final expression of $\langle n^B\rangle$: 5 marks
(total: 20 marks)